Part II, Chapter 7

Simplicial finite elements

This chapter deals with finite elements (K, P, Σ) where K is a triangle in $\mathbb{R}^2$, a tetrahedron in $\mathbb{R}^3$, and more generally a simplex in $\mathbb{R}^d$, $d \geq 2$. The degrees of freedom (dofs) Σ are either nodal values in K or integrals over the faces or the edges of K, and P is the space $\mathbb{P}_{k,d}$ composed of multivariate polynomials of total degree at most $k \geq 0$. We focus our attention on scalar-valued finite elements. The results extend to the vector-valued case by reasoning componentwise.

7.1 Simplices

Definition 7.1 (Simplex, vertices, normal). *Let $d \geq 1$. Let $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$ be a set of points in $\mathbb{R}^d$ such that the vectors $\{\boldsymbol{z}_1 - \boldsymbol{z}_0, \ldots, \boldsymbol{z}_d - \boldsymbol{z}_0\}$ are linearly independent. The convex hull of these points is called* simplex *in $\mathbb{R}^d$, say $K := \mathrm{conv}(\{\boldsymbol{z}_i\}_{i\in\{0:d\}})$. By definition, K is a closed set. The points $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$ are called* vertices *of K. The outward unit normal vector on ∂K is denoted by $\boldsymbol{n}_K$.*

Example 7.2 ($d \in \{1, 2, 3\}$). A simplex is a compact interval if $d = 1$, a triangle if $d = 2$, and a tetrahedron if $d = 3$ (see Figure 5.2). □

Example 7.3 (Unit simplex). The unit simplex in $\mathbb{R}^d$ is the set $\{\boldsymbol{x} \in \mathbb{R}^d \mid 0 \leq x_i \leq 1, \forall i \in \{1:d\}, \sum_{i\in\{0:d\}} x_i \leq 1\}$. This corresponds to setting $\boldsymbol{z}_0 := \boldsymbol{0}$ and $\boldsymbol{z}_i - \boldsymbol{z}_0 := \boldsymbol{e}_i$ for all $i \in \{1:d\}$, where $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. The unit simplex has volume $\frac{1}{d!}$. □

Definition 7.4 (Faces, edges). *The convex hull of the set $\{\boldsymbol{z}_0, \ldots, \boldsymbol{z}_d\} \backslash \{\boldsymbol{z}_i\}$ is denoted by F_i for all $i \in \{0:d\}$ and is called the* face *of K opposite to the vertex $\boldsymbol{z}_i$. For all $l \in \{0:d-1\}$, an l-face of K is the convex hull of a subset of $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$ of cardinality $(l + 1)$ (i.e., usual faces are $(d - 1)$-faces). By definition, l-faces are closed sets and are subsets of an affine subspace of $\mathbb{R}^d$*

of codimension $(d-l)$. The 0-faces of K are the vertices of K. The 1-faces of K are called edges. *In dimension $d = 2$, the notions of edge and face coincide. In dimension $d = 1$, the notions of vertex, edge, and face coincide.*

Example 7.5 (Number of faces and edges). The number of l-faces in a simplex in $\mathbb{R}^d$ is equal to $\binom{d+1}{l+1}$, e.g., there are $(d+1)$ faces and vertices, and for $d \geq 2$, there are $\frac{d(d+1)}{2}$ edges. □

Remark 7.6 (Geometric identities). Let $\boldsymbol{n}_{K|F_i}$ be the value of $\boldsymbol{n}_K$ on F_i for all $i \in \{0:d\}$. Then $\{\boldsymbol{n}_{K|F_i}\}_{i\in\{1:d\}}$ is a basis of $\mathbb{R}^d$. Let $\boldsymbol{c}_{F_i}$ be the barycenter of F_i, $\boldsymbol{c}_K$ that of K, and $\mathbb{I}_d$ the identity matrix in $\mathbb{R}^{d\times d}$. We have

$$\sum_{i\in\{0:d\}} |F_i|\boldsymbol{n}_{K|F_i} = \boldsymbol{0}, \qquad \sum_{i\in\{0:d\}} |F_i|\boldsymbol{n}_{K|F_i} \otimes (\boldsymbol{c}_{F_i} - \boldsymbol{c}_K) = |K|\mathbb{I}_d. \tag{7.1}$$

See Exercise 7.2. These identities hold true for any polyhedron in $\mathbb{R}^d$. □

7.2 Barycentric coordinates, geometric mappings

Let K be a simplex in $\mathbb{R}^d$ with vertices $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$. For all $\boldsymbol{x} \in \mathbb{R}^d$ and all $i \in \{1:d\}$, we denote by $\lambda_i(\boldsymbol{x})$ the components of the vector $\boldsymbol{x} - \boldsymbol{z}_0$ in the basis $(\boldsymbol{z}_1 - \boldsymbol{z}_0, \ldots, \boldsymbol{z}_d - \boldsymbol{z}_0)$, i.e.,

$$\boldsymbol{x} - \boldsymbol{z}_0 = \sum_{i\in\{1:d\}} \lambda_i(\boldsymbol{x})(\boldsymbol{z}_i - \boldsymbol{z}_0). \tag{7.2}$$

Differentiating (7.2) twice, we infer that $\sum_{i\in\{0:d\}} D^2\lambda_i(\boldsymbol{x})(\boldsymbol{h}_1, \boldsymbol{h}_2)(\boldsymbol{z}_i - \boldsymbol{z}_0) = 0$ for all $\boldsymbol{h}_1, \boldsymbol{h}_2 \in \mathbb{R}^d$. The vectors $\{\boldsymbol{z}_i - \boldsymbol{z}_0\}_{i\in\{1:d\}}$ being linearly independent, this implies that $D^2\lambda_1(\boldsymbol{x})(\boldsymbol{h}_1, \boldsymbol{h}_2) = \ldots = D^2\lambda_d(\boldsymbol{x})(\boldsymbol{h}_1, \boldsymbol{h}_2) = 0$. Hence, λ_i is an affine function of $\boldsymbol{x}$, i.e., there exist $\gamma_i \in \mathbb{R}$ and $\boldsymbol{g}_i \in \mathbb{R}^d$ such that $\lambda_i(\boldsymbol{x}) = \gamma_i + \boldsymbol{g}_i{\cdot}\boldsymbol{x}$ for all $\boldsymbol{x} \in \mathbb{R}^d$, where $\boldsymbol{a}{\cdot}\boldsymbol{b}$ denotes the inner product in $\mathbb{R}^d$. Note that $D\lambda_i$ is independent of $\boldsymbol{x}$ and $D\lambda_i(\boldsymbol{h}) = \boldsymbol{g}_i{\cdot}\boldsymbol{h}$ for all $\boldsymbol{h} \in \mathbb{R}^d$. In other words, we have $\nabla\lambda_i = \boldsymbol{g}_i$.

