Part III, Chapter 11

Local interpolation on affine meshes

We have seen in the previous chapter how to build finite elements and local interpolation operators in each cell K of a mesh $\mathcal{T}_h$. In this chapter, we analyze the local interpolation error for smooth $\mathbb{R}^q$-valued functions, $q \geq 1$. We restrict the material to affine meshes and to transformations ψ_K s.t.

$$\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K), \tag{11.1}$$

where $\mathbb{A}_K$ is a matrix in $\mathbb{R}^{q\times q}$. Nonaffine meshes are treated in Chapter 13. We introduce the notion of shape-regular families of affine meshes, we study the transformation of Sobolev norms using (11.1), and we present important approximation results collectively known as the Bramble–Hilbert lemmas. We finally prove the main result of this chapter, which is an upper bound on the local interpolation error over each mesh cell for smooth functions.

11.1 Shape regularity for affine meshes

Let $\mathcal{T}_h$ be an affine mesh. Let $K \in \mathcal{T}_h$. Since the geometric mapping $\boldsymbol{T}_K$ is affine, its Jacobian matrix $\mathbb{J}_K \in \mathbb{R}^{d\times d}$ defined in (8.3) is such that

$$\boldsymbol{T}_K(\widehat{\boldsymbol{x}}) - \boldsymbol{T}_K(\widehat{\boldsymbol{y}}) = \mathbb{J}_K(\widehat{\boldsymbol{x}} - \widehat{\boldsymbol{y}}), \qquad \forall \widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}} \in \widehat{K}. \tag{11.2}$$

The matrix $\mathbb{J}_K$ is invertible since the mapping $\boldsymbol{T}_K$ is bijective. Moreover, the (Fréchet) derivative of the geometric mapping is such that $D\boldsymbol{T}_K(\widehat{\boldsymbol{x}})(\widehat{\boldsymbol{h}}) = \mathbb{J}_K\widehat{\boldsymbol{h}}$ for all $\widehat{\boldsymbol{h}} \in \mathbb{R}^d$ (see Appendix B). We denote the Euclidean norm in $\mathbb{R}^d$ by $\|\cdot\|_{\ell^2(\mathbb{R}^d)}$, or $\|\cdot\|_{\ell^2}$ when the context is unambiguous. We abuse the notation by using the same symbol for the induced matrix norm.

Lemma 11.1 (Bound on $\mathbb{J}_K$). *Let $\mathcal{T}_h$ be an affine mesh and let $K \in \mathcal{T}_h$. Let ρ_K be the diameter of the largest ball that can be inscribed in K and let h_K be the diameter of K, as shown in Figure* 11.1. *Let $\widehat{\rho}_{\widehat{K}}$ and $\widehat{h}_{\widehat{K}}$ be defined* *similarly. The following holds true:*

$$|\det(\mathbb{J}_K)| = \frac{|K|}{|\widehat{K}|}, \qquad \|\mathbb{J}_K\|_{\ell^2} \le \frac{h_K}{\rho_{\widehat{K}}}, \qquad \|\mathbb{J}_K^{-1}\|_{\ell^2} \le \frac{h_{\widehat{K}}}{\rho_K}. \tag{11.3}$$

Proof. The first equality results from the fact that

$$|K| = \int_K \mathrm{d}x = \int_{\widehat{K}} |\det(\mathbb{J}_K)|\,\mathrm{d}\widehat{x} = |\det(\mathbb{J}_K)|\,|\widehat{K}|.$$

Regarding the bound on $\|\mathbb{J}_K\|_{\ell^2}$, we observe that

$$\|\mathbb{J}_K\|_{\ell^2} = \sup_{\widehat{\boldsymbol{h}}\neq 0} \frac{\|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2}}{\|\widehat{\boldsymbol{h}}\|_{\ell^2}} = \frac{1}{\rho_{\widehat{K}}} \sup_{\|\widehat{\boldsymbol{h}}\|_{\ell^2}=\rho_{\widehat{K}}} \|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2}.$$

Any $\widehat{\boldsymbol{h}} \in \mathbb{R}^d$ such that $\|\widehat{\boldsymbol{h}}\|_{\ell^2} = \rho_{\widehat{K}}$ can be written as $\widehat{\boldsymbol{h}} = \widehat{\boldsymbol{x}}_1 - \widehat{\boldsymbol{x}}_2$ with $\widehat{\boldsymbol{x}}_1, \widehat{\boldsymbol{x}}_2 \in \widehat{K}$. We infer that $\mathbb{J}_K\widehat{\boldsymbol{h}} = \boldsymbol{T}_K(\widehat{\boldsymbol{x}}_1) - \boldsymbol{T}_K(\widehat{\boldsymbol{x}}_2) = \boldsymbol{x}_1 - \boldsymbol{x}_2$, which in turn proves that $\|\mathbb{J}_K\widehat{\boldsymbol{h}}\|_{\ell^2} \le h_K$. This establishes the bound on $\|\mathbb{J}_K\|_{\ell^2}$. The bound on $\|\mathbb{J}_K^{-1}\|_{\ell^2}$ is obtained by exchanging the roles of K and $\widehat{K}$. □

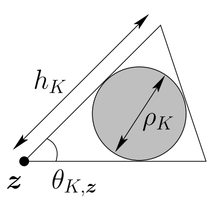

Fig. 11.1 Triangular cell K with vertex $\boldsymbol{z}$, angle $\theta_{K,\boldsymbol{z}}$, and largest inscribed ball.

Since the analysis of the interpolation error (implicitly) invokes sequences of successively refined meshes, we henceforth denote by $(\mathcal{T}_h)_{h\in\mathcal{H}}$ a sequence of meshes discretizing a domain D in $\mathbb{R}^d$, where the index h takes values in a countable set $\mathcal{H}$ having zero as the only accumulation point.

Definition 11.2 (Shape regularity). *A sequence of affine meshes* $(\mathcal{T}_h)_{h\in\mathcal{H}}$ *is said to be* shape-regular *if there is* $\sigma_\sharp$ *such that*

$$\sigma_K := \frac{h_K}{\rho_K} \le \sigma_\sharp, \qquad \forall K \in \mathcal{T}_h,\ \forall h \in \mathcal{H}. \tag{11.4}$$

Occasionally, when the context is unambiguous, we will say that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is regular instead of shape-regular. Owing to Lemma 11.1, a shape-regular sequence of affine meshes satisfies

$$\|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2} \le \sigma_\sharp\sigma_{\widehat{K}}, \qquad \forall K \in \mathcal{T}_h,\ \forall h \in \mathcal{H}. \tag{11.5}$$

Example 11.3 (Dimension 1). Every sequence of one-dimensional meshes is shape-regular, since $h_K = \rho_K$ when $d = 1$. □

Example 11.4 (Triangulations). A shape-regular sequence of affine triangulations can be obtained from an initial triangulation by connecting all the edge midpoints and repeating this procedure as many times as needed. □

