Last name: ***name:***

HW9

Show all your work in the blank space you are given on the exam sheet. Always justify your answer. Answers with **no justification will not be graded**.

Question 1: Consider the modified KdV equation:

$$\partial_t u - (6u + 6u^2)\partial_x u + \partial_{xxx} u = 0. \tag{1}$$

(a) Assume that the equation has a wave solution of the form $u(x,t) = v(x-ct)$. What is the wave speed of this solution?

The wave speed of the wave described by the function $v(x - ct)$ is c.

(b) Assume that $u(-\infty, t) = 0$, $\partial_x u(-\infty, t) = 0$, and $\partial_{xx} u(-\infty, t) = 0$ for all $t > 0$. Assume that the equation has a wave solution of the form $u(x,t) = v(x - t)$. Show that there exists a function g so that v solves a nonlinear ODE of the form $(\partial_\xi v)^2 = g(v)$.

We insert the ansatz $u(x,t) = v(x - t)$ in the PDE. Using the Liebniz product rule we obtain

$$-\partial_\xi v - \partial_\xi(3v^2 + 2v^3) + \partial_{\xi\xi\xi} v = 0.$$

Integrating once, we obtain

$$-v - 3v^2 - 2v^3 + \partial_{\xi\xi} v = A.$$

The integration constant A is equal to zero because $v(-\infty) = 0$ and $\partial_{\xi\xi} v(-\infty) = 0$. We now multiply the result by $\partial_\xi v$ and apply the Liebniz product rule

$$\partial_\xi(-\frac{1}{2}v^2 - v^3 - \frac{1}{2}v^4) + \partial_\xi(\partial_\xi v)^2 = 0.$$

Integrating once we obtain

$$-\frac{1}{2}(v^2 + 2v^3 + v^4) + \frac{1}{2}(\partial_\xi v)^2 = B.$$

The integration constant B is equal to zero because $v(-\infty) = 0$ and $\partial_\xi v(-\infty) = 0$. Hence,

$$(\partial_\xi v)^2 = v^2(1 + v)^2.$$

We have obtained the desired result with $g(v) = v^2(1 + v)^2$.

(c) What are the two roots of g, say $v_1 < v_2$?

We clearly have $g(v) = (v - v_1)^2(v_2 - v)^2$ with $v_1 = -1$ and $v_2 = 0$.

(d) What is the Taylor expansion of g at 0. Use the notation $T_0(g)(v)$?

By definition, the Taylor expansion of g at 0

$$T_0(g)(v) = \sum_{n\in\mathbb{N}} \frac{1}{n!}\partial_\xi^n(g)(0)v^n.$$

But g is a polynomial, so $T_0(g)(v) = g(v)$ for all $v \in \mathbb{R}$. Hence,

$$\boxed{T_0(g)(v) = v^2 + 2v^3 + v^4.}$$

(e) What is the Taylor expansion of g at -1. Use the notation $T_{-1}(g)(v)$?

By definition, the Taylor expansion of g at -1

$$T_{-1}(g)(v) = \sum_{n\in\mathbb{N}} \frac{1}{n!}\partial_\xi^n(g)(-1)v^n.$$

But g is a polynomial, so $T_{-1}(g)(v) = g(v)$ for all $v \in \mathbb{R}$. Moreover,

$$\begin{aligned}
\partial_\xi g(v) &= 2v + 6v^2 + 4v^3\\
\partial_\xi^2 g(v) &= 2 + 12v + 12v^2\\
\partial_\xi^3 g(v) &= 12 + 24v\\
\partial_\xi^4 g(v) &= 24.
\end{aligned}$$

Hence, $g(-1)\partial_\xi = 0$, $\partial_\xi g(-1) = 0$, $\partial_\xi^2 g(-1) = 2$, $\partial_\xi^3 g(-1) = -12$, $\partial_\xi^4 g(-1) = 24$.

$$T_{-1}(g)(v) = \frac{2}{2}(v+1)^2 - \frac{12}{6}(v+1)^3 + \frac{24}{24}(v+1)^4$$
$$\boxed{T_{-1}(g)(v) = (v+1)^2 - 2(v+1)^3 + (v+1)^4.}$$

(f) Using the little-o notation to answer, what is the asymptotic behavior of g in the neighborhood of v_1 and in the neighborhood of v_2?

