Appendix C

Bijective operators in Banach spaces

The goal of this appendix is to recall fundamental results on linear operators (that is, bounded linear maps) in Banach and Hilbert spaces, and in particular, to state conditions allowing us to assert the bijectivity of these operators. The results collected herein provide a theoretical framework for the mathematical analysis of the finite element method. We refer the reader to Aubin [22], Brezis [63], Lax [214], Rudin [260], Yosida [307], Zeidler [308] for further reading.

C.1 Injection, surjection, bijection

Since we are interested in asserting the bijectivity of bounded linear maps in Banach and Hilbert spaces, let us first recall some basic notions concerning injectivity, surjectivity, and bijectivity, as well as left and right inverses.

Definition C.1 (Injection, surjection, bijection). *Let E and G be two nonempty sets. A function (or map) $f : E \to G$ is said to be* injective *if every element of the codomain (i.e., G) is mapped to by at most one element of the domain (i.e., E). The function is said to be* surjective *if every element of the codomain is mapped to by at least one element of the domain. Finally f is said to be* bijective *if every element of the codomain is mapped to by exactly one element of the domain (i.e., f is both injective and surjective).*

Definition C.2 (Left and right inverse). *Let E and G be two nonempty sets and let $f : E \to G$ be a function. We say that $f^{\ddagger} : G \to E$ is a* left inverse *of f if $(f^{\ddagger} \circ f)(e) = e$ for all $e \in E$, and that $f^{\dagger} : G \to E$ is a* right inverse *of f if $(f \circ f^{\dagger})(g) = g$ for all $g \in G$.*

A map with a left inverse is necessarily injective. Conversely, if the map $f : E \to G$ is injective, the following holds true: (i) The map $\tilde{f} : E \to f(E)$ such that $\tilde{f}(e) = f(e)$ for all $e \in E$ has necessarily a unique left inverse;

(ii) One can construct a left inverse $f^\ddagger : G \to E$ of f by setting $f^\ddagger(g) := e$ (with $e \in E$ arbitrary) if $g \notin f(E)$ and $f^\ddagger(g) := (\tilde{f})^\ddagger(g)$ otherwise; (iii) If E, G are vector spaces and the map f is linear, the left inverse of $\tilde{f}$ is also linear. A map with a right inverse is necessarily surjective. Conversely one can construct right inverse maps for every surjective map by invoking the axiom of choice.

C.2 Banach spaces

Basic properties of Banach and Hilbert spaces are collected in Appendix A. In this section we recall these properties and give more details. To stay general we consider complex vector spaces, i.e., vector spaces over the field $\mathbb{C}$ of complex numbers. The case of real vector spaces is recovered by replacing the field $\mathbb{C}$ by $\mathbb{R}$, by removing the real part symbol $\Re(\cdot)$ and the complex conjugate symbol $\bar{\cdot}$, and by interpreting $|\cdot|$ as the absolute value instead of the complex modulus. Recall that a complex vector space V equipped with a norm $\|\cdot\|_V$ is said to be a Banach space if every Cauchy sequence in V has a limit in V.

Let V, W be complex vector spaces. The complex vector space composed of the bounded linear maps from V to W is denoted $\mathcal{L}(V; W)$. Members of $\mathcal{L}(V; W)$ are often called *operators*. This space is equipped with the norm

$$\|A\|_{\mathcal{L}(V;W)} := \sup_{v \in V} \frac{\|A(v)\|_W}{\|v\|_V} < \infty, \qquad \forall A \in \mathcal{L}(V; W). \tag{C.1}$$

In this book we systematically abuse the notation by implicitly assuming that the argument in this type of supremum or infimum is nonzero. If W is a Banach space, then $\mathcal{L}(V; W)$ equipped with the above norm is also a Banach space (see Rudin [260, p. 87], Yosida [307, p. 111]).

Theorem C.3 (Banach–Steinhaus). *Let V, W be Banach spaces and let $\{A_i\}_{i \in \mathcal{I}}$ be a family (not necessarily countable) in $\mathcal{L}(V; W)$. Assume that $\sup_{i \in \mathcal{I}} \|A_i(v)\|_W$ is a finite number for all $v \in V$. Then there is a real number C such that*

$$\sup_{i \in \mathcal{I}} \|A_i(v)\|_W \le C\|v\|_V, \qquad \forall v \in V. \tag{C.2}$$

Proof. See Brezis [63, p. 32], Lax [214, Chap. 10]. □

Corollary C.4 (Pointwise convergence). *Let V, W be Banach spaces. Let $(A_n)_{n \in \mathbb{N}}$ be a sequence in $\mathcal{L}(V; W)$ such that for all $v \in V$, the sequence $(A_n(v))_{n \in \mathbb{N}}$ converges as $n \to \infty$ to a limit in W denoted $A(v)$ (one says that the sequence $(A_n)_{n \in \mathbb{N}}$ converges pointwise to A). The following holds true:*

(i) $\sup_{n \in \mathbb{N}} \|A_n\|_{\mathcal{L}(V;W)} < \infty$.
(ii) $A \in \mathcal{L}(V; W)$.

(iii) $\|A\|_{\mathcal{L}(V;W)} \leq \liminf_{n\to\infty} \|A_n\|_{\mathcal{L}(V;W)}$.

Proof. The statement (i) follows from Theorem C.3. Owing to (C.2), we infer that $\|A_n(v)\|_W \leq C\|v\|_V$ for all $v \in V$ and all $n \in \mathbb{N}$. Letting $n \to \infty$, we obtain $\|A(v)\|_W \leq C\|v\|_V$, and since A is obviously linear, we infer that the statement (ii) holds true. Finally the statement (iii) results from the fact that $\|A_n(v)\|_W \leq \|A_n\|_{\mathcal{L}(V;W)}\|v\|_V$ for all $v \in V$ and all $n \in \mathbb{N}$. □

Remark C.5 (Uniform convergence on compact sets). Corollary C.4 does not claim that $(A_n)_{n\in\mathbb{N}}$ converges to A in $\mathcal{L}(V;W)$, i.e., uniformly on bounded sets. A standard argument shows however that $(A_n)_{n\in\mathbb{N}}$ converges uniformly to A on compact sets. Let indeed $K \subset V$ be a compact set. Let $\epsilon > 0$. Set $C := \sup_{n\in\mathbb{N}} \|A_n\|_{\mathcal{L}(V;W)}$. We notice that C is finite owing to Corollary C.4(i). The set K being compact, there is a finite set of points $\{x_i\}_{i\in\mathcal{I}}$ in K such that for all $v \in K$, there is $i \in \mathcal{I}$ such that $\|v - x_i\|_V \leq (3C)^{-1}\epsilon$. Owing to the pointwise convergence of $(A_n)_{n\in\mathbb{N}}$ to A, there is N_i such that $\|A_n(x_i) - A(x_i)\|_W \leq \frac{1}{3}\epsilon$ for all $n \geq N_i$. Using the triangle inequality and the statement (iii) above, we infer that

$$\|A_n(v)-A(v)\|_W \leq \|A_n(v-x_i)\|_W + \|A_n(x_i)-A(x_i)\|_W + \|A(v-x_i)\|_W \leq \epsilon,$$

for all $v \in K$ and all $n \geq \max_{i\in\mathcal{I}} N_i$. □

C.3 Hilbert spaces

Let V be a complex vector space equipped with an inner product $(\cdot,\cdot)_V : V \times V \to \mathbb{C}$. Recall that the inner product is linear w.r.t. its first argument and antilinear w.r.t. its second argument, i.e., $(\lambda v, w)_V = \lambda(v,w)_V$ and $(v, \lambda w)_V = \overline{\lambda}(v,w)_V$ for all $\lambda \in \mathbb{C}$ and all $v, w \in V$, and that Hermitian symmetry means that $(v,w)_V = \overline{(w,v)_V}$. The space V is said to be a Hilbert space if it is a Banach space when equipped with the induced norm $\|v\|_V := (v,v)_V^{\frac{1}{2}}$ for all $v \in V$. Recall the Cauchy–Schwarz inequality

$$|(v,w)_V| \leq \|v\|_V\|w\|_V, \qquad \forall v, w \in V. \tag{C.3}$$

Notice that we obtain an equality in (C.3) iff v and w are collinear. This follows from $\|v\|_V\|w\|_V - \Re(\xi(v,w)_V) = \frac{\|v\|_V\|w\|_V}{2}\left\|\frac{v}{\|v\|_V} - \overline{\xi}\frac{w}{\|w\|_V}\right\|_V^2$ for all nonzero $v, w \in V$ and all $\xi \in \mathbb{C}$ with $|\xi| = 1$.