To allow all the vertices of K to play a symmetric role, we introduce the additional function $\lambda_0(\boldsymbol{x}) := 1 - \sum_{i\in\{1:d\}} \lambda_i(\boldsymbol{x})$. Then we have

$$\sum_{i\in\{0:d\}} \lambda_i(\boldsymbol{x}) = 1 \qquad \text{and} \qquad \boldsymbol{x} = \sum_{i\in\{0:d\}} \lambda_i(\boldsymbol{x})\boldsymbol{z}_i, \tag{7.3}$$

for all $\boldsymbol{x} \in \mathbb{R}^d$. A consequence of the above definitions is that $\lambda_i(\boldsymbol{z}_j) = \delta_{ij}$ for all $i, j \in \{0:d\}$. This implies that the functions $\{\lambda_i\}_{i\in\{0:d\}}$ are linearly independent: if the linear combination $\sum_{i\in\{0:d\}} \beta_i\lambda_i(\boldsymbol{x})$ vanishes identically, evaluating it at the vertex $\boldsymbol{z}_j$ yields $\beta_j = 0$ for all $j \in \{0:d\}$. Moreover, since

K is the convex hull of $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$, we infer that $0 \le \lambda_i(\boldsymbol{x}) \le 1$ for all $\boldsymbol{x} \in K$ and all $i \in \{0:d\}$.

Definition 7.7 (Barycentric coordinates). *The functions* $\{\lambda_i\}_{i\in\{0:d\}}$ *are called* barycentric coordinates *in* K.

It is shown below that the barycentric coordinates are also the shape functions of the $\mathbb{P}_{1,d}$ Lagrange finite element.

Example 7.8 (Unit simplex). Since $\boldsymbol{x} = \sum_{i\in\{1:d\}} x_i\boldsymbol{e}_i$, (7.2) shows that the barycentric coordinates in the unit simplex of $\mathbb{R}^d$ are $\lambda_0(\boldsymbol{x}) := 1 - \sum_{i\in\{1:d\}} x_i$ and $\lambda_i(\boldsymbol{x}) := x_i$ for all $i \in \{1:d\}$. $\square$

The following construction plays an important role in the rest of the book. Let $\widehat{S}^l := \mathrm{conv}(\{\widehat{\boldsymbol{z}}_j\}_{j\in\{0:l\}})$ be the unit simplex in $\mathbb{R}^l$ with barycentric coordinates $\{\widehat{\lambda}_j\}_{j\in\{0:l\}}$ (see Example 7.8).

Proposition 7.9 (Geometric mapping). *Let K be a simplex in $\mathbb{R}^d$, let $l \in \{1:d\}$, and let $\sigma : \{0:l\} \to \{0:d\}$ be an injective map, i.e., σ chooses $(l+1)$ distinct integers in $\{0:d\}$. Let $S := \mathrm{conv}(\{\boldsymbol{z}_{\sigma(j)}\}_{j\in\{0:l\}})$ be an l-face of K or K itself if $l = d$. Let $\boldsymbol{T}_S : \widehat{S}^l \to \mathbb{R}^d$ be the geometric mapping s.t. $\boldsymbol{T}_S(\widehat{\boldsymbol{x}}) = \sum_{j\in\{0:l\}} \widehat{\lambda}_j(\widehat{\boldsymbol{x}})\boldsymbol{z}_{\sigma(j)}$ for all $\widehat{\boldsymbol{x}} \in \widehat{S}^l$. Then $S = \boldsymbol{T}_S(\widehat{S}^l)$, and the mapping $\boldsymbol{T}_S$ is a smooth diffeomorphism.*

Proof. We first notice that $\boldsymbol{T}_S(\widehat{\boldsymbol{z}}_j) = \boldsymbol{z}_{\sigma(j)}$ for all $j \in \{0:l\}$ and that $\boldsymbol{T}_S$ is an affine mapping since $\boldsymbol{T}_S(\widehat{\boldsymbol{x}}) = \boldsymbol{z}_{i_0} + \sum_{j\in\{1:l\}} \widehat{x}_j(\boldsymbol{z}_{\sigma(j)} - \boldsymbol{z}_{i_0})$. Let $\{\theta_j\}_{j\in\{0:l\}}$ be any nonnegative numbers s.t. $\sum_{j\in\{0:l\}} \theta_j = 1$. We have

$$\sum_{j\in\{0:l\}} \theta_j \boldsymbol{z}_{\sigma(j)} = \sum_{j\in\{0:l\}} \theta_j \boldsymbol{T}_S(\widehat{\boldsymbol{z}}_j) = \boldsymbol{T}_S\bigg(\sum_{j\in\{0:l\}} \theta_j \widehat{\boldsymbol{z}}_j\bigg).$$

Since $S = \mathrm{conv}(\{\boldsymbol{z}_{\sigma(j)}\}_{j\in\{0:l\}})$ and $\widehat{S}^l = \mathrm{conv}(\{\widehat{\boldsymbol{z}}_j\}_{j\in\{0:l\}})$, this proves that $S = \boldsymbol{T}_S(\widehat{S}^l)$. Moreover, the mapping $\boldsymbol{T}_S$ is of class C^∞ since it is linear. We now show that the linear mapping $D\boldsymbol{T}_S : \mathbb{R}^l \to \mathbb{R}^l$ is invertible by verifying the injectivity. Let $\widehat{\boldsymbol{h}} \in \mathbb{R}^l$ be such that $D\boldsymbol{T}_S(\widehat{\boldsymbol{h}}) = \boldsymbol{0}$. Writing $\widehat{\boldsymbol{h}} = \sum_{j\in\{1:l\}} \widehat{h}_j(\widehat{\boldsymbol{z}}_j - \widehat{\boldsymbol{z}}_0)$ and since $D\boldsymbol{T}_S(\widehat{\boldsymbol{z}}_j - \widehat{\boldsymbol{z}}_0) = \boldsymbol{T}_S(\widehat{\boldsymbol{z}}_j) - \boldsymbol{T}_S(\widehat{\boldsymbol{z}}_0) = \boldsymbol{z}_{\sigma(j)} - \boldsymbol{z}_{\sigma(0)}$, we infer that $\boldsymbol{0} = \sum_{j\in\{1:l\}} \widehat{h}_j(\boldsymbol{z}_{\sigma(j)} - \boldsymbol{z}_{\sigma(0)})$, implying that $\boldsymbol{h} = \boldsymbol{0}$. $\square$

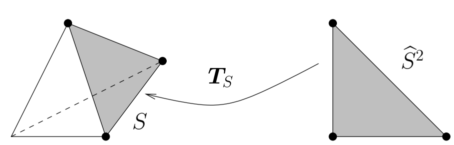

Fig. 7.1 Geometric mapping $\boldsymbol{T}_S$ ($d = 3$, $l = 2$). The face S of K is highlighted in gray, and the vertices of both $\widehat{S}^2$ and S are indicated by bullets.