From the Taylor expansion of g at 0 we obtain

$$g(v) = v^2 + o(v^2)$$

because $\lim_{v\to 0} |v^3(2+v)|/|v^2| = 0$. Similarly, from the Taylor expansion of g at -1 we obtain

$$g(v) = (1+v)^2 + o((1+v)^2).$$

(g) Deduce from the previous question the structure of the wave: is it a traveling front, a solitary wave, or a periodic wave? Draw an approximate picture of the wave clearly showing its behavior at $\xi = -\infty$ and $\xi = +\infty$. Do not solve the nonlinear ODE $(\partial_\xi v)^2 = g(v)$ but use the results from the lecture notes (without reproving them).

We have $\alpha = 1$ and $\beta = 1$. According to what we did in class and the lecture notes, the wave is a traveling front. The function $v(\xi)$ is monotone and takes values in the interval $(-1, 0)$. Since $v(-\infty) = 0$, the function is monotone decreasing and converges to -1 at $\xi = +\infty$.

(h) Consider the nonlinear ODE $(\partial_\xi v)^2 = v^2(1+v)^2$. We are interested in the solution that is monotone decreasing and takes values in the interval $(-1, 0)$. Which of the following two ODEs $\partial_\xi v = v(1+v)$ or $\partial_\xi v = -v(1+v)$ satisfies these constraints?

Since $v \in (-1, 0)$, we have $v(1+v) < 0$ and $-v(1+v) > 0$. The solution that is monotone decreasing is the one that solves $\partial_\xi v = v(1+v)$.

(i) Solve the ODE $\partial_\xi v = v(1+v)$ assuming that $v \in (-1, 0)$.

We have

$$0 = \frac{\partial_\xi v}{v(1+v)} = \frac{\partial_\xi v}{v} - \frac{\partial_\xi v}{1+v} = \partial_\xi \big(\log(|v|) - \log(|1+v|) \big).$$

Using that $v \in (0, 1)$, this is equivalent to

$$0 = \frac{\partial_\xi v}{v(1+v)} = \frac{\partial_\xi v}{v} - \frac{\partial_\xi v}{1+v} = \partial_\xi \big(\log(-v) - \log(1+v) \big).$$

Integrating once, we obtain

$$\xi - \xi_0 = \log(-v) - \log(1+v)\big),$$

where ξ_0 is any integrating constant in $\mathbb{R}$. Exponentiating this identity gives

$$-v = (1+v)e^{\xi - \xi_0}.$$

Hence,

$$v(\xi) = \frac{-e^{\xi-\xi_0}}{1+e^{\xi-\xi_0}} = \frac{-1}{1+e^{-\xi+\xi_0}}$$

(j) Give the expression of the wave solving the PDE (1) in terms of x and t.

We have

$$u(x,t) = \frac{-1}{1+e^{-x+t+\xi_0}}$$

Question 2: Consider the following PDE which models the behavior of weakly compressible granular materials:

$$\partial_{tt}u - \partial_{xx}\left(u^{\frac{3}{2}} + u^{\frac{1}{4}}\partial_{xx}u^{\frac{5}{4}}\right) = 0. \tag{2}$$

(b) Assume that the equation has a wave solution of the form $u(x,t) = v(x-t)$ with wave speed equal to 1. Write the nonlinear ODE solved by v (do not expand the derivatives).

Inserting the ansatz in the PDE gives

$$c^2\partial_{\xi\xi}v - \partial_{\xi\xi}\left(v^{\frac{3}{2}} + v^{\frac{1}{4}}\partial_{\xi\xi}v^{\frac{5}{4}}\right) = 0.$$

(b) Show that there is a function h so that $\partial_{\xi\xi}v^{\frac{5}{4}} = h(v)$. (Hint: Integrate twice the ODE taking the two integration constants equal to zero).