Remark 11.5 (Angles). Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine simplicial meshes. Assume that $d = 2$, let K be a triangle in $\mathcal{T}_h$ and let $\boldsymbol{z}$ be a vertex of K. Then the angle $\theta_{K,\boldsymbol{z}} \in (0, 2\pi)$ formed by the two edges of K sharing $\boldsymbol{z}$ is uniformly bounded away from zero. Indeed, the angular sector centered at $\boldsymbol{z}$ of angle $\theta_{K,\boldsymbol{z}}$ and radius h_K covers the ball of diameter ρ_K that is inscribed in K (see Figure 11.1). Hence, $\frac{1}{2}h_K^2\theta_{K,\boldsymbol{z}} \geq \frac{1}{4}\pi\rho_K^2$, which in turn implies that $\theta_{K,\boldsymbol{z}} \geq \frac{1}{2}\pi\sigma_\sharp^{-2}$. Assume now that $d = 3$, let K be a tetrahedron, and let $\boldsymbol{z}$ be a vertex of K. Then the solid angle $\omega_{K,\boldsymbol{z}} \in (0, 4\pi)$ formed by the three faces of K sharing $\boldsymbol{z}$ is uniformly bounded away from zero. Reasoning as above, with volumes instead of surfaces, leads to $\frac{1}{3}h_K^3\omega_{K,\boldsymbol{z}} \geq \frac{1}{6}\pi\rho_K^3$, so that $\omega_{K,\boldsymbol{z}} \geq \frac{1}{2}\pi\sigma_\sharp^{-3}$. □

We close this section with a useful result on matching meshes. Recall from §8.2 the notion of mesh faces, edges, and vertices in a matching mesh (assuming $d = 3$). For every mesh vertex $\boldsymbol{z} \in \mathcal{V}_h$, we denote

$$\mathcal{T}_{\boldsymbol{z}} := \{K \in \mathcal{T}_h \mid \boldsymbol{z} \in K\} \tag{11.6}$$

the collection of the mesh cells sharing $\boldsymbol{z}$. Similarly, recall from (10.3) that for every mesh edge $E \in \mathcal{E}_h$ and every mesh face $F \in \mathcal{F}_h$, $\mathcal{T}_E := \{K \in \mathcal{T}_h \mid E \subset K\}$ and $\mathcal{T}_F := \{K \in \mathcal{T}_h \mid F \subset K\}$ are the collection of the mesh cells sharing E and F, respectively.

Proposition 11.6 (Neighboring cells). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of matching affine meshes. Then the cardinality of the set $\mathcal{T}_{\boldsymbol{z}}$ is uniformly bounded for all $\boldsymbol{z} \in \mathcal{V}_h$ and all $h \in \mathcal{H}$, and the sizes of all the cells in $\mathcal{T}_{\boldsymbol{z}}$ are uniformly equivalent w.r.t. $h \in \mathcal{H}$. The same assertion holds true for the sets $\mathcal{T}_E$ and $\mathcal{T}_F$.*

Proof. It suffices to prove the assertions for $\mathcal{T}_{\boldsymbol{z}}$. The bound on $\text{card}(\mathcal{T}_{\boldsymbol{z}})$ follows from Remark 11.5. Concerning the sizes of the cells in $\mathcal{T}_{\boldsymbol{z}}$, we first observe that if $K', K'' \in \mathcal{T}_{\boldsymbol{z}}$, $K' \neq K''$, share a common face, say F with diameter h_F, then $h_{K'} \leq \sigma_\sharp\rho_{K'} \leq \sigma_\sharp h_F \leq \sigma_\sharp h_{K''}$, and similarly, $h_{K''} \leq \sigma_\sharp h_{K'}$. This shows that the sizes of K' and K'' are uniformly equivalent. Now, for all K' and K'' in $\mathcal{T}_{\boldsymbol{z}}$, there is a finite path of cells linking K' to K'' s.t. any two consecutive mesh cells in the path share a common face. The number of cells composing the path cannot exceed $\text{card}(\mathcal{T}_{\boldsymbol{z}})$, so that it is uniformly bounded. Hence, the sizes of K' and K'' are uniformly equivalent. □

11.2 Transformation of Sobolev seminorms

The question we investigate now is the following: given a function $v \in W^{m,p}(K;\mathbb{R}^q)$, how does the seminorm of $\psi_K(v)$ in $W^{m,p}(\widehat{K};\mathbb{R}^q)$ compare to that of v in $W^{m,p}(K;\mathbb{R}^q)$ with ψ_K is defined in (11.1)?

Lemma 11.7 (Norm scaling by ψ_K). *Let $\mathcal{T}_h$ be an affine mesh. Let $s \in [0,\infty)$ and $p \in [1,\infty]$ (with $z^{\pm\frac{1}{p}} := 1$, $\forall z > 0$ if $p = \infty$). There exists c, depending only on s and d, such that the following bounds hold true for all $v \in W^{s,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$:*

$$|\psi_K(v)|_{W^{s,p}(\widehat{K};\mathbb{R}^q)} \le c\,\gamma_K^{\frac{1}{p}} \|\mathbb{A}_K\|_{\ell^2} \|\mathbb{J}_K\|_{\ell^2}^s \,|\det(\mathbb{J}_K)|^{-\frac{1}{p}}\, |v|_{W^{s,p}(K;\mathbb{R}^q)}, \quad (11.7a)$$

$$|v|_{W^{s,p}(K;\mathbb{R}^q)} \le c\,\delta_K^{\frac{1}{p}} \|\mathbb{A}_K^{-1}\|_{\ell^2} \|\mathbb{J}_K^{-1}\|_{\ell^2}^s \,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\, |\psi_K(v)|_{W^{s,p}(\widehat{K};\mathbb{R}^q)}, \quad (11.7b)$$

where $\gamma_K = \delta_K := 1$ if $s \in \mathbb{N}$ and $\gamma_K := |\det(\mathbb{J}_K)|^{-1}\|\mathbb{J}_K\|_{\ell^2}^d$, $\delta_K := |\det(\mathbb{J}_K)|\,\|\mathbb{J}_K^{-1}\|_{\ell^2}^d$ otherwise (the real numbers γ_K and δ_K are uniformly bounded w.r.t. $h \in \mathcal{H}$ on shape-regular mesh sequences).

Proof. We start by assuming $s = m \in \mathbb{N}$. The bounds are obvious for $m = 0$. For $m \ge 1$, let α be a multi-index with length $|\alpha| = m$, i.e., $\alpha := (\alpha_1, \dots, \alpha_d) \in \mathbb{N}^d$ with $\alpha_1 + \ldots + \alpha_d = m$. Let $\widehat{\boldsymbol{x}} \in \widehat{K}$. Owing to (B.6), we infer that

$$\partial^\alpha(\psi_K(v))(\widehat{\boldsymbol{x}}) = \mathbb{A}_K D^m(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\underbrace{\boldsymbol{e}_1, \dots, \boldsymbol{e}_1}_{\alpha_1 \text{ times}}, \dots, \underbrace{\boldsymbol{e}_d, \dots, \boldsymbol{e}_d}_{\alpha_d \text{ times}}),$$

where $D^m(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})$ is the m-th Fréchet derivative of $v \circ \boldsymbol{T}_K$ at $\widehat{\boldsymbol{x}}$ and $\{\boldsymbol{e}_1, \dots, \boldsymbol{e}_d\}$ is the canonical Cartesian basis of $\mathbb{R}^d$. We now apply the chain rule (see Lemma B.4) to $v \circ \boldsymbol{T}_K$. Since $\boldsymbol{T}_K$ is affine, the Fréchet derivative of $\boldsymbol{T}_K$ is independent of $\widehat{\boldsymbol{x}}$ and its higher-order Fréchet derivatives vanish. Hence, we have

