Part I, Chapter 2

Weak derivatives and Sobolev spaces

We investigate in this chapter the notion of differentiation for Lebesgue integrable functions. We introduce an extension of the classical concept of derivative and partial derivative which is called weak derivative. This notion will be used throughout the book. It is particularly useful when one tries to differentiate finite element functions that are continuous and piecewise polynomial. In that case, one does not need to bother about the points where the classical derivative is multivalued to define the weak derivative. We also introduce the concept of Sobolev spaces. These spaces are useful to study the well-posedness of partial differential equations and their approximation using finite elements.

2.1 Differentiation

We study here the concept of differentiation for Lebesgue integrable functions.

2.1.1 Lebesgue points

Theorem 2.1 (Lebesgue points). *Let $f \in L^1(D)$. Let $B(\boldsymbol{x},h)$ be the ball of radius $h>0$ centered at $\boldsymbol{x} \in D$. The following holds true for a.e. $\boldsymbol{x} \in D$:*

$$\lim_{h\downarrow 0} \frac{1}{|B(\boldsymbol{x},h)|} \int_{B(\boldsymbol{x},h)} |f(\boldsymbol{y}) - f(\boldsymbol{x})| \,\mathrm{d}y = 0. \tag{2.1}$$

Points $\boldsymbol{x} \in D$ where (2.1) *holds true are called* Lebesgue points *of f.*

Proof. See, e.g., Rudin [170, Thm. 7.6]. □

This result says that for a.e. $\boldsymbol{x} \in D$, the averages of $|f(\cdot) - f(\boldsymbol{x})|$ are small over small balls centered at $\boldsymbol{x}$, i.e., f does not oscillate too much in the neighborhood of $\boldsymbol{x}$. Notice that if the function f is continuous at $\boldsymbol{x}$, then $\boldsymbol{x}$ is a Lebesgue point of f (recall that a continuous function is uniformly continuous over compact sets).

Let $\mathcal{H} \subset \mathbb{R}$ be is a countable set with 0 as unique accumulation point (the sign of the members of $\mathcal{H}$ is unspecified). Let $F : \mathbb{R} \to \mathbb{R}$. We say that F is strongly differentiable at x if the sequence $(\frac{F(x+h)-F(x)}{h})_{h\in\mathcal{H}}$ converges.

Theorem 2.2 (Lebesgue's differentiation). *Let $f \in L^1(\mathbb{R})$. Let $F(x) := \int_{-\infty}^{x} f(t)\,\mathrm{d}t$. Then F is strongly differentiable at every Lebesgue point x of f, and at these points we have $F'(x) = f(x)$.*

Proof. See Exercise 2.2. □

In the above theorem, we have $F'(x) = f(x)$ for a.e. x in $\mathbb{R}$. Thus, it is tempting to move away from the classical sense of differentiation and view F' as a function in $L^1(\mathbb{R})$. If we could make sense of F' in $L^1(\mathbb{R})$, then $F(x) = \int_{-\infty}^{x} F'(t)\,\mathrm{d}t$ would be an extension of the fundamental theorem of calculus in Lebesgue spaces. As an example of this possibility, let $f := \mathbb{1}_{[0,\infty)}$ be the Heaviside function (i.e., $f(x) := 1$ if $x \geq 0$ and $f(x) := 0$ otherwise). Notice that $f \notin L^1(\mathbb{R})$ but $f \in L^1_{\mathrm{loc}}(\mathbb{R})$ (see Definition 1.29), and $F(x) := \int_{-\infty}^{x} f(t)\,\mathrm{d}t$ is well defined. Then $F(x) = 0$ if $x < 0$ and $F(x) = x$ if $x > 0$ (notice that 0 is not a Lebesgue point of f; see Exercise 2.1). We would like to say that $F' = f$ in $L^1_{\mathrm{loc}}(\mathbb{R})$. The objective of the rest of this section is to make sense of the above argument.

2.1.2 Weak derivatives

Definition 2.3 (Weak derivative). *Let D be an open set in $\mathbb{R}^d$. Let $u, v \in L^1_{\mathrm{loc}}(D)$. Let $i \in \{1{:}d\}$. We say that v is the* weak partial derivative *of u in the direction i if*

$$\int_D u\partial_i\varphi\,\mathrm{d}x = -\int_D v\varphi\,\mathrm{d}x, \qquad \forall\varphi \in C_0^\infty(D), \tag{2.2}$$

and we write $\partial_i u := v$. Let $\alpha \in \mathbb{N}^d$ be a multi-index. We say that v is the weak α-th partial derivative of u and we write $\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} u := v$ if

$$\int_D u\partial_1^{\alpha_1} \dots \partial_d^{\alpha_d}\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_D v\varphi\,\mathrm{d}x, \qquad \forall\varphi \in C_0^\infty(D), \tag{2.3}$$

where $|\alpha| := \alpha_1 + \ldots + \alpha_d$. Finally, we write $\partial^\alpha u := \partial_1^{\alpha_1} \dots \partial_d^{\alpha_d} u$, and we set $\partial^{(0,\ldots,0)} u := u$.

Lemma 2.4 (Uniqueness). *Let $u \in L^1_{\mathrm{loc}}(D)$. If u has a weak α-th partial derivative, then it is uniquely defined.*

Proof. Let $v_1, v_2 \in L^1_{\mathrm{loc}}(D)$ be two weak α-th derivatives of u. We have

$$\int_D v_1\varphi\,\mathrm{d}x = (-1)^{|\alpha|}\int_D u\partial^\alpha\varphi\,\mathrm{d}x = \int_D v_2\varphi\,\mathrm{d}x, \qquad \forall\varphi \in C_0^\infty(D).$$

Hence, $\int_D (v_1 - v_2)\varphi \, \mathrm{d}x = 0$. The vanishing integral theorem (Theorem 1.32) implies that $v_1 = v_2$ a.e. in D. $\square$

If $u \in C^{|\alpha|}(D)$, then the usual and the weak α-th partial derivatives are identical. Moreover, it can be shown that if $\alpha, \beta \in \mathbb{N}^d$ are multi-indices such that $\alpha_i \geq \beta_i$ for all $i \in \{1{:}d\}$, then if the α-th weak derivative of u exists in $L^1_{\mathrm{loc}}(D)$, so does the β-th weak derivative. For instance, with $d = 1$ (writing ∂_x instead of ∂_1), if $\partial_{xx} u$ exists in $L^1_{\mathrm{loc}}(D)$, so does $\partial_x u$; see Exercise 2.4.