Remark C.6 (Arithmetic-geometric and Young's inequalities). Let $x_1, \ldots, x_n$ be nonnegative real numbers. Using the convexity of the function $x \mapsto e^x$, one can show the following arithmetic-geometric inequality:

$$(x_1 x_2 \ldots x_n)^{\frac{1}{n}} \leq \frac{1}{n}(x_1 + \ldots + x_n). \tag{C.4}$$

Moreover Young's inequality states that for every positive real number $\gamma > 0$,

$$|(v, w)_V| \le \frac{\gamma}{2}\|v\|_V^2 + \frac{1}{2\gamma}\|w\|_V^2, \qquad \forall v, w \in V. \tag{C.5}$$

This follows from the Cauchy–Schwarz inequality and (C.4) with $n = 2$, $x_1 = \gamma^{\frac{1}{2}}\|v\|_V$, and $x_2 = \gamma^{-\frac{1}{2}}\|w\|_V$. □

Definition C.7 (Hilbert basis). *A sequence $(e_n)_{n\in\mathbb{N}}$ in V is said to be a* Hilbert basis *of V if it satisfies the following two properties:*

(i) $(e_m, e_n)_V = \delta_{mn}$ *for all* $m, n \in \mathbb{N}$.
(ii) *The linear space composed of all the finite linear combinations of the vectors in $(e_n)_{n\in\mathbb{N}}$ is dense in V.*

The existence of Hilbert bases is not a natural consequence of the Hilbert space structure, but the question of the existence of Hilbert bases can be given a positive answer by introducing the notion of separability.

Definition C.8 (Separability). *A Hilbert space V is said to be* separable *if it admits a countable dense subset $(v_n)_{n\in\mathbb{N}}$.*

Not every Hilbert space is separable, but all the Hilbert spaces encountered in this book are separable (or by default are always assumed to be separable). The main motivation for the notion of separability is the following result.

Theorem C.9 (Separability and Hilbert basis). *Every separable Hilbert space has a Hilbert basis.*

Proof. See [63, Thm. 5.11].

Lemma C.10 (Pareseval). *Let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of V. For all $u \in V$, set $u_n := \sum_{k\in\{0:n\}}(u, e_k)_V e_k$. The following holds true:*

$$\lim_{n\to\infty}\|u - u_n\|_V = 0 \quad \text{and} \quad \|u\|_V^2 = \sum_{k\in\mathbb{N}}|(u, e_k)_V|^2. \tag{C.6}$$

Conversely let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^2(\mathbb{C})$ and set $u_{\alpha,n} := \sum_{k\in\{0:n\}}\alpha_k e_k$. Then the sequence $(u_{\alpha,n})_{n\in\mathbb{N}}$ converges to some u_α in V such that $(u_\alpha, e_n)_V = \alpha_n$ for all $n \in \mathbb{N}$, and we have $\|u_\alpha\|_V^2 = \lim_{n\to\infty}\sum_{k\in\{0:n\}}\alpha_k^2$.

Proof. See Brezis [63, Thm. 5.9]. □

A striking consequence of Lemma C.10 is that all separable Hilbert spaces are isomorphic and isometric with $\ell^2(\mathbb{C})$.

Remark C.11 (Space $V_{\mathbb{R}}$). Let V be a complex vector space. By restricting the scaling operation $(\lambda, v) \mapsto \lambda v$ to $(\lambda, v) \in \mathbb{R}\times V$, V can also be equipped with a vector space structure over $\mathbb{R}$, which we denote $V_{\mathbb{R}}$ (V and $V_{\mathbb{R}}$ are the same sets, but they are equipped with different vector space structures). For instance, if $V = \mathbb{C}^m$, then $\dim(V) = m$ but $\dim(V_{\mathbb{R}}) = 2m$. Moreover the canonical set $\{\boldsymbol{e}_k\}_{k\in\{1:m\}}$, where the Cartesian components of $\boldsymbol{e}_k$ in $\mathbb{C}^m$ are $e_{k,l} = \delta_{kl}$ (the Kronecker symbol) for all $l \in \{1:m\}$, is a basis of V, whereas the set $\{\boldsymbol{e}_k, \mathrm{i}\boldsymbol{e}_k\}_{k\in\{1:m\}}$ with $\mathrm{i}^2 = -1$ is a basis of $V_{\mathbb{R}}$. Finally, if V is a complex Hilbert space with inner product $(\cdot,\cdot)_V$, then the space $V_{\mathbb{R}}$ is a real Hilbert space with inner product $\Re(\cdot,\cdot)_V$. □

C.4 Duality, reflexivity, and adjoint operators

Let V be a complex Banach space. Its dual space V' is composed of all the antilinear forms $A : V \to \mathbb{C}$ that are bounded. The reason we consider antilinear forms is that we employ the complex conjugate of test functions in the weak formulation of complex-valued PDEs. The action of $A \in V'$ on $v \in V$ is denoted $\langle A, v\rangle_{V',V} \in \mathbb{C}$ (and sometimes also $A(v)$). Equipped with the norm

$$\|A\|_{V'} := \sup_{v\in V} \frac{|\langle A, v\rangle_{V',V}|}{\|v\|_V}, \qquad \forall A \in V', \tag{C.7}$$

V' is a Banach space. In the real case the absolute value can be omitted from the numerators since $\pm v$ can be considered in the supremizing set. In the complex case the modulus can be replaced by the real part since v can be multiplied by any unit complex number.

Remark C.12 (Linear vs. antilinear form). If $A : V \to \mathbb{C}$ is an antilinear form, then $\overline{A}$ (defined by $\overline{A}(v) := \overline{A(v)} \in \mathbb{C}$ for all $v \in V$) is a linear form. □

C.4.1 Fundamental results in Banach spaces

Theorem C.13 (Hahn–Banach). *Let V be a normed vector space over $\mathbb{C}$ and let W be a subspace of V. Let $B \in W'$. There exists $A \in V'$ that extends B, i.e., $A(w) = B(w)$ for all $w \in W$, and such that $\|A\|_{V'} = \|B\|_{W'}$.*

Proof. For the real case, see Brezis [63, p. 3], Lax [214, Chap. 3], Rudin [260, p. 56], Yosida [307, p. 102]. The above statement is a simplified version of the actual Hahn–Banach theorem. For the complex case, see Lax [214, p. 27], Brezis [63, Prop. 11.23]. □

Corollary C.14 (Norm by duality). *The following holds true:*

$$\|v\|_V = \sup_{A\in V'} \frac{|A(v)|}{\|A\|_{V'}} = \sup_{A\in V'} \frac{|\langle A, v\rangle_{V'V}|}{\|A\|_{V'}}, \tag{C.8}$$

for all $v \in V$, and the supremum is attained.

Proof. Assume $v \neq 0$ (the assertion is obvious for $v = 0$). We first observe that $\sup_{A \in V'} \frac{|A(v)|}{\|A\|_{V'}} \leq \|v\|_V$ since $|A(v)| \leq \|A\|_{V'}\|v\|_V$. Let $W := \mathrm{span}(v)$ and let $B \in W'$ be defined as $B(\lambda v) := \overline{\lambda}\|v\|_V$ for all $\lambda \in \mathbb{C}$. By construction $B \in W'$ and $\|B\|_{W'} = 1$. Owing to the Hahn–Banach theorem, there exists $A \in V'$ such that $\|A\|_{V'} = 1$ and $A(v) = B(v) = \|v\|_V$. □

Corollary C.15 (Characterization of density). *Let V be a normed vector space over $\mathbb{C}$ and W be a subspace of V such that $\overline{W} \neq V$ (i.e., W is not dense in V). Then there exists $f \in V'\backslash\{0\}$ such that $f(w) = 0$ for all $w \in W$.*

Proof. See Brezis [63, p. 8], Rudin [260, Thm. 5.19]. □

Definition C.16 (Double dual). *The double dual of a Banach space V is denoted V'' and is defined to be the dual space of its dual space V'.*

Proposition C.17 (Isometry into double dual). *The bounded linear map $J_V : V \to V''$ such that*

$$\langle J_V(v), \phi' \rangle_{V'',V'} = \overline{\langle \phi', v \rangle_{V',V}}, \qquad \forall (v, \phi') \in V \times V', \tag{C.9}$$

is an isometry.