$$D^m(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})(\boldsymbol{h}_1, \dots, \boldsymbol{h}_m) = \sum_{\sigma \in \mathcal{S}_m} \frac{1}{m!} (D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))(D\boldsymbol{T}_K(\boldsymbol{h}_{\sigma(1)}), \dots, D\boldsymbol{T}_K(\boldsymbol{h}_{\sigma(m)})),$$

for all $\boldsymbol{h}_1, \dots, \boldsymbol{h}_m \in \mathbb{R}^d$, where $\mathcal{S}_m$ is the set of the permutations of $\{1{:}m\}$. Since $D\boldsymbol{T}_K(\boldsymbol{h}) = \mathbb{J}_K \boldsymbol{h}$ for all $\boldsymbol{h} \in \mathbb{R}^d$ owing to (11.2), we infer that

$$|\partial^\alpha(v \circ \boldsymbol{T}_K)(\widehat{\boldsymbol{x}})| \le \|\mathbb{J}_K\|_{\ell^2}^m \,\|(D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\mathcal{M}_m(\mathbb{R}^d,\dots,\mathbb{R}^d;\mathbb{R}^q)},$$

with $\|A\|_{\mathcal{M}_m(\mathbb{R}^d,\dots,\mathbb{R}^d;\mathbb{R}^q)} := \sup_{(\boldsymbol{y}_1,\dots,\boldsymbol{y}_m) \in \mathbb{R}^d \times \dots \times \mathbb{R}^d} \frac{\|A(\boldsymbol{y}_1,\dots,\boldsymbol{y}_m)\|_{\ell^2}}{\|\boldsymbol{y}_1\|_{\ell^2} \cdots \|\boldsymbol{y}_m\|_{\ell^2}}$ for every multilinear map $A \in \mathcal{M}_m(\mathbb{R}^d, \dots, \mathbb{R}^d; \mathbb{R}^q)$. Owing to the multilinearity of $D^m v$ and using again (B.6), we infer that (see Exercise 11.1)

$$\|(D^m v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R}^q)} \le c \sum_{|\beta|=m} \|(\partial^\beta v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\ell^2},$$

where c only depends on m and d. As a result, we have

$$\|\partial^\alpha(\psi_K(v))(\widehat{\boldsymbol{x}})\|_{\ell^2} \le c\,\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{J}_K\|_{\ell^2}^m \sum_{|\beta|=m} \|(\partial^\beta v)(\boldsymbol{T}_K(\widehat{\boldsymbol{x}}))\|_{\ell^2},$$

and (11.7a) follows by taking the $L^p(\widehat{K})$-norm on both sides of the inequality. The proof of (11.7b) is similar. We refer to Exercise 11.7 when $s \notin \mathbb{N}$. □

Remark 11.8 (Seminorms). The upper bounds in (11.7a) and (11.7b) involve only seminorms because the geometric mappings are affine. □

11.3 Bramble–Hilbert lemmas

This section contains an important result for the analysis of the approximation properties of finite elements. We consider scalar-valued functions. The result extends to vector-valued functions by reasoning componentwise.

Lemma 11.9 ($\mathbb{P}_k$-Bramble–Hilbert/Deny–Lions). *Let S be a Lipschitz domain in $\mathbb{R}^d$. Let $p \in [1,\infty]$. Let $k \in \mathbb{N}$. There is c (depending on k, p, S) s.t. for all $v \in W^{k+1,p}(S)$,*

$$\inf_{q\in\mathbb{P}_{k,d}} \|v-q\|_{W^{k+1,p}(S)} \le c\,|v|_{W^{k+1,p}(S)}. \tag{11.8}$$

Proof. (1) Consider the bounded linear forms $f_\alpha : W^{k+1,p}(S) \to \mathbb{R}$ s.t.

$$f_\alpha(v) := \ell_S^{|\alpha|-d} \int_S \partial^\alpha v \,\mathrm{d}x, \qquad \forall \alpha \in \mathcal{A}_{k,d},$$

where $\mathcal{A}_{k,d} := \{\alpha := (\alpha_1,\ldots,\alpha_d) \in \mathbb{N}^d \mid |\alpha| \le k\}$ and $\ell_S := \mathrm{diam}(S)$ (the factor $\ell_S^{|\alpha|-d}$ is introduced for dimensional consistency). Let us set $N_{k,d} := \mathrm{card}(\mathcal{A}_{k,d}) = \binom{k+d}{d}$. Let us consider the map $\Phi_{k,d} : W^{k+1,p}(S) \to \mathbb{R}^{N_{k,d}}$ s.t.

$$\Phi_{k,d}(q) := (f_\alpha(q))_{\alpha\in\mathcal{A}_{k,d}},$$

and let us prove that the restriction of this map to $\mathbb{P}_{k,d}$ is an isomorphism. To prove this, we observe that $\dim(\mathbb{P}_{k,d}) = N_{k,d}$, so that it is sufficient to prove that $\Phi_{k,d}$ is injective, which we do by induction on k. For $k=0$, if $q \in \mathbb{P}_0$ satisfies $\Phi_{0,d}(q) = 0$, then $\int_S q\,\mathrm{d}x = q|S| = 0$ so that $q = 0$. Let us assume now that $k \ge 1$ and let $q \in \mathbb{P}_{k,d}$ be such that $\Phi_{k,d}(q) = 0$. Let us write $q(\boldsymbol{x}) = \sum_{\alpha\in\mathcal{A}_{k,d}} a_\alpha \boldsymbol{x}^\alpha$. Whenever $|\alpha| = k$, we obtain $\partial^\alpha q(\boldsymbol{x}) = a_\alpha \alpha_1! \ldots \alpha_d!$ so that $f_\alpha(q) = 0$ implies that $a_\alpha = 0$. Since this property is satisfied for all α

such that $|\alpha| = k$, we infer that $q \in \mathbb{P}_{k-1,d}$ and conclude from the induction assumption that $q = 0$.
(2) Let us prove that there is $c > 0$, depending on S, k, and p, such that

$$c\,\|v\|_{W^{k+1,p}(S)} \le \ell_S^{k+1}|v|_{W^{k+1,p}(S)} + \ell_S^{\frac{d}{p}}\|\Phi_{k,d}(v)\|_{\ell^1(\mathbb{R}^{N_{k,d}})}, \tag{11.9}$$

for all $v \in W^{k+1,p}(S)$, with $\|f\|_{\ell^1(\mathbb{R}^{N_{k,d}})} := \sum_{\alpha\in N_{k,d}} |f_\alpha|$. Reasoning by contradiction, let $(v_n)_{n\in\mathbb{N}}$ be a sequence s.t.

$$\|v_n\|_{W^{k+1,p}(S)} = 1, \quad \lim_{n\to\infty} |v_n|_{W^{k+1,p}(S)} = 0, \quad \lim_{n\to\infty} \Phi_{k,d}(v_n) = 0. \tag{11.10}$$

