# Part II, Chapter 6

## One-dimensional finite elements and tensorization

This chapter presents important examples of finite elements, first in dimension one, then in multiple dimensions using tensor-product techniques. Important computational issues related to the manipulation of high-order polynomial bases are addressed. We also show how to approximate integrals over intervals using the roots of the Legendre and Jacobi polynomials.

### 6.1 Legendre and Jacobi polynomials

Legendre and related polynomials are useful tools to design high-order finite elements. Their roots are also important to construct nodal finite element bases and to devise approximate integration rules called quadratures.

**Definition 6.1 (Legendre polynomials).** *The* Legendre polynomials *are univariate polynomial functions $\mathbb{R} \to \mathbb{R}$ defined for every integer $m \geq 0$ by*

$$L_m(t) := \frac{(-1)^m}{2^m m!} \frac{\mathrm{d}^m}{\mathrm{d}t^m}\big((1-t)^m(1+t)^m\big). \tag{6.1}$$

**Proposition 6.2 ($L^2$-orthogonality).** *The Legendre polynomials are $L^2$-orthogonal over the interval $(-1,1)$, and the following holds true:*

$$\int_{-1}^{1} L_m(t)L_n(t)\,\mathrm{d}t = \frac{2}{2m+1}\delta_{mn}, \qquad \forall m,n \geq 0. \tag{6.2}$$

Legendre polynomials satisfy many useful properties. The most important ones are that $L_m$ is a polynomial of degree $m$, $L_m$ is an even function if $m$ is even and an odd function if $m$ is odd, $L_m(-1) = (-1)^m$ and $L_m(1) = 1$, $L'_m(-1) = \frac{1}{2}(-1)^{m+1}m(m+1)$, and $L'_m(1) = \frac{1}{2}m(m+1)$. We also have for all $m \geq 1$,

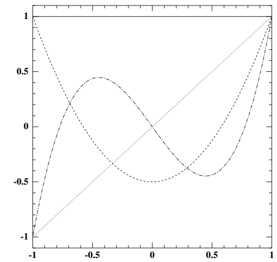
$$\frac{1}{m}(t^2-1)L'_m(t) = tL_m(t) - L_{m-1}(t), \tag{6.3a}$$

$$L'_m(t) = mL_{m-1}(t) + tL'_{m-1}(t), \tag{6.3b}$$

$$(1-t^2)L''_m(t) - 2tL'_m(t) + m(m+1)L_m(t) = 0, \tag{6.3c}$$

and finally $(m+1)L_{m+1}(t) = (2m+1)tL_m(t) - mL_{m-1}(t)$ (Bonnet's recursion formula). The first four Legendre polynomials are the following:

$$\begin{cases} L_0(t) = 1 \\ L_1(t) = t \\ L_2(t) = \frac{1}{2}(3t^2-1) \\ L_3(t) = \frac{1}{2}(5t^3-3t) \end{cases}$$




**Definition 6.3 (Jacobi polynomials).** *Let $\alpha, \beta \in \mathbb{R}$ be such that $\alpha > -1$ and $\beta > -1$. The* Jacobi polynomials *are univariate polynomial functions $\mathbb{R} \to \mathbb{R}$ defined for every integer $m \geq 0$ by*

$$J_m^{\alpha,\beta}(t) := \frac{(-1)^m}{2^m m!}(1-t)^{-\alpha}(1+t)^{-\beta}\frac{\mathrm{d}^m}{\mathrm{d}t^m}\big((1-t)^{\alpha+m}(1+t)^{\beta+m}\big). \tag{6.4}$$

The Jacobi polynomials are orthogonal w.r.t. to the $L^2$-inner product in the interval $(-1,1)$ weighted by the function $(1-t)^\alpha(1+t)^\beta$:

$$\int_{-1}^{1}(1-t)^\alpha(1+t)^\beta J_m^{\alpha,\beta}(t)J_n^{\alpha,\beta}(t)\,\mathrm{d}t = c_{m,\alpha,\beta}\delta_{mn}, \qquad \forall m,n \geq 0, \tag{6.5}$$

with $c_{m,\alpha,\beta} := \frac{2^{\alpha+\beta+1}}{2m+\alpha+\beta+1}\frac{\Gamma(m+\alpha+1)\Gamma(m+\beta+1)}{m!\Gamma(m+\alpha+\beta+1)}$, where $\Gamma$ is the Gamma function (s.t. $\Gamma(n+1) = n!$ for every natural number $n$). The Jacobi polynomials satisfy the following recursion formula for all $m \geq 1$:

$$\begin{aligned} &2(m+1)(m+\alpha+\beta+1)(2m+\alpha+\beta)J_{m+1}^{\alpha,\beta}(t) = \\ &\quad (2m+\alpha+\beta+1)((2m+\alpha+\beta+2)(2m+\alpha+\beta)t + \alpha^2 - \beta^2)J_m^{\alpha,\beta}(t) \\ &\quad - 2(m+\alpha)(m+\beta)(2m+\alpha+\beta+2)J_{m-1}^{\alpha,\beta}(t). \end{aligned}$$

$J_m^{\alpha,\beta}$ is a polynomial of degree $m$ and $J_m^{\alpha,\beta}(-1) = (-1)^m\binom{m+\beta}{m}$, $J_m^{\alpha,\beta}(1) = \binom{m+\alpha}{m}$. The Legendre polynomials are Jacobi polynomials with parameters $\alpha = \beta = 0$, i.e., $L_m(t) = J_m^{0,0}(t)$ for all $m \geq 0$. The first three Jacobi polynomials corresponding to the parameters $\alpha = \beta = 1$ are

$$J_0^{1,1}(t) = 1, \qquad J_1^{1,1}(t) = 2t, \qquad J_2^{1,1}(t) = \frac{3}{4}(5t^2-1).$$

The Jacobi polynomials $J_m^{1,1}$ are related to the integrated Legendre polynomials as follows (see Exercise 6.1):

$$\int_{-1}^{t} L_m(s)\,\mathrm{d}s = -\frac{1}{2m}(1-t^2)J_{m-1}^{1,1}(t), \qquad \forall m \geq 1. \tag{6.6}$$

We refer the reader to Abramowitz and Stegun [1, Chap. 22] for further results on the Legendre and Jacobi polynomials.