Integrating twice the nonlinear ODE we obtain

$$c^2 v - \left(v^{\frac{3}{2}} + v^{\frac{1}{4}}\partial_{\xi\xi}v^{\frac{5}{4}}\right) = A + B\xi.$$

Taking the two integration constants equal to zero gives $v^{\frac{1}{4}}\partial_{\xi\xi}v^{\frac{5}{4}} = c^2 v - v^{\frac{3}{2}}$, i.e.,

$$\partial_{\xi\xi}v^{\frac{5}{4}} = c^2 v^{\frac{3}{4}} - v^{\frac{5}{4}},$$

whence the assertion with $h(v) := c^2 v^{\frac{3}{4}} - v^{\frac{5}{4}}$.

(c) Multiplying the equation $\partial_{\xi\xi}v^{\frac{5}{4}} = h(v)$ by $\partial_\xi v^{\frac{5}{4}}$, show that there exists a function k so that $(\partial_\xi v^{\frac{5}{4}})^2 = k(v)$ (take the integration constant equal to zero).

Multiplying the equation $\partial_{\xi\xi}v^{\frac{5}{4}} = c^2 v^{\frac{3}{4}} - v^{\frac{5}{4}}$ by $\partial_\xi v^{\frac{5}{4}}$, we obtain

$$\begin{aligned}\partial_\xi v^{\frac{5}{4}}\partial_{\xi\xi}v^{\frac{5}{4}} &= \partial_\xi \frac{1}{2}(\partial_\xi v^{\frac{5}{4}})^2 = c^2 v^{\frac{3}{4}}\partial_\xi v^{\frac{5}{4}} - v^{\frac{5}{4}}\partial_\xi v^{\frac{5}{4}} \\ &= \frac{5}{4}c^2 v\partial_\xi v - \frac{5}{4}v^{\frac{6}{4}}\partial_\xi v = \frac{5}{4}\frac{1}{2}c^2\partial_\xi v^2 - \frac{5}{4}\frac{2}{5}\partial_\xi v^{\frac{5}{2}} \\ &= \frac{5}{8}c^2\partial_\xi v^2 - \frac{1}{2}\partial_\xi v^{\frac{5}{2}}.\end{aligned}$$

Integrating once and choosing the integration constant equal to zero, we obtain

$$\left(\partial_\xi v^{\frac{5}{4}}\right)^2 = \frac{5}{4}c^2 v^2 - v^{\frac{5}{2}} = v^2(\frac{5}{4}c^2 - v^{\frac{1}{2}}),$$

whence the assertion with $k(v) := v^2(\frac{5}{4}c^2 - v^{\frac{1}{2}})$,

(d) Using the chain rule, show that there exists a function g so that $(\partial_\xi v)^2 = g(v)$.

Using the chain rule, we obtain

$$\left(\partial_\xi v^{\frac{5}{4}}\right)^2 = \left(\frac{5}{4} v^{\frac{1}{4}} \partial_\xi v\right)^2 = \frac{25}{16} v^{\frac{1}{2}} (\partial_\xi v)^2 = v^2 (\frac{5}{4} c^2 - v^{\frac{1}{2}}).$$

Hence,

$$(\partial_\xi v)^2 = \frac{16}{25} v^{\frac{3}{2}} (\frac{5}{4} c^2 - v^{\frac{1}{2}}).$$

whence the assertion with $g(v) = \frac{16}{25} v^{\frac{3}{2}} (\frac{5}{4} c^2 - v^{\frac{1}{2}})$.

(e) What are the two roots of g, say $v_1 < v_2$?

We clearly have

$$v_1 = 0, \quad v_2 = \frac{25}{16} c^4.$$

(e) Using the little-o notation to answer, what is the asymptotic behavior of g in the neighborhood of v_1 and in the neighborhood of v_2?