Owing to the Rellich–Kondrachov theorem (Theorem 2.35), we infer that, up to a subsequence (not renumbered for simplicity), the sequence $(v_n)_{n\in\mathbb{N}}$ converges strongly to a function v in $W^{k,p}(S)$. Moreover, $(v_n)_{n\in\mathbb{N}}$ is a Cauchy sequence in $W^{k+1,p}(S)$ since

$$\|v_n - v_m\|_{W^{k+1,p}(S)} \le \|v_n - v_m\|_{W^{k,p}(S)} + \ell_S^{k+1}|v_n - v_m|_{W^{k+1,p}(S)},$$

and $|v_n - v_m|_{W^{k+1,p}(S)} \to 0$ by assumption. Hence, $(v_n)_{n\in\mathbb{N}}$ converges to v strongly in $W^{k+1,p}(S)$ (that the limit is indeed v comes from the uniqueness of the limit in $W^{k,p}(S)$). Owing to (11.10), we infer that $\|v\|_{W^{k+1,p}(S)} = 1$, $|v|_{W^{k+1,p}(S)} = 0$, and $\Phi_{k,d}(v) = 0$. Repeated applications of Lemma 2.11 (stating that in an open connected set S, $\nabla v = 0$ implies that v is constant on S) allow us to infer from $|v|_{W^{k+1,p}(S)} = 0$ that $v \in \mathbb{P}_{k,d}$. Since we have established in Step (1) that the restriction of $\Phi_{k,d}$ to $\mathbb{P}_{k,d}$ is an isomorphism, this yields $v = 0$, which contradicts $\|v\|_{W^{k+1,p}(S)} = 1$.
(3) Let $v \in W^{k+1,p}(S)$ and define $\pi(v) \in \mathbb{P}_{k,d}$ such that $\Phi_{k,d}(\pi(v)) = \Phi_{k,d}(v)$. This is possible since the restriction of $\Phi_{k,d}$ to $\mathbb{P}_{k,d}$ is an isomorphism. Then

$$\begin{aligned} c \inf_{q\in\mathbb{P}_{k,d}} \|v - q\|_{W^{k+1,p}(S)} &\le c\|v - \pi(v)\|_{W^{k+1,p}(S)} \\ &\le \ell_S^{k+1}|v - \pi(v)|_{W^{k+1,p}(S)} + \|\Phi_{k,d}(v - \pi(v))\|_{\ell^1(\mathbb{R}^{N_{k,d}})} \\ &= \ell_S^{k+1}|v|_{W^{k+1,p}(S)}, \end{aligned}$$

since $\partial^\alpha \pi(v) = 0$ for all $\alpha \in \mathbb{N}^d$ such that $|\alpha| = k + 1$. □

Remark 11.10 (Peetre–Tartar lemma). Step (2) in the above proof is similar to the Peetre–Tartar lemma (Lemma A.20). Define $X := W^{k+1,p}(S)$, $Y := [L^p(D)]^{N_{k+1,d}-N_{k,d}} \times \mathbb{R}^{N_{k,d}}$, $Z := W^{k,p}(S)$, and the operator

$$A : X \ni v \longmapsto ((\partial^\alpha v)_{|\alpha|=k+1}, \Phi_{k,d}(v)) \in Y.$$

Since A is bounded and injective, and the embedding $X \hookrightarrow Z$ is compact, the property (11.9) results from the Peetre–Tartar lemma. □

Corollary 11.11 ($\mathbb{P}_k$-Bramble–Hilbert for linear functionals). *Under the hypotheses of Lemma 11.9, there is c s.t. the following holds true for all* $f \in (W^{k+1,p}(S))' := \mathcal{L}(W^{k+1,p}(S);\mathbb{R})$ *vanishing on* $\mathbb{P}_{k,d}$,

$$|f(v)| \le c\,\|f\|_{(W^{k+1,p}(S))'}\ell_S^{k+1}|v|_{W^{k+1,p}(S)}, \qquad \forall v \in W^{k+1,p}(S). \tag{11.11}$$

Proof. Left as an exercise. □

Remark 11.12 (Literature). The estimate (11.8) is proved in Bramble and Hilbert [40, Thm. 1] and in Ciarlet and Raviart [79, Lem. 7]; see also Deny and Lions [90]. The estimate (11.11) is proved in Bramble and Hilbert [40, Thm. 2] and in Ciarlet and Raviart [79, Lem. 6]. There is some variability in the literature regarding the terminology for these results. For instance, Lemma 11.9 is called Bramble–Hilbert lemma in Brenner and Scott [47, Lem. 4.3.8] and Ciarlet and Raviart [78, p. 219], whereas it is called Deny–Lions lemma in Ciarlet [77, p. 111], and it is not given any name in Braess [39, p. 77]. Corollary 11.11 is called Bramble–Hilbert lemma in Ciarlet [77, p. 192] and Braess [39, p. 78]. Incidentally, there are two additional results that are the counterparts of Lemma 11.9 and Corollary 11.11 for $\mathbb{Q}_{k,d}$ polynomials; see Lemma 13.8 and Corollary 13.9. □

11.4 Local finite element interpolation

This section contains our main result on local finite element interpolation. Recall the construction of §9.1 to generate a finite element and a local interpolation operator in each mesh cell $K \in \mathcal{T}_h$. Our goal is now to estimate the interpolation error $v - \mathcal{I}_K(v)$ for every smooth function v. The key point is that we want this bound to depend on K only through its size h_K under the assumption that the mesh sequence is shape-regular. The Bramble–Hilbert/Deny–Lions lemma cannot be used directly on K since this would give a constant depending on the shape of K. The crucial idea is then to use the fact that $\mathcal{I}_K = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ owing to Proposition 9.3 and to apply Lemma 11.9 on the fixed reference cell $\widehat{K}$.

Theorem 11.13 (Local interpolation). *Let* $\widehat{P}$ *be finite-dimensional,* $\mathcal{I}_{\widehat{K}} \in \mathcal{L}(V(\widehat{K});\widehat{P})$, $p \in [1,\infty]$, $k,l \in \mathbb{N}$, *and assume that the following holds true:*

(i) $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)$.
(ii) $[\mathbb{P}_{k,d}]^q$ *is pointwise invariant under* $\mathcal{I}_{\widehat{K}}$.
(iii) $W^{l,p}(\widehat{K};\mathbb{R}^q) \hookrightarrow V(\widehat{K})$, *i.e.,* $\|\widehat{v}\|_{V(\widehat{K})} \le \widehat{c}\,\|\widehat{v}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)}$ *for all* $\widehat{v} \in V(\widehat{K})$.