Example 2.5 (1D). Let us revisit the heuristic argument at the end of §2.1.1. Let $D := (-1, 1)$. (i) Let us first consider a continuous function $u \in C^0(D; \mathbb{R})$, e.g., $u(x) := 0$ if $x < 0$ and $u(x) := x$ otherwise. Then u has a weak derivative. Indeed, let $v \in L^1(D)$ be s.t. $v(x) := 0$ if $x < 0$ and $v(x) := 1$ otherwise. Let $\varphi \in C_0^\infty(D)$. We have

$$\int_{-1}^{1} u \partial_x \varphi \, \mathrm{d}x = \int_0^1 x \partial_x \varphi \, \mathrm{d}x = -\int_0^1 \varphi \, \mathrm{d}x = -\int_{-1}^{1} v\varphi \, \mathrm{d}x.$$

Hence, v is the weak derivative of u. (Notice that $\tilde{v}$ defined by $\tilde{v}(x) := 0$ if $x < 0$, $\tilde{v}(0) := \frac{1}{2}$ and $\tilde{v}(x) := 1$ if $x > 0$ is also a weak derivative of u, but $v = \tilde{v}$ a.e. in D, i.e., v and $\tilde{v}$ coincide in the Lebesgue sense.) (ii) Let us now consider a function $u \in L^1(D; \mathbb{R})$ that is piecewise smooth but exhibits a jump at $x = 0$, e.g., $u(x) := -1$ if $x < 0$ and $u(x) := x$ otherwise. Then u does not have a weak derivative. Let us prove this statement by contradiction. Assume that there is $v \in L^1_{\mathrm{loc}}(D)$ s.t. $\partial_x u = v$. We have

$$\int_{-1}^{1} v\varphi \, \mathrm{d}x = -\int_{-1}^{1} u \partial_x \varphi \, \mathrm{d}x = \int_{-1}^{0} \partial_x \varphi \, \mathrm{d}x - \int_0^1 x \partial_x \varphi \, \mathrm{d}x = \varphi(0) + \int_0^1 \varphi \, \mathrm{d}x,$$

for all $\varphi \in C_0^\infty(D)$. Let $\{\varphi_n\}_{n \in \mathbb{N}}$ be a sequence of functions in $C_0^\infty(D)$ s.t. $0 \leq \varphi_n(x) \leq 1$ for all $x \in D$, $\varphi_n(0) = 1$, and $\varphi_n \to 0$ a.e. in D. Lebesgue's dominated convergence theorem implies that $1 = \lim_{n\to\infty}(\int_{-1}^{1} v\varphi_n \, \mathrm{d}x - \int_0^1 \varphi_n \, \mathrm{d}x) = 0$, which is a contradiction. $\square$

Lemma 2.6 (Passing to the limit). *Let $\{v_n\}_{n \in \mathbb{N}}$ be a sequence in $L^p(D)$, $p \in [1, \infty]$, with weak α-th partial derivatives $\{\partial^\alpha v_n\}_{n \in \mathbb{N}}$ in $L^p(D)$. Assume that $v_n \to v$ in $L^p(D)$ and $\partial^\alpha v_n \to g_\alpha$ in $L^p(D)$. Then v has a weak α-th partial derivative and $\partial^\alpha v = g_\alpha$.*

Proof. The assumptions imply that $\lim_{n\to\infty} \int_D \partial^\alpha v_n \varphi \, \mathrm{d}x = \int_D g_\alpha \varphi \, \mathrm{d}x$ and

$$\lim_{n\to\infty} \int_D \partial^\alpha v_n \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \lim_{n\to\infty} \int_D v_n \partial^\alpha \varphi \, \mathrm{d}x = (-1)^{|\alpha|} \int_D v \partial^\alpha \varphi \, \mathrm{d}x,$$

for all $\varphi \in C_0^\infty(D)$. The conclusion follows readily. $\square$

A function $v \in L^1_{\mathrm{loc}}(D)$ is said to be *locally Lipschitz* in D if for all $\boldsymbol{x} \in D$, there is a neighborhood $\mathcal{N}_{\boldsymbol{x}}$ of $\boldsymbol{x}$ in D and a constant $L_{\boldsymbol{x}}$ such that $|v(\boldsymbol{z}) - v(\boldsymbol{y})| \leq L_{\boldsymbol{x}} \|\boldsymbol{z} - \boldsymbol{y}\|_{\ell^2(\mathbb{R}^d)}$ for all $\boldsymbol{y}, \boldsymbol{z} \in \mathcal{N}_{\boldsymbol{x}}$.

Theorem 2.7 (Rademacher). *Let D be an open set in $\mathbb{R}^d$. Let f be a locally Lipschitz function in D. Then f is differentiable in the classical sense a.e. in D. The function f is also weakly differentiable, and the classical and weak derivatives of f coincide a.e. in D.*

Proof. See [99, p. 280], [138, p. 44]. □

2.2 Sobolev spaces

In this section, we introduce integer-order and fractional-order Sobolev spaces. The scale of Sobolev spaces plays a central role in the finite element error analysis to quantify the decay rate of the approximation error.

2.2.1 Integer-order spaces

Definition 2.8 ($W^{m,p}(D)$). *Let $m \in \mathbb{N}$ and $p \in [1, \infty]$. Let D be an open set in $\mathbb{R}^d$. We define the Sobolev space*

$$W^{m,p}(D) := \{v \in L^1_{\text{loc}}(D) \mid \partial^\alpha v \in L^p(D), \forall \alpha \in \mathbb{N}^d \text{ s.t. } |\alpha| \leq m\}, \tag{2.4}$$

where the derivatives are weak partial derivatives. We write $W^{m,p}(D; \mathbb{R}^q)$, $q \geq 1$, for the space composed of $\mathbb{R}^q$-valued functions whose components are all in $W^{m,p}(D)$, and we write $\boldsymbol{W}^{m,p}(D)$ whenever $q = d$.