7.3 The polynomial space $\mathbb{P}_{k,d}$

The real vector space $\mathbb{P}_{k,d}$ is composed of d-variate polynomial functions $p : \mathbb{R}^d \to \mathbb{R}$ of total degree at most k. Thus, we have

$$\mathbb{P}_{k,d} := \operatorname{span}\{x_1^{\alpha_1}\dots x_d^{\alpha_d},\ 0 \le \alpha_1,\dots,\alpha_d \le k,\ \alpha_1+\dots+\alpha_d \le k\}. \tag{7.4}$$

The importance of the polynomial space $\mathbb{P}_{k,d}$ is rooted in the fact that the Taylor expansion of order k of any d-variate function belongs to $\mathbb{P}_{k,d}$. Another important fact is that for every smooth function $v : \mathbb{R}^d \to \mathbb{R}$,

$$[\, v \in \mathbb{P}_{k,d}\,] \iff [\, D^{k+1}v(\boldsymbol{x}) = 0,\ \ \forall \boldsymbol{x} \in \mathbb{R}^d\,]. \tag{7.5}$$

The vector space $\mathbb{P}_{k,d}$ has dimension (see Exercise 7.4)

$$\dim \mathbb{P}_{k,d} = \binom{k+d}{d} = \begin{cases} k+1 & \text{if } d=1,\\ \frac{1}{2}(k+1)(k+2) & \text{if } d=2,\\ \frac{1}{6}(k+1)(k+2)(k+3) & \text{if } d=3.\end{cases} \tag{7.6}$$

We omit the subscript d and write $\mathbb{P}_k$ when the context is unambiguous.

An element $\alpha := (\alpha_1,\dots,\alpha_d)$ of $\mathbb{N}^d$ is called *multi-index*, and its length is defined as $|\alpha| := \alpha_1+\dots+\alpha_d$. We define the multi-index set $\mathcal{A}_{k,d} := \{\alpha \in \mathbb{N}^d \mid |\alpha| \le k\}$. Note that $\operatorname{card}(\mathcal{A}_{k,d}) = \dim(\mathbb{P}_{k,d}) = \binom{k+d}{d}$. Any polynomial function $p \in \mathbb{P}_{k,d}$ can be written in the form

$$p(\boldsymbol{x}) = \sum_{\alpha \in \mathcal{A}_{k,d}} a_\alpha \boldsymbol{x}^\alpha, \quad \text{with} \ \ \boldsymbol{x}^\alpha := x_1^{\alpha_1}\dots x_d^{\alpha_d} \text{ and } a_\alpha \in \mathbb{R}. \tag{7.7}$$

Let H be an affine subspace in $\mathbb{R}^d$ of dimension $l \in \{1{:}d{-}1\}$. Given a polynomial $p \in \mathbb{P}_{k,d}$, the following result gives a characterization of the trace of p on H which will be used repeatedly in the book.

Lemma 7.10 (Trace space). *Let H be an affine subspace in $\mathbb{R}^d$ of dimension $l \in \{1{:}d{-}1\}$. Then $p_{|H} \circ \boldsymbol{T}_H \in \mathbb{P}_{k,l}$ for all $p \in \mathbb{P}_{k,d}$ and every affine bijective mapping $\boldsymbol{T}_H : \mathbb{R}^l \to H$. Moreover, $q \circ \boldsymbol{T}_{\mathbb{R}^l} \in \mathbb{P}_{k,d}$ for all $q \in \mathbb{P}_{k,l}$ and every affine mapping $\boldsymbol{T}_{\mathbb{R}^l} : \mathbb{R}^d \to \mathbb{R}^l$.*

Proof. We observe that $D^{k+1}(p_{|H} \circ \boldsymbol{T}_H)(\boldsymbol{y}) = 0$ for all $\boldsymbol{y} \in \mathbb{R}^l$ by using the chain rule and the fact that $\boldsymbol{T}_H$ is affine. Then we apply (7.5). The second statement is proved similarly. $\square$

7.4 Lagrange (nodal) finite elements

We begin with a simple example where we set $k := 1$; see Table 7.1.

Proposition 7.11 (Simplicial Lagrange, $k := 1$). *Let K be a simplex in $\mathbb{R}^d$ with vertices $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$. Let $P := \mathbb{P}_{1,d}$. Let $\Sigma := \{\sigma_i\}_{i\in\{0:d\}}$ be the linear forms on P such that $\sigma_i(p) := p(\boldsymbol{z}_i)$ for all $i \in \{0:d\}$. Then (K, P, Σ) is a Lagrange finite element and the shape functions are $\theta_i := \lambda_i$.*

Proof. Let $p \in P$. We use (7.2), i.e., $\boldsymbol{x} - \boldsymbol{z}_0 = \sum_{i\in\{1:d\}} \lambda_i(\boldsymbol{x})(\boldsymbol{z}_i - \boldsymbol{z}_0)$, that p is affine, the linearity of Dp, and the first identity in (7.3) to infer that

$$\begin{aligned} p(\boldsymbol{x}) &= p(\boldsymbol{z}_0) + Dp(\boldsymbol{x} - \boldsymbol{z}_0) = p(\boldsymbol{z}_0) + \sum_{i\in\{1:d\}} \lambda_i(\boldsymbol{x}) Dp(\boldsymbol{z}_i - \boldsymbol{z}_0) \\ &= \sum_{i\in\{0:d\}} \big(\lambda_i(\boldsymbol{x})(p(\boldsymbol{z}_0) + Dp(\boldsymbol{z}_i - \boldsymbol{z}_0))\big) = \sum_{i\in\{0:d\}} \lambda_i(\boldsymbol{x}) p(\boldsymbol{z}_i), \end{aligned}$$

for all $\boldsymbol{x} \in \mathbb{R}^d$. Now we use Remark 5.3. We have $\dim P = d + 1 = \operatorname{card} \Sigma$, and the above identity shows that any polynomial in P vanishing at the $(d+1)$ vertices of K vanishes identically. Hence, (K, P, Σ) is a finite element. Finally, owing to the above identity applied with $p := \theta_j$, we have $\theta_j(\boldsymbol{x}) = \sum_{i\in\{0:d\}} \lambda_i(\boldsymbol{x})\theta_j(\boldsymbol{z}_i) = \sum_{i\in\{0:d\}} \lambda_i(\boldsymbol{x})\delta_{ij} = \lambda_j(\boldsymbol{x})$ for all $\boldsymbol{x} \in K$. This proves that $\theta_j = \lambda_j$ for all $j \in \{0:d\}$. $\square$

We now extend the above construction to any polynomial order $k \geq 1$ using equidistributed nodes in the simplex K. Other choices are discussed in Remark 7.14.