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ *be a shape-regular sequence of affine meshes, let the transformation* ψ_K *be defined in* (11.1) *for all* $K \in \mathcal{T}_h$, *and assume that there is* γ *s.t. for all* $K \in \mathcal{T}_h$ *and all* $h \in \mathcal{H}$,

$$\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \leq \gamma\, \|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}. \tag{11.12}$$

Define the operator

$$\mathcal{I}_K := \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K. \tag{11.13}$$

There is c s.t. the following local interpolation error estimates hold true:
(i) *If $l \leq k+1$, then for every integers $r \in \{l{:}k+1\}$ and $m \in \{0{:}r\}$, all $v \in W^{r,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$|v - \mathcal{I}_K(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c\, h_K^{r-m}|v|_{W^{r,p}(K;\mathbb{R}^q)}. \tag{11.14}$$

(ii) *If $l > k+1$, then for every integer $m \in \{0{:}k+1\}$, all $v \in W^{l,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$|v - \mathcal{I}_K(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c \sum_{n\in\{k+1:l\}} h_K^{n-m}|v|_{W^{n,p}(K;\mathbb{R}^q)}. \tag{11.15}$$

Proof. We present a unified proof of (11.14) and (11.15). Let

$$\overline{r} \in \{l{:}\max(l,k+1)\}, \quad \underline{r} = \min(\overline{r}, k+1), \quad m \in \{0{:}\underline{r}\}.$$

If $l \leq k+1$, then $\overline{r} \in \{l{:}k+1\}$, $\underline{r} = \overline{r} =: r$, $m \in \{0{:}r\}$, whereas if $l > k+1$, then $\overline{r} = l$, $\underline{r} = k+1$, $m \in \{0{:}k+1\}$. Thus, proving (11.14) and (11.15) is equivalent to prove that for all $v \in W^{\overline{r},p}(K;\mathbb{R}^q)$,

$$|v - \mathcal{I}_K(v)|_{W^{m,p}(K;\mathbb{R}^q)} \leq c \sum_{n\in\{\underline{r}:\overline{r}\}} h_K^{n-m}|v|_{W^{n,p}(K;\mathbb{R}^q)}.$$

Let c denote a generic constant whose value can change at each occurrence as long as it is independent of v, K, and h. We take $\ell_{\widehat{K}} := 1$.
(1) For all $\widehat{v} \in W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)$, we set $\mathcal{G}(\widehat{v}) := \widehat{v} - \mathcal{I}_{\widehat{K}}(\widehat{v})$. Since all the norms are equivalent in $\widehat{P}$, there is a constant $c_{\widehat{P}}$ such that

$$\|\widehat{p}\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \leq c_{\widehat{P}}\|\widehat{p}\|_{V(\widehat{K})}, \qquad \forall \widehat{p} \in \widehat{P}.$$

Using $m \leq \underline{r} \leq \overline{r}$, the above bound applied to $\widehat{p} := \mathcal{I}_{\widehat{K}}(\widehat{v})$, $\mathcal{I}_{\widehat{K}} \in \mathcal{L}(V(\widehat{K}))$, and Assumption (iii), we infer that

$$\begin{aligned}
\|\mathcal{G}(\widehat{v})\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} &\leq \|\widehat{v}\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} + \|\mathcal{I}_{\widehat{K}}(\widehat{v})\|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq \|\widehat{v}\|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} + c_{\widehat{P}}\|\mathcal{I}_{\widehat{K}}(\widehat{v})\|_{V(\widehat{K})} \\
&\leq \|\widehat{v}\|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} + c_{\widehat{P}}\|\mathcal{I}_{\widehat{K}}\|_{\mathcal{L}(V(\widehat{K}))}\|\widehat{v}\|_{V(\widehat{K})} \\
&\leq \|\widehat{v}\|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} + c_{\widehat{P}}\|\mathcal{I}_{\widehat{K}}\|_{\mathcal{L}(V(\widehat{K}))}\widehat{c}\,\|\widehat{v}\|_{W^{l,p}(\widehat{K};\mathbb{R}^q)}.
\end{aligned}$$

Since $l \leq \overline{r}$, this shows that $\mathcal{G} \in \mathcal{L}(W^{\overline{r},p}(\widehat{K};\mathbb{R}^q); W^{m,p}(\widehat{K};\mathbb{R}^q))$.
(2) Let us prove that

$$|\widehat{v} - \mathcal{I}_{\widehat{K}}(\widehat{v})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \leq c\left(|\widehat{v}|_{W^{\underline{r},p}(\widehat{K};\mathbb{R}^q)} + \ldots + |\widehat{v}|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)}\right). \quad (11.16)$$

The estimate is trivial if $\underline{r} = 0$. Assume now that $\underline{r} \geq 1$. Then $0 \leq \underline{r} - 1 \leq k$, so that $\mathbb{P}_{\underline{r}-1,d} \subset \mathbb{P}_{k,d}$, which implies that $[\mathbb{P}_{\underline{r}-1,d}]^q$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$. Hence, the operator $\mathcal{G}$ vanishes on $\mathbb{P}_{\underline{r}-1,d}$. We then infer that

$$\begin{aligned}
|\widehat{v} - \mathcal{I}_{\widehat{K}}(\widehat{v})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} &= |\mathcal{G}(\widehat{v})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} = \inf_{\widehat{p}\in[\mathbb{P}_{\underline{r}-1,d}]^q} |\mathcal{G}(\widehat{v}-\widehat{p})|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq \|\mathcal{G}\|_{\mathcal{L}(W^{\overline{r},p}(\widehat{K};\mathbb{R}^q);W^{m,p}(\widehat{K};\mathbb{R}^q))} \inf_{\widehat{p}\in[\mathbb{P}_{\underline{r}-1,d}]^q} \|\widehat{v}-\widehat{p}\|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c \inf_{\widehat{p}\in[\mathbb{P}_{\underline{r}-1,d}]^q} \|\widehat{v}-\widehat{p}\|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} \leq c\left(|\widehat{v}|_{W^{\underline{r},p}(\widehat{K};\mathbb{R}^q)} + \ldots + |\widehat{v}|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)}\right),
\end{aligned}$$

since $\|\widehat{v}-\widehat{p}\|^p_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)} = \|\widehat{v}-\widehat{p}\|^p_{W^{\underline{r},p}(\widehat{K};\mathbb{R}^q)} + \sum_{n\in\{\underline{r}+1:\overline{r}\}} |\widehat{v}|^p_{W^{n,p}(\widehat{K};\mathbb{R}^q)}$ for $\underline{r} < \overline{r}$, and owing to the estimate (11.8) from the Bramble–Hilbert/Deny–Lions lemma applied componentwise to $\|\widehat{v}-\widehat{p}\|^p_{W^{\underline{r},p}(\widehat{K};\mathbb{R}^q)}$. This proves (11.16).

(3) Finally, let $v \in W^{\overline{r},p}(K;\mathbb{R}^q)$. We infer that

$$\begin{aligned}
&|v - \mathcal{I}_K(v)|_{W^{m,p}(K;\mathbb{R}^q)} \\
&\leq c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|^m_{\ell^2}\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,|\psi_K(v-\mathcal{I}_K(v))|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|^m_{\ell^2}\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\,|\psi_K(v)-\mathcal{I}_{\widehat{K}}(\psi_K(v))|_{W^{m,p}(\widehat{K};\mathbb{R}^q)} \\
&\leq c\,\|\mathbb{A}_K^{-1}\|_{\ell^2}\|\mathbb{J}_K^{-1}\|^m_{\ell^2}\,|\det(\mathbb{J}_K)|^{\frac{1}{p}}\left(|\psi_K(v)|_{W^{\underline{r},p}(\widehat{K};\mathbb{R}^q)} + \ldots + |\psi_K(v)|_{W^{\overline{r},p}(\widehat{K};\mathbb{R}^q)}\right) \\
&\leq c\,\|\mathbb{J}_K^{-1}\|^m_{\ell^2}\left(\|\mathbb{J}_K\|^{\underline{r}}_{\ell^2}|v|_{W^{\underline{r},p}(K;\mathbb{R}^q)} + \ldots + \|\mathbb{J}_K\|^{\overline{r}}_{\ell^2}|v|_{W^{\overline{r},p}(K;\mathbb{R}^q)}\right),
\end{aligned}$$

where we used the bound (11.7b) in the first line, the linearity of ψ_K and $\mathcal{I}_K = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ in the second line, the bound (11.16) in the third line, and the bound (11.7a) together with (11.12) in the fourth line. The expected error estimate follows by using (11.3) and the fact that $\sigma_K := \frac{h_K}{\rho_K}$ is uniformly bounded owing to the shape-regularity of the mesh sequence. □