## 6.2 One-dimensional Gauss quadrature

A quadrature formula on, say, the reference interval $K := [-1,1]$ allows one to approximate the integral of functions $\phi$ in $C^0(K)$ as follows:

$$\int_{-1}^{1} \phi(t)\,\mathrm{d}t \approx \sum_{l\in\{1:m\}} \omega_l \phi(\xi_l), \tag{6.7}$$

for some integer $m \geq 1$. The points $\{\xi_l\}_{l\in\{1:m\}}$ are called *quadrature nodes*, are all in $K$, and are all distinct. The real numbers $\{\omega_l\}_{l\in\{1:m\}}$ are called *quadrature weights*. By a change of variables, the quadrature (6.7) can be used on any interval $[a,b]$. Letting $c := \frac{1}{2}(a+b)$ and $\delta := b-a$, (6.7) implies

$$\int_{a}^{b} \phi(t)\,\mathrm{d}t \approx \sum_{l\in\{1:m\}} \tfrac{1}{2}\delta\omega_l \phi(c+\tfrac{1}{2}\delta\xi_l). \tag{6.8}$$

The largest integer $k_{\mathcal{Q}}$ such that equality holds true in (6.7) for every polynomial in $\mathbb{P}_{k_{\mathcal{Q}}}$ is called *quadrature order*, that is, we have

$$\int_{-1}^{1} p(t)\,\mathrm{d}t = \sum_{l\in\{1:m\}} \omega_l p(\xi_l), \qquad \forall p \in \mathbb{P}_{k_{\mathcal{Q}}}, \tag{6.9}$$

and there is $q \in \mathbb{P}_{k_{\mathcal{Q}}+1}$ s.t. $\int_{-1}^{1} q(t)\,\mathrm{d}t \neq \sum_{l\in\{1:m\}} \omega_l q(\xi_l)$. At this stage, it suffices to know that the higher the quadrature order, the more accurate the quadrature (6.7). We refer the reader to Chapter 30 for estimates on the quadrature error and for quadratures in multiple dimensions.

**Lemma 6.4 (Quadrature order).** *Let $\{\xi_l\}_{l\in\{1:m\}}$ be $m$ distinct points in $K$. Let $\{\mathcal{L}_l\}_{l\in\{1:m\}}$ be the associated Lagrange interpolation polynomials, i.e., $\mathcal{L}_l(\xi_j) = \delta_{lj}$ for all $l,j \in \{1:m\}$. Set $\omega_l := \int_{-1}^{1} \mathcal{L}_l(t)\,\mathrm{d}t$ for all $l \in \{1:m\}$. Then the quadrature (6.7) is at least of order $(m-1)$ and at most of order $(2m-1)$, i.e., $m-1 \leq k_{\mathcal{Q}} \leq 2m-1$.*

*Proof.* Let $p \in \mathbb{P}_{m-1}$. Since the $m$ quadrature nodes are all distinct, the Lagrange interpolation polynomials $\{\mathcal{L}_l\}_{l\in\{1:m\}}$ form a basis of $\mathbb{P}_{m-1}$. Thus, we can write $p(t) = \sum_{l\in\{1:m\}} p(\xi_l)\mathcal{L}_l(t)$, whence we infer that

$$\int_{-1}^{1} p(t)\,\mathrm{d}t = \sum_{l\in\{1:m\}} p(\xi_l)\int_{-1}^{1}\mathcal{L}_l(t)\,\mathrm{d}t = \sum_{l\in\{1:m\}} \omega_l p(\xi_l),$$

owing to the linearity of the integral and the definition of the weights. Hence, $k_{\mathcal{Q}} \geq m-1$. Moreover, the polynomial $q(t) := \prod_{l\in\{1:m\}}(t-\xi_l)^2$ is of degree $2m$ and is not integrated exactly by the quadrature (which approximates its integral by zero). Hence, $k_{\mathcal{Q}} \leq 2m-1$. □

For all $m \geq 1$, one can show that the $m$ roots of the Legendre polynomial $L_m$ are distinct and are all in the open interval $(-1,1)$. The most important example of quadrature is the one based on these roots, which we henceforth call *Gauss–Legendre nodes*.

**Proposition 6.5 (Gauss–Legendre).** *Let $m \geq 1$. Let $\{\xi_l\}_{l\in\{1:m\}}$ be the $m$ roots of the Legendre polynomial $L_m(t)$ (all distinct and in $(-1,1)$). Let the weights be defined as in Lemma 6.4. Then the quadrature (6.7) is of order $k_{\mathcal{Q}} = 2m-1$. Moreover, all the weights are positive and are given by*

$$\omega_l = \frac{2}{(1-\xi_l^2)L_m'(\xi_l)^2}, \qquad \forall l \in \{1:m\}. \tag{6.10}$$

*Proof.* (i) Order of the quadrature. We already know from Lemma 6.4 that $m-1 \leq k_{\mathcal{Q}} \leq 2m-1$. Consider a polynomial $p \in \mathbb{P}_{2m-1}$. The Euclidean division of polynomials shows that there are $p_1, p_2 \in \mathbb{P}_{m-1}$ such that $p = p_1 L_m + p_2$. Using that $k_{\mathcal{Q}} \geq m-1$, the $L^2$-orthogonality of Legendre polynomials, and the identity $p(\xi_l) = p_2(\xi_l)$ (since $L_m(\xi_l) = 0$ for all $l \in \{1:m\}$), we infer that

$$\int_{-1}^{1} p(t)\,\mathrm{d}t = \int_{-1}^{1} p_2(t)\,\mathrm{d}t = \sum_{l\in\{1:m\}} \omega_l p_2(\xi_l) = \sum_{l\in\{1:m\}} \omega_l p(\xi_l).$$

This shows that $k_{\mathcal{Q}} \geq 2m-1$. Hence, $k_{\mathcal{Q}} = 2m-1$.
(ii) Let us prove (6.10) for all $l \in \{1:m\}$. Let $\mathcal{L}_l \in \mathbb{P}_{m-1}$ be the Lagrange interpolation polynomial associated with the node $\xi_l$, i.e., $\mathcal{L}_l(\xi_j) = \delta_{lj}$ for all $l, j \in \{1:m\}$. Since the polynomial $\mathcal{L}_l(t)(1-t)L_m'(t)$ is of degree $(2m-1)$, it is integrated exactly by the quadrature. Hence, we have

$$\int_{-1}^{1} \mathcal{L}_l(t)(1-t)L_m'(t)\,\mathrm{d}t = \omega_l(1-\xi_l)L_m'(\xi_l).$$

Moreover, integrating by parts and since $\int_{-1}^{1}\big(\mathcal{L}_l(t)(1-t)\big)' L_m(t)\,\mathrm{d}t = 0$ owing to the $L^2$-orthogonality of the Legendre polynomials, we obtain

$$\int_{-1}^{1} \mathcal{L}_l(t)(1-t)L_m'(t)\,\mathrm{d}t = -2\mathcal{L}_l(-1)L_m(-1).$$

Next, we observe that $\mathcal{L}_l(t) = \frac{L_m(t)}{t-\xi_l}\frac{1}{L_m'(\xi_l)}$ since both functions are polynomials in $\mathbb{P}_{m-1}$ having the same roots and taking the same value 1 at $\xi_l$ owing to l'Hôpital's rule. Thus, $\mathcal{L}_l(-1) = -\frac{L_m(-1)}{1+\xi_l}\frac{1}{L_m'(\xi_l)}$. Combining the above identities leads to

$$\omega_l(1-\xi_l)L_m'(\xi_l) = 2\frac{L_m(-1)^2}{1+\xi_l}\frac{1}{L_m'(\xi_l)},$$

which proves the claim since $L_m(-1)^2 = 1$. □

In some situations, it is interesting to use quadratures with nodes including one or the two endpoints of the interval $[-1,1]$. The Gauss–Lobatto quadrature corresponds to the case where both endpoints are included. The nodes of this quadrature for $m \geq 3$ are $\{-1,1\}$ plus the $(m-2)$ roots of the polynomial $L_{m-1}'(t)$, which can be shown to be all distinct and contained in the open interval $(-1,1)$.