Proposition 7.12 (Simplicial Lagrange). *Let K be a simplex in $\mathbb{R}^d$. Let $k \geq 1$, $P := \mathbb{P}_{k,d}$, and $\mathcal{A}_{k,d} := \{\alpha \in \mathbb{N}^d \mid |\alpha| \leq k\}$. Set $n_{\rm sh} := \binom{k+d}{d}$ and consider the set of nodes $\{\boldsymbol{a}_\alpha\}_{\alpha\in\mathcal{A}_{k,d}}$ s.t. $\boldsymbol{a}_\alpha - \boldsymbol{z}_0 := \sum_{i\in\{1:d\}} \frac{\alpha_i}{k}(\boldsymbol{z}_i - \boldsymbol{z}_0)$. Let $\Sigma := \{\sigma_\alpha\}_{\alpha\in\mathcal{A}_{k,d}}$ be the linear forms on P s.t. $\sigma_\alpha(p) := p(\boldsymbol{a}_\alpha)$ for all $\alpha \in \mathcal{A}_{k,d}$. Then (K, P, Σ) is a Lagrange finite element.*

Proof. We use Remark 5.3. Since $\operatorname{card} \Sigma = \operatorname{card} \mathcal{A}_{k,d} = \binom{k+d}{d} = \dim \mathbb{P}_{k,d}$, we need to prove the following property which we call $[\mathcal{P}_{k,d}]$: Any polynomial $p \in \mathbb{P}_{k,d}$ vanishing at all the Lagrange nodes $\{\boldsymbol{a}_\alpha\}_{\alpha\in\mathcal{A}_{k,d}}$ of any simplex in $\mathbb{R}^d$ vanishes identically. Property $[\mathcal{P}_{k,1}]$ holds true for all $k \geq 1$ owing to Proposition 6.8. Assume now that $d \geq 2$ and that $[\mathcal{P}_{k,d-1}]$ holds true for all $k \geq 1$ and let us prove that $[\mathcal{P}_{k,d}]$ holds true for all $k \geq 1$. Assume that $p \in \mathbb{P}_{k,d}$ vanishes at all the Lagrange nodes of a simplex K. Let F_0 be the face of K opposite to the vertex $\boldsymbol{z}_0$ and consider an affine bijective mapping $\boldsymbol{T}_{H_0} : \mathbb{R}^{d-1} \to H_0$, where H_0 is the affine hyperplane supporting F_0. Then $p_0 := p \circ \boldsymbol{T}_{H_0}$ is in $\mathbb{P}_{k,d-1}$ owing to Lemma 7.10, and by assumption, $p_0(\boldsymbol{T}_{H_0}^{-1}(\boldsymbol{a}_\alpha)) = p(\boldsymbol{a}_\alpha) = 0$ for all $\boldsymbol{a}_\alpha \in F_0$. Moreover, $\boldsymbol{a}_\alpha \in F_0$ iff $|\alpha| = k$. Let us set $\tilde{\beta} := (k - |\beta|, \beta_1, \ldots, \beta_{d-1})$ for all $\beta \in \mathcal{A}_{k,d-1}$, so that $\tilde{\beta} \in \mathcal{A}_{k,d}$ and $|\tilde{\beta}| = k$. Setting $\boldsymbol{b}_\beta := \boldsymbol{T}_{H_0}^{-1}(\boldsymbol{a}_{\tilde{\beta}})$ for all $\beta \in \mathcal{A}_{k,d-1}$, we obtain all the Lagrange nodes of the simplex $\boldsymbol{T}_{H_0}^{-1}(F)$ in $\mathbb{R}^{d-1}$. Since $p_0(\boldsymbol{b}_\beta) = p(\boldsymbol{a}_{\tilde{\beta}}) = 0$ for all $\beta \in \mathcal{A}_{k,d-1}$, we infer owing to $[\mathcal{P}_{k,d-1}]$ that $p_0 = 0$. Since $\boldsymbol{T}_{H_0}$ is bijective, we obtain $p_{|F_0} = 0$. Denoting by $\lambda_0 \in \mathbb{P}_{1,d}$ the barycentric coordinate associated with $\boldsymbol{z}_0$, this implies that there is $q \in \mathbb{P}_{k-1,d}$ s.t. $p = \lambda_0 q$ (see Exercise 7.4(iv)). Let us prove by induction on k that $q = 0$. For $k = 1$, we have already proved $[\mathcal{P}_{1,d}]$ in Proposition 7.11. Let us now assume that $[\mathcal{P}_{k-1,d}]$ holds true for $k \geq 2$. Since $k \geq 2$, q vanishes at all the Lagrange nodes $\boldsymbol{a}_\alpha$ s.t. $|\alpha| < k$ (since $\lambda_0(\boldsymbol{a}_\alpha) \neq 0$ at these nodes), i.e., $|\alpha| \leq k - 1$. Hence, q vanishes at all the Lagrange nodes $\boldsymbol{a}_\alpha$, $\alpha \in \mathcal{A}_{k-1,d}$. Since these nodes belong again to a simplex, $[\mathcal{P}_{k-1,d}]$ implies $q = 0$. $\square$

We have established the following result in the proof of Proposition 7.12.

Lemma 7.13 (Face unisolvence). *Let F be one of the $(d+1)$ faces of the simplex $K \subset \mathbb{R}^d$. Let $\mathcal{N}_F$ be the collection of the indices of the Lagrange nodes on F. The following holds true for all $p \in \mathbb{P}_{k,d}$:*

$$[\,\sigma_j(p) = 0,\ \forall j \in \mathcal{N}_F\,] \iff [\,p_{|F} = 0\,]. \tag{7.8}$$

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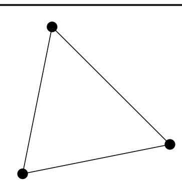	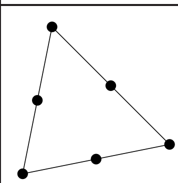	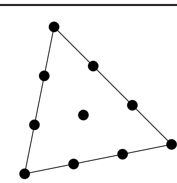
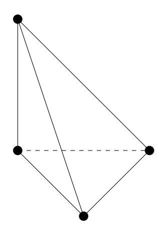	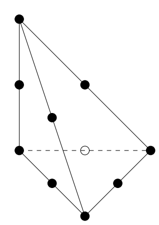	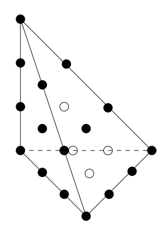
λ_i	$\lambda_i(2\lambda_i - 1)$ $4\lambda_i\lambda_j$	$\frac{1}{2}\lambda_i(3\lambda_i - 1)(3\lambda_i - 2)$ $\pm\frac{9}{2}\lambda_i(3\lambda_i - \frac{3}{2} \pm \frac{1}{2})\lambda_j$ $27\lambda_i\lambda_j\lambda_k$

Table 7.1 Two- and three-dimensional $\mathbb{P}_1$, $\mathbb{P}_2$, and $\mathbb{P}_3$ Lagrange elements. Visible degrees of freedom are shown in black, hidden degrees of freedom are in white, and hidden edges are represented with dashed lines. The shape functions are expressed in terms of the barycentric coordinates. The first, second, and third lines list shape functions associated with the vertices ($i \in \{0{:}d\}$), the edges ($i, j \in \{0{:}d\}$, $i < j$), and the faces ($i, j, k \in \{0{:}d\}$, $i < j < k$).

Table 7.1 presents examples of node locations and shape functions for $k \in \{1, 2, 3\}$ in dimension $d \in \{2, 3\}$. The bullets conventionally indicate the location of the nodes; see Exercise 7.5 for some properties of these nodes.

Possible choices for the domain of the interpolation operator are $V(K) := C^0(K)$ or $V(K) := W^{s,p}(K)$ with $p \in [1, \infty]$ and $sp > d$ (or $s \geq d$ if $p = 1$); see §5.4.1.