Whenever it is possible to identify a length scale ℓ_D associated with D, e.g., its diameter $\ell_D := \text{diam}(D)$ if D is bounded, we equip $W^{m,p}(D)$ with the following norm and seminorm: If $p \in [1, \infty)$, we set

$$\|v\|^p_{W^{m,p}(D)} := \sum_{|\alpha| \leq m} \ell_D^{|\alpha|p} \|\partial^\alpha v\|^p_{L^p(D)}, \qquad |v|^p_{W^{m,p}(D)} := \sum_{|\alpha|=m} \|\partial^\alpha v\|^p_{L^p(D)},$$

and if $p = \infty$, we set

$$\|v\|_{W^{m,\infty}(D)} := \max_{|\alpha| \leq m} \ell_D^{|\alpha|} \|\partial^\alpha v\|_{L^\infty(D)}, \qquad |v|_{W^{m,\infty}(D)} := \max_{|\alpha|=m} \|\partial^\alpha v\|_{L^\infty(D)},$$

where the sums and the maxima run over multi-indices $\alpha \in \mathbb{N}^d$. The advantage of using the factor ℓ_D is that all the terms in the sums or maxima have the same dimension (note that $\|\cdot\|_{W^{m,p}(D)}$ and $|\cdot|_{W^{m,p}(D)}$ have a different scaling w.r.t. ℓ_D). If there is no length scale available or if one works with dimensionless space variables, one sets $\ell_D := 1$ in the above definitions.

Proposition 2.9 (Banach space). *$W^{m,p}(D)$ equipped with the $\|\cdot\|_{W^{m,p}(D)}$-norm is a Banach space. For $p = 2$, the space*

$$H^m(D) := W^{m,2}(D) \tag{2.5}$$

is a real Hilbert space when equipped with the inner product $(v,w)_{H^m(D)} := \sum_{|\alpha|\le m}\int_D \partial^\alpha v\, \partial^\alpha w \,\mathrm{d}x$. *Similarly,* $H^m(D;\mathbb{C})$ *is a complex Hilbert space when equipped with the inner product* $(v,w)_{H^m(D;\mathbb{C})} := \sum_{|\alpha|\le m}\int_D \partial^\alpha v\, \overline{\partial^\alpha w}\,\mathrm{d}x$.

Proof. We are going to do the proof for $m = 1$. See e.g., [3, Thm. 3.3], [99, p. 249], or [189, Lem. 5.2] for the general case. Let $\{v_n\}_{n\in\mathbb{N}}$ be a Cauchy sequence in $W^{1,p}(D)$. Then $\{v_n\}_{n\in\mathbb{N}}$ is a Cauchy sequence in $L^p(D)$ and the sequences of weak partial derivatives $\{\partial_i v_n\}_{n\in\mathbb{N}}$ are also Cauchy sequences in $L^p(D)$. Hence, there is $v \in L^p(D)$ and there are $g_1,\ldots,g_d \in L^p(D)$ such that $v_n \to v$ in $L^p(D)$ and $\partial_i v_n \to g_i$ in $L^p(D)$. We conclude by invoking Lemma 2.6. □

Example 2.10 ($H^1(D)$). Taking $m := 1$ and $p := 2$ we have

$$H^1(D) := \{v \in L^2(D) \mid \partial_i v \in L^2(D),\ \forall i \in \{1{:}d\}\},$$

(notice that $L^2(D) \subset L^1_{\mathrm{loc}}(D)$) and

$$\|v\|_{H^1(D)} := \Big(\|v\|^2_{L^2(D)} + \ell_D^2 |v|^2_{H^1(D)}\Big)^{\frac12}, \quad |v|^2_{H^1(D)} := \sum_{i\in\{1:d\}} \|\partial_i v\|^2_{L^2(D)}.$$

Let ∇v be the column vector in $\mathbb{R}^d$ whose components are the directional weak derivatives $\partial_i v$ of v. Then a more compact notation is $H^1(D) := \{v \in L^2(D) \mid \nabla v \in \boldsymbol{L}^2(D)\}$ and $|v|_{H^1(D)} := \|\nabla v\|_{\boldsymbol{L}^2(D)}$. □

Lemma 2.11 (Kernel of ∇). *Let D be open and connected set in $\mathbb{R}^d$. Let $v \in W^{1,p}(D)$, $p \in [1,\infty]$. Then $\nabla v = \mathbf{0}$ a.e. on D iff v is constant.*

Proof. We prove the result for $D := (-1,1)$ and we refer the reader to [138, p. 24], [189, Lem. 6.4] for the general case. Let $u \in L^1_{\mathrm{loc}}(D)$ be such that $\partial_x u = 0$. Fix a function $\rho \in C_0^\infty(D)$ such that $\int_D \rho\,\mathrm{d}x = 1$ and set $c_\rho := \int_D u\rho\,\mathrm{d}x$. Let now $\varphi \in C_0^\infty(D)$ and set $c_\varphi := \int_D \varphi\,\mathrm{d}x$. Then the function $\psi(x) := \int_{-1}^x (\varphi(y) - c_\varphi \rho(y))\,\mathrm{d}y$ is by construction in $C_0^\infty(D)$, and we have $\partial_x \psi(x) = \varphi(x) - c_\varphi\rho(x)$. Since $\int_D u\partial_x\psi\,\mathrm{d}x = -\int_D (\partial_x u)\psi\,\mathrm{d}x = 0$ by assumption on $\partial_x u$, we infer that

$$\int_D u\varphi\,\mathrm{d}x = \int_D u(\partial_x\psi + c_\varphi\rho)\,\mathrm{d}x = c_\varphi \int_D u\rho\,\mathrm{d}x = c_\rho \int_D \varphi\,\mathrm{d}x,$$

for all $\varphi \in C_0^\infty(D)$. Theorem 1.32 shows that $u = c_\rho$. □

Remark 2.12 (Lipschitz functions). Let D be an open set in $\mathbb{R}^d$. The space of Lipschitz functions $C^{0,1}(D)$ is closely related to the Sobolev space $W^{1,\infty}(D)$. Indeed, $C^{0,1}(D)\cap L^\infty(D)$ is continuously embedded into $W^{1,\infty}(D)$. Conversely, if $v \in W^{1,\infty}(D)$, then $|v(\boldsymbol{y}) - v(\boldsymbol{z})| \le d_D(\boldsymbol{y},\boldsymbol{z})\|\nabla v\|_{\boldsymbol{L}^\infty(D)}$ for all $\boldsymbol{y},\boldsymbol{z} \in D$, where $d_D(\boldsymbol{y},\boldsymbol{z})$ denotes the geodesic distance of $\boldsymbol{y}$ to $\boldsymbol{z}$ in D, i.e., the shortest length of a smooth path connecting $\boldsymbol{y}$ to $\boldsymbol{z}$ in D (if D is convex,

