Part III, Chapter 14

$\boldsymbol{H}(\mathrm{div})$ finite elements

The goal of this chapter is to construct $\mathbb{R}^d$-valued finite elements $(K, \boldsymbol{P}, \Sigma)$ with $d \geq 2$ such that (i) $\mathbb{P}_{k,d} := [\mathbb{P}_{k,d}]^d \subset \boldsymbol{P}$ for some $k \geq 0$ and (ii) the degrees of freedom (dofs) in Σ fully determine the normal components of the polynomials in $\boldsymbol{P}$ on all the faces of K. The first requirement is key for proving convergence rates on the interpolation error. The second one is key for constructing $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces (see Chapter 19). The finite elements introduced in this chapter are used, e.g., in Chapter 51 to approximate Darcy's equations which constitute a fundamental model for porous media flows. The focus here is on defining a reference element and generating finite elements on the mesh cells. The estimation of the interpolation error is done in Chapters 16 and 17. We detail the construction for the simplicial Raviart–Thomas finite elements. Some alternative elements are outlined at the end of the chapter.

14.1 The lowest-order case

We start by considering the lowest-order *Raviart–Thomas finite element.* Let $d \geq 2$ be the space dimension, and define the polynomial space

$$\mathbb{RT}_{0,d} := \mathbb{P}_{0,d} \oplus \boldsymbol{x}\, \mathbb{P}_{0,d}. \tag{14.1}$$

Since the above sum is indeed direct, $\mathbb{RT}_{0,d}$ is a vector space of dimension $\dim(\mathbb{RT}_{0,d}) = d + 1$. A basis of $\mathbb{RT}_{0,2}$ is $\{\binom{1}{0}, \binom{0}{1}, \binom{x_1}{x_2}\}$. The space $\mathbb{RT}_{0,d}$ has several interesting properties. (a) One has $\mathbb{P}_{0,d} \subset \mathbb{RT}_{0,d}$ in agreement with the first requirement stated above. (b) If $\boldsymbol{v} \in \mathbb{RT}_{0,d}$ is divergence-free, then $\boldsymbol{v}$ is constant. (c) If H is an affine hyperplane of $\mathbb{R}^d$ with normal vector $\boldsymbol{\nu}_H$, then the function $\boldsymbol{v}\cdot\boldsymbol{\nu}_H$ is constant on H for all $\boldsymbol{v} \in \mathbb{RT}_{0,d}$. Writing $\boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{a} + b\boldsymbol{x}$ with $\boldsymbol{a} \in \mathbb{R}^d$ and $b \in \mathbb{R}$, we indeed have $(\boldsymbol{v}(\boldsymbol{x}_1) - \boldsymbol{v}(\boldsymbol{x}_2))\cdot\boldsymbol{\nu}_H = b(\boldsymbol{x}_1 - \boldsymbol{x}_2)\cdot\boldsymbol{\nu}_H = 0$ for all $\boldsymbol{x}_1, \boldsymbol{x}_2 \in H$.

Let K be a simplex in $\mathbb{R}^d$ and let $\mathcal{F}_K$ be the collection of the faces of K. Each face $F \in \mathcal{F}_K$ is oriented by a fixed unit normal vector $\boldsymbol{n}_F$, and we set $\boldsymbol{\nu}_F := |F|\boldsymbol{n}_F$. Let Σ be the collection of the following linear forms acting on $\mathbb{RT}_{0,d}$:

$$\sigma_F^{\mathrm{f}}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)\,\mathrm{d}s, \qquad \forall F \in \mathcal{F}_K. \tag{14.2}$$

Since $\boldsymbol{v}\cdot\boldsymbol{\nu}_F$ is constant on F, $\sigma_F^{\mathrm{f}}(\boldsymbol{v}) = 0$ implies that $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0$ in agreement with the second requirement stated above. Note that we could have written more simply $\sigma_F^{\mathrm{f}}(\boldsymbol{v}) := \int_F (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,\mathrm{d}s$, but the expression (14.2) is introduced to be consistent with later notation. In any case, the unit of $\sigma_F^{\mathrm{f}}(\boldsymbol{v})$ is a surface times the dimension of $\boldsymbol{v}$. A graphic representation of the dofs is shown in Figure 14.1.