**Proposition 6.6 (Gauss–Lobatto).** *Let $m \geq 2$. Let $\{\xi_l\}_{l\in\{1:m\}}$ be the* Gauss–Lobatto nodes*, i.e., the $m$ roots of the polynomial $(1-t^2)L_{m-1}'(t)$ (they are all distinct and in $[-1,1]$). Let the weights be defined as in Lemma 6.4. Then the quadrature (6.7) is of order $k_{\mathcal{Q}} = 2m-3$. Moreover, all the weights are positive and are given by*

$$\omega_l = \frac{2}{m(m-1)}\frac{1}{L_{m-1}(\xi_l)^2}, \qquad \forall l \in \{1{:}m\}. \tag{6.11}$$

*In particular, we have $\omega_1 = \omega_m = \frac{2}{m(m-1)}$.*

*Proof.* See Exercise 6.2. □

The case where one keeps only one of the two endpoints leads to the Gauss–Radau quadrature. For brevity, we focus on the right-sided version which keeps the right endpoint $\xi_m = 1$. The left-sided version keeping the left endpoint $\xi_1 = -1$ can be derived from symmetry arguments. The nodes of the right-sided quadrature are the $m$ roots of the polynomial $L_m(t) - L_{m-1}(t)$, which can be shown to be all distinct and contained in $(-1,1]$ (notice that 1 is a root of this polynomial).

**Proposition 6.7 (Gauss–Radau, right-sided).** *Let $m \geq 1$. Let $\{\xi_l\}_{l\in\{1:m\}}$ be the* Gauss–Radau nodes*, i.e., the $m$ roots of the polynomial $L_m(t) - L_{m-1}(t)$ (they are all distinct and in $(-1,1]$). Let the weights be defined as in Lemma 6.4. Then the quadrature (6.7) is of order $k_{\mathcal{Q}} = 2m-2$. Moreover, all the weights are positive and are given by*

$$\omega_l = \frac{1}{(1+\xi_l)L_{m-1}'(\xi_l)^2}, \quad \forall l \in \{1{:}m-1\}, \qquad \omega_m = \frac{2}{m^2}. \tag{6.12}$$

*Proof.* See Exercise 6.3. □

Examples of quadratures on the reference interval $[-1,1]$ are presented in Table 6.1. The Gauss–Legendre quadrature of order 1 is called *midpoint rule*, the Gauss–Lobatto quadrature of order 1 is called *trapezoidal rule* and that of order 3 *Simpson's rule*. For quadratures of higher order, we refer the reader, e.g., to Karniadakis and Sherwin [123, §B.2].

| type | order | nodes | weights |
|---|---|---|---|
| G-Rad | 0 | $1$ | $2$ |
| G-Leg | 1 | $0$ | $2$ |
| G-Lob | 1 | $-1, 1$ | $1, 1$ |
| G-Rad | 2 | $-\frac{1}{3}, 1$ | $\frac{3}{2}, \frac{1}{2}$ |
| G-Leg | 3 | $-\frac{\sqrt{3}}{3}, \frac{\sqrt{3}}{3}$ | $1, 1$ |
| G-Lob | 3 | $-1, 0, 1$ | $\frac{1}{3}, \frac{4}{3}, \frac{1}{3}$ |
| G-Rad | 4 | $\frac{-1-\sqrt{6}}{5}, \frac{-1+\sqrt{6}}{5}, 1$ | $\frac{16+\sqrt{6}}{18}, \frac{16-\sqrt{6}}{18}, \frac{2}{9}$ |
| G-Leg | 5 | $-\frac{\sqrt{15}}{5}, 0, \frac{\sqrt{15}}{5}$ | $\frac{5}{9}, \frac{8}{9}, \frac{5}{9}$ |
| G-Lob | 5 | $-1, -\frac{\sqrt{5}}{5}, \frac{\sqrt{5}}{5}, 1$ | $\frac{1}{6}, \frac{5}{6}, \frac{5}{6}, \frac{1}{6}$ |

**Table 6.1** One-dimensional quadratures on the reference interval $[-1,1]$. G-Leg: Gauss–Legendre, G-Rad: Gauss–Radau, G-Lob: Gauss–Lobatto.

## 6.3 One-dimensional finite elements

In this section, we present important examples of one-dimensional finite elements. Recall that $\mathbb{P}_k$, $k \geq 0$, is the real vector space composed of univariate polynomial functions of degree at most $k$. For convenience, degrees of freedom (dofs) and shape functions of one-dimensional finite elements using the polynomial space $\mathbb{P}_k$ are numbered from 0 to $k$.

### 6.3.1 Lagrange (nodal) finite elements

Following Definition 5.11, the dofs for Lagrange finite elements are chosen as the values at some set of nodes.

**Proposition 6.8 (Lagrange finite element).** *Let $k \geq 0$. Let $K$ be a compact interval with nonempty interior, and let $P := \mathbb{P}_k$. Consider a set of $n_{\text{sh}} := k+1$ distinct nodes $\{a_l\}_{l\in\{0:k\}}$ in $K$. Let $\Sigma := \{\sigma_l\}_{l\in\{0:k\}}$ be the linear forms on $P$ such that $\sigma_l(p) := p(a_l)$ for all $l \in \{0:k\}$. Then $(K, P, \Sigma)$ is a Lagrange finite element.*

*Proof.* We use Remark 5.3. We observe that $\dim P = k+1 = n_{\rm sh} = \operatorname{card} \Sigma$. Moreover, let $p \in P$ be such that $\sigma_l(p) = p(a_l) = 0$ for all $l \in \{0{:}k\}$. Then $p = 0$ since $p$ is of degree at most $k$ and has $(k+1)$ distinct roots. □

The shape functions of a one-dimensional Lagrange finite element are the Lagrange interpolation polynomials $\{\mathcal{L}_l^{[a]}\}_{i\in\{0:k\}}$ defined as in (5.2). Following (5.10), the Lagrange interpolation operator acts as follows:

$$\mathcal{I}_K^{\rm L}(v)(t) := \sum_{l\in\{0:k\}} v(a_l)\mathcal{L}_l^{[a]}(t), \qquad \forall t \in K, \tag{6.13}$$

and possible choices for the domain of $\mathcal{I}_K^{\rm L}$ are $V(K) := C^0(K)$ or $V(K) := W^{1,1}(K)$; see Exercise 5.7.