In the neighborhood of $v_1 = 0$ we have

$$g(v) - \frac{16}{25} v^{\frac{3}{2}} \frac{5}{4} c^2 = \frac{16}{25} v^2 = o(v^{\frac{3}{2}}).$$

Hence

$$g(v) = \frac{4}{5} v^{\frac{3}{2}} c^2 + o(v^{\frac{3}{2}}).$$

Similarly, using the Taylor expansion of $v^{\frac{1}{2}}$ and $v^{\frac{3}{2}}$ at v_2 we have

$$v^{\frac{1}{2}} = v_2^{\frac{1}{2}} + (v - v_2) \frac{1}{2} v_2^{-\frac{1}{2}} + o(|v - v_2|),$$
$$v^{\frac{3}{2}} = v_2^{\frac{3}{2}} + (v - v_2) \frac{3}{2} v_2^{\frac{1}{2}} + o(|v - v_2|),$$

Hence,

$$g(v) = \frac{16}{25} v^{\frac{3}{2}} (v_2^{\frac{1}{2}} - v^{\frac{1}{2}}) = \frac{16}{25} \left(v_2^{\frac{3}{2}} + (v - v_2) \frac{3}{2} v_2^{\frac{1}{2}} + o(|v - v_2|)\right) \left(\frac{v_2 - v}{2 v_2^{\frac{1}{2}}} + o(|v - v_2|)\right).$$

This proves that

$$g(v) - \frac{16}{25} v_2^{\frac{3}{2}} \frac{v_2 - v}{2 v_2^{\frac{1}{2}}} = o(|v - v_2|).$$

Hence,

$$g(v) = \frac{8}{25} v_2 (v_2 - v) + o(|v - v_2|).$$

(f) From the previous question deduce the structure of the wave: is it a traveling front, a solitary wave, or a periodic wave? Draw a qualitative picture of the wave. Do not solve the nonlinear ODE $(\partial_\xi v)^2 = g(v)$ but use the results from the lecture notes (without reproving them).

The roots of g are $v_1 = 0$, $v_2 = \frac{25}{16} c^4$, and the asymptotic exponents are $\alpha = \frac{3}{4}$ in the neighborhood of v_1, and $\beta = \frac{1}{2}$ in the neighborhood of v_2 and β. Hence the solution is a periodic wave with values in $(0, v_2)$.

(g) Let $v(\xi) := \frac{25}{16}c^4\cos^4(\frac{\xi-\xi_0}{5})$ where ξ_0 is any real number. Prove that $u(x,t) = v(x-ct)$ solves the PDE (2). (*Hint*: verify that v solves $(\partial_\xi v)^2 = g(v)$.)

We have

$$\partial_\xi v(\xi) = -\frac{4}{5}\frac{25}{16}c^4\sin(\frac{\xi-\xi_0}{5})\cos^3(\frac{\xi-\xi_0}{5}) = .$$

Hence

$$(\partial_\xi v(\xi))^2 = \frac{16}{25}\left(\frac{25}{16}c^4\right)^2\sin^2(\frac{\xi-\xi_0}{5})\cos^6(\frac{\xi-\xi_0}{5}).$$

Moreover,

$$\begin{aligned}\frac{16}{25}v^{\frac{3}{2}}(\frac{5}{4}c^2 - v^{\frac{1}{2}}) &= \frac{16}{25}\left(\frac{25}{16}c^4\right)^{\frac{3}{2}}\cos^6(\frac{\xi-\xi_0}{5})\frac{5}{4}c^2\left(1-\cos^2(\frac{\xi-\xi_0}{5})\right)\\ &= \frac{16}{25}\left(\frac{25}{16}c^4\right)^2\cos^6(\frac{\xi-\xi_0}{5})\sin^2(\frac{\xi-\xi_0}{5}).\end{aligned}$$

Hence,

$$(\partial_\xi v)^2 = g(v),$$

thereby proving the assertion.