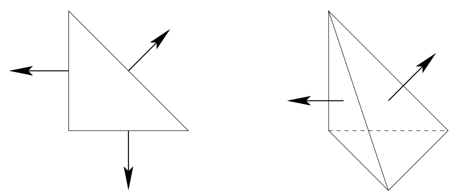

Fig. 14.1 $\mathbb{RT}_{0,d}$ finite element in dimensions two (left) and three (right). Only visible degrees of freedom are shown in dimension three. (The arrows have been drawn outward under the assumption that the vectors $\boldsymbol{\nu}_F$ point outward. The orientation of the arrows must be changed if some vectors $\boldsymbol{\nu}_F$ point inward.)

Proposition 14.1 (Finite element). *$(K, \mathbb{RT}_{0,d}, \Sigma)$ is a finite element.*

Proof. Since $\dim(\mathbb{RT}_{0,d}) = \mathrm{card}(\Sigma) = d+1$, we just need to prove that the only function $\boldsymbol{v} \in \mathbb{RT}_{0,d}$ that annihilates the dofs in Σ is zero. Since $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F$ is constant and has zero mean-value on F, we have $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0$ for all $F \in \mathcal{F}_K$. Moreover, the divergence theorem implies that $\int_K (\nabla\cdot\boldsymbol{v})\,\mathrm{d}x = \sum_{F\in\mathcal{F}_K}\int_F (\boldsymbol{v}\cdot\boldsymbol{n}_F)\,\mathrm{d}s = 0$. Since $\nabla\cdot\boldsymbol{v} \in \mathbb{P}_{0,d}$, we infer that $\nabla\cdot\boldsymbol{v}$ is zero, so that $\boldsymbol{v} \in \mathbb{P}_{0,d}$. Hence, $\boldsymbol{v}\cdot\boldsymbol{\nu}_F$ vanishes identically in K for all $F \in \mathcal{F}_K$. Since $\mathrm{span}\{\boldsymbol{\nu}_F\}_{F\in\mathcal{F}_K} = \mathbb{R}^d$ (see Exercise 7.3(iv)), we conclude that $\boldsymbol{v} = \boldsymbol{0}$. □

Since the volume of a simplex is $|K| = \frac{1}{d}|F|h_F^{\perp}$ for all $F \in \mathcal{F}_K$ where $h_F^{\perp}$ is the height of K measured from the vertex $\boldsymbol{z}_F$ opposite to F, one readily verifies that the shape functions are

$$\boldsymbol{\theta}_F^{\mathrm{f}}(\boldsymbol{x}) := \frac{\iota_{F,K}}{d|K|}(\boldsymbol{x} - \boldsymbol{z}_F), \qquad \forall \boldsymbol{x} \in \mathbb{R}^d,\ \forall F \in \mathcal{F}_K, \tag{14.3}$$

where $\iota_{F,K} := 1$ if $\boldsymbol{\nu}_F$ points outward and $\iota_{F,K} := -1$ otherwise (i.e., $\iota_{F,K} = \boldsymbol{n}_F\cdot\boldsymbol{n}_K$ where $\boldsymbol{n}_K$ is the outward unit normal to K). The normal component of $\boldsymbol{\theta}_F^{\mathrm{f}}$ is constant on each of the $(d+1)$ faces of K (as expected), it is equal to

1 on F and to 0 on the other faces. See Exercise 14.1 for additional properties of the $\mathbb{RT}_{0,d}$ shape functions.

14.2 The polynomial space $\mathbb{RT}_{k,d}$

We now generalize the construction of §14.1 to an arbitrary polynomial order $k \in \mathbb{N}$. Let $d \geq 2$ be the space dimension. Recall from §7.3 the multi-index set $\mathcal{A}_{k,d} := \{\alpha \in \mathbb{N}^d \mid |\alpha| \leq k\}$ where $|\alpha| := \alpha_1 + \ldots + \alpha_d$. We additionally introduce the subset $\mathcal{A}^{\mathrm{H}}_{k,d} := \{\alpha \in \mathcal{A}_{k,d} \mid |\alpha| = k\}$. For instance, $\mathcal{A}_{1,2} = \{(0,0),(1,0),(0,1)\}$ and $\mathcal{A}^{\mathrm{H}}_{1,2} = \{(1,0),(0,1)\}$.