The Lagrange interpolation polynomials based on $(k+1)$ equidistant nodes (including both endpoints) in the interval $K := [-1,1]$ are henceforth denoted by $\{\mathcal{L}_l^k\}_{l\in\{0:k\}}$. The graphs of these polynomials are illustrated in Figure 5.1 for $k \in \{1,2,3\}$. Explicit expressions are as follows:

$$\begin{aligned}
&\mathcal{L}_0^1(t) := \tfrac12(1-t), && \mathcal{L}_0^2(t) := \tfrac12 t(t-1), && \mathcal{L}_0^3(t) := \tfrac{9}{16}(t+\tfrac13)(t-\tfrac13)(1-t),\\
&\mathcal{L}_1^1(t) := \tfrac12(1+t), && \mathcal{L}_1^2(t) := (t+1)(1-t), && \mathcal{L}_1^3(t) := \tfrac{27}{16}(t+1)(t-\tfrac13)(t-1),\\
& && \mathcal{L}_2^2(t) := \tfrac12(t+1)t, && \mathcal{L}_2^3(t) := \tfrac{27}{16}(t+1)(t+\tfrac13)(1-t),\\
& && && \mathcal{L}_3^3(t) := \tfrac{9}{16}(t+1)(t+\tfrac13)(t-\tfrac13).
\end{aligned}$$

Although the choice of equidistant nodes appears somewhat natural, it is appropriate only when working with low-degree polynomials; see §6.3.5. An alternative choice is to consider the Gauss–Lobatto nodes. The corresponding Lagrange interpolation polynomials for $k=3$ (four nodes) are illustrated in the left panel of Figure 6.1.

### 6.3.2 Modal finite elements

Let us illustrate the construction of §5.4.2 in the one-dimensional setting.

**Proposition 6.9 (Legendre finite element).** *Let $k \geq 0$. Let $K := [-1,1]$, $P := \mathbb{P}_k$, and $\Sigma := \{\sigma_l\}_{l\in\{0:k\}}$ be the $n_{\rm sh} := k+1$ linear forms on $P$ s.t.*

$$\sigma_l(p) := \frac{2l+1}{2}\int_{-1}^{1} L_l(t)p(t)\,\mathrm{d}t, \quad \forall l \in \{0{:}k\}, \tag{6.14}$$

*where $L_l$ is the Legendre polynomial of order $l$. Then $(K,P,\Sigma)$ is a finite element, and the shape functions are $\theta_l := L_l$ for all $l \in \{0{:}k\}$.*

*Proof.* Use (6.2) and Proposition 5.12. □

Following (5.13), the Legendre interpolation operator acts as follows: $\mathcal{I}_K^{\rm m}(v)(t) := \sum_{l\in\{0:k\}} \left(\frac{2l+1}{2}\int_{-1}^1 L_l(s)v(s)\,\mathrm{d}s\right) L_l(t)$, for all $t \in K$ and all $v \in V(K) := L^1(K)$.

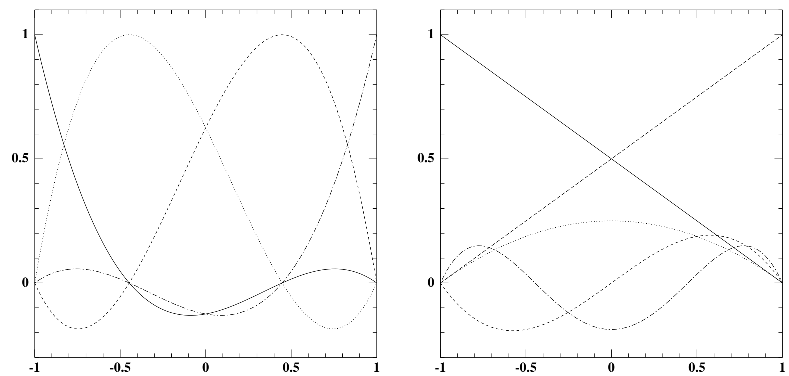



**Fig. 6.1** Left: Lagrange interpolation polynomials for Gauss–Lobatto nodes and $k = 3$. Right: Hybrid nodal/modal shape functions for $k = 4$ (see §6.3.3).

### 6.3.3 Canonical hybrid finite element

Hybrid finite elements mix nodal and modal dofs. When constructing $H^1$-conforming approximation spaces (see Chapter 19), it is convenient that all the basis functions but one vanish at each endpoint of the interval, say $K := [-1, 1]$. This calls for using the values at $\pm 1$ as nodal dofs on $\mathbb{P}_k$. For $k \geq 2$, some or all of the remaining dofs can be taken to be of modal type. Taking all of them to be moments against polynomials in $\mathbb{P}_{k-2}$ gives a finite element called *canonical hybrid finite element*. A multidimensional extension is presented in §7.6.

**Proposition 6.10 (Canonical hybrid finite element).** *Let $k \geq 1$. Set $K := [-1, 1]$ and $P := \mathbb{P}_k$. Define $\sigma_0(p) := p(-1)$, $\sigma_k(p) := p(1)$, and, if $k \geq 2$, let $\{\mu_l\}_{l \in \{1:k-1\}}$ be a basis of $\mathbb{P}_{k-2}$ and define $\sigma_l(p) := \int_K p\mu_l \, \mathrm{d}s$ for all $l \in \{1:k-1\}$. Set $\Sigma := \{\sigma_l\}_{l \in \{0:k\}}$. Then $(K, P, \Sigma)$ is a finite element.*

*Proof.* See Exercise 6.6. □

The corresponding interpolation operator is denoted by $\mathcal{I}_K^{\mathrm{g}}$ (the superscript is consistent with the notation introduced in §16.2 where the letter "g" refers to the gradient operator). Its action on functions $v \in V(K) := W^{1,1}(K)$ is s.t. $\mathcal{I}_K^{\mathrm{g}}(v)(\pm 1) = v(\pm 1)$ and $\int_{-1}^{1} (\mathcal{I}_K^{\mathrm{g}}(v) - v) q \, \mathrm{d}s = 0$ for all $q \in \mathbb{P}_{k-2}$.