Remark 7.14 (High-order). Other sets of Lagrange nodes can be used. For instance, the Fekete points from §6.3.5 can be extended to simplices, although finding Fekete points on simplices for high polynomial degrees is a difficult problem. We refer the reader to Chen and Babuška [66] and Taylor et al. [190] for results on triangles with degrees up to $k = 13$ and $k = 19$, respectively; see also Canuto et al. [58, p. 112]. A comparison of various nodal sets on triangles and tetrahedra can be found in Blyth et al. [26]. □

Remark 7.15 (Modal and hybrid simplicial elements). A hierarchical basis of $\mathbb{P}_{k,d}$ can be built by combining a hierarchical univariate basis of $\mathbb{P}_{k,1}$ with the barycentric coordinates; see Ainsworth and Coyle [6] and Exercise 7.6. One can also introduce a nonlinear transformation mapping the simplex to a cuboid and use tensor products of one-dimensional basis functions in the cuboid; see Proriol [162], Dubiner [91], Owens [154], Karniadakis and Sherwin [123, §3.2]. Another possibility is to use *Bernstein polynomials*, i.e., the basis $\{\binom{p}{m} t^m (1-t)^{p-m}\}_{m \in \{0:p\}}$ if $d = 1$; see Ainsworth et al. [7], Kirby [125] for scalar-valued polynomials and Kirby [126] for the extension to the de Rham complex (see also §16.3). □

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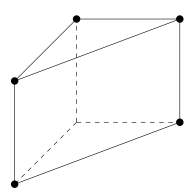	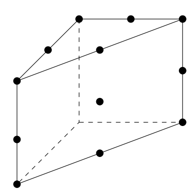	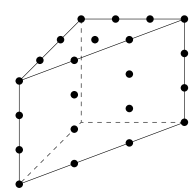

Table 7.2 Nodes for prismatic Lagrange finite elements of degree 1, 2, and 3. The bullets indicate the location of the nodes. Only visible nodes are shown.

Remark 7.16 (Prismatic Lagrange elements). Let $d \geq 3$ and set $\boldsymbol{x}' := (x_1, \ldots, x_{d-1})$ for all $\boldsymbol{x} \in \mathbb{R}^d$. Let K' be a simplex in $\mathbb{R}^{d-1}$ and $[z_d^-, z_d^+]$ be an interval with $z_d^- < z_d^+$. The set $K := \{\boldsymbol{x} \in \mathbb{R}^d \mid \boldsymbol{x}' \in K',\ x_d \in [z_d^-, z_d^+]\}$ is called *prism* in $\mathbb{R}^d$. Let $k \geq 1$ and let $\mathbb{PR}_k := \operatorname{span}\{p(\boldsymbol{x}) = p_1(\boldsymbol{x}')\, p_2(x_d) \mid p_1 \in \mathbb{P}_{k,d-1},\ p_2 \in \mathbb{P}_{k,1}\}$. Examples of prismatic Lagrange elements based on K and $\mathbb{PR}_k$ with equidistributed nodes are shown in Table 7.2 for $k \in \{1, 2, 3\}$. □

7.5 Crouzeix–Raviart finite element

The Crouzeix–Raviart finite element is based on the polynomial space $\mathbb{P}_{1,d}$. It has been introduced in [86] to approximate the Stokes equations. Let K be a simplex in $\mathbb{R}^d$ with vertices $\{\boldsymbol{z}_i\}_{i\in\{0:d\}}$. Recall that the face of K opposite to $\boldsymbol{z}_i$ is denoted by F_i.

Proposition 7.17 (Finite element). *Let K be a simplex in $\mathbb{R}^d$, set $P := \mathbb{P}_{1,d}$, and define the following dofs on P:*

$$\sigma_i^{\mathrm{CR}}(p) := \frac{1}{|F_i|}\int_{F_i} p\,\mathrm{d}s, \qquad \forall i \in \{0:d\}. \tag{7.9}$$

Set $\Sigma := \{\sigma_i^{\mathrm{CR}}\}_{i\in\{0:d\}}$. Then (K,P,Σ) is a finite element.

Proof. Since $\operatorname{card}\Sigma = \dim P = d+1$, it suffices to verify that any polynomial p in P satisfying $\sigma_i^{\mathrm{CR}}(p) = \frac{1}{|F_i|}\int_{F_i} p\,\mathrm{d}s = 0$ for all $i \in \{0:d\}$ vanishes identically. Since $p \in \mathbb{P}_{1,d}$, we have $p = \sum_{j\in\{0:d\}} p(\boldsymbol{z}_j)\lambda_j$, where $\{\lambda_j\}_{j\in\{0:d\}}$ are the barycentric coordinates in K. Owing to Exercise 7.3(iii), we infer that $\sigma_i^{\mathrm{CR}}(p) = \sum_{j\in\{0:d\}} p(\boldsymbol{z}_j)\sigma_i^{\mathrm{CR}}(\lambda_j) = \frac{1}{d}\sum_{j\neq i} p(\boldsymbol{z}_j)$ since $\sigma_i^{\mathrm{CR}}(\lambda_i) = 0$ and $\sigma_i^{\mathrm{CR}}(\lambda_j) = \frac{1}{d}|F_i|$ for all $j \neq i$. Hence, $\sum_{j\neq i} p(\boldsymbol{z}_j) = 0$ for all $i \in \{0:d\}$. This implies that $0 = \sum_{j\neq i} p(\boldsymbol{z}_j) - \sum_{j\neq i'} p(\boldsymbol{z}_j) = p(\boldsymbol{z}_i) - p(\boldsymbol{z}_{i'})$ for every pair (i,i') such that $i \neq i'$. Hence, p takes a constant value at all the vertices of K, and this value must be zero since, say, $\sum_{j\neq 0} p(\boldsymbol{z}_j) = 0$. $\square$

Using the barycentric coordinates $\{\lambda_i\}_{i\in\{0:d\}}$ in K, one can verify that the shape functions are $\theta_i^{\mathrm{CR}}(\boldsymbol{x}) := 1 - d\lambda_i(\boldsymbol{x})$ for all $i \in \{0:d\}$ and all $\boldsymbol{x} \in K$. Note that $\theta_{i|F_i}^{\mathrm{CR}} = 1$ and $\theta_i^{\mathrm{CR}}(\boldsymbol{z}_i) = 1-d$. The Crouzeix–Raviart interpolation operator acts as follows:

$$\mathcal{I}_K^{\mathrm{CR}}(v)(\boldsymbol{x}) := \sum_{i\in\{0:d\}} \sigma_{K,i}^{\mathrm{CR}}(v)\theta_{K,i}^{\mathrm{CR}}(\boldsymbol{x}) = \sum_{i\in\{0:d\}} \left(\frac{1}{|F_i|}\int_{F_i} v\,\mathrm{d}s\right)\theta_i^{\mathrm{CR}}(\boldsymbol{x}), \tag{7.10}$$

for all $\boldsymbol{x} \in K$. A possible choice for the domain of $\mathcal{I}_K^{\mathrm{CR}}$ is $V(K) := W^{1,1}(K)$ since the trace theorem (Theorem 3.10) applied with $p := 1$ implies that any function in $W^{1,1}(K)$ has a trace in $L^1(\partial K)$. The two- and three-dimensional Crouzeix–Raviart elements are shown in Table 7.3.