Definition 14.2 (Homogeneous polynomials). *A polynomial $p \in \mathbb{P}_{k,d}$ is said to be* homogeneous *of degree k if $p(\boldsymbol{x}) = \sum_{\alpha\in\mathcal{A}^{\mathrm{H}}_{k,d}} a_\alpha \boldsymbol{x}^\alpha$ with real coefficients a_α. The real vector space composed of homogeneous polynomials is denoted by $\mathbb{P}^{\mathrm{H}}_{k,d}$ or $\mathbb{P}^{\mathrm{H}}_{k}$ when the context is unambiguous.*

Lemma 14.3 (Properties of $\mathbb{P}^{\mathrm{H}}_{k,d}$). *We have $\boldsymbol{x}{\cdot}\nabla q = kq$ (Euler's identity) and $\nabla{\cdot}(\boldsymbol{x}q) = (k+d)q$ for all $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$.*

Proof. By linearity, it suffices to verify the assertion with $q(\boldsymbol{x}) := \boldsymbol{x}^\alpha$ for all $\alpha \in \mathcal{A}^{\mathrm{H}}_{k,d}$. We have $\boldsymbol{x}{\cdot}\nabla q = \sum_{i\in\{1:d\}} \alpha_i x_i x_1^{\alpha_1} \ldots x_i^{\alpha_i-1} \ldots x_d^{\alpha_d} = (\sum_{i\in\{1:d\}} \alpha_i) q = kq$. Moreover, the assertion for $\nabla{\cdot}(\boldsymbol{x}q)$ follows from the observation that $\nabla{\cdot}\boldsymbol{x} = d$ and $\nabla{\cdot}(\boldsymbol{x}q) = q\nabla{\cdot}\boldsymbol{x} + \boldsymbol{x}{\cdot}\nabla q$. □

Definition 14.4 ($\mathbb{RT}_{k,d}$). *Let $k \in \mathbb{N}$ and let $d \geq 2$. We define the following real vector space of $\mathbb{R}^d$-valued polynomials:*

$$\mathbb{RT}_{k,d} := \mathbb{P}_{k,d} \oplus \boldsymbol{x}\,\mathbb{P}^{\mathrm{H}}_{k,d}. \tag{14.4}$$

The above sum is direct since polynomials in $\boldsymbol{x}\,\mathbb{P}^{\mathrm{H}}_{k,d}$ are members of $\mathbb{P}^{\mathrm{H}}_{k+1,d}$, whereas the degree of any polynomial in $\mathbb{P}_{k,d}$ does not exceed k.

Example 14.5 ($k = 1$, $d = 2$). $\dim(\mathbb{RT}_{1,2}) = 8$ and $\left\{ \binom{1}{0}, \binom{x_1}{0}, \binom{x_2}{0}, \binom{0}{1}, \binom{0}{x_1}, \binom{0}{x_2}, \binom{x_1^2}{x_1x_2}, \binom{x_1x_2}{x_2^2} \right\}$ is a basis of $\mathbb{RT}_{1,2}$. □

Lemma 14.6 (Dimension of $\mathbb{RT}_{k,d}$). $\dim(\mathbb{RT}_{k,d}) = (k+d+1)\binom{k+d-1}{k}$*, in particular* $\dim(\mathbb{RT}_{k,2}) = (k+1)(k+3)$ *and* $\dim(\mathbb{RT}_{k,3}) = \frac{1}{2}(k+1)(k+2)(k+4)$.