**Proposition 6.11 (Commuting with derivative).** *Let $k \geq 0$. Let $\mathcal{I}_K^{\mathrm{g}}$ be the interpolation operator built from the canonical hybrid finite element of order $(k + 1)$. Let $\mathcal{I}_K^{\mathrm{m}}$ be the interpolation operator built from any modal finite element of order $k$. Then $\mathcal{I}_K^{\mathrm{g}}(v)' = \mathcal{I}_K^{\mathrm{m}}(v')$ for all $v \in W^{1,1}(K)$.*

*Proof.* Integrating by parts, using the properties of $\mathcal{I}_K^{\mathrm{g}}$ (i.e., $\int_{-1}^{1} (\mathcal{I}_K^{\mathrm{g}}(v) - v) r \, \mathrm{d}s = 0$ for all $r \in \mathbb{P}_{k-1}$ and $\mathcal{I}_K^{\mathrm{g}}(v)(\pm 1) = v(\pm 1)$) and recalling that $\mathcal{I}_K^{\mathrm{m}}$ is the $L^2$-orthogonal projection onto $\mathbb{P}_k$ (see §5.4.2 and Exercise 5.2), we infer that for all $q \in \mathbb{P}_k$,

$$\int_{-1}^{1} \mathcal{I}_K^{\mathrm{g}}(v)' q \,\mathrm{d}t = -\int_{-1}^{1} \mathcal{I}_K^{\mathrm{g}}(v) q' \,\mathrm{d}t + \Big[\mathcal{I}_K^{\mathrm{g}}(v) q\Big]_{-1}^{1}$$
$$= -\int_{-1}^{1} v q' \,\mathrm{d}t + \Big[vq\Big]_{-1}^{1} = \int_{-1}^{1} v' q \,\mathrm{d}t = \int_{-1}^{1} \mathcal{I}_K^{\mathrm{m}}(v') q \,\mathrm{d}t.$$

This proves that $\mathcal{I}_K^{\mathrm{g}}(v)' = \mathcal{I}_K^{\mathrm{m}}(v')$ since both functions are in $\mathbb{P}_k$. □

The shape functions of the canonical hybrid finite element can be computed explicitly once a choice for the basis functions $\{\mu_l\}_{l\in\{1:k-1\}}$ of $\mathbb{P}_{k-2}$ is made. An example is proposed in Exercise 6.6 using Jacobi polynomials (see the right panel of Figure 6.1 for an illustration).

### 6.3.4 Hierarchical bases

The notion of hierarchical polynomial bases is important when working with high-order polynomials. It is particularly convenient in simulations where the degree $k$ varies from one element to the other.

**Definition 6.12 (Hierarchical basis).** *A sequence of polynomials* $(P_k)_{k\in\mathbb{N}}$ *is said to be a* hierarchical polynomial basis *if the set* $\{P_l\}_{l\in\{0:k\}}$ *is a basis of* $\mathbb{P}_k$ *for all* $k \in \mathbb{N}$.

The monomial basis (i.e., $P_k(t) = t^k$) is the simplest example of hierarchical polynomial basis. Another example is the Jacobi polynomials introduced in §6.1. They form a hierarchical basis with the additional property to be $L^2$-orthogonal with respect to the weight $(1-t)^\alpha(1+t)^\beta$.

The Lagrange shape functions do not form a hierarchical basis, i.e., increasing $k$ to $(k+1)$ requires to recompute the whole basis of shape functions. The shape functions of modal elements form, by construction, a hierarchical basis. Finally, the shape functions of the canonical hybrid finite element do not form in general a hierarchical basis. One can obtain a hierarchical basis though by slightly modifying the dofs. For instance, the following shape functions form a hierarchical basis:

$$\theta_0(t) := \tfrac{1}{2}(1-t), \tag{6.15a}$$
$$\theta_l(t) := \tfrac{1}{4}(1-t)(1+t)J_{l-1}^{1,1}(t), \quad \forall l \in \{1{:}k-1\}, \quad k \geq 2, \tag{6.15b}$$
$$\theta_k(t) := \tfrac{1}{2}(1+t). \tag{6.15c}$$

The corresponding dofs are $\sigma_0(p) := p(-1)$, $\sigma_k(p) := p(1)$, and $\sigma_l(p) := \alpha_l \int_{-1}^{1} p J_{l-1}^{1,1}\,\mathrm{d}t + \beta_l^- p(-1) + \beta_l^+ p(1)$ for all $l \in \{1{:}k-1\}$, where $\alpha_l := 4c_{l-1,1,1}^{-1}$, $\beta_l^\pm := -2c_{l-1,1,1}^{-1}\int_{-1}^{1}(1\pm t)J_{l-1}^{1,1}\,\mathrm{d}t$, and $c_{l-1,1,1}$ defined in (6.5).

### 6.3.5 High-order Lagrange elements

The Lagrange polynomials oscillate more and more as the number of interpolation nodes grows. This phenomenon is often referred to as the Runge phenomenon [171] (see also Meray [142]). A classic example illustrating this phenomenon consists of considering the function $f(x) := (1+x^2)^{-1}$, $x \in [-5,5]$. The Lagrange interpolation polynomial of $f$ using $n$ equidistant points over $[-5,5]$ converges uniformly to $f$ in the interval $(-x_c, x_c)$ with $x_c \sim 3.63$ and diverges outside this interval.

The approximation quality in the maximum norm of the one-dimensional Lagrange interpolation operator using $n_{\text{sh}} := k+1$ distinct nodes is quantified by the Lebesgue constant; see Theorem 5.14. Since the Lebesgue constant is invariant under any linear transformation of the interval $K$, we henceforth restrict the discussion to $K := [-1,1]$. It can be shown that the Lebesgue constant for equidistant nodes grows exponentially with $k$. More precisely, $\Lambda_k \sim \frac{2^k}{ek(\ln(k)+\gamma)}$ as $k \to \infty$ where $\gamma := 0.5772\cdots$ is the Euler constant; see Trefethen and Weideman [191]. A lower bound for the Lebesgue constant for every set of points is $\frac{2}{\pi}\ln(k) - C$ for some positive $C$; see Erdős [94]. If the Chebyshev nodes $\big\{a_l := \cos\big(\frac{(2l-1)\pi}{2n_{\text{sh}}}\big)\big\}_{l\in\{1:n_{\text{sh}}\}}$ are used instead of the equidistant nodes, the Lebesgue constant behaves as $\frac{2}{\pi}\ln(k) + C + \alpha_k$, with $C := \frac{2}{\pi}(\gamma + \ln(\frac{8}{\pi})) = 0.9625\cdots$ and $\alpha_k \to 0$ as $k \to \infty$, showing that this choice is asymptotically optimal; see Luttmann and Rivlin [136] and Rivlin [167, Chap. 4]. The Gauss–Lobatto nodes, which include the two endpoints, lead to an asymptotically optimal Lebesgue constant with upper bound $\frac{2}{\pi}\ln(k) + C$, with $C \sim 0.685$; see Hesthaven [114, Conj. 3.2] and Hesthaven et al. [115, p. 106]. Note that the Lagrange polynomial bases using the Gauss–Lobatto nodes are not hierarchical since the set of $n_{\text{sh}}$ Gauss–Lobatto nodes is not included in the set of $(n_{\text{sh}}+1)$ nodes.