Definition 11.14 (Degree of a finite element). *The largest integer k such that $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)$ is called* degree *of the finite element.*

Remark 11.15 (Assumptions). The assumption (i) in Theorem 11.13 is easy to satisfy for finite elements since $\widehat{P}$ is in general composed of polynomial functions. If $\widehat{P}$ and $\mathcal{I}_{\widehat{K}}$ are generated from a finite element construction, then the assumption (ii) follows from (i) since $\widehat{P}$ is then pointwise invariant under $\mathcal{I}_{\widehat{K}}$. The assumption (iii) requires a bit more care since it amounts to finding an integer l s.t. $\mathcal{I}_{\widehat{K}} : W^{l,p}(\widehat{K};\mathbb{R}^q) \to \widehat{P}$ is bounded, i.e., the assumption (iii) is a stability property of the reference interpolation operator. □

Remark 11.16 (Fractional order). For simplicity, the interpolation error estimates from Theorem 11.13 are derived for functions in Sobolev spaces of integer order. We refer the reader to Chapter 22 for interpolation error estimates in Sobolev spaces of fractional order. □

11.5 Some examples

In this section, we present some examples of the application of Theorem 11.13 where ψ_K is the pullback by the geometric mapping. We refer the reader to Chapter 16 for applications of Theorem 11.13 to vector-valued finite elements when ψ_K is one of the Piola transformations from Definition 9.8.

11.5.1 Lagrange elements

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine meshes. For Lagrange elements, we have seen in Example 9.4 that the transformation ψ_K is the pullback by the geometric mapping, i.e., $\psi_K(v) := \psi_K^{\mathrm{g}}(v) := v \circ \boldsymbol{T}_K$ (see (9.9a)). Hence, the choice (11.1) with $\mathbb{A}_K := 1$ for ψ_K is legitimate, and (11.12) trivially holds true (with $\gamma := 1$). Proposition 9.3 shows that $\mathcal{I}_K^{\mathrm{L}} = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}}^{\mathrm{L}} \circ \psi_K$, where $\mathcal{I}_{\widehat{K}}^{\mathrm{L}}$ and $\mathcal{I}_K^{\mathrm{L}}$ are, respectively, the Lagrange interpolation operator in the reference cell $\widehat{K}$ and in a generic mesh cell $K \in \mathcal{T}_h$. Hence, (11.13) holds true. Furthermore, Assumption (i) in Theorem 11.13 holds true with k being the degree of the Lagrange element. Assumption (ii) also holds true since $\widehat{P}$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$. It remains to verify Assumption (iii). This assumption is satisfied if we take l to be the smallest integer such that $l > \frac{d}{p}$ (or $l \geq d$ if $p = 1$). This indeed implies that $W^{l,p}(\widehat{K}) \hookrightarrow C^0(\widehat{K})$ owing to Theorem 2.35. Assuming that $k+1 > \frac{d}{p}$ (so that $k+1 \geq l$), the estimate (11.14) implies that there is c s.t. for every integers $r \in \{l{:}k+1\}$ and $m \in \{0{:}r\}$, all $v \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,

$$|v - \mathcal{I}_K^{\mathrm{L}}(v)|_{W^{m,p}(K)} \leq c\, h_K^{r-m} |v|_{W^{r,p}(K)}. \tag{11.17}$$

If $k+1 \leq \frac{d}{p}$, the more general estimate (11.15) has to be used. For instance, assume that $k=1$, $d=3$, and $p \in [1, \frac{3}{2}]$, so that $k+1 = 2 \leq \frac{3}{p}$. In the range $p \in [1, \frac{3}{2}]$, we can take $l = 3$ in Assumption (iii) (since either $3 > \frac{3}{p}$ for $p > 1$ or $3 \geq \frac{3}{1}$). For $m = 0$, we get

$$\|v - \mathcal{I}_K^{\mathrm{L}}(v)\|_{L^p(K)} \leq c\, h_K^2 (|v|_{W^{2,p}(K)} + h_K |v|_{W^{3,p}(K)}). \tag{11.18}$$

Remark 11.17 (Quadrangular meshes). When working on quadrangular (or hexahedral meshes), the geometric mapping is affine if and only if all the cells are parallelograms (or parallelotopes). If one wants to work with more general meshes, nonaffine geometric mappings need to be considered. This case is treated in §13.5. □

11.5.2 Modal elements

Consider now a modal finite element of degree k and let $\mathcal{I}_{\widehat{K}}^{\mathrm{m}}$ and $\mathcal{I}_K^{\mathrm{m}}$ be the modal interpolation operators in the reference cell $\widehat{K}$ and in a generic mesh cell $K \in \mathcal{T}_h$, respectively. We have seen in Example 9.5 that the choice $\psi_K(v) := \psi_K^{\mathrm{g}}(v) := v \circ \boldsymbol{T}_K$ is legitimate, that is, we take $\mathbb{A}_K := 1$ in (11.1) to define ψ_K, so that (11.12) trivially holds true (with $\gamma := 1$). Proposition 9.3 shows that $\mathcal{I}_K^{\mathrm{m}} = \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}}^{\mathrm{m}} \circ \psi_K$. Hence, (11.13) holds true. As for Lagrange elements, Assumptions (i) and (ii) in Theorem 11.13 are easy to verify. Concerning Assumption (iii), it is legitimate to take $l = 0$ since $V(\widehat{K}) = L^1(\widehat{K};\mathbb{R}^q)$. Hence, the estimate (11.14) can always be used, i.e., there is c s.t. for every integers $r \in \{0{:}k+1\}$ and $m \in \{0{:}r\}$, all $v \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,

$$|v - \mathcal{I}_K^{\mathrm{m}}(v)|_{W^{m,p}(K)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K)}. \tag{11.19}$$

11.5.3 L^2-orthogonal projection

Let $\widehat{P}$ be a finite-dimensional space such that $\mathbb{P}_{k,d} \subset \widehat{P} \subset W^{k+1,\infty}(\widehat{K})$. The L^2-orthogonal projection onto $\widehat{P}$ is the linear operator $\mathcal{I}_{\widehat{K}}^{\mathrm{b}} : L^1(\widehat{K}) \to \widehat{P}$ such that for all $\widehat{v} \in L^1(\widehat{K})$, $\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v})$ is the unique element in $\widehat{P}$ s.t.

$$\int_{\widehat{K}} (\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v}) - \widehat{v})\widehat{q}\,\mathrm{d}\widehat{x} = 0, \qquad \forall \widehat{q} \in \widehat{P}. \tag{11.20}$$

Since $\widehat{v} - \mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v})$ and $\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v}) - \widehat{q}$ are L^2-orthogonal for all $\widehat{q} \in \widehat{P}$, the Pythagorean identity gives

$$\|\widehat{v} - \widehat{q}\|_{L^2(\widehat{K})}^2 = \|\widehat{v} - \mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v})\|_{L^2(\widehat{K})}^2 + \|\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v}) - \widehat{q}\|_{L^2(\widehat{K})}^2. \tag{11.21}$$

This implies that

$$\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v}) = \arg\min_{\widehat{q} \in \widehat{P}} \|\widehat{v} - \widehat{q}\|_{L^2(\widehat{K})}. \tag{11.22}$$

Hence, $\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\widehat{v})$ is the element in $\widehat{P}$ that is the closest to $\widehat{v}$ in the L^2-norm, and $\widehat{P}$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}^{\mathrm{b}}$.