$d_D(\boldsymbol{y},\boldsymbol{z}) = \|\boldsymbol{y}-\boldsymbol{z}\|_{\ell^2}$); see [189, Lem. 7.8]. A set $D \subset \mathbb{R}^d$ is said to be quasiconvex if there exists $C \geq 1$ s.t. every pair of points $\boldsymbol{x},\boldsymbol{y} \in D$ can be joined by a curve γ in D with length$(\gamma) \leq C\|\boldsymbol{x}-\boldsymbol{y}\|_{\ell^2}$. If D is a quasiconvex open set, then $W^{1,\infty}(D) = C^{0,1}(D) \cap L^\infty(D)$, and if D is also bounded, then $W^{1,\infty}(D) = C^{0,1}(D)$; see Heinonen [113, Thm. 4.1]. □

Remark 2.13 (Broken seminorms). Let $D \subset \mathbb{R}^d$ be an open set and let $\{D_i\}_{i\in\{1:I\}}$ be a partition of D, i.e., all the subsets D_i are open, mutually disjoint, and $D \setminus \bigcup_{i\in\{1:I\}} D_i$ has zero Lebesgue measure. Let $v \in W^{1,p}(D)$ and $p \in [1,\infty)$. Then one can write $|v|^p_{W^{1,p}(D)} = \sum_{i\in\{1:I\}} \|(\nabla v)_{|D_i}\|^p_{\boldsymbol{L}^p(D_i)}$. In this book, we are going to abuse the notation by writing $|v|^p_{W^{1,p}(D)} = \sum_{i\in\{1:I\}} \|\nabla v\|^p_{\boldsymbol{L}^p(D_i)}$. This abuse is justified by observing that $(\nabla v)_{|D_i} = \nabla(v_{|D_i})$ for all $v \in W^{1,1}_{\mathrm{loc}}(D)$. We stress that it is important that the weak derivative of v exists to make sense of the above identities. For instance, letting H be the Heaviside function, we have $\|\nabla(H_{|(-1,0)})\|^p_{\boldsymbol{L}^p(-1,0)} + \|\nabla(H_{|(0,1)})\|^p_{\boldsymbol{L}^p(0,1)} = 0$, but $H \notin W^{1,p}(D)$; see Exercise 2.8. □

2.2.2 Fractional-order spaces

Definition 2.14 ($W^{s,p}(D)$). *Let $s \in (0,1)$ and $p \in [1,\infty]$. Let D be an open set in $\mathbb{R}^d$. We define $W^{s,p}(D) := \left\{v \in L^p(D) \mid |v|_{W^{s,p}(D)} < \infty\right\}$, where*

$$|v|_{W^{s,p}(D)} := \left(\int_D\int_D \frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|^p}{\|\boldsymbol{x}-\boldsymbol{y}\|^{sp+d}_{\ell^2}}\,\mathrm{d}x\,\mathrm{d}y\right)^{\frac{1}{p}}, \quad p < \infty, \tag{2.6}$$

and $|v|_{W^{s,\infty}(D)} := \operatorname{ess\,sup}_{\boldsymbol{x},\boldsymbol{y}\in D} \frac{|v(\boldsymbol{x})-v(\boldsymbol{y})|}{\|\boldsymbol{x}-\boldsymbol{y}\|^s_{\ell^2}}$. Letting now $s > 1$, we define

$$W^{s,p}(D) := \{v \in W^{m,p}(D) \mid \partial^\alpha v \in W^{\sigma,p}(D),\ \forall\alpha,\ |\alpha| = m\}, \tag{2.7}$$

where $m := \lfloor s \rfloor$ and $\sigma := s - m$. Finally, we denote $H^s(D) := W^{s,2}(D)$. We write $W^{s,p}(D;\mathbb{R}^q)$, $q \geq 1$, for the space composed of $\mathbb{R}^q$-valued functions whose components are all in $W^{s,p}(D)$, and we write $\boldsymbol{W}^{s,p}(D)$ whenever $q = d$.

Definition 2.15 (Sobolev–Slobodeckij norm). *Let $s = m + \sigma$ with $m := \lfloor s \rfloor$ and $\sigma := s - m \in (0,1)$. For all $p \in [1,\infty)$ and all $v \in W^{s,p}(D)$, we set $\|v\|^p_{W^{s,p}(D)} := \|v\|^p_{W^{m,p}(D)} + \ell_D^{sp}|v|^p_{W^{s,p}(D)}$ with seminorm $|v|^p_{W^{s,p}(D)} := \sum_{|\alpha|=m} |\partial^\alpha v|^p_{W^{\sigma,p}(D)}$. We also set*

$$\|v\|_{W^{s,\infty}(D)} := \max(\|v\|_{W^{m,\infty}(D)}, \ell_D^s|v|_{W^{s,\infty}(D)}),$$

with seminorm $|v|_{W^{s,\infty}(D)} := \max_{|\alpha|=m} |\partial^\alpha v|_{W^{\sigma,\infty}(D)}$. Equipped with this norm $W^{s,p}(D)$ is a Banach space (and a Hilbert space if $p = 2$).