Another important class of sets of nodes is that consisting of the *Fekete points*. These points are defined from a maximization problem. Let $\{a_i\}_{i\in\mathcal{N}}$ be a set of nodes in $K := [-1,1]$ and let $\{\phi_i\}_{i\in\mathcal{N}}$ be a basis of $\mathbb{P}_k$ (recall that $\mathcal{N} := \{1{:}n_{\text{sh}}\}$). Recall the (generalized) $n_{\text{sh}}\times n_{\text{sh}}$ Vandermonde matrix $\mathcal{V}$ with entries $\mathcal{V}_{ij} := \phi_i(a_j)$ for all $i,j \in \mathcal{N}$. Since the Lagrange polynomials can be expressed as $\mathcal{L}_i^{[a]}(t) = \sum_{j\in\mathcal{N}}(\mathcal{V}^{-1})_{ij}\phi_j(t)$ (see Proposition 5.5), a reasonable criterion for selecting the interpolation nodes is to maximize the determinant of $\mathcal{V}$ with respect to $\{a_i\}_{i\in\mathcal{N}}$ (observe that the solution to this problem does not depend on the chosen basis, since a change of basis only multiplies the determinant by a factor equal to the determinant of the change of basis matrix). It is shown in Fejér [101] that the Fekete points and the Gauss–Lobatto nodes coincide on any interval. The notion of Fekete points extends naturally to any dimension, but the concept of Gauss–Lobatto nodes can be extended to higher dimension only by invoking tensor-product techniques as we show in the next section.

## 6.4 Multidimensional tensor-product elements

We show in this section that the one-dimensional finite elements presented in §6.3 can be extended to higher dimension by using tensor-product techniques when $K \subset \mathbb{R}^d$ is a *cuboid*, $d \geq 2$, i.e., when $K$ has the Cartesian product structure $K := \prod_{i=1}^{d}[z_i^-, z_i^+]$ where $z_i^\pm \in \mathbb{R}$ are such that $z_i^- < z_i^+$ for all $i \in \{1{:}d\}$.

### 6.4.1 The polynomial space $\mathbb{Q}_{k,d}$

Tensor-product finite elements in $\mathbb{R}^d$ make use of the polynomial space

$$\mathbb{Q}_{k,d} := \underbrace{\mathbb{P}_k \otimes \ldots \otimes \mathbb{P}_k}_{d \text{ times}}. \tag{6.16}$$

This space is composed of $d$-variate polynomial functions $q : \mathbb{R}^d \to \mathbb{R}$ of partial degree at most $k$ with respect to each variable. Thus, we have

$$\mathbb{Q}_{k,d} = \operatorname{span}\left\{ x_1^{\beta_1} \ldots x_d^{\beta_d},\ 0 \leq \beta_1, \ldots, \beta_d \leq k \right\}. \tag{6.17}$$

We omit the subscript $d$ and simply write $\mathbb{Q}_k$ when the context is unambiguous. Let $\beta := (\beta_1, \ldots, \beta_d) \in \mathbb{N}^d$ be a multi-index, define $\|\beta\|_{\ell^\infty} := \max_{i\in\{1:d\}} \beta_i$, and consider the multi-index set $\mathcal{B}_{k,d} := \{\beta \in \mathbb{N}^d \mid \|\beta\|_{\ell^\infty} \leq k\}$. Polynomial functions $q \in \mathbb{Q}_{k,d}$ can be written in the generic form

$$q(\boldsymbol{x}) = \sum_{\beta\in\mathcal{B}_{k,d}} a_\beta \boldsymbol{x}^\beta, \qquad \boldsymbol{x}^\beta := x_1^{\beta_1} \ldots x_d^{\beta_d}, \tag{6.18}$$

with real numbers $a_\beta$. Note that $\operatorname{card}(\mathcal{B}_{k,d}) = \dim(\mathbb{Q}_{k,d}) = (k+1)^d$.

A direct verification leads to the following useful characterization of the trace of polynomials in $\mathbb{Q}_{k,d}$.

**Lemma 6.13 (Trace space).** *Let $H$ be an affine subspace of $\mathbb{R}^d$ of codimension $l \in \{1{:}d-1\}$. Let $\boldsymbol{T}_H : \mathbb{R}^{d-l} \to H$ be an affine bijective mapping. Then $q_{|H} \circ \boldsymbol{T}_H \in \mathbb{Q}_{k,d-l}$ for all $q \in \mathbb{Q}_{k,d}$.*

### 6.4.2 Tensor-product construction of finite elements

We begin with tensor-product Lagrange finite elements with nodes obtained by invoking the tensor product of nodes along each Cartesian direction. This leads to the following construction.

**Proposition 6.14 (Tensor-product Lagrange).** *Let $K := \prod_{i=1}^{d}[z_i^-, z_i^+]$ be a cuboid in $\mathbb{R}^d$. Let $P := \mathbb{Q}_{k,d}$ for some integer $k \geq 1$. For all $i \in \{1{:}d\}$, consider $(k+1)$ distinct nodes $\{a_{i,l}\}_{l\in\{0:k\}}$ in $[z_i^-, z_i^+]$. For every multi-index $\beta := (\beta_1, \ldots, \beta_d) \in \mathcal{B}_{k,d}$, let $\boldsymbol{a}_\beta$ be the node in $K$ with Cartesian components*

$(a_{i,\beta_i})_{i\in\{1:d\}}$. *Let* $\Sigma := \{\sigma_\beta\}_{\beta\in\mathcal{B}_{k,d}}$ *be the degrees of freedom (dofs) on* $P$ *s.t.* $\sigma_\beta(p) := p(\boldsymbol{a}_\beta)$ *for all* $\beta \in \mathcal{B}_{k,d}$. *Then* $(K, P, \Sigma)$ *is a Lagrange finite element.*

*Proof.* See Exercise 6.7. □

The following property (see Exercise 6.9) is important for the construction of $H^1$-conforming finite elements spaces using tensor-product finite elements.

**Lemma 6.15 (Face unisolvence).** *Consider the cuboid* $K := \prod_{i=1}^{d}[z_i^-, z_i^+]$. *Assume that* $a_{i,0} = z_i^-$ *and* $a_{i,k} = z_i^+$ *for all* $i \in \{1:d\}$. *Let* $F$ *be one of the faces of* $K$. *Let* $\mathcal{N}_F$ *be the collection of the indices of the Lagrange nodes on* $F$. *The following holds true for all* $p \in \mathbb{Q}_{k,d}$*:*

$$[\,\sigma_j(p) = 0,\ \forall j \in \mathcal{N}_F\,] \iff [\,p_{|F} = 0\,]. \tag{6.19}$$

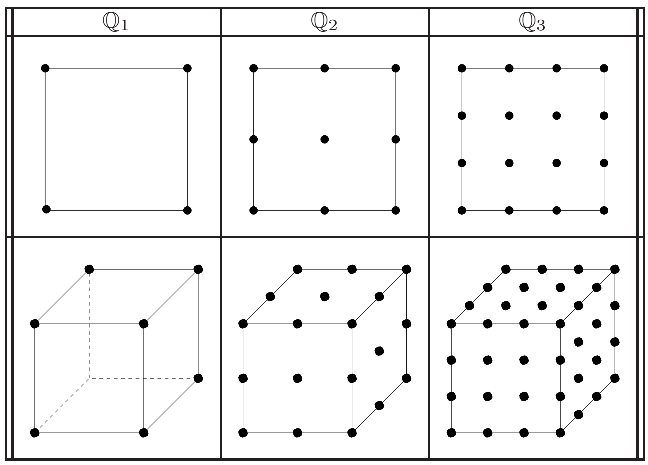



**Table 6.2** Two- and three-dimensional Lagrange finite elements $\mathbb{Q}_1$, $\mathbb{Q}_2$, and $\mathbb{Q}_3$. Only visible degrees of freedom are shown in dimension three.