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine meshes. Let $K \in \mathcal{T}_h$. Let ψ_K^{g} be the pullback by the geometric mapping $\boldsymbol{T}_K$, i.e., $\psi_K^{\mathrm{g}}(v) := v \circ \boldsymbol{T}_K$, and set $P_K := (\psi_K^{\mathrm{g}})^{-1}(\widehat{P})$. The L^2-orthogonal projection onto P_K is the linear operator $\mathcal{I}_K^{\mathrm{b}} : L^1(K) \to P_K$ such that for all $v \in L^1(K)$, $\mathcal{I}_K^{\mathrm{b}}(v)$ is the unique element in P_K s.t.

$$\int_K (\mathcal{I}_K^{\mathrm{b}}(v) - v)q\,\mathrm{d}x = 0, \qquad \forall q \in P_K. \tag{11.23}$$

As above, $\mathcal{I}_K^{\mathrm{b}}(v)$ is the element in P_K that is the closest to v in the L^2-norm, and P_K is pointwise invariant under $\mathcal{I}_K^{\mathrm{b}}$.

Lemma 11.18 (L^2-projection). *Let $p \in [1,\infty]$. There is c s.t. for every integers $r \in \{0{:}k+1\}$ and $m \in \{0{:}r\}$, all $v \in W^{r,p}(K)$, all $K \in \mathcal{T}_h$, and all*

$h \in \mathcal{H}$,
$$|v - \mathcal{I}_K^{\mathrm{b}}(v)|_{W^{m,p}(K)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K)}. \tag{11.24}$$

Proof. We apply Theorem 11.13. Recall from (9.9d) the Piola transformation $\psi_K^{\mathrm{b}}(v) := \det(\mathbb{J}_K)(v \circ \boldsymbol{T}_K)$. Observe that $(\psi_K^{\mathrm{b}})^{-1}(\widehat{P}) = P_K$. The map ψ_K^{b} is of the general form (11.1), i.e., $\psi_K^{\mathrm{b}}(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$ where $\mathbb{A}_K := \det(\mathbb{J}_K)$ is a $1{\times}1$ matrix (i.e., a real number) that trivially satisfies (11.12) (with $\gamma := 1$). For all $q \in P_K$ with $q = \widehat{q} \circ \boldsymbol{T}_K^{-1}$, we have

$$\begin{aligned}
\int_K (\psi_K^{\mathrm{b}})^{-1}(\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\psi_K^{\mathrm{b}}(v)))q \,\mathrm{d}x &= \int_K \det(\mathbb{J}_K)^{-1}(\mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\psi_K^{\mathrm{b}}(v)) \circ \boldsymbol{T}_K^{-1}) q \,\mathrm{d}x \\
&= \int_{\widehat{K}} \epsilon_K \mathcal{I}_{\widehat{K}}^{\mathrm{b}}(\psi_K^{\mathrm{b}}(v))\widehat{q} \,\mathrm{d}\widehat{x} \\
&= \int_{\widehat{K}} \epsilon_K \psi_K^{\mathrm{b}}(v)\widehat{q} \,\mathrm{d}\widehat{x} = \int_K vq \,\mathrm{d}x,
\end{aligned}$$

with $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|}$, which proves that $\mathcal{I}_K^{\mathrm{b}} = (\psi_K^{\mathrm{b}})^{-1} \circ \mathcal{I}_{\widehat{K}}^{\mathrm{b}} \circ \psi_K^{\mathrm{b}}$ since $(\psi_K^{\mathrm{b}})^{-1}(\widehat{P}) = P_K$, i.e., (11.13) holds true with $\psi_K := \psi_K^{\mathrm{b}}$. It remains to verify the assumptions (i), (ii), and (iii). Assumption (i) follows from our assumption on $\widehat{P}$. Assumption (ii) follows from $\widehat{P}$ being pointwise invariant under $\mathcal{I}_{\widehat{K}}^{\mathrm{b}}$. Finally, Assumption (iii) holds true with $l := 0$. Since $l \le k+1$, we can apply the estimate (11.14), which is nothing but (11.24). □

Remark 11.19 (Beyond finite elements). The above example shows that Theorem 11.13 can be understood more generally as an approximation result for the operator $\mathcal{I}_K$ defined by $\mathcal{I}_K := \psi_K^{-1} \circ \mathcal{I}_{\widehat{K}} \circ \psi_K$ without directly invoking any finite element structure to build the operator $\mathcal{I}_{\widehat{K}}$. Given the affine geometric mapping $\boldsymbol{T}_K : \widehat{K} \to K$ and the transformation $\psi_K(v) := \mathbb{A}_K(v \circ \boldsymbol{T}_K)$, the key requirements are that $\mathcal{I}_{\widehat{K}} : W^{l,p}(\widehat{K};\mathbb{R}^q) \to \widehat{P}$ is bounded, $\mathbb{P}_{k,d}$ is pointwise invariant under $\mathcal{I}_{\widehat{K}}$, and ψ_K is such that $\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \le \gamma\, \|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}$. In conclusion, the finite element construction of §9.1 is sufficient to apply Theorem 11.13 but not necessary. □

Exercises

Exercise 11.1 (High-order derivative). Let two integers $m, d \ge 2$. Consider the map $\Phi : \{1{:}d\}^m \ni \mathtt{j} \longmapsto (\Phi_1(\mathtt{j}), \ldots, \Phi_d(\mathtt{j})) \in \mathbb{N}^d$, where $\Phi_i(\mathtt{j}) := \mathrm{card}\{k \in \{1{:}m\} \mid \mathtt{j}_k = i\}$ for all $i \in \{1{:}d\}$, so that $|\Phi(\mathtt{j})| = m$ by construction. Let $C_{m,d} := \max_{\alpha\in\mathbb{N}^d, |\alpha|=m} \mathrm{card}\{\mathtt{j} \in \{1{:}d\}^m \mid \Phi(\mathtt{j}) = \alpha\}$. Let v be a smooth (scalar-valued) function. (i) Show that $\|D^m v\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R})} \le C_{m,d}^{\frac12}\left(\sum_{\alpha\in\mathbb{N}^d,|\alpha|=m} |\partial^\alpha v|^2\right)^{\frac12}$. (ii) Show that $C_{m,2} = \max_{0\le l\le m}\binom{m}{l} = 2^m$.