Proof. Since $\dim(\mathbb{P}_{k,d}) = \binom{k+d}{k}$, $\dim(\mathbb{P}^{\mathrm{H}}_{k,d}) = \binom{k+d-1}{k}$, and the sum in (14.4) is direct, $\dim(\mathbb{RT}_{k,d}) = d\binom{k+d}{k} + \binom{k+d-1}{k} = (k+d+1)\binom{k+d-1}{k}$. □

Lemma 14.7 (Trace space). *Let H be an affine hyperplane in $\mathbb{R}^d$ with normal vector $\boldsymbol{n}_H$, and let $\boldsymbol{T}_H : \mathbb{R}^{d-1} \to H$ be an affine bijective mapping. Then $\boldsymbol{v}_{|H}{\cdot}\boldsymbol{n}_H \in \mathbb{P}_{k,d-1} \circ \boldsymbol{T}_H^{-1}$ for all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$.*

Proof. Let $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ with $\boldsymbol{v} = \boldsymbol{p} + \boldsymbol{x}q$, $\boldsymbol{p} \in \mathbb{P}_{k,d}$, and $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$. Let $\boldsymbol{x} \in H$ and set $\boldsymbol{y} := \boldsymbol{T}_H^{-1}(\boldsymbol{x})$. Since the quantity $\boldsymbol{x}{\cdot}\boldsymbol{n}_H$ is constant, say $\boldsymbol{x}{\cdot}\boldsymbol{n}_H =: c_H$, we infer that $(\boldsymbol{v}_{|H}{\cdot}\boldsymbol{n}_H)(\boldsymbol{x}) = (\boldsymbol{p}_{|H}{\cdot}\boldsymbol{n}_H)(\boldsymbol{x}) + (\boldsymbol{x}{\cdot}\boldsymbol{n}_H)q(\boldsymbol{x}) = ((\boldsymbol{p} \circ \boldsymbol{T}_H){\cdot}\boldsymbol{n}_H)(\boldsymbol{y}) + c_H(q \circ \boldsymbol{T}_H)(\boldsymbol{y})$. Hence, $(\boldsymbol{v}_{|H}{\cdot}\boldsymbol{n}_H) \circ \boldsymbol{T}_H = (\boldsymbol{p} \circ \boldsymbol{T}_H){\cdot}\boldsymbol{n}_H + c_H(q \circ \boldsymbol{T}_H)$, and both terms in the sum are in $\mathbb{P}_{k,d-1}$ by virtue of Lemma 7.10. □

Remark 14.8 ($\boldsymbol{T}_H$). Consider a second affine bijective mapping $\tilde{\boldsymbol{T}}_H : \mathbb{R}^{d-1} \to H$. Since $\boldsymbol{S} := \boldsymbol{T}_H^{-1} \circ \tilde{\boldsymbol{T}}_H$ is an affine bijective mapping from $\mathbb{R}^{d-1}$ onto itself, we have $\mathbb{P}_{k,d-1} \circ \boldsymbol{S} = \mathbb{P}_{k,d-1}$. Hence, $\mathbb{P}_{k,d-1} \circ \boldsymbol{T}_H^{-1} = \mathbb{P}_{k,d-1} \circ \boldsymbol{S} \circ \tilde{\boldsymbol{T}}_H^{-1} = \mathbb{P}_{k,d-1} \circ \tilde{\boldsymbol{T}}_H^{-1}$. This proves that the assertion of Lemma 14.7 is independent of the mapping $\boldsymbol{T}_H$. □

Lemma 14.9 (Divergence). *$\nabla{\cdot}\boldsymbol{v} \in \mathbb{P}_{k,d}$ for all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$, and if the function $\boldsymbol{v}$ is divergence-free, then $\boldsymbol{v} \in \mathbb{P}_{k,d}$.*

Proof. That $\nabla{\cdot}\boldsymbol{v} \in \mathbb{P}_{k,d}$ follows from $v_i \in \mathbb{P}_{k+1,d}$ for all $i \in \{1{:}d\}$. Let $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ be divergence-free. Since $\boldsymbol{v} \in \mathbb{RT}_{k,d}$, there are $\boldsymbol{p} \in \mathbb{P}_{k,d}$ and $q \in \mathbb{P}^{\mathrm{H}}_{k,d}$ such that $\boldsymbol{v} = \boldsymbol{p} + \boldsymbol{x}q$. Owing to Lemma 14.3, we infer that $\nabla{\cdot}\boldsymbol{p} + (k+d)q = 0$, which implies that $q = 0$ since $\mathbb{P}^{\mathrm{H}}_{k,d} \cap \mathbb{P}_{k-1,d} = \{0\}$ if $k \geq 1$. The argument for $k = 0$ is trivial. Hence, $\boldsymbol{v} = \boldsymbol{p} \in \mathbb{P}_{k,d}$. □