Remark 7.18 (Definition as a Lagrange element). The mean-value over a face of a polynomial in $\mathbb{P}_{1,d}$ is equal to the value this polynomial takes at the barycenter of the face. Another possible choice for the dofs is therefore to take the values at the barycenter of all the faces. The resulting finite element is a Lagrange finite element (see Definition 5.11), and $W^{1,1}(K)$ is no longer a legitimate domain for the interpolation operator. One possible choice is the smaller space $V(K) := C^0(K)$. $\square$

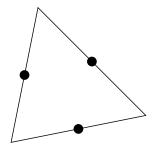	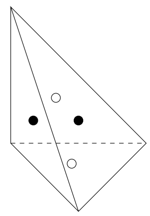
$1-2\lambda_i$	$1-3\lambda_i$

Table 7.3 $\mathbb{P}_1$ Crouzeix–Raviart elements in dimensions two and three. Visible degrees of freedom are shown in black, hidden degrees of freedom are in white, and hidden edges are represented with dashed lines. The shape functions are expressed in terms of the barycentric coordinates.

7.6 Canonical hybrid finite element

We now present a finite element based on the polynomial space $\mathbb{P}_{k,d}$ whose dofs combine values at the vertices of the simplex K with integrals over the l-faces of K for $l \geq 1$ (hence the name hybrid). It is a useful alternative to Lagrange elements that has interesting commuting properties, which will be invoked in §16.3 in the context of the discrete de Rham complex (hence the name canonical).

Let K be a tetrahedron in $\mathbb{R}^3$. Let $\mathcal{V}_K$, $\mathcal{E}_K$, and $\mathcal{F}_K$ be the collections of the vertices, edges, and faces of K, respectively. Let $\boldsymbol{T}_E : \widehat{S}^1 \to E$ for all $E \in \mathcal{E}_K$, and $\boldsymbol{T}_F : \widehat{S}^2 \to F$ for all $F \in \mathcal{F}_K$, be affine bijective mappings (see Proposition 7.9), where $\widehat{S}^1$ and $\widehat{S}^2$ are the unit simplices in $\mathbb{R}$ and $\mathbb{R}^2$. Let $k \geq 1$ be the polynomial degree. The canonical hybrid finite element involves vertex dofs, edge dofs if $k \geq 2$, surface (or face) dofs if $k \geq 3$, and volume (or cell) dofs if $k \geq 4$. We consider the following dofs:

$$\sigma^{\mathrm{v}}_{\boldsymbol{z}}(p) := p(\boldsymbol{z}), \qquad \boldsymbol{z} \in \mathcal{V}_K, \tag{7.11a}$$

$$\sigma^{\mathrm{e}}_{E,m}(p) := \frac{1}{|E|}\int_E (\mu_m \circ \boldsymbol{T}_E^{-1}) p \,\mathrm{d}l, \qquad E \in \mathcal{E}_K,\ m \in \{1{:}n^{\mathrm{e}}_{\mathrm{sh}}\}, \tag{7.11b}$$

$$\sigma^{\mathrm{f}}_{F,m}(p) := \frac{1}{|F|}\int_F (\zeta_m \circ \boldsymbol{T}_F^{-1}) p \,\mathrm{d}s, \qquad F \in \mathcal{F}_K,\ m \in \{1{:}n^{\mathrm{f}}_{\mathrm{sh}}\}, \tag{7.11c}$$

$$\sigma^{\mathrm{c}}_{m}(p) := \frac{1}{|K|}\int_K \psi_m p \,\mathrm{d}x, \qquad m \in \{1{:}n^{\mathrm{c}}_{\mathrm{sh}}\}, \tag{7.11d}$$

where $\{\mu_m\}_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-2,1}$ with $n^{\mathrm{e}}_{\mathrm{sh}} := \binom{k-1}{1}$ if $k \geq 2$, $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-3,2}$ with $n^{\mathrm{f}}_{\mathrm{sh}} := \binom{k-1}{2}$ if $k \geq 3$, and $\{\psi_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-4,3}$ with $n^{\mathrm{c}}_{\mathrm{sh}} := \binom{k-1}{3}$ if $k \geq 4$. The above construction is possible in any dimension. If $d = 2$ for instance, the vertex dofs are defined in (7.11a), the edge (face) ones in (7.11b) if $k \geq 2$, and the cell ones in (7.11d), where $\{\psi_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{P}_{k-3,2}$ with $n^{\mathrm{c}}_{\mathrm{sh}} := \binom{k-1}{2}$ if $k \geq 3$.

Proposition 7.19 (Canonical hybrid finite element). *Let $k \geq 1$. Let K be a simplex in $\mathbb{R}^d$, let $P := \mathbb{P}_{k,d}$, and let $\Sigma := \{\sigma_i\}_{i\in\mathcal{N}}$ be the collection of all the dofs defined in* (7.11). *Then (K, P, Σ) is a finite element.*

Proof. We use Remark 5.3. Since we use polynomials in $\mathbb{P}_{k-l-1,l}$ to define the dofs of the l-faces, and the number of l-faces is $\binom{d+1}{l+1} = \binom{d+1}{d-l}$ (see Example 7.5), the total number of dofs for all the l-faces is $\binom{k-1}{l}\binom{d+1}{d-l}$. Vandermonde's convolution identity implies that

$$n_{\mathrm{sh}} = \sum_{j\in\{0:d\}} \binom{k-1}{j}\binom{d+1}{d-j} = \binom{k+d}{d} = \dim(\mathbb{P}_{k,d}).$$

It remains to prove that if $p \in \mathbb{P}_{k,d}$ is such that $\sigma_i(p) = 0$ for all $i \in \mathcal{N}$, then p vanishes identically. First, p vanishes at all the vertices of K. If $k = 1$, this concludes the proof. If $k \geq 2$, fix an edge E of K. Since $p \circ \boldsymbol{T}_E$ vanishes at the two endpoints of E, $p \circ \boldsymbol{T}_E = \lambda_0\lambda_1 q$, where $\lambda_0, \lambda_1 \in \mathbb{P}_{1,1}$ are the local barycentric coordinates over $\widehat{S}^1$ and $q \in \mathbb{P}_{k-2,1}$. Since the dofs of p attached to E vanish, we infer that $\int_{\widehat{S}^1} \lambda_0\lambda_1 q^2 \,\mathrm{d}l = 0$, which implies that $q = 0$. Hence, p is identically zero on all edges of K. If $k = 2$, this completes the proof since all the Lagrange nodes for $k = 2$ are located at the edges of K. If $k \geq 3$, we proceed similarly by fixing a face F of K and showing that p is identically zero on all faces of K. If $k = 3$, this completes the proof since all the Lagrange nodes for $k = 3$ are located at the faces of K. For $k \geq 4$, we finally infer that $p = \lambda_0 \ldots \lambda_d q_K$ where $\{\lambda_i\}_{i\in\{0:d\}}$ are the barycentric coordinates of K and $q_K \in \mathbb{P}_{k-4,d}$. Since the dofs of p attached to K vanish, we infer that $\int_K \lambda_0 \ldots \lambda_d q_K^2 \,\mathrm{d}x = 0$, which implies that $q_K = 0$, i.e., $p = 0$. $\square$