Proof. The claim follows from Corollary C.14 since

$$\|J_V(v)\|_{V''} = \sup_{\phi' \in V'} \frac{|\langle J_V(v), \phi' \rangle_{V'',V'}|}{\|\phi'\|_{V'}} = \sup_{\phi' \in V'} \frac{|\langle \phi', v \rangle_{V',V}|}{\|\phi'\|_{V'}} = \|v\|_V. \qquad \square$$

Definition C.18 (Reflexivity). *A Banach space V is said to be* reflexive *if J_V is an isomorphism.*

Remark C.19 (Map J_V). Since J_V is an isometry, it is injective. Thus V can be identified with the subspace $J_V(V) \subset V''$. It may happen that the map J_V is not surjective. In this case V is a proper subspace of V''. □

Example C.20 (Lebesgue spaces). One important consequence of Theorem 1.37 is that the Lebesgue space $L^p(D)$ is reflexive for all $p \in (1, \infty)$. However $L^1(D)$ and $L^\infty(D)$ are not reflexive. Indeed $L^\infty(D) = L^1(D)'$, but $L^1(D) \subsetneq L^\infty(D)'$ with strict inclusion; see §1.4 and Brezis [63, p. 102]. □

Remark C.21 (Space $V_{\mathbb{R}}$). Let V be a complex vector space and let $V_{\mathbb{R}}$ be defined in Remark C.11. Let $V'_{\mathbb{R}}$ be the dual space of $V_{\mathbb{R}}$, i.e., the normed real vector space composed of the bounded $\mathbb{R}$-linear maps from V to $\mathbb{R}$. Then the map $I : V' \to V'_{\mathbb{R}}$ s.t. for all $\ell \in V'$, $I(\ell)(v) := \Re(\ell(v))$, for all $v \in V$, is a bijective isometry; see [63, Prop. 11.22]. □

Definition C.22 (Weak convergence). *Let V be a Banach space. The sequence $(v_n)_{n \in \mathbb{N}}$ in V is said to* converge weakly *to $v \in V$ if*

$$\langle A, v_n \rangle_{V',V} \to \langle A, v \rangle_{V',V}, \qquad \forall A \in V'. \tag{C.10}$$

It is shown in Brezis [63, Prop. 3.5] that if the sequence $(v_n)_{n\in\mathbb{N}}$ converges strongly to v (that is, in the norm topology, i.e., $\|v_n - v\|_V \to 0$ as $n \to \infty$), then it also converges weakly to v. The converse is true if V is finite-dimensional (see [63, Prop. 3.6]). Furthermore, if the sequence $(v_n)_{n\in\mathbb{N}}$ converges weakly to v, then it is bounded and $\|v\|_V \le \liminf_{n\to\infty} \|v_n\|_V$. One important result on weak convergence is the following (see [63, Thm. 3.18]).

Theorem C.23 (Reflexivity and weak compactness). *Let V be a reflexive Banach space. Then from every bounded sequence $(v_n)_{n\in\mathbb{N}}$ of V, there exists a subsequence $(v_{n_k})_{k\in\mathbb{N}}$ that is weakly convergent.*

C.4.2 Further results in Hilbert spaces

Theorem C.24 (Riesz–Fréchet). *The operator $J_V^{\mathrm{RF}} : V \to V'$ such that*

$$\langle J_V^{\mathrm{RF}}(v), w\rangle_{V',V} := (v, w)_V, \qquad \forall v, w \in V, \tag{C.11}$$

is a linear isometric isomorphism.

Proof. See Brezis [63, Thm. 5.5], Lax [214, p. 56], Yosida [307, p. 90], or Exercise 25.1. □

Remark C.25 (Riesz–Fréchet representation). Theorem C.24 is often called Riesz–Fréchet representation theorem. It states that for every antilinear form $v' \in V'$, there exists a unique vector $v \in V$ such that $v' = J_V^{\mathrm{RF}}(v)$. The vector $(J_V^{\mathrm{RF}})^{-1}(v') \in V$ is called *Riesz–Fréchet representative* of the antilinear form $v' \in V'$. The action of v' on V is represented by $(J_V^{\mathrm{RF}})^{-1}(v')$ with the identity $\langle v', w\rangle_{V',V} = \big((J_V^{\mathrm{RF}})^{-1}(v'), w\big)_V$ for all $w \in V$. □

Remark C.26 (Linear vs. antilinear). Notice that J_V^{RF} is a linear operator. If we had adopted the convention that dual spaces were composed of linear forms, we would have had to define J_V^{RF} by setting $\langle J_V^{\mathrm{RF}}(v), w\rangle_{V',V} := \overline{(v, w)_V}$ for all $v, w \in V$, or equivalently $\langle v', w\rangle_{V',V} := \overline{((J_V^{\mathrm{RF}})^{-1}(v'), w)_V}$ for all $w \in V$ and $v' \in V'$. In this case J_V^{RF} would have been antilinear. □

Corollary C.27 (Reflexivity). *Hilbert spaces are reflexive.*

Owing to the Riesz–Fréchet theorem, the notion of weak convergence (see Definition C.22) notion can be reformulated as follows in Hilbert spaces.

Definition C.28 (Weak convergence). *Let V be a Hilbert space. The sequence $(v_n)_{n\in\mathbb{N}}$ in V is said to* converge weakly *to $v \in V$ if $(w, v_n)_V \to (w, v)_V$ as $n \to \infty$, for all $w \in V$.*

A useful connection between weak and strong convergence in Hilbert spaces is that if the sequence $(v_n)_{n\in\mathbb{N}}$ converges weakly to $v \in V$ and if additionally, $\|v_n\|_V \to \|v\|_V$ as $n \to \infty$, then the sequence $(v_n)_{n\in\mathbb{N}}$ converges strongly to v, i.e., $\|v_n - v\|_V \to 0$ as $n \to \infty$ (see, e.g., Brezis [63, Prop. 3.32]).

C.4.3 Adjoint

Definition C.29 (Adjoint operator). *Let V, W be complex Banach spaces. Let $A \in \mathcal{L}(V;W)$. The* adjoint operator *of A is the bounded linear operator $A^* \in \mathcal{L}(W';V')$ such that*

$$\langle A^*(w'), v\rangle_{V',V} := \langle w', A(v)\rangle_{W',W}, \qquad \forall (v,w') \in V \times W'. \tag{C.12}$$

Note that $(\lambda A)^* = \overline{\lambda} A^*$ for all $\lambda \in \mathbb{C}$.

Lemma C.30 (Norm of adjoint). *Let $A \in \mathcal{L}(V;W)$ and let $A^* \in \mathcal{L}(W';V')$ be its adjoint. Then $\|A^*\|_{\mathcal{L}(W';V')} = \|A\|_{\mathcal{L}(V;W)}$.*

Proof. We have

$$\begin{aligned}\|A^*\|_{\mathcal{L}(W';V')} &= \sup_{w' \in W'} \frac{\|A^*(w')\|_{V'}}{\|w'\|_{W'}} = \sup_{w' \in W'} \sup_{v \in V} \frac{|\langle A^*(w'), v\rangle_{V',V}|}{\|v\|_V \|w'\|_{W'}} \\ &= \sup_{v \in V} \sup_{w' \in W'} \frac{|\langle w', A(v)\rangle_{W',W}|}{\|v\|_V \|w'\|_{W'}} = \sup_{v \in V} \frac{\|A(v)\|_W}{\|v\|_V} = \|A\|_{\mathcal{L}(V;W)},\end{aligned}$$

where we used that $\sup_{w' \in W'} \sup_{v \in V} = \sup_{v \in V} \sup_{w' \in W'}$, the definition of A^*, and Corollary C.14. □

Definition C.31 (Self-adjoint operator). *Let V be a reflexive Banach space. Let $A \in \mathcal{L}(V;V')$, so that $A^* \in \mathcal{L}(V'';V')$. The operator A is said to be* self-adjoint *if $A = A^* \circ J_V$, i.e., if the following holds true:*

$$\langle A(v), w\rangle_{V',V} = \overline{\langle A(w), v\rangle_{V',V}}, \qquad \forall v, w \in V. \tag{C.13}$$

In particular, $\langle A(v), v\rangle_{V',V}$ takes real values if A is self-adjoint. If the spaces V and V'' are actually identified, we write $A^* \in \mathcal{L}(V;V')$ and say that A is self-adjoint if $A = A^*$.

Remark C.32 (Hermitian transpose). If V and W are finite-dimensional and after choosing one basis for V and one for W, A can be represented by a matrix with complex-valued entries. Then A^* is represented in the same bases by the Hermitian transpose of this matrix. Self-adjoint operators are represented by Hermitian matrices. □

C.5 Open mapping and closed range theorems

Let V, W be complex Banach spaces. For $A \in \mathcal{L}(V;W)$, we denote by $\ker(A)$ its kernel and by $\mathrm{im}(A)$ its range. The operator A being bounded, $\ker(A)$ is closed in V. Hence the quotient of V by $\ker(A)$, $V/\ker(A)$, can be defined. This space is composed of equivalence classes $\check{v}$ such that v and w are in the same class $\check{v}$ if and only if $v - w \in \ker(A)$, i.e., $A(v) = A(w)$.