Example 2.16 (Power functions). Let $D := (0,1)$ and consider the function $v(x) := x^\alpha$ with $\alpha \in \mathbb{R}$. One can verify that $v \in L^2(D)$ if $\alpha > -\frac{1}{2}$, $v \in H^1(D)$ if $\alpha > \frac{1}{2}$, and, more generally $v \in H^s(D)$ if $\alpha > s - \frac{1}{2}$. □

Example 2.17 (Hölder functions). If D is bounded and $p \in [1, \infty)$, then $C^{0,\alpha}(D) \hookrightarrow W^{s,p}(D)$ provided $0 \le s < \alpha \le 1$; see Exercise 2.9. □

Example 2.18 (Step function). Let $D := (-1, 1)$ and consider $v(x) := 0$ if $x < 0$ and $v(x) := 1$ if $x \ge 0$. Then $v \in W^{s,p}(D)$ iff $sp < 1$ as shown by the following computation (notice that $sp > 0$):

$$|v|^p_{W^{s,p}} = 2 \int_{-1}^{0} \int_{0}^{1} \frac{1}{|y-x|^{sp+1}} \, \mathrm{d}x \, \mathrm{d}y = 2 \int_{-1}^{0} -\frac{1}{sp} \left(\frac{1}{(1-x)^{sp}} - \frac{1}{|x|^{sp}} \right) \mathrm{d}x.$$

The integral $\int_{-1}^{0} \frac{1}{|x|^{sp}} \, \mathrm{d}x$ is convergent if and only if $sp < 1$. □

Remark 2.19 (Limits $s\downarrow 0$ and $s\uparrow 1$). The expression (2.6), which is usually adopted in the literature to define $|v|_{W^{s,p}(D)}$, gives $|v|_{W^{s,p}(D)} \to \infty$ as $s\uparrow 1$ even if $v \in W^{1,p}(D)$. A remedy to this deficiency has been proposed in Bourgain et al. [38], Maz'ya and Shaposhnikova [140]. It is shown in [38] that by redefining $|v|^*_{W^{s,p}} := (1-s)^{\frac{1}{p}} |v|_{W^{s,p}}$ for all $s \in (0, 1)$, and setting $|v|^*_{W^{1,p}(D)} := |v|_{W^{1,p}(D)}$, there exists c, s.t. for all σ, s with $0 < \sigma < s \le 1$ and all $v \in W^{\sigma,p}(D)$, one has $|v|^*_{W^{\sigma,p}(D)} \le c|v|^*_{W^{s,p}(D)}$ and $\lim_{s\to 1} |v|^*_{W^{s,p}(D)} = |v|_{W^{1,p}(D)}$ (see Borthagaray and Ciarlet [34, Rmk. 2.3]). It has been proposed [140, Thm. 3] to redefine $|v|^*_{W^{s,p}(D)} := (s(1-s))^{\frac{1}{p}} |v|_{W^{s,p}(D)}$ to improve also the behavior of the seminorm when $s\downarrow 0$. It is shown therein that if there is $\sigma > 0$ s.t. $v \in \overline{C_0^\infty(\mathbb{R}^d)}^{W^{\sigma,p}}$, then $\lim_{s\downarrow 0} s|v|^p_{W^{s,p}(\mathbb{R}^d)} = 2p^{-1}|S^{d-1}| \|v\|^p_{L^p(\mathbb{R}^d)}$, where $|S^{d-1}|$ is the measure of the unit sphere in $\mathbb{R}^d$. □

Remark 2.20 (Definition by interpolation). Fractional-order Sobolev spaces can also be defined by means of the interpolation theory between Banach spaces (see §A.5). Let $p \in [1, \infty)$ and $s \in (0, 1)$. Then we have

$$W^{s,p}(D) = [L^p(D), W^{1,p}(D)]_{s,p},$$

and more generally $W^{m+s,p}(D) = [W^{m,p}(D), W^{m+1,p}(D)]_{s,p}$ for all $m \in \mathbb{N}$, with equivalent norms in all the cases; see Tartar [189, Lem. 36.1]. Using the interpolation theory may not be convenient in finite element analysis if one is interested in local approximation properties. Unless specified otherwise we use the Sobolev–Slobodeckij norm in the book. □

2.3 Key properties: density and embedding

This section reviews some key properties of Sobolev spaces: the density of smooth functions and the (compact) embedding into Lebesgue spaces or into spaces composed of Hölder continuous functions.

2.3.1 Density of smooth functions

Theorem 2.21 (Meyers–Serrin). *Let D be an open set in $\mathbb{R}^d$. Let $s \geq 0$ and $p \in [1, \infty)$. Then $C^\infty(D) \cap W^{s,p}(D)$ is dense in $W^{s,p}(D)$.*

Proof. See Meyers and Serrin [143] and Adams and Fournier [3, Thm. 3.17]; see also Evans [99, p. 251] for bounded D. □

Remark 2.22 ($p = \infty$). Let $m \in \mathbb{N}$. The closure of $C^\infty(D) \cap W^{m,\infty}(D)$ with respect to the Sobolev norm $\|\cdot\|_{W^{m,\infty}(D)}$ differs from $W^{m,\infty}(D)$ since it is composed of functions whose derivatives up to order m are continuous and bounded on D. □

The density of smooth functions in Sobolev spaces allows one to derive many useful results. We list here some of the most important ones.

Corollary 2.23 (Differentiation of a product). *Let D be an open subset of $\mathbb{R}^d$. Then we have $uv \in W^{1,p}(D) \cap L^\infty(D)$ and $\nabla(uv) = v\nabla u + u\nabla v$ for all $u, v \in W^{1,p}(D) \cap L^\infty(D)$ and all $p \in [1, \infty]$.*

Proof. See, e.g., [48, Prop. 9.4, p. 269]. □

Corollary 2.24 (Differentiation of a composition). *Let $D \subset \mathbb{R}^d$ be an open set. Let $G \in C^1(\mathbb{R})$. Assume that $G(0) = 0$ and there is $M < \infty$ such that $|G'(t)| \leq M$ for all $t \in \mathbb{R}$. Then we have $G(u) \in W^{1,p}(D)$ and $\nabla(G(u)) = G'(u)\nabla u$ for all $u \in W^{1,p}(D)$ and all $p \in [1, \infty]$.*