The shape functions associated with the vertices, the edges, the faces, and K are denoted by $\{\tilde{\xi}_{\boldsymbol{z}}\}_{\boldsymbol{z}\in\mathcal{V}_K}$, $\{\tilde{\mu}_{E,m}\}_{E\in\mathcal{E}_K, m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}}$, $\{\tilde{\zeta}_{F,m}\}_{F\in\mathcal{F}_K, m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$, and $\{\tilde{\psi}_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$, respectively. All these functions are in $\mathbb{P}_{k,d}$ and form a basis thereof. Recalling Proposition 5.5 the shape functions are computed by inverting the generalized Vandermonde matrix $\mathcal{V}$ after choosing a basis of $\mathbb{P}_{k,d}$. A basis of $\mathbb{P}_{k,d}$ with a structure close to that of the above shape functions can be found in Fuentes et al. [103, §7.1]. The proposed basis can be organized into functions attached to the vertices of K, the edges of K, the faces of K, and to K itself, and the associated generalized Vandermonde matrix $\mathcal{V}$ is block-triangular. The interpolation operator has domain $V(K) := C^0(K)$ (or $V(K) := W^{s,p}(K)$ with $sp > d$, $p \in [1, \infty]$ or $s \geq d$, $p = 1$) and it acts as follows:

$$\begin{aligned}\mathcal{I}^{\mathrm{g}}_K(v)(\boldsymbol{x}) := &\sum_{\boldsymbol{z}\in\mathcal{V}_K} \sigma^{\mathrm{v}}_{\boldsymbol{z}}(v)\tilde{\xi}_{\boldsymbol{z}}(\boldsymbol{x}) + \sum_{E\in\mathcal{E}_K}\sum_{m\in\{1:n^{\mathrm{e}}_{\mathrm{sh}}\}} \sigma^{\mathrm{e}}_{E,m}(v)\tilde{\mu}_{E,m}(\boldsymbol{x}) \\ &+ \sum_{F\in\mathcal{F}_K}\sum_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}} \sigma^{\mathrm{f}}_{F,m}(v)\tilde{\zeta}_{F,m}(\boldsymbol{x}) + \sum_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}} \sigma^{\mathrm{c}}_m(v)\tilde{\psi}_m(\boldsymbol{x}).\end{aligned}$$

Remark 7.20 (Dofs). The interpolation operator $\mathcal{I}_K^{\mathrm{g}}$ is independent of the bases $\{\mu_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{e}}\}}$, $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$, and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ (this follows from Exercise 5.2). It is also independent of the choice of the mappings $\boldsymbol{T}_E$ and $\boldsymbol{T}_F$. Let for instance $\boldsymbol{T}_F$ and $\tilde{\boldsymbol{T}}_F$ be two geometric mappings associated with the face F. Then $\boldsymbol{T}_F^{-1}\circ\tilde{\boldsymbol{T}}_F$ is affine and bijective from $\mathbb{R}^2$ to $\mathbb{R}^2$. Hence, $\zeta_m\circ(\boldsymbol{T}_F^{-1}\circ\tilde{\boldsymbol{T}}_F)\in\mathbb{P}_{k,2}$ for all $m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}$, so that $\zeta_m\circ(\boldsymbol{T}_F^{-1}\circ\tilde{\boldsymbol{T}}_F)=\sum_{n\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}\mathcal{S}_{mn}\zeta_n$ for some real numbers $\mathcal{S}_{mn}$, i.e.,

$$\zeta_m\circ\boldsymbol{T}_F^{-1}=\sum_{n\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}\mathcal{S}_{mn}(\zeta_n\circ\tilde{\boldsymbol{T}}_F^{-1}).$$

Since the mappings $\boldsymbol{T}_F$ and $\tilde{\boldsymbol{T}}_F$ are bijective, the matrix $\mathcal{S}\in\mathbb{R}^{n_{\mathrm{sh}}^{\mathrm{f}}\times n_{\mathrm{sh}}^{\mathrm{f}}}$ is invertible, and we use again Exercise 5.2 to conclude. □

Exercises

Exercise 7.1 (Lagrange interpolation). Let $\mathcal{I}_K$ be the $\mathbb{P}_1$ Lagrange interpolation operator on a simplex K. Prove that $\|\mathcal{I}_K(v)\|_{C^0(K)}\le\|v\|_{C^0(K)}$ for all $v\in C^0(K)$. (*Hint*: use the convexity of K and recall that K is closed.) Does this property hold true for $\mathbb{P}_2$ Lagrange elements?

Exercise 7.2 (Geometric identities). Prove the statements in Remark 7.6. (*Hint*: use the divergence theorem to prove (7.1).)

Exercise 7.3 (Barycentric coordinates). Let K be a simplex in $\mathbb{R}^d$. (i) Prove that $\lambda_i(\boldsymbol{x})=1-\frac{|F_i|}{d|K|}\boldsymbol{n}_{K|F_i}\cdot(\boldsymbol{x}-\boldsymbol{z}_i)$ for all $\boldsymbol{x}\in K$ and all $i\in\{0:d\}$, and that $\nabla\lambda_i=-\frac{|F_i|}{d|K|}\boldsymbol{n}_{K|F_i}$. (ii) For all $\boldsymbol{x}\in K$, let $K_i(\boldsymbol{x})$ be the simplex obtained by joining $\boldsymbol{x}$ to the d vertices $\boldsymbol{z}_j$ with $j\neq i$. Show that $\lambda_i(\boldsymbol{x})=\frac{|K_i(\boldsymbol{x})|}{|K|}$. (iii) Prove that $\int_K\lambda_i\,\mathrm{d}x=\frac{1}{d+1}|K|$ for all $i\in\{0:d\}$, and that $\int_{F_j}\lambda_i\,\mathrm{d}s=\frac{1}{d}|F_j|$ for all $j\in\{0:d\}$ with $j\neq i$, and $\int_{F_i}\lambda_i\,\mathrm{d}s=0$. (*Hint*: consider an affine mapping from K to the unit simplex.) (iv) Prove that if $\boldsymbol{h}\in\mathbb{R}^d$ satisfies $D\lambda_i(\boldsymbol{h})=0$ for all $i\in\{1:d\}$, then $\boldsymbol{h}=\boldsymbol{0}$.

Exercise 7.4 (Space $\mathbb{P}_{k,d}$). (i) Give a basis for $\mathbb{P}_{2,d}$ for $d\in\{1,2,3\}$. (ii) Show that any polynomial $p\in\mathbb{P}_{k,d}$ can be written in the form $p(x_1,\ldots,x_d)=r(x_1,\ldots,x_{d-1})+x_dq(x_1,\ldots,x_d)$, with unique polynomials $r\in\mathbb{P}_{k,d-1}$ and $q\in\mathbb{P}_{k-1,d}$. (iii) Determine the dimension of $\mathbb{P}_{k,d}$. (*Hint*: by induction on d.) (iv) Let K be a simplex in $\mathbb{R}^d$. Let F_0 be the face of K opposite to the vertex $\boldsymbol{z}_0$. Prove that if $p\in\mathbb{P}_{k,d}$ satisfies $p_{|F_0}=0$, then there is $q\in\mathbb{P}_{k-1,d}$ s.t. $p=\lambda_0q$. (*Hint*: write the Taylor expansion of p at $\boldsymbol{z}_d$ and use (7.2) with $\boldsymbol{z}_d$ playing the role of $\boldsymbol{z}_0$.) (v) Prove that $\{\lambda_0^{\beta_0}\ldots\lambda_d^{\beta_d}\mid\beta_0+\ldots+\beta_d=k\}$ is a basis of $\mathbb{P}_{k,d}$.