Theorem C.33 (Quotient space). *The space* $V/\ker(A)$ *is a Banach space when equipped with the norm* $\|\check{v}\| := \inf_{v \in \check{v}} \|v\|_V$*. Moreover the operator* $\check{A} : V/\ker(A) \to \mathrm{im}(A)$ *s.t.* $\check{A}(\check{v}) := A(v)$*, for all* v *in* $\check{v}$*, is an isomorphism.*

Proof. See Brezis [63, §11.2], Yosida [307, p. 60]. □

For subspaces $M \subset V$ and $N \subset V'$, we define the *annihilators* of M and N as follows:

$$M^\perp := \{v' \in V' \mid \forall m \in M,\ \langle v', m\rangle_{V',V} = 0\}, \tag{C.14a}$$

$$N^\perp := \{v \in V \mid \forall n' \in N,\ \langle n', v\rangle_{V',V} = 0\}. \tag{C.14b}$$

Let $\overline{M}$ denote the closure of the subspace M in V. A characterization of $\ker(A)$ and $\mathrm{im}(A)$ is given by the following result.

Lemma C.34 (Kernel and range). *Let* $A \in \mathcal{L}(V;W)$*. The following holds true:*

(i) $\ker(A) = (\mathrm{im}(A^*))^\perp$.
(ii) $\ker(A^*) = (\mathrm{im}(A))^\perp$.
(iii) $\overline{\mathrm{im}(A)} = (\ker(A^*))^\perp$.
(iv) $\overline{\mathrm{im}(A^*)} \subset (\ker(A))^\perp$.

Proof. See Brezis [63, Cor. 2.18], Yosida [307, pp. 202-209]. □

Showing that the range of an operator is closed is a crucial step towards proving that this operator is surjective. This is the purpose of the following fundamental theorem.

Theorem C.35 (Banach or closed range). *Let* $A \in \mathcal{L}(V;W)$*. The following statements are equivalent:*

(i) $\mathrm{im}(A)$ *is closed.*
(ii) $\mathrm{im}(A^*)$ *is closed.*
(iii) $\mathrm{im}(A) = (\ker(A^*))^\perp$.
(iv) $\mathrm{im}(A^*) = (\ker(A))^\perp$.

Proof. See Brezis [63, Thm. 2.19], Yosida [307, p. 205]. □

We now put in place the second keystone of the edifice.

Theorem C.36 (Open mapping). *If* $A \in \mathcal{L}(V;W)$ *is surjective and* U *is an open set in* V*, then* $A(U)$ *is an open set in* W*.*

Proof. See Brezis [63, Thm. 2.6], Lax [214, p. 168], Rudin [260, p. 47], Yosida [307, p. 75]. □

Theorem C.36, also due to Banach, has far-reaching consequences. In particular it leads to the following characterization of the closedness of $\mathrm{im}(A)$.

Lemma C.37 (Characterization of closed range). *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $\mathrm{im}(A)$ *is closed in* W.

(ii) *A has a bounded right inverse map $A^\dagger : \mathrm{im}(A) \to V$, i.e., $(A \circ A^\dagger)(w) = w$ for all $w \in \mathrm{im}(A)$, and there exists $\alpha > 0$ such that $\alpha\|A^\dagger(w)\|_V \leq \|w\|_W$ for all $w \in \mathrm{im}(A)$ ($A^\dagger$ is not necessarily linear).*

Proof. (i) $\Rightarrow$ (ii). Since $\mathrm{im}(A)$ is closed in W, $\mathrm{im}(A)$ is a Banach space. Applying the open mapping theorem to $A : V \to \mathrm{im}(A)$ and $U = B_V(0,1)$ (the open unit ball in V) proves that $A(B_V(0,1))$ is open in $\mathrm{im}(A)$. Since $0 \in A(B_V(0,1))$, there is $\gamma > 0$ s.t. $B_W(0,\gamma) \subset A(B_V(0,1))$. Let $w \in \mathrm{im}(A)$. Since $\frac{\gamma}{2}\frac{w}{\|w\|_W} \in B_W(0,\gamma)$, there is $z \in B_V(0,1)$ s.t. $A(z) = \frac{\gamma}{2}\frac{w}{\|w\|_W}$. Setting $A^\dagger(w) := \frac{2\|w\|_W}{\gamma} z$ leads to $A(A^\dagger(w)) = w$ and $\frac{\gamma}{2}\|A^\dagger(w)\|_V \leq \|w\|_W$.
(ii) $\Rightarrow$ (i). Let $(w_n)_{n\in\mathbb{N}}$ be a sequence in $\mathrm{im}(A)$ that converges to some $w \in W$. The sequence $(v_n := A^\dagger(w_n))_{n\in\mathbb{N}}$ in V is such that $A(v_n) = w_n$ and $\alpha\|v_n\|_V \leq \|w_n\|_W$. Then $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V. Since V is a Banach space, $(v_n)_{n\in\mathbb{N}}$ converges to a certain $v \in V$. Owing to the boundedness of A, $(A(v_n))_{n\in\mathbb{N}}$ converges to $A(v)$. Hence $w = A(v) \in \mathrm{im}(A)$. $\square$

Corollary C.38 (Bounded inverse). *If $A \in \mathcal{L}(V;W)$ is bijective, then $A^{-1} \in \mathcal{L}(W;V)$.*

Proof. Since A is bijective, $\mathrm{im}(A) = W$ is closed. Moreover the right inverse $A^\dagger$ is necessarily equal to A^{-1} (apply A^{-1} to $A \circ A^\dagger = I_W$). Lemma C.37(ii) shows that $A^{-1} \in \mathcal{L}(W;V)$ with $\|A^{-1}\|_{\mathcal{L}(W;V)} \leq \alpha^{-1}$. $\square$

C.6 Characterization of surjectivity

As a consequence of the closed range theorem and of the open mapping theorem, we deduce two characterizations of surjective operators.

Lemma C.39 (Surjectivity of A^*). *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) *$A^* : W' \to V'$ is surjective.*

(ii) *$A : V \to W$ is injective and* $\mathrm{im}(A)$ *is closed in* W.

(iii) *There exists $\alpha > 0$ such that*

$$\|A(v)\|_W \geq \alpha\|v\|_V, \qquad \forall v \in V, \tag{C.15}$$

or, equivalently, there exists $\alpha > 0$ such that

$$\inf_{v\in V}\sup_{w'\in W'} \frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}\|v\|_V} \geq \alpha. \tag{C.16}$$

Proof. (i) $\Rightarrow$ (iii). Since the map A^* is surjective, Lemma C.37 implies that A^* has a bounded right inverse map $A^{*\dagger} : V' \to W'$. In particular $A^*(A^{*\dagger}(v')) = v'$ for all $v' \in V'$, and there is $\alpha > 0$ such that $\alpha\|A^{*\dagger}(v')\|_{W'} \leq \|v'\|_{V'}$. Let now $v \in V$. We infer that

$$\frac{|\langle v', v\rangle_{V',V}|}{\|v'\|_{V'}} = \frac{|\langle A^*(A^{*\dagger}(v')), v\rangle_{V',V}|}{\|v'\|_{V'}} \leq \alpha^{-1}\frac{|\langle A^{*\dagger}(v'), A(v)\rangle_{W',W}|}{\|A^{*\dagger}(v')\|_{W'}}$$
$$\leq \alpha^{-1}\sup_{w'\in W'}\frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}}.$$

Since $\|v\|_V = \sup_{v'\in V'}\frac{|\langle v',v\rangle_{V',V}|}{\|v'\|_{V'}}$, taking the supremum with respect to $v' \in V'$ in the above bound proves (C.16).
(iii) $\Rightarrow$ (ii). The bound (C.15) implies that A is injective. Consider a sequence $(v_n)_{n\in\mathbb{N}}$ such that $(A(v_n))_{n\in\mathbb{N}}$ is a Cauchy sequence in W. Then (C.15) implies that $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in V. Let v be its limit. A being bounded implies that $A(v_n) \to A(v)$. Hence $\mathrm{im}(A)$ is closed.
(ii) $\Rightarrow$ (i). Since $\mathrm{im}(A)$ is closed, we use Theorem C.35(iv) together with the injectivity of A to infer that $\mathrm{im}(A^*) = (\ker(A))^\perp = \{0\}^\perp = V'$. □

Lemma C.40 (Surjectivity of A). *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $A : V \to W$ *is surjective.*
(ii) $A^* : W' \to V'$ *is injective and* $\mathrm{im}(A^*)$ *is closed in* V'.
(iii) *There exists $\alpha > 0$ such that*

$$\|A^*(w')\|_{V'} \geq \alpha\|w'\|_{W'}, \qquad \forall w' \in W', \tag{C.17}$$

or, equivalently, there exists $\alpha > 0$ such that

$$\inf_{w'\in W'}\sup_{v\in V}\frac{|\langle A^*(w'), v\rangle_{V',V}|}{\|w'\|_{W'}\|v\|_V} \geq \alpha. \tag{C.18}$$