Proof. See, e.g., [48, Prop. 9.5, p. 270]. □

Corollary 2.25 (Change of variable). *Let D, D' be two open subsets of $\mathbb{R}^d$. Assume that there exists a bijection $\boldsymbol{T} : D' \to D$ s.t. $\boldsymbol{T} \in C^1(D'; D)$, $\boldsymbol{T}^{-1} \in C^1(D; D')$, $D\boldsymbol{T} \in L^\infty(D'; \mathbb{R}^{d\times d})$, and $D\boldsymbol{T}^{-1} \in L^\infty(D; \mathbb{R}^{d\times d})$, where $D\boldsymbol{T}$ and $D\boldsymbol{T}^{-1}$ are the Jacobian matrices of $\boldsymbol{T}$ and $\boldsymbol{T}^{-1}$, respectively. Then we have $u \circ T \in W^{1,p}(D')$ for all $u \in W^{1,p}(D)$ and all $p \in [1, \infty]$, and $\partial_{x'_i}(u \circ \boldsymbol{T})(\boldsymbol{x}') = \sum_{j\in\{1:d\}} \partial_{x_j} u(T(\boldsymbol{x}))\partial_{x'_i}\boldsymbol{T}(\boldsymbol{x}')$ for all $i \in \{1:d\}$ and $\boldsymbol{x}' \in D'$.*

Proof. See, e.g., [48, Prop. 9.6, p. 270]. □

2.3.2 Embedding

We use the notation $V \hookrightarrow W$ to mean that the embedding of V into W is continuous, i.e., there is c such that $\|v\|_W \leq c\|v\|_V$ for all $v \in V$ (see §A.2). The main idea of the results in this section is that functions in the Sobolev space $W^{s,p}(D)$ with differentiability index $s > 0$ do have an integrability index larger than p (i.e., they belong to some Lebesgue space $L^q(D)$ with $q > p$), and if s is sufficiently large, for all $u \in W^{s,p}(D)$ (recall that u is actually a class of functions that coincide almost everywhere in D), there is a representative of u that is continuous (or even Hölder continuous). How large s must be for these properties to hold true depends on the space dimension. The case $d = 1$ is particularly simple since $W^{1,1}(\mathbb{R}) \hookrightarrow C^0(\mathbb{R})$ and $W^{1,1}(D) \hookrightarrow C^0(\overline{D})$ for every bounded interval D; see [189, Lem. 8.5] (see also Exercise 5.7). In the rest of this section, we assume that $d \geq 2$. We first consider the case where $D := \mathbb{R}^d$.

Theorem 2.26 (Embedding of $W^{1,p}(\mathbb{R}^d)$). *Let $d \geq 2$ and let $p \in [1, \infty]$. The following holds true:*

(i) *(Gagliardo–Nirenberg–Sobolev): If $p \in [1, d)$, then*

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \qquad \forall q \in [p, p^*],\ p^* := \frac{pd}{d-p}. \tag{2.8}$$

In particular, $\|u\|_{L^{p^*}(\mathbb{R}^d)} \leq \frac{p^*}{1^*}\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}$ with $1^* := \frac{d}{d-1}$ for all $u \in W^{1,p}(\mathbb{R}^d)$. Hence, $W^{1,p}(\mathbb{R}^d) \hookrightarrow L^{p^*}(\mathbb{R}^d)$, and the embedding into $L^q(\mathbb{R}^d)$ for all $q \in [p, p^*)$ follows from Corollary 1.43*.*

(ii) *If $p = d$, then*

$$W^{1,d}(\mathbb{R}^d) \hookrightarrow L^q(\mathbb{R}^d), \qquad \forall q \in [d, \infty). \tag{2.9}$$

(iii) *(Morrey): If $p \in (d, \infty]$, then*

$$W^{1,p}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d), \qquad \alpha := 1 - \frac{d}{p}. \tag{2.10}$$

Proof. See [48, Thm. 9.9, Cor. 9.11, Thm. 9.12], [99, p. 263–266], [180, §I.7.4, §I.8.2], [189, Chap. 8-9]. □

Remark 2.27 (Continuous function). The embedding (2.10) means that there is c, only depending on p and d, such that

$$|u(\boldsymbol{x}) - u(\boldsymbol{y})| \leq c\|\boldsymbol{x} - \boldsymbol{y}\|^\alpha_{\ell^2(\mathbb{R}^d)}\|\nabla u\|_{\boldsymbol{L}^p(\mathbb{R}^d)}, \quad \text{for a.e. } \boldsymbol{x}, \boldsymbol{y} \in \mathbb{R}^d, \tag{2.11}$$

for all $u \in W^{1,p}(\mathbb{R}^d)$. In other words, there is a continuous function $v \in C^{0,\alpha}(\mathbb{R}^d)$ such that $u = v$ almost everywhere. It is then possible to replace u by its continuous representative v. We will systematically do this replacement in this book when a continuous embedding in a space of continuous functions is invoked. □

The above results extend to Sobolev spaces of arbitrary order.

Theorem 2.28 (Embedding of $W^{s,p}(\mathbb{R}^d)$). *Let $d \geq 2$, $s > 0$, and $p \in [1, \infty]$. The following holds true:*

$$W^{s,p}(\mathbb{R}^d) \hookrightarrow \begin{cases} L^q(\mathbb{R}^d) & \forall q \in [p, \frac{pd}{d-sp}], \text{ if } sp < d, \\ L^q(\mathbb{R}^d) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(\mathbb{R}^d) \cap C^{0,\alpha}(\mathbb{R}^d) & \alpha := 1 - \frac{d}{sp}, \text{ if } sp > d. \end{cases} \tag{2.12}$$

Moreover, $W^{d,1}(\mathbb{R}^d) \hookrightarrow L^\infty(\mathbb{R}^d) \cap C^0(\mathbb{R}^d)$ *(case* $s = d$ *and* $p = 1$*).*

Proof. See [110, Thm. 1.4.4.1], [88, Thm. 4.47] for $p \in (1, \infty)$. For $s = d$ and $p = 1$, see, e.g., Ponce and Van Schaftingen [160] and Campos Pinto [55, Prop. 3.4] (if $d = 2$). $\square$

Our aim is now to generalize Theorem 2.28 to the space $W^{s,p}(D)$, where D is an open set in $\mathbb{R}^d$. A rather generic way to proceed is to use the concept of extension.