Table 6.2 presents examples for $k \in \{1, 2, 3\}$ in dimensions $d \in \{2, 3\}$ with equidistant nodes in each Cartesian direction. The bullets conventionally indicate the location of the nodes. It is useful to use the tensor product of Gauss–Lobatto nodes when $k$ is large, since it can be shown that the Fekete points in cuboids are the tensor products of the one-dimensional Gauss–Lobatto nodes; see Bos et al. [35].

The shape functions of a tensor-product Lagrange finite element are products of the one-dimensional Lagrange polynomials defined in (5.2):

$$\theta_\beta(\boldsymbol{x}) := \prod_{i\in\{1:d\}} \mathcal{L}_{\beta_i}^{[a_i]}(x_i), \qquad \forall \boldsymbol{x} \in K, \tag{6.20}$$

for all $\beta \in \mathcal{B}_{k,d}$. The Lagrange interpolation operator acts as follows:

$$\mathcal{I}_K^{\mathrm{L}}(v)(\boldsymbol{x}) := \sum_{\beta \in \mathcal{B}_{k,d}} v(\boldsymbol{a}_\beta)\theta_\beta(\boldsymbol{x}), \qquad \forall \boldsymbol{x} \in K, \tag{6.21}$$

and possible choices for its domain are $V(K) := C^0(K)$ or $V(K) := W^{s,p}(K)$ with real numbers $p \in [1, \infty]$ and $sp > d$ (or $s \geq d$ if $p = 1$). Note that in general $\mathcal{I}_K^{\mathrm{L}}(v)(\boldsymbol{x})$ cannot be factored as a product of univariate functions, except when the function $v$ has the separated form $v(\boldsymbol{x}) = \prod_{i \in \{1:d\}} v_i(x_i)$ with $v_i \in C^0([x_i^-, x_i^+])$, in which case $\mathcal{I}_K^{\mathrm{L}}(v)(\boldsymbol{x}) = \prod_{i \in \{1:d\}} \mathcal{I}_{[x_i^-, x_i^+]}^{\mathrm{L}}(v_i)(x_i)$.

The tensor-product technique can also be used to build modal and hybrid nodal/modal finite elements in cuboids; see [35]. Finite element methods based on nodal bases using tensor products are often referred to as *spectral element methods*; see Patera [156].

### 6.4.3 Serendipity finite elements

It is possible to remove some nodes inside the cuboid while maintaining the approximation properties of the full tensor product. This is the idea of the serendipity finite elements. The corresponding polynomial space, $\mathbb{S}_k$, is then a proper subspace of $\mathbb{Q}_k$. The main motivation is to reduce computational costs without sacrificing the possibility to build $H^1$-conforming finite element spaces and without sacrificing the accuracy of the interpolation operator. Classical two-dimensional examples consist of using the 8 boundary nodes if $k = 2$ and the 12 boundary nodes if $k = 3$ (see Table 6.3). If $k = 4$ one uses the 16 boundary nodes plus the barycenter of $K$. A systematic construction of the serendipity finite elements for all dimensions and all polynomial degrees is devised in Arnold and Awanou [10].

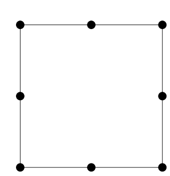
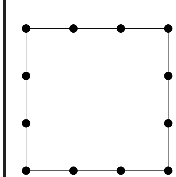
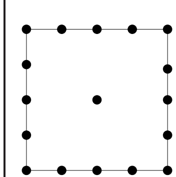
| $\mathbb{S}_2$ | $\mathbb{S}_3$ | $\mathbb{S}_4$ |
|---|---|---|
|  |  |  |

**Table 6.3** Two-dimensional serendipity Lagrange finite elements $\mathbb{S}_2$, $\mathbb{S}_3$, and $\mathbb{S}_4$.

## Exercises

**Exercise 6.1 (Integrated Legendre polynomials).** Let $k \geq 2$ and set $\mathbb{P}_k^{(0)} := \{p \in \mathbb{P}_k \mid p(\pm 1) = 0\}$. Show that a basis for $\mathbb{P}_k^{(0)}$ are the integrated

Legendre polynomials $\{\int_{-1}^{t} L_l(s)\,\mathrm{d}s\}_{l\in\{1:k-1\}}$. Prove (6.6). (*Hint*: consider moments against polynomials in $\mathbb{P}_{m-2}$ and the derivative at $t = 1$.)

**Exercise 6.2 (Gauss–Lobatto).** The goal of this exercise is to prove Proposition 6.6. (i) Prove that $k_{\mathcal{Q}} = 2m-3$. (*Hint*: for all $p \in \mathbb{P}_{2m-3}$, $m \geq 3$, write $p = p_1(1-t^2)L'_{m-1} + p_2$ with $p_1 \in \mathbb{P}_{m-3}$ and $p_2 \in \mathbb{P}_{m-1}$.) (ii) Prove that $\omega_1 = \omega_m = \frac{2}{m(m-1)}$. (*Hint*: compute $\int_{-1}^{1} L'_{m-1}(t)(1+t)L'_{m-1}(t)\,\mathrm{d}t$ using the quadrature and by integrating by parts.) (iii) Assume $m \geq 3$ and let $l \in \{2{:}m{-}1\}$. Prove that $L'_{m-2}(\xi_l) = (1-m)L_{m-1}(\xi_l)$ and $(1-\xi_l^2)L''_{m-1}(\xi_l)+m(m-1)L_{m-1}(\xi_l) = 0$. (*Hint*: use (6.3).) Let $\mathcal{L}_l \in \mathbb{P}_{m-3}$ be the Lagrange interpolation polynomial s.t. $\mathcal{L}_l(\xi_j) = \delta_{lj}$, for all $l, j \in \{2{:}m{-}1\}$ (i.e., $\xi_1$ and $\xi_m$ are excluded). Prove that $\mathcal{L}_l(t) = \frac{L'_{m-1}(t)}{t-\xi_l}\frac{1}{L''_{m-1}(\xi_l)}$. (*Hint*: compare the degree of the polynomials, their roots, and their value at $\xi_l$.) Finally, prove (6.11). (*Hint*: integrate the polynomial $\mathcal{L}_l(t)(1-t)L'_{m-2}(t)$.) (iv) Let $\|p\|_\xi^2 := \sum_{l\in\{1:m\}} \omega_l p(\xi_l)^2$. Verify that $\|\cdot\|_\xi$ defines a norm on $\mathbb{P}_k$ with $k := m-1$, and prove that $\|p\|_{L^2(K)} \leq \|p\|_\xi \leq (\frac{2k+1}{k})^{\frac{1}{2}}\|p\|_{L^2(K)}$ for all $p \in \mathbb{P}_k$, with $K := (-1, 1)$. (*Hint*: write $p = p_{k-1} + \lambda L_k$ with $p_{k-1} \in \mathbb{P}_{k-1}$ and $\lambda \in \mathbb{R}$, and compute $\|p\|^2_{L^2(K)}$ and $\|p\|^2_\xi$.)