Proof. We only detail the proof that (i) $\Rightarrow$ (iii) since the other two implications are shown as above. Since the map A is surjective, Lemma C.37 implies that A has a bounded right inverse map $A^\dagger : W \to V$. In particular $A(A^\dagger(w)) = w$ for all $w \in W$, and there is $\alpha > 0$ such that $\alpha\|A^\dagger(w)\|_V \leq \|w\|_W$. Then for all $w' \in W'$, we have

$$\|A^*(w')\|_{V'} = \sup_{v\in V}\frac{|\langle A^*(w'), v\rangle_{V',V}|}{\|v\|_V} \geq \sup_{w\in W}\frac{|\langle A^*(w'), A^\dagger(w)\rangle_{V',V}|}{\|A^\dagger(w)\|_V}$$
$$= \sup_{w\in W}\frac{|\langle w', w\rangle_{W',W}|}{\|A^\dagger(w)\|_V} \geq \alpha\sup_{w\in W}\frac{|\langle w', w\rangle_{W',W}|}{\|w\|_W} = \alpha\|w'\|_{W'}. \qquad □$$

Remark C.41 (Lions' theorem). The assertion (i) $\Leftrightarrow$ (iii) in Lemma C.40 is sometimes called Lions' theorem. It means that establishing the a priori estimate (C.17) is a necessary and sufficient condition to prove that the problem $A(u) = f$ has at least one solution $u \in V$ for all $f \in W$. $\square$

Lemma C.42 (Right inverse). *Let V, W be Banach spaces and let $A \in \mathcal{L}(V; W)$ be a surjective operator. Assume that V is reflexive. Then A has a bounded right inverse $A^\dagger : W \to V$ satisfying $\alpha\|A^\dagger(w)\|_V \le \|w\|_W$, where α is the same constant as in the equivalent statements* (C.17) *and* (C.18).

Proof. The proof is inspired from ideas by P. Azerad (private communication). Lemma C.34(ii) shows that the adjoint operator $A^* : W' \to V'$ is injective. Let us equip the subspace $R := \mathrm{im}(A^*) \subset V'$ with the norm $\|\cdot\|_{V'}$. The injectivity of A^* implies the existence of a linear left inverse $A^{*\ddagger} : R \to W'$. Consider its adjoint $A^{*\ddagger *} : W'' \to R'$. Let $E^{\mathrm{HB}}_{R'V''}$ be one Hahn–Banach extension operator from R' to V'' (see Theorem C.13). Let us set

$$A^\dagger := J_V^{-1} \circ E^{\mathrm{HB}}_{R'V''} \circ A^{*\ddagger *} \circ J_W : W \to V,$$

and let us verify that $A^\dagger$ satisfies the expected properties. Note that we used here the reflexivity of V to invoke the inverse of the canonical isometry $J_V : V \to V''$. We have for all $(w', w) \in W' \times W$,

$$\begin{aligned}
\langle w', A(A^\dagger(w))\rangle_{W',W} &= \langle A^*(w'), A^\dagger(w)\rangle_{V',V} \\
&= \overline{\langle E^{\mathrm{HB}}_{R'V''}(A^{*\ddagger *}(J_W(w))), A^*(w')\rangle_{V'',V'}} \\
&= \overline{\langle A^{*\ddagger *}(J_W(w)), A^*(w')\rangle_{R',R}} = \overline{\langle J_W(w), A^{*\ddagger}(A^*(w'))\rangle_{W'',W'}} \\
&= \overline{\langle J_W(w), w'\rangle_{W'',W'}} = \langle w', w\rangle_{W',W},
\end{aligned}$$

where to pass from the first to the second line we used that $A^*(w') \in R$. Moreover we observe that for all $w \in W$,

$$\begin{aligned}
\|A^\dagger(w)\|_V = \|A^{*\ddagger *}(J_W(w))\|_{R'} &= \sup_{w' \in W'} \frac{|\langle A^{*\ddagger *}(J_W(w)), A^*(w')\rangle_{R',R}|}{\|A^*(w')\|_{V'}} \\
&= \sup_{w' \in W'} \frac{|\langle J_W(w), w'\rangle_{W'',W'}|}{\|A^*(w')\|_{V'}} \le \sup_{w' \in W'} \frac{\|w'\|_{W'}}{\|A^*(w')\|_{V'}} \|w\|_W.
\end{aligned}$$

We conclude from (C.17) that $\|A^\dagger(w)\|_V \le \alpha^{-1}\|w\|_W$. $\square$

Remark C.43 (Counterexample). The assumption that V be reflexive in Lemma C.42 cannot be removed if one insists on having the bound $\alpha \sup_{\|w\|_W=1} \|A^\dagger(w)\|_V \le 1$. Let us consider the real sequence spaces ℓ^p, $p \in [1, \infty]$. Since $V := \ell^1$ is not reflexive, there exists a linear form $A : \ell^1 \to W := \mathbb{R}$ that does not attain its norm on the unit ball of V (this is James's theorem [197, Thm. 1]). Using ℓ^2 as pivot space, it is well known that ℓ^∞ can be identified with the dual of V (see e.g., Brezis [63, Thm. 4.14]). Let t be the nonzero sequence in ℓ^∞ such that $A(v) = (v, t)_{\ell^2} := \sum_{i\ge 1} v_i t_i$ for all $v \in V$. A simple computation shows that the adjoint $A^* : \mathbb{R} \to V' \equiv \ell^\infty$ is such that

$A^*(s) = st$ for all $s \in \mathbb{R}$. Let us define $\alpha := \inf_{w' \in \mathbb{R}} \sup_{v \in \ell^1} \frac{|(A^*(w'),v)_{\ell^2}|}{|w'|\|v\|_{\ell^1}}$. We have $\alpha = \sup_{v \in \ell^1} \frac{|(t,v)_{\ell^2}|}{\|v\|_{\ell^1}} = \|A\|_{V'}$. Let $A^\dagger$ be a right inverse of A. Then for all $s \in \mathbb{R}$, we have $s = (A \circ A^\dagger)(s) = (t, A^\dagger(s))_{\ell^2}$. For all $s \in \mathbb{R}\backslash\{0\}$, $\frac{A^\dagger(s)}{\|A^\dagger(s)\|_V}$ is in the unit ball of V. Since A does not attain its norm on this ball by assumption, we infer that $|A(\frac{A^\dagger(s)}{\|A^\dagger(s)\|_V})| < \alpha$. Since $A(\frac{A^\dagger(s)}{\|A^\dagger(s)\|_V}) = \frac{1}{\|A^\dagger(s)\|_V} s$, we can rewrite the above bound as $\frac{1}{\alpha}|s| < \|A^\dagger(s)\|_V$ for all $s \in \mathbb{R}\backslash\{0\}$, that is, $1 < \alpha \sup_{\|w\|_W=1} \|A^\dagger(w)\|_V$. □

We observe that nothing is said in Lemma C.42 on the linearity of the right inverse $A^\dagger$. A slightly different construction of $A^\dagger$ that guarantees linearity is possible in the Hilbertian setting.

Lemma C.44 (Right inverse in Hilbert spaces). *Let Y, Z be two non-trivial Hilbert spaces. Let $B : Y \to Z'$ be a bounded linear operator such that there exists $\beta > 0$ s.t.*

$$\|B(y)\|_{Z'} \geq \beta \|y\|_Y, \qquad \forall y \in Y. \tag{C.19}$$

Then $B^* : Z \to Y'$ has a linear right inverse $B^{*\dagger} : Y' \to Z$ such that $\|B^{*\dagger}\|_{\mathcal{L}(Y';Z)} \leq \beta^{-1}$.

Proof. Owing to Lemma C.39, the assumption (C.19) is equivalent to $B^* : Z \to Y'$ being surjective. Let us set $M := \ker(B^*)^\perp \subset Z$, where the orthogonality is defined using the inner product of Z (note that $M \neq \{0\}$ since otherwise $Y = \{0\}$ would be trivial). Let $J : M \to Z$ be the canonical injection, and note that $J^* : Z' \to M'$ is s.t. for all $z' \in Z'$ and all $m \in M$,

$$\langle J^*(z'), m\rangle_{M',M} = \langle z', J(m)\rangle_{Z',Z} = \langle z', m\rangle_{Z',Z}.$$