Exercise 7.5 (Nodes of simplicial Lagrange FE). Let K be a simplex in $\mathbb{R}^d$, and consider the set of nodes $\{\boldsymbol{a}_i\}_{i\in\mathcal{N}}$ with barycentric coordinates $\left(\frac{i_0}{k},\ldots,\frac{i_d}{k}\right)$, $\forall i_0,\ldots,i_d \in \{0{:}k\}$ with $i_0+\ldots+i_d=k$. (i) Prove that the number of nodes located on any one-dimensional edge of K is $(k+1)$ in any dimension $d\geq 2$. (ii) Prove that the number of nodes located on any $(d-1)$-dimensional face of K is the dimension of $\mathbb{P}_{k,d-1}$. (iii) Prove that if $k\leq d$, all the nodes are located on the boundary of K.

Exercise 7.6 (Hierarchical basis). Let $k\geq 1$ and let $\{\theta_0,\ldots,\theta_k\}$ be a hierarchical basis of $\mathbb{P}_{k,1}$. Let $\{\lambda_0,\ldots,\lambda_d\}$ be a basis of $\mathbb{P}_{1,d}$ and assume that $\lambda_i:\mathbb{R}^d\to\mathbb{R}$ is surjective for all $i\in\{0{:}d\}$ (i.e., λ_i is not constant). (i) Show that the functions (mapping $\mathbb{R}^d$ to $\mathbb{R}$) $\{\theta_0(\lambda_i),\ldots,\theta_k(\lambda_i)\}$ are linearly independent for all $i\in\{0{:}d\}$. (*Hint*: consider a linear combination $\sum_{l\in\{0:k\}}\alpha_l\theta_l(\lambda_i)\in\mathbb{P}_{k,d}$ and prove that the polynomial $\sum_{l\in\{0:k\}}\alpha_l\theta_l\in\mathbb{P}_{k,1}$ vanishes at $(k+1)$ distinct points.) (ii) Show that the functions (mapping $\mathbb{R}^d$ to $\mathbb{R}$) from the set $S_{k,d}:=\{\theta_{\alpha_1}(\lambda_1)\ldots\theta_{\alpha_d}(\lambda_d)\mid(\alpha_1,\ldots\alpha_d)\in\mathbb{N}^d,\ |\alpha|\leq k\}$ are linearly independent. (*Hint*: by induction on d.) (iii) Show that $(S_{k,d})_{k\geq 0}$ is a hierarchical polynomial basis, i.e., $S_{k,d}\subset S_{k+1,d}$ and $S_{k,d}$ is basis of $\mathbb{P}_{k,d}$. (*Note*: the $(d+1)$ vertices of K do not play here the same role.)

Exercise 7.7 (Cubic Hermite triangle). Let K be a triangle with vertices $\{\boldsymbol{z}_0,\boldsymbol{z}_1,\boldsymbol{z}_2\}$. Set $\Sigma:=\{p(\boldsymbol{z}_i),\partial_{x_1}p(\boldsymbol{z}_i),\partial_{x_2}p(\boldsymbol{z}_i)\}_{0\leq i\leq 2}\cup\{p(\boldsymbol{a}_K)\}$, where $\boldsymbol{a}_K$ is a point inside K. Show that $(K,\mathbb{P}_{3,2},\Sigma)$ is a finite element. (*Hint*: show that any $p\in\mathbb{P}_{3,2}$ for which all the dofs vanish is identically zero on the three edges of K and infer that $p=c\lambda_0\lambda_1\lambda_2$ for some $c\in\mathbb{R}$.)

Exercise 7.8 ($\mathbb{P}_{2,d}$ canonical hybrid FE). Compute the shape functions of the $\mathbb{P}_{2,d}$ canonical hybrid finite element for the unit simplex for $d=1$ and $d=2$ (provide an expression using the Cartesian coordinates and another one using the barycentric coordinates).

Exercise 7.9 ($\mathbb{P}_{4,2}$ Lagrange). Using the Lagrange nodes defined as in Proposition 7.11, give the expression of the $\mathbb{P}_{4,2}$ Lagrange shape functions in terms of the barycentric coordinates.

Exercise 7.10 (Quadratic Crouzeix–Raviart). Let K be the unit simplex. Let $\alpha\in(0,1)$. Let $\boldsymbol{g}_1:=(\alpha,0)$, $\boldsymbol{g}_2:=(1-\alpha,0)$, $\boldsymbol{g}_3:=(1-\alpha,\alpha)$, $\boldsymbol{g}_4:=(\alpha,1-\alpha)$, $\boldsymbol{g}_5:=(0,1-\alpha)$, $\boldsymbol{g}_6:=(0,\alpha)$. (i) Compute $\lambda_0(\boldsymbol{g}_j)^2+\lambda_1(\boldsymbol{g}_j)^2+\lambda_2(\boldsymbol{g}_j)^2$ for all $j\in\{1{:}6\}$, where $\lambda_0,\lambda_1,\lambda_2$ are the barycentric coordinates of K. (ii) Let $\sigma_j\in\mathcal{L}(\mathbb{P}_{2,2};\mathbb{R})$ be defined by $\sigma_j(p):=p(\boldsymbol{g}_j)$ for all $p\in\mathbb{P}_{2,2}$ and $j\in\{1{:}6\}$. Let $\Sigma:=\{\sigma_j\}_{j\in\{1:6\}}$. Is the triple $(K,\mathbb{P}_{2,2},\Sigma)$ a finite element? (iii) Let F_i, $i\in\{0{:}2\}$, be one of the three faces of K. Let $\boldsymbol{T}_{F_i}:[-1,1]\to F_i$ be one of the two affine mappings that realize a bijection between $[-1,1]$ and F_i. Let $\{q_0,q_1\}$ be a basis of $\mathbb{P}_{1,1}$. Let $\varpi_{2i+k}\in\mathcal{L}(\mathbb{P}_{2,2};\mathbb{R})$, $i\in\{0{:}2\}$, $k\in\{0{:}1\}$, be defined by $\varpi_{2i+k}(p):=\frac{1}{|F_i|}\int_{F_i}(q_k\circ\boldsymbol{T}_{F_i}^{-1})p\,\mathrm{d}s$ for all $p\in\mathbb{P}_{2,2}$. Let $\Sigma:=\{\varpi_j\}_{j\in\{0:5\}}$. Is the triple $(K,\mathbb{P}_{2,2},\Sigma)$ a finite element? (*Hint*: consider the points $\boldsymbol{T}_{F_i}(\xi_k)$, $i\in\{0{:}2\}$, $k\in\{0{:}1\}$, where ξ_0, ξ_1 are the two nodes of the Gauss–Legendre quadrature of order 3, then use Step (ii).)