(iii) Evaluate $C_{m,3}$ and $m \in \{2,3\}$. (iv) Show that $\sum_{\alpha\in\mathbb{N}^d,|\alpha|=m} |\partial^\alpha v| \leq \binom{d+m-1}{d-1} \|D^m v\|_{\mathcal{M}_m(\mathbb{R}^d,\ldots,\mathbb{R}^d;\mathbb{R})}$.

Exercise 11.2 (Flat triangle). Let K be a triangle with vertices $(0,0)$, $(1,0)$ and $(-1,\epsilon)$ with $0 < \epsilon \ll 1$. Consider the function $v(x_1,x_2) := x_1^2$. Evaluate the $\mathbb{P}_1$ Lagrange interpolant $\mathcal{I}_K^{\mathrm{L}}(v)$ (see (9.7)) and show that $|v - \mathcal{I}_K^{\mathrm{L}}(v)|_{H^1(K)} \geq \epsilon^{-1}|v|_{H^2(K)}$. (*Hint*: use a direct calculation of $\mathcal{I}_K^{\mathrm{L}}(v)$.)

Exercise 11.3 (Barycentric coordinate). Let K be a simplex with barycentric coordinates $\{\lambda_i\}_{i\in\{0:d\}}$. Prove that $|\lambda_i|_{W^{1,\infty}(K)} \leq \rho_K^{-1}$ for all $i \in \{0:d\}$.

Exercise 11.4 (Bramble–Hilbert). Prove Corollary 11.11. (*Hint*: use the Bramble–Hilbert/Deny–Lions lemma.)

Exercise 11.5 (Taylor polynomial). Let K be a convex cell. Consider a Lagrange finite element of degree $k \geq 1$ with nodes $\{\boldsymbol{a}_i\}_{i\in\mathcal{N}}$ and associated shape functions $\{\theta_i\}_{i\in\mathcal{N}}$. Consider a sufficiently smooth function v. For all $\boldsymbol{x},\boldsymbol{y} \in K$, consider the Taylor polynomial of order k and the exact remainder defined as follows:

$$\mathbb{T}_k(\boldsymbol{x},\boldsymbol{y}) := v(\boldsymbol{x}) + Dv(\boldsymbol{x})(\boldsymbol{y}-\boldsymbol{x}) + \ldots + \frac{1}{k!}D^k v(\boldsymbol{x})(\underbrace{\boldsymbol{y}-\boldsymbol{x},\ldots,\boldsymbol{y}-\boldsymbol{x}}_{k \text{ times}}),$$

$$R_k(v)(\boldsymbol{x},\boldsymbol{y}) := \frac{1}{(k+1)!}D^{k+1}v(\eta\boldsymbol{x} + (1-\eta)\boldsymbol{y})(\underbrace{\boldsymbol{y}-\boldsymbol{x},\ldots,\boldsymbol{y}-\boldsymbol{x}}_{(k+1) \text{ times}}),$$

so that $v(\boldsymbol{y}) = \mathbb{T}_k(\boldsymbol{x},\boldsymbol{y}) + R_k(v)(\boldsymbol{x},\boldsymbol{y})$ for some $\eta \in [0,1]$. (i) Prove that $v(\boldsymbol{x}) = \mathcal{I}_K^{\mathrm{L}}(v)(\boldsymbol{x}) - \sum_{i\in\mathcal{N}} R_k(v)(\boldsymbol{x},\boldsymbol{a}_i)\theta_i(\boldsymbol{x})$, where $\mathcal{I}_K^{\mathrm{L}}$ is the Lagrange interpolant defined in (9.7). (*Hint*: interpolate with respect to $\boldsymbol{y}$.) (ii) Prove that $D^m v(\boldsymbol{x}) = D^m(\mathcal{I}_K^{\mathrm{L}}(v))(\boldsymbol{x}) - \sum_{i\in\mathcal{N}} R_k(v)(\boldsymbol{x},\boldsymbol{a}_i)D^m\theta_i(\boldsymbol{x})$ for all $m \leq k$. (*Hint*: proceed as in (i), take m derivatives with respect to $\boldsymbol{y}$ at $\boldsymbol{x}$, and observe that $v(\boldsymbol{x}) = \mathbb{T}_k(\boldsymbol{x},\boldsymbol{x})$.) (iii) Deduce that $|v - \mathcal{I}_K^{\mathrm{L}}(v)|_{W^{m,\infty}(K)} \leq c\sigma_K^m h_K^{k+1-m}|v|_{W^{k+1,\infty}(K)}$ with $c := \frac{1}{(k+1)!}c_* h_{\widehat{K}}^m \sum_{i\in\mathcal{N}} |\widehat{\theta}_i|_{W^{m,\infty}(\widehat{K})}$, where c_* comes from (11.7b) with $s = m$ and $p = \infty$.

Exercise 11.6 (L^p-stability of Lagrange interpolant). Let $\alpha \in (0,1)$. Consider the Lagrange $\mathbb{P}_1$ shape functions $\theta_1(x) := 1 - x$ and $\theta_2(x) := x$. Consider the sequence of continuous functions $\{u_n\}_{n\in\mathbb{N}\setminus\{0\}}$ defined over the interval $K := [0,1]$ as $u_n(x) := n^\alpha - 1$ if $0 \leq x \leq \frac{1}{n}$ and $u_n(x) := x^{-\alpha} - 1$ otherwise. (i) Prove that the sequence is uniformly bounded in $L^p(0,1)$ for all p such that $p\alpha < 1$. (ii) Compute $\mathcal{I}_K^{\mathrm{L}}(u_n)$. Is the operator $\mathcal{I}_K^{\mathrm{L}}$ stable in the L^p-norm? (iii) Is the operator $\mathcal{I}_K^{\mathrm{L}}$ stable in any L^r-norm with $r \in [1,\infty)$?

Exercise 11.7 (Norm scaling, $s \notin \mathbb{N}$). Complete the proof of Lemma 11.7 for the case $s \notin \mathbb{N}$. (*Hint*: use (2.6) with $s = m + \sigma$, $m := \lfloor s \rfloor$, $\sigma := s - m \in (0,1)$.)

Exercise 11.8 (Morrey's polynomial). Let U be a nonempty open set in $\mathbb{R}^d$. Let $k \in \mathbb{N}$ and $p \in [1, \infty]$. Let $u \in W^{k,p}(U)$. Show that there is a unique polynomial $q \in \mathbb{P}_{k,d}$ s.t. $\int_U \partial^\alpha (u - q)\,\mathrm{d}x = 0$ for all $\alpha \in \mathbb{N}^d$ of length at most k. (*Hint*: see the proof of Lemma 11.9 and also Morrey [148, Thm. 3.6.10].)

Exercise 11.9 (Fractional Sobolev norm). Let $r \in (0, 1)$. Let $(\mathcal{T}_h)_{h \in \mathcal{H}}$ be an shape-regular affine mesh sequence and let $\widehat{K}$ be the reference element. Let K be an affine cell in $\mathcal{T}_h$. Using the notation $\widehat{v} := v \circ \boldsymbol{T}_K$, show that there is c such that $\|\widehat{v}\|_{H^r(\widehat{K})} \le c h_K^{r - \frac{d}{2}} |v|_{H^r(K)}$ for all $v \in H^r(K)$ such that $\int_K v\,\mathrm{d}x = 0$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$. (*Hint*: use Lemma 3.26.)