Let us set $S := J^* \circ B : Y \to M'$. Let $y' \in Y'$. The surjectivity of B^* together with the definition of M implies that there is $z := m + m^\perp \in M \oplus M^\perp = Z$ s.t. $y' = B^*(z) = B^*(m) = B^*(J(m)) = S^*(m)$, which proves that S^* is surjective. Let $m \in M := \ker(B^*)^\perp$ and assume that $0 = S^*(m) = B^*(J(m)) = B^*(m)$. Then $m \in \ker(B^*) \cap \ker(B^*)^\perp$, i.e., $m = 0$, which proves that $S^* : M \to Y'$ is injective. Hence S^* and S are isomorphisms. Moreover we have

$$\begin{aligned}\|(S^*)^{-1}\|^{-1}_{\mathcal{L}(Y';M)} &= \inf_{m\in M} \frac{\|S^*(m)\|_{Y'}}{\|m\|_Z} = \inf_{m\in M} \sup_{y\in Y} \frac{|\langle S^*(m), y\rangle_{Y',Y}|}{\|m\|_Z \|y\|_Y} \\ &= \inf_{y\in Y} \sup_{m\in M} \frac{|\langle S(y), m\rangle_{M',M}|}{\|y\|_Y \|m\|_Z} = \inf_{y\in Y} \sup_{m\in M} \frac{|\langle B(y), m\rangle_{Z',Z}|}{\|y\|_Y \|m\|_Z},\end{aligned}$$

where the first equality on the second line follows from (C.25) below and the bijectivity of S. Using that $Z = M \oplus M^\perp$, we obtain

$$\|(S^*)^{-1}\|^{-1}_{\mathcal{L}(Y';M)} = \inf_{y\in Y}\sup_{m\in M}\frac{|\langle B(y),m\rangle_{Z',Z}|}{\|y\|_Y\|m\|_Z}$$
$$\geq \inf_{y\in Y}\sup_{m+m^\perp\in M\oplus M^\perp}\frac{|\langle B(y),m+m^\perp\rangle_{Z',Z}|}{\|y\|_Y(\|m\|_Z^2+\|m^\perp\|_Z^2)^{1/2}} = \beta,$$

which proves that $\|(S^*)^{-1}\|_{\mathcal{L}(Y';M)} \leq \beta^{-1}$. (Note that we actually have $\|(S^*)^{-1}\|_{\mathcal{L}(Y';M)} = \beta^{-1}$, since $\sup_{m\in M}\frac{|\langle B(y),m\rangle_{Z',Z}|}{\|m\|_Z} \leq \sup_{z\in Z}\frac{|\langle B(y),z\rangle_{Z',Z}|}{\|z\|_Z}$.) Let us now set

$$B^{*\dagger} := J\circ(B^*\circ J)^{-1} = J\circ(S^*)^{-1} : Y' \to Z.$$

Then $B^*\circ B^{*\dagger} = B^*\circ J\circ(S^*)^{-1} = I_{Y'}$, which proves that $B^{*\dagger}$ is indeed a right inverse of B^*. Moreover $\|B^{*\dagger}\|_{\mathcal{L}(Y',Z)} = \|J\circ(S^*)^{-1}\|_{\mathcal{L}(Y',Z)} \leq \|(S^*)^{-1}\|_{\mathcal{L}(Y',Z)} = \beta^{-1}$. □

Remark C.45 (Lemma C.44 vs. Lemma C.42). Without the statement on the linearity of $B^{*\dagger}$, Lemma C.44 would be a direct consequence of Lemma C.42 applied with $A := B^*$, $V := Z$, and $W := Y'$. Indeed the condition (C.19) implies that A is a surjective operator satisfying the inf-sup condition (C.18) with constant β. □

Remark C.46 (Left inverse). The operator $B^{*\ddagger} := (J^*\circ B)^{-1}\circ J^* = S^{-1}\circ J^* : Z' \to Y$ is a left inverse of B s.t. $\|B^{*\ddagger}\|_{\mathcal{L}(Z';Y)} \leq \beta^{-1}$. □

Finally let us recall that compactness can be invoked to give a sufficient condition for the range of an injective operator to be closed.

Lemma C.47 (Peetre–Tartar). *Let X, Y, Z be Banach spaces. Let $A \in \mathcal{L}(X;Y)$ be injective and let $T \in \mathcal{L}(X;Z)$ be compact. Assume that there is $c > 0$ such that $c\|x\|_X \leq \|A(x)\|_Y + \|T(x)\|_Z$ for all $x \in X$. Then $\mathrm{im}(A)$ is closed. Equivalently there is $\alpha > 0$ such that*

$$\alpha\|x\|_X \leq \|A(x)\|_Y, \qquad \forall x \in X. \tag{C.20}$$

Proof. Owing to Lemma C.39 and since A is injective, $\mathrm{im}(A)$ is closed iff (C.20) holds true. This inequality has already been proved in Lemma A.20 (see (A.6)). □

Theorem C.48 (Schauder). *A bounded linear operator between Banach spaces is compact if and only if its adjoint is compact.*

Proof. See Brezis [63, Thm. 6.4]. □

C.7 Characterization of bijectivity

The following theorem provides the theoretical foundation of the BNB theorem stated in §25.3 and which is often invoked in this book.

Theorem C.49 (Bijectivity of A). *Let $A \in \mathcal{L}(V;W)$. The following statements are equivalent:*

(i) $A : V \to W$ *is bijective.*
(ii) A *is injective,* $\mathrm{im}(A)$ *is closed, and* $A^* : W' \to V'$ *is injective.*
(iii) A^* *is injective and there exists* $\alpha > 0$ *such that*

$$\|A(v)\|_W \geq \alpha \|v\|_V, \qquad \forall v \in V, \tag{C.21}$$

or, equivalently, A^* *is injective and*

$$\inf_{v\in V} \sup_{w'\in W'} \frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}\|v\|_V} =: \alpha > 0. \tag{C.22}$$

Proof. (1) The statements (ii) and (iii) are equivalent since (C.21) is equivalent to A injective and $\mathrm{im}(A)$ closed owing to Lemma C.39.
(2) Let us first prove that (i) implies (ii). Since A is surjective, $\ker(A^*) = \mathrm{im}(A)^\perp = \{0\}$, i.e., A^* is injective. Since $\mathrm{im}(A) = W$ is closed and A is injective, this yields (ii). Finally, to prove that (ii) implies (i), we only need to prove that (ii) implies the surjectivity of A. The injectivity of A^* implies $\overline{\mathrm{im}(A)} = (\ker(A^*))^\perp = W$. Since $\mathrm{im}(A)$ is closed, $\mathrm{im}(A) = W$, i.e., A is surjective. □

Corollary C.50 (Self-adjoint bijective operator). *Assume that V is reflexive. Let $A \in \mathcal{L}(V;V')$ be a self-adjoint operator. Then A is bijective iff there is a real number $\alpha > 0$ such that*

$$\|A(v)\|_{V'} \geq \alpha \|v\|_V, \qquad \forall v \in V. \tag{C.23}$$

Proof. Owing to Theorem C.49, the bijectivity of A implies that A satisfies the inequality (C.23). Conversely (C.23) means that A is injective. It follows that A^* is injective since $A^* = A \circ J_V^{-1}$ owing to the reflexivity hypothesis. The bijectivity of A then follows from Theorem C.49(iii). □

Let $A \in \mathcal{L}(V;W)$ be a bijective operator. We have seen in Corollary C.38 that $A^{-1} \in \mathcal{L}(W;V)$. We can now characterize more precisely the constants associated with the boundedness of A^{-1} and the closedness of its range.

Lemma C.51 (Bounds on A^{-1}). *Let $A \in \mathcal{L}(V;W)$ be a bijective operator. Then $\|A^{-1}\|_{\mathcal{L}(W;V)} = \alpha^{-1}$ with α defined in* (C.22)*, and*

$$\inf_{w\in W} \frac{\|A^{-1}(w)\|_V}{\|w\|_W} = \inf_{w\in W} \sup_{v'\in V'} \frac{|\langle v', A^{-1}(w)\rangle_{V',V}|}{\|v'\|_{V'}\|w\|_W} = \|A\|^{-1}_{\mathcal{L}(V;W)}. \tag{C.24}$$

Proof. (1) Using the bijectivity of A, we have

$$\left(\sup_{w\in W}\frac{\|A^{-1}(w)\|_V}{\|w\|_W}\right)^{-1} = \left(\sup_{v\in V}\frac{\|v\|_V}{\|A(v)\|_W}\right)^{-1}$$
$$= \inf_{v\in V}\frac{\|A(v)\|_W}{\|v\|_V} = \inf_{v\in V}\sup_{w'\in W'}\frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}\|v\|_V},$$

which shows using (C.22) that $\|A^{-1}\|_{\mathcal{L}(W;V)} = \alpha^{-1}$.
(2) Similarly we have

$$\left(\inf_{w\in W}\frac{\|A^{-1}(w)\|_V}{\|w\|_W}\right)^{-1} = \sup_{v\in V}\frac{\|A(v)\|_W}{\|v\|_V} = \|A\|_{\mathcal{L}(V;W)},$$

which leads to the inf-sup condition (C.24) once we observe that $\|A^{-1}(w)\|_V = \sup_{v'\in V'}\frac{|\langle v', A^{-1}(w)\rangle_{V',V}|}{\|v'\|_{V'}}$ owing to Corollary C.14. □

Let us finish this section with some useful results concerning the bijectivity of the adjoint operator and some bounds on its inverse.