Definition 2.29 ((s,p)-extension). *Let* $s > 0$ *and* $p \in [1, \infty]$*. Let* D *be an open set in* $\mathbb{R}^d$*. The set* D *is said to have the* (s,p)-extension property *if there is a bounded linear operator* $E : W^{s,p}(D) \to W^{s,p}(\mathbb{R}^d)$ *such that* $E(u)|_D = u$ *for all* $u \in W^{s,p}(D)$*.*

Theorem 2.28 can be restated by replacing $\mathbb{R}^d$ with any open set D in $\mathbb{R}^d$ that has the (s,p)-extension property. A rather general class of sets that we consider in this book is that of Lipschitz sets in $\mathbb{R}^d$. A precise definition is given in the next chapter. At this stage, it suffices to know that the boundary of a Lipschitz set can be viewed as being composed of a finite collection of epigraphs of Lipschitz functions.

Theorem 2.30 (Extension from Lipschitz sets). *Let* $s > 0$ *and* $p \in [1, \infty]$*. Let* D *be an open, bounded subset of* $\mathbb{R}^d$*. If* D *is a Lipschitz set, then it has the* (s,p)*-extension property.*

Proof. See Calderón [54], Stein [181, p. 181] (for $s \in \mathbb{N}$), [110, Thm. 1.4.3.1] and [88, Prop. 4.43] (for $p \in (1, \infty)$), [141, Thm. A.1&A.4] (for $s \in [0, 1]$, $p \in [1, \infty]$ and $s > 0$, $p = 2$) [189, Lem. 12.4] (for $s = 1$). $\square$

Theorem 2.31 (Embedding of $W^{s,p}(D)$). *Let* $d \geq 2$*,* $s > 0$*, and* $p \in [1, \infty]$*. Let* D *be an open, bounded subset of* $\mathbb{R}^d$*. If* D *is a Lipschitz set, then we have*

$$W^{s,p}(D) \hookrightarrow \begin{cases} L^q(D) & \forall q \in [p, \frac{pd}{d-sp}], \text{ if } sp < d, \\ L^q(D) & \forall q \in [p, \infty), \text{ if } sp = d, \\ L^\infty(D) \cap C^{0,\alpha}(D) & \alpha := 1 - \frac{d}{sp}, \text{ if } sp > d. \end{cases} \tag{2.13}$$

Moreover, $W^{d,1}(D) \hookrightarrow L^\infty(D) \cap C^0(D)$ *(case* $s = d$ *and* $p = 1$*).*

Remark 2.32 (Bounded set). Note that $W^{s,p}(D) \hookrightarrow L^q(D)$ for $sp \leq d$ and all $q \in [1, p]$ since D is bounded. The boundedness of D also implies that $W^{s,p}(D) \hookrightarrow C^{0,\alpha}(\overline{D})$, with $sp > d$ and $\alpha := 1 - \frac{d}{sp}$, and $W^{d,1}(D) \hookrightarrow C^0(\overline{D})$, i.e., there is (Hölder-)continuity up to the boundary. $\square$

Example 2.33 (Embedding into continuous functions). In dimension one, functions in $H^1(D)$ are bounded and continuous, whereas this may not be the case in dimension $d \geq 2$ (see Exercise 2.10). In dimension $d \in \{2, 3\}$, Theorem 2.31 says that functions in $H^2(D)$ are bounded and continuous. $\square$

Example 2.34 (Boundary smoothness). Let $\alpha > 1$, $p \in [1,2)$, and $D := \{(x_1,x_2) \in \mathbb{R}^2 \mid x_1 \in (0,1),\ x_2 \in (0,x_1^\alpha)\}$. Let $u(x_1,x_2) := x_1^\beta$ with $1-\frac{1+\alpha}{p} < \beta < 0$ (this is possible since $p < 2 < 1+\alpha$). Then $u \in W^{1,p}(D)$ and $u \in L^q(D)$ for all q such that $1 \le q < p_\alpha$ where $\frac{1}{p_\alpha} := \frac{1}{p} - \frac{1}{1+\alpha}$. Let us set $\frac{1}{p^*} := \frac{1}{p} - \frac{1}{2}$, $\epsilon := \frac{\beta-1}{1+\alpha} + \frac{1}{p} > 0$, and $\frac{1}{p_\beta} := \frac{1}{p_\alpha} - \epsilon$. Notice that $p_\alpha < p^*$ and also $p_\alpha < p_\beta$ since $\epsilon > 0$. Since one can choose β s.t. ϵ is arbitrarily close to zero, we pick β so that $p_\beta < p^*$. Hence, $p_\beta \in (p_\alpha, p^*)$. But $u \notin L^q(D)$ for all $q \in (p_\beta, p^*)$, which would contradict Theorem 2.31 if the Lipschitz property had been omitted. Hence, D cannot be a Lipschitz set in $\mathbb{R}^2$ ($\alpha > 1$ means that D has a cusp at the origin). This counterexample shows that some smoothness assumption on D is needed for Theorem 2.31 to hold true. $\square$

We conclude this section with important compactness results. Recall from §A.4 that the embedding $V \hookrightarrow W$ between two Banach spaces is compact iff from every bounded sequence in V, one can extract a converging subsequence in W.

Theorem 2.35 (Rellich–Kondrachov). *Let $s > 0$ and $p \in [1,\infty]$. Let D be an open, bounded subset of $\mathbb{R}^d$. If D is a Lipschitz set, then the following embeddings are compact:*

(i) *If $sp \le d$, $W^{s,p}(D) \hookrightarrow L^q(D)$ for all $q \in [1, \frac{pd}{d-sp})$.*
(ii) *If $sp > d$, $W^{s,p}(D) \hookrightarrow C^0(\overline{D})$.*
(iii) *$W^{s,p}(D) \hookrightarrow W^{s',p}(D)$ for all $s > s'$.*

Proof. See [3, Thm. 6.3], [48, Thm. 9.16], [99, p. 272], [138, p. 35], [110, Thm. 1.4.3.2]. $\square$

Exercises

Exercise 2.1 (Lebesgue point). Let $a \in \mathbb{R}$. Let $f : \mathbb{R} \to \mathbb{R}$ be defined by $f(x) := 0$ if $x < 0$, $f(0) := a$, and $f(x) := 1$ if $x > 0$. Show that 0 is not a Lebesgue point of f for all a.