**Exercise 6.3 (Gauss–Radau).** The goal is to prove Proposition 6.7. (i) Prove that $k_{\mathcal{Q}} = 2m-2$. (*Hint*: for all $p \in \mathbb{P}_{2m-2}$, write $p = p_1(L_m - L_{m-1}) + p_2$ with $p_1 \in \mathbb{P}_{m-2}$ and $p_2 \in \mathbb{P}_{m-1}$.) (ii) Prove that $\omega_m = \frac{2}{m^2}$. (*Hint*: integrate the polynomial $\frac{L_m(t)-L_{m-1}(t)}{t-1}L'_{m-1}(t)$.) (iii) Assume $m \geq 2$ and let $l \in \{1{:}m{-}1\}$. Prove that $L'_m(\xi_l) = -L'_{m-1}(\xi_l)$. (*Hint*: use (6.3a) and (6.3b).) Let $\mathcal{L}_l \in \mathbb{P}_{m-2}$ be the Lagrange interpolation polynomial s.t. $\mathcal{L}_l(\xi_j) = \delta_{lj}$ for all $l, j \in \{1{:}m{-}1\}$ (i.e., $\xi_m$ is excluded). Prove that $\mathcal{L}_l(t) = \frac{L_m(t)-L_{m-1}(t)}{(1-t)(t-\xi_l)}\frac{1-\xi_l}{-2L'_{m-1}(\xi_l)}$. (*Hint*: compare the degree of the polynomials, their roots, and their value at $\xi_l$.) Finally prove (6.12). (*Hint*: integrate the polynomial $\mathcal{L}_l(t)(1-t)L'_{m-1}(t)$.)

**Exercise 6.4 (Inverse trace inequality).** Let $K := [-1, 1]^d$. Let $m \geq 3$ and let $\{\xi_l\}_{l\in\{1:m\}}$ be the Gauss–Lobatto (GL) nodes in $[-1, 1]$. Set $I_{m,d} := \{1\ldots m\}^d$ and $I^0_{m,d} := \{2{:}(m-1)\}^d$. For any $\alpha \in I_{m,d}$, let $\boldsymbol{a}_\alpha \in K$ be the node with Cartesian coordinates $(a_\alpha)_i := \xi_{\alpha_i}$ for all $i \in \{1{:}d\}$. The set $(\boldsymbol{a}_\alpha)_{\alpha\in I_{m,d}}$ consists of the tensorized GL nodes in $K$. Let $k := m-1$ and define the polynomial space $\mathbb{Q}^0_{k,d} := \{q \in \mathbb{Q}_{k,d} \mid q(\boldsymbol{a}_\alpha) = 0, \forall\alpha \in I^0_{m,d}\}$, i.e., polynomials in $\mathbb{Q}^0_{k,d}$ vanish at all the tensorized GL nodes that are located inside $K$. Prove that

$$\|v\|_{L^2(K)} \leq \left(\frac{2d}{k(k+1)}(2+\tfrac{1}{k})^{d-1}\frac{|K|}{|\partial K|}\right)^{\frac{1}{2}} \|v\|_{L^2(\partial K)},$$

for all $v \in \mathbb{Q}^0_{k,d}$. (*Hint*: use Exercise 6.2.)

**Exercise 6.5 (Lagrange mass matrix).** Let $\mathcal{M} \in \mathbb{R}^{n_{\mathrm{sh}} \times n_{\mathrm{sh}}}$ be the mass matrix with entries $\mathcal{M}_{ij} := \int_{-1}^{1} \mathcal{L}_{i-1}^{[a]}(t)\mathcal{L}_{j-1}^{[a]}(t)\,\mathrm{d}t$ for all $i, j \in \mathcal{N}$. Prove that $\mathcal{M} = (\mathcal{V}^{\mathsf{T}}\mathcal{V})^{-1}$, where $\mathcal{V} \in \mathbb{R}^{n_{\mathrm{sh}} \times n_{\mathrm{sh}}}$ is the (generalized) Vandermonde matrix with entries $\mathcal{V}_{ij} := (\frac{2i-1}{2})^{\frac{1}{2}} L_{i-1}(a_j)$. (*Hint*: see Proposition 5.5.)

**Exercise 6.6 (Canonical hybrid element).** Prove Proposition 6.10. (*Hint*: use Remark 5.3.) Compute the shape functions when $\mu_l := J_{l-1}^{1,1}$ for all $l \in \{1{:}k-1\}$. (*Hint*: consider the polynomials $J_{k-1}^{1,0}$, $J_{l-1}^{1,1}$ for all $l \in \{1{:}k-1\}$, and $J_{k-1}^{0,1}$.)

**Exercise 6.7 ($\mathbb{Q}_{k,d}$ Lagrange).** Prove Proposition 6.14. (*Hint*: observe that any polynomial $q \in \mathbb{Q}_{k,d}$ is such that $q(\boldsymbol{x}) = \sum_{i_d \in \{0:k\}} q_{i_d}(x_1, \ldots, x_{d-1}) x_d^{i_d}$ and use induction on $d$.)

**Exercise 6.8 (Bicubic Hermite).** Let $K$ be a rectangle with vertices $\{\boldsymbol{z}_i\}_{1 \le i \le 4}$, $P := \mathbb{Q}_{3,2}$, and $\Sigma := \{p(\boldsymbol{z}_i), \partial_{x_1} p(\boldsymbol{z}_i), \partial_{x_2} p(\boldsymbol{z}_i), \partial^2_{x_1 x_2} p(\boldsymbol{z}_i)\}_{1 \le i \le 4}$. Show that $(K, P, \Sigma)$ is a finite element. (*Hint*: write $p \in \mathbb{Q}_{3,2}$ in the form $p(\boldsymbol{x}) = \sum_{i,j \in \{1:4\}} \gamma_{ij} \theta_i(x_1) \theta_j(x_2)$, where $\{\theta_1, \ldots, \theta_4\}$ are the shape functions of the one-dimensional Hermite finite element; see Exercise 5.4.)

**Exercise 6.9 (Face unisolvence).** Prove Lemma 6.15. (*Hint*: use the hint from Exercise 6.7.)