Corollary C.52 (Bijectivity of A^*). *Let $A \in \mathcal{L}(V;W)$ and consider its adjoint $A^* \in \mathcal{L}(W';V')$. Then A is bijective if and only if A^* is bijective.*

Proof. We observe that the statement (ii) in Theorem C.49 is equivalent to A^* injective and A^* surjective owing to the equivalence of the statements (i) and (ii) from Lemma C.39. □

Lemma C.53 (Inf-sup condition). *Let $A \in \mathcal{L}(V;W)$ be a bijective operator. Assume that V is reflexive. The following holds true:*

$$\inf_{v\in V}\sup_{w'\in W'}\frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}\|v\|_V} = \inf_{w'\in W'}\sup_{v\in V}\frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}\|v\|_V}. \tag{C.25}$$

In other words the inf-sup constant of $A \in \mathcal{L}(V;W)$ on $V \times W'$ is equal to the inf-sup constant of $A^* \in \mathcal{L}(W';V')$ on $W' \times V$.

Proof. The left-hand side, l, and the right-hand side, r, of (C.25) are two positive finite numbers since A is a bijective bounded operator. The left-hand side being equal to l means that l is the largest number such that $\|A(v)\|_W \geq l\,\|v\|_V$ for all v in V. Let $w' \in W'$ and $w \in W$. Since A is surjective, we can consider its right inverse $A^\dagger$, and the previous statement regarding l implies that $l\,\|A^\dagger(w)\|_V \leq \|w\|_W$. Since $A(A^\dagger(w)) = w$, this implies that

$$\|w'\|_{W'} = \sup_{w\in W}\frac{|\langle w', w\rangle_{W',W}|}{\|w\|_W} = \sup_{w\in W}\frac{|\langle A^*(w'), A^\dagger(w)\rangle_{V',V}|}{\|w\|_W}$$
$$\leq \|A^*(w')\|_{V'}\sup_{w\in W}\frac{\|A^\dagger(w)\|_V}{\|w\|_W} \leq \frac{1}{l}\|A^*(w')\|_{V'},$$

so that $l \leq r$. The other inequality $r \leq l$ is proved similarly by working with W' in lieu of V, V' in lieu of W and A^* in lieu of A, leading to

$$\inf_{w' \in W'} \sup_{v'' \in V''} \frac{|\langle v'', A^*(w') \rangle_{V'',V'}|}{\|v''\|_{V''} \|w'\|_{W'}} \leq \inf_{v'' \in V''} \sup_{w' \in W'} \frac{|\langle v'', A^*(w') \rangle_{V'',V'}|}{\|v''\|_{V''} \|w'\|_{W'}},$$

and we conclude by using the reflexivity of V. $\square$

Remark C.54 (Counterexample). The identity (C.25) can fail if $A \neq 0$ is not bijective. For instance, if $A : (x_1, x_2, x_3 \ldots) \mapsto (0, x_1, x_2, x_3, \ldots)$ is the right shift operator in ℓ^2, then $A^* : (x_1, x_2, x_3 \ldots) \mapsto (x_2, x_3, x_4, \ldots)$ is the left shift operator. It can be verified that A is injective but not surjective, whereas A^* is surjective but not injective. Using the notation of the proof of Lemma C.53, it can also be shown that $l = 1$ and $r = 0$. $\square$

Lemma C.55 (Bounds on A^{-*}). *Let $A \in \mathcal{L}(V; W)$ be a bijective operator. Assume that V is reflexive. Let $A^* \in \mathcal{L}(W'; V')$ be the adjoint of A and let $A^{-*} \in \mathcal{L}(V'; W')$ denote its inverse. Then $\|A^{-*}\|_{\mathcal{L}(V';W')} = \alpha^{-1}$ with α defined in* (C.22)*, and*

$$\inf_{v' \in V'} \frac{\|A^{-*}(v')\|_{W'}}{\|v'\|_{V'}} = \inf_{v' \in V'} \sup_{w \in W} \frac{|\langle A^{-*}(v'), w \rangle_{W',W}|}{\|v'\|_{V'} \|w\|_W} = \frac{1}{\|A\|_{\mathcal{L}(V;W)}}. \quad \text{(C.26)}$$

Proof. Notice that the notation A^{-*} is meant to reflect that $(A^{-1})^* = (A^*)^{-1}$. Combining the results from Lemma C.30 and Lemma C.51, we infer that $\|A^{-*}\|_{\mathcal{L}(V';W')} = \|A^{-1}\|_{\mathcal{L}(W;V)} = \alpha^{-1}$. Moreover (C.26) follows from (C.24) since $\langle A^{-*}(v'), w \rangle_{W',W} = \langle v', A^{-1}(w) \rangle_{V',V}$. $\square$

C.8 Coercive operators

We now focus on the more specific class of coercive operators. The notion of coercivity plays a central role in the analysis of PDEs involving the Laplace operator, and more generally elliptic operators (see Chapter 31).

Definition C.56 (Coercive operator). *Let V be a complex Banach space. The operator $A \in \mathcal{L}(V; V')$ is said to be a* coercive *if there exist a real number $\alpha > 0$ and a complex number $\xi \in \mathbb{C}$ with $|\xi| = 1$ such that*

$$\Re\left(\xi \langle A(v), v \rangle_{V',V}\right) \geq \alpha \|v\|_V^2, \qquad \forall v \in V. \quad \text{(C.27)}$$

In the real case we have either $\xi = 1$ or $\xi = -1$.

Remark C.57 (Self-adjoint case). Let A be a coercive operator and assume that A is self-adjoint (see Definition C.31), so that $\langle A(v), v \rangle_{V',V}$ is real for all $v \in V$. Then coercivity means that $\Re(\xi) \langle A(v), v \rangle_{V',V} \geq \alpha \|v\|_V^2$. Hence up to rescaling α, one can always take either $\xi = 1$ or $\xi = -1$ when A is self-adjoint. $\square$

The coercivity condition is sometimes defined as follows: There exists a real number $\alpha > 0$ such that $|\langle A(v), v\rangle_{V',V}| \geq \alpha \|v\|_V^2$ for all $v \in V$. Although this variant looks slightly more general since $\Re(\xi\langle A(v), v\rangle_{V',V}) \leq |\langle A(v), v\rangle_{V',V}|$, it is equivalent to (C.27). More precisely we have the following result.

Lemma C.58 (Real part vs. module). *Let $\alpha > 0$ and let V be a Hilbert space. The following two statements are equivalent:*
(i) $|\langle A(v), v\rangle_{V',V}| \geq \alpha \|v\|_V^2$ *for all* $v \in V$.
(ii) *There is* $\xi \in \mathbb{C}$ *with* $|\xi| = 1$ *s.t.* (C.27) *holds true.*

Proof. Let us prove the claim in the real case. It suffices to show that the statement (i) implies that $\langle A(v), v\rangle_{V',V}$ has always the same sign for all nonzero $v \in V$. Reasoning by contradiction, if there are nonzero $v, w \in V$ such that $\langle A(v), v\rangle_{V',V} < 0$ and $\langle A(w), w\rangle_{V',V} > 0$, then the second-order polynomial $\mathbb{R} \ni \lambda \mapsto \langle A(v + \lambda w), v + \lambda w\rangle_{V',V} \in \mathbb{R}$ has at least one root $\lambda_* \in \mathbb{R}$. The statement (i) yields $v + \lambda_* w = 0$, so that $\langle A(v), v\rangle_{V',V} = \lambda_*^2 \langle A(w), w\rangle_{V',V} > 0$, which contradicts $\langle A(v), v\rangle_{V',V} < 0$. We refer the reader to Brezis [63, p. 366] for the proof in the complex case (see also Exercise 45.10 for a proof of the Hausdorff–Toeplitz theorem). □

It turns out that the notion of coercivity is relevant only in *Hilbert spaces*.

Proposition C.59 (Hilbert structure). *Let V be a Banach space. V can be equipped with a Hilbert structure with the same topology if and only if there is a coercive operator in $\mathcal{L}(V; V')$.*

Proof. Setting $(\!(v, w)\!)_V := \frac{1}{2}(\xi\langle A(v), w\rangle_{V',V} + \overline{\xi\langle A(w), v\rangle_{V',V}})$, we define a sesquilinear form on $V \times V$ that is Hermitian. The coercivity and boundedness of A imply that

$$\alpha \,\|v\|_V^2 \leq (\!(v, v)\!)_V \leq \|A\|_{\mathcal{L}(V;V')} \,\|v\|_V^2,$$

for all $v \in V$. This shows positive definiteness (so that $(\!(\cdot, \cdot)\!)_V$ is an inner product in V) and that the induced norm is equivalent to $\|\cdot\|_V$. □

Let us now examine the connections between coercivity and bijectivity.