Exercise 2.2 (Lebesgue differentiation). The goal is to prove Theorem 2.2. (i) Let $h \in \mathcal{H}$ (the sign of h is unspecified). Show that $R(x,h) := \frac{F(x+h)-F(x)}{h} - f(x) = \frac{1}{h}\int_x^{x+h}(f(t)-f(x))\,\mathrm{d}t$. (ii) Conclude.

Exercise 2.3 (Lebesgue measure and weak derivative). Let $D := (0,1)$. Let C_∞ be the Cantor set (see Example 1.5). Let $f : D \to \mathbb{R}$ be defined by $f(x) := x$ if $x \notin C_\infty$, and $f(x) := 2-5x$ if $x \in C_\infty$. (i) Is f measurable? (*Hint*: use Corollary 1.11.) (ii) Compute $\sup_{x\in D} f(x)$, $\operatorname{ess\,sup}_{x\in D} f(x)$, $\inf_{x\in D} f(x)$, $\operatorname{ess\,inf}_{x\in D} f(x)$, and $\|f\|_{L^\infty(D)}$. (iii) Show that f is weakly differentiable and compute $\partial_x f(x)$. (iv) Compute $f(x) - \int_0^x \partial_t f(t)\,\mathrm{d}t$ for all $x \in D$. (iv) Identify the function $f^{\mathrm{c}} \in C^0(\overline{D})$ that satisfies $f = f^{\mathrm{c}}$ a.e. on D? Compute $f^{\mathrm{c}}(x) - \int_0^x \partial_t f(t)\,\mathrm{d}t$ for all $x \in D$.

Exercise 2.4 (Weak derivative). Let $D := (-1,1)$. Prove that if $u \in L^1_{\mathrm{loc}}(D)$ has a second-order weak derivative, it also has a first-order weak derivative. (*Hint*: consider $\psi(x) := \int_{-1}^{x}(\varphi(t) - c_\varphi\rho(t))\,\mathrm{d}t$ for all $\varphi \in C_0^\infty(D)$, with $c_\varphi := \int_D \varphi\,\mathrm{d}x$, $\rho \in C_0^\infty(D)$, and $\int_D \rho\,\mathrm{d}x = 1$.)

Exercise 2.5 (Clairaut's theorem). Let $v \in L^1_{\mathrm{loc}}(D)$. Let $\alpha, \beta \in \mathbb{N}^d$ and assume that the weak derivatives $\partial^\alpha v$, $\partial^\beta v$ exist and that the weak derivative $\partial^\alpha(\partial^\beta v)$ exists. Prove that $\partial^\beta(\partial^\alpha v)$ exists and $\partial^\alpha(\partial^\beta v) = \partial^\beta(\partial^\alpha v)$.

Exercise 2.6 (Weak and classical derivatives). Let $k \in \mathbb{N}$, $k \geq 1$, and let $v \in C^k(D)$. Prove that, up to the order k, the weak derivatives and the classical derivatives of v coincide.

Exercise 2.7 ($H^1(D)$). (i) Let $D := (-1,1)$ and $u : D \to \mathbb{R}$ s.t. $u(x) := |x|^{\frac{3}{2}} - 1$. Determine whether u is a member of $H^1(D;\mathbb{R})$. (ii) Let $u_1 \in C^1((-1,0];\mathbb{R})$ and $u_2 \in C^1([0,1);\mathbb{R})$ and assume that $u_1(0) = u_2(0)$. Let u be such that $u_{|(-1,0)} := u_1$ and $u_{|(0,1)} := u_2$. Determine whether u is a member of $H^1(D;\mathbb{R})$. Explain why $u \notin H^1(D;\mathbb{R})$ if $u_1(0) \neq u_2(0)$.

Exercise 2.8 (Broken seminorm). Let D be an open set in $\mathbb{R}^d$. Let $\{D_1, \ldots, D_n\}$ be a partition of D as in Remark 2.13. (i) Show that $(\nabla v)_{|D_i} = \nabla(v_{|D_i})$ for all $i \in \{1{:}n\}$ and all $v \in W^{1,1}_{\mathrm{loc}}(D)$. (ii) Let $p \in [1,\infty)$ and $v \in W^{1,p}(D)$. Show that $\sum_{i\in\{1:n\}} |v_{|D_i}|^p_{W^{1,p}(D_i)} = |v|^p_{W^{1,p}(D)}$. (iii) Let $s \in (0,1)$, $p \in [1,\infty)$, and $v \in W^{s,p}(D)$. Prove that $\sum_{i\in\{1:n\}} |v_{|D_i}|^p_{W^{s,p}(D_i)} \leq |v|^p_{W^{s,p}(D)}$.

Exercise 2.9 ($W^{s,p}$). Let D be a bounded open set in $\mathbb{R}^d$. Let $\alpha \in (0,1]$. Show that $C^{0,\alpha}(D;\mathbb{R}) \hookrightarrow W^{s,p}(D;\mathbb{R})$ for all $p \in [1,\infty)$ if $s \in [0,\alpha)$.

Exercise 2.10 (Unbounded function in $H^1(D)$). Let $D := B(\mathbf{0}, \frac{1}{2}) \subset \mathbb{R}^2$ be the ball centered at 0 and of radius $\frac{1}{2}$. (i) Show that the (unbounded) function $u(\boldsymbol{x}) := \ln\big(-\ln(\|\boldsymbol{x}\|_{\ell^2})\big)$ has weak partial derivatives. (*Hint*: work on $D\backslash B(\mathbf{0},\epsilon)$ with $\epsilon \in (0, \frac{1}{2})$, and use Lebesgue's dominated convergence theorem.) (ii) Show that u is in $H^1(D)$.

Exercise 2.11 (Equivalent norm). Let $m \in \mathbb{N}$, $m \geq 2$, and let $p \in [1,\infty)$. Prove that the norm $\|v\| := (\|v\|^p_{L^p} + \ell_D^{mp}|v|^p_{W^{m,p}(D)})^{\frac{1}{p}}$ is equivalent to the canonical norm in $W^{m,p}(D)$. (*Hint*: use the Peetre–Tartar lemma (Lemma A.20) and invoke the compact embeddings from Theorem 2.35.)