Corollary C.60 (Coericivty as a sufficient condition). *If the operator $A \in \mathcal{L}(V; V')$ is coercive, then it is bijective.*

Proof. This is the Lax–Milgram lemma which is proved in §25.2. □

Definition C.61 (Monotone operator). *The operator $A \in \mathcal{L}(V; V')$ is said to be* monotone *if*

$$\Re(\langle A(v), v\rangle_{V',V}) \geq 0, \qquad \forall v \in V. \tag{C.28}$$

Corollary C.62 (Coercivity as a necessary and sufficient condition). *Assume that V is reflexive. Let $A \in \mathcal{L}(V; V')$ be a monotone self-adjoint operator. Then A is bijective iff it is coercive (with $\xi = 1$).*

Proof. See Exercise 25.7. □

If the operator $A \in \mathcal{L}(V;V')$ is coercive (and therefore bijective), its inverse $A^{-1} \in \mathcal{L}(V';V)$ turns out to be coercive as well. Indeed using the coercivity of A and the lower bound on A^{-1} resulting from (C.24), we infer that for all $\phi \in V'$,

$$\begin{aligned}\Re(\xi\langle \phi, A^{-1}(\phi)\rangle_{V',V}) &= \Re(\xi\langle A(A^{-1}(\phi)), A^{-1}(\phi)\rangle_{V',V}) \\ &\geq \alpha\,\|A^{-1}(\phi)\|_V^2 \geq \frac{\alpha}{\|A\|^2}\|\phi\|_{V'}^2,\end{aligned} \tag{C.29}$$

with the shorthand notation $\|A\| := \|A\|_{\mathcal{L}(V;V')}$. The following results provide more precise characterizations of the coercivity constant of A^{-1}.

Lemma C.63 (Coercivity of A^{-1}, self-adjoint case). *Let $A \in \mathcal{L}(V;V')$ be a self-adjoint coercive operator (i.e.,* (C.27) *holds true with either $\xi = 1$ or $\xi = -1$ according to Remark* C.57*). Then A^{-1} is coercive with coercivity constant $\|A\|^{-1}$, and we have more precisely*

$$\inf_{\phi\in V'} \frac{\xi\langle \phi, A^{-1}(\phi)\rangle_{V',V}}{\|\phi\|_{V'}^2} = \frac{1}{\|A\|}. \tag{C.30}$$

Proof. Assume that $\xi = 1$ (the case $\xi = -1$ is identical). The coercivity of A together with $A = A^*$ implies that $(\!(v,w)\!)_A := \langle A(v), w\rangle_{V',V}$ is an inner product on V. Let $v \in V$ and $\phi \in V'$. Since $\langle \phi, v\rangle_{V',V} = (\!(A^{-1}(\phi), v)\!)_A$, the Cauchy–Schwarz inequality implies that

$$\begin{aligned}\Re(\langle \phi, v\rangle_{V',V}) &\leq (\!(v,v)\!)_A^{\frac12} (\!(A^{-1}(\phi), A^{-1}(\phi))\!)_A^{\frac12} \\ &= \langle A(v), v\rangle_{V',V}^{\frac12} \langle \phi, A^{-1}(\phi)\rangle_{V',V}^{\frac12} \\ &\leq \|A\|^{\frac12}\,\|v\|_V\,\langle \phi, A^{-1}(\phi)\rangle_{V',V}^{\frac12},\end{aligned}$$

where we used the boundedness of A. This implies that

$$\|\phi\|_{V'} = \sup_{v\in V} \frac{|\langle \phi, v\rangle_{V',V}|}{\|v\|_V} \leq \|A\|^{\frac12}\,\langle \phi, A^{-1}(\phi)\rangle_{V',V}^{\frac12}.$$

Taking the supremum over $\phi \in V'$, we infer that

$$\frac{1}{\|A\|} \leq \inf_{\phi\in V'} \frac{\langle \phi, A^{-1}(\phi)\rangle_{V',V}}{\|\phi\|_{V'}^2} \leq \inf_{\phi\in V'}\sup_{\psi\in V'} \frac{|\langle \psi, A^{-1}(\phi)\rangle_{V',V}|}{\|\psi\|_{V'}\|\phi\|_{V'}} = \frac{1}{\|A\|},$$

where the last equality follows from (C.24). Thus all the terms are equal, and this concludes the proof. □

Let us now consider the case where the operator $A \in \mathcal{L}(V;V')$ is not necessarily self-adjoint. Since Hilbert spaces are reflexive, the adjoint of A is $A^* \in \mathcal{L}(V;V')$, and we have $\langle A^*(v), w\rangle_{V',V} = \overline{\langle A(w), v\rangle_{V',V}}$.

Lemma C.64 (Coercivity of A^{-1}, general case). *Let $A \in \mathcal{L}(V;V')$ be a coercive operator with parameters $\alpha > 0$ and $\xi \in \mathbb{C}$ with $|\xi| = 1$. Let the self-adjoint part of ξA be defined as $(\xi A)_{\mathrm{s}} := \frac{1}{2}(\xi A + (\xi A)^*) = \frac{1}{2}(\xi A + \overline{\xi} A^*)$. The following holds true:*

$$\frac{\alpha}{\|A\|^2} \leq \inf_{\phi \in V'} \frac{\Re(\xi \langle \phi, A^{-1}(\phi)\rangle_{V',V})}{\|\phi\|_{V'}^2} \leq \frac{1}{\|(\xi A)_{\mathrm{s}}\|}, \tag{C.31}$$

and the upper bound is attained if and only if ξA is self-adjoint (and the value of the upper bound is then $\frac{1}{\|A\|}$).

Proof. The lower bound in (C.31) is a restatement of (C.29). To establish the upper bound, let us set $B := \xi A$. Then B are B_{s} are coercive (and therefore invertible) operators, since

$$\langle B_{\mathrm{s}}(v), v\rangle_{V',V} = \Re(\langle B(v), v\rangle_{V',V}) = \Re(\xi \langle A(v), v\rangle_{V',V}) \geq \alpha \|v\|_V^2,$$

for all $v \in V$. A direct calculation shows that

$$\begin{aligned}
&B^{-1}(B - B_{\mathrm{s}})B_{\mathrm{s}}^{-1}(B^* - B_{\mathrm{s}})B^{-*} \\
&= (B_{\mathrm{s}}^{-1} - B^{-1})(I - B_{\mathrm{s}}B^{-*}) = B_{\mathrm{s}}^{-1} - B^{-1} - B^{-*} + B^{-1}B_{\mathrm{s}}B^{-*} \\
&= B_{\mathrm{s}}^{-1} - B^{-1} - B^{-*} + \frac{1}{2}B^{-1}(B + B^*)B^{-*} = B_{\mathrm{s}}^{-1} - \frac{1}{2}(B^{-1} + B^{-*}).
\end{aligned}$$

This implies that for all $\phi \in V'$,

$$\begin{aligned}
&\langle \phi, B_{\mathrm{s}}^{-1}(\phi)\rangle_{V',V} \\
&= \frac{1}{2}\langle \phi, (B^{-1} + B^{-*})(\phi)\rangle_{V',V} + \langle \phi, B^{-1}(B - B_{\mathrm{s}})B_{\mathrm{s}}^{-1}(B^* - B_{\mathrm{s}})B^{-*}(\phi)\rangle_{V',V} \\
&= \Re(\langle \phi, B^{-1}(\phi)\rangle_{V',V}) + \langle \psi, B_{\mathrm{s}}^{-1}(\psi)\rangle_{V',V} \geq \Re(\langle \phi, B^{-1}(\phi)\rangle_{V',V}),
\end{aligned}$$

with $\psi := (B^* - B_{\mathrm{s}})B^{-*}(\phi)$ and where we used that $\langle \psi, B_{\mathrm{s}}^{-1}(\psi)\rangle_{V',V} \geq 0$. Applying Lemma C.63 to B_{s} which is coercive and self-adjoint, we conclude that

$$\frac{1}{\|B_{\mathrm{s}}\|} = \inf_{\phi \in V'} \frac{\langle \phi, B_{\mathrm{s}}^{-1}(\phi)\rangle_{V',V}}{\|\phi\|_{V'}^2} \geq \inf_{\phi \in V'} \frac{\Re(\langle \phi, B^{-1}(\phi)\rangle_{V',V})}{\|\phi\|_{V'}^2}.$$

Since $\langle \phi, B^{-1}(\phi)\rangle_{V',V} = (\overline{\xi})^{-1}\langle \phi, A^{-1}(\phi)\rangle_{V',V}$ and $(\overline{\xi})^{-1} = \xi$, this proves the upper bound in (C.31). Let us finally show that A being self-adjoint is a necessary and sufficient condition for attaining the upper bound. If A is self-adjoint, Lemma C.63 proves that the upper bound is attained. Conversely, if the upper bound is attained, the above identity relating B_{s}^{-1} and B^{-1}, together with the coercivity of B_{s}^{-1}, implies that $\psi = 0$, that is, $(B^* - B_{\mathrm{s}})B^{-*}(\phi) = 0$ for all $\phi \in V'$. Since B^{-*} is invertible, this means that $B^* = B_{\mathrm{s}}$, i.e., $B = B_{\mathrm{s}}$, proving that B is self-adjoint. $\square$

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