14.3 Simplicial Raviart–Thomas elements

Let $k \in \mathbb{N}$ and let $d \geq 2$. Let K be a simplex in $\mathbb{R}^d$. Each face $F \in \mathcal{F}_K$ of K is oriented by the normal vector $\boldsymbol{\nu}_F := |F|\boldsymbol{n}_F$ (so that $\|\boldsymbol{\nu}_F\|_{\ell^2} = |F|$). The simplex K itself is oriented by the d vectors $\{\boldsymbol{\nu}_{K,j} := |F_j|\boldsymbol{n}_{F_j}\}_{j \in \{1:d\}}$ where $\{F_j\}_{j \in \{1:d\}}$ are the d faces of K sharing the vertex with the lowest index. Note that $\{\boldsymbol{\nu}_{K,j}\}_{j \in \{1:d\}}$ is a basis of $\mathbb{R}^d$ (see Exercise 7.3(iv)), and this basis coincides with the canonical Cartesian basis of $\mathbb{R}^d$ when K is the unit simplex. The dofs of the $\mathbb{RT}_{k,d}$ finite element involve integrals over the faces of K or over K itself (for $k \geq 1$). Since the face dofs require to evaluate moments against $(d-1)$-variate polynomials, we introduce an affine bijective mapping $\boldsymbol{T}_F : \widehat{S}^{d-1} \to F$ for all $F \in \mathcal{F}_K$, where $\widehat{S}^{d-1}$ is the unit simplex of $\mathbb{R}^{d-1}$; see Figure 14.2. For instance, after enumerating the d vertices of $\widehat{S}^{d-1}$ and the $(d+1)$ vertices of K, we can define $\boldsymbol{T}_F$ such that the d vertices of $\widehat{S}^{d-1}$ are mapped to the d vertices of F with increasing indices.

Definition 14.10 (dofs). *We denote by Σ the collection of the following linear forms acting on $\mathbb{RT}_{k,d}$:*

$$\sigma^{\mathrm{f}}_{F,m}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}{\cdot}\boldsymbol{\nu}_F)(\zeta_m \circ \boldsymbol{T}_F^{-1})\,\mathrm{d}s, \quad \forall F \in \mathcal{F}_K, \tag{14.5a}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}{\cdot}\boldsymbol{\nu}_{K,j})\psi_m\,\mathrm{d}x, \quad \forall j \in \{1{:}d\}, \tag{14.5b}$$

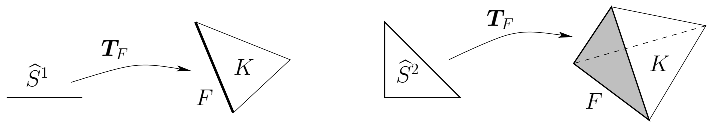

Fig. 14.2 Reference face $\widehat{S}^{d-1}$ and mapping $\boldsymbol{T}_F$ for $d=2$ (left, the face F is indicated in bold) and $d=3$ (right, the face F is highlighted in gray).

where $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ *is a basis of* $\mathbb{P}_{k,d-1}$ *with* $n^{\mathrm{f}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k,d-1}) = \binom{d+k-1}{k}$ *and* $\{\psi_m\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ *is a basis of* $\mathbb{P}_{k-1,d}$ *with* $n^{\mathrm{c}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k-1,d}) = \binom{d+k-1}{k-1}$ *if* $k \geq 1$. *We regroup the dofs as follows:*

$$\Sigma^{\mathrm{f}}_F := \{\sigma^{\mathrm{f}}_{F,m}\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}, \qquad \forall F \in \mathcal{F}_K, \tag{14.6a}$$

$$\Sigma^{\mathrm{c}} := \{\sigma^{\mathrm{c}}_{j,m}\}_{(j,m)\in\{1:d\}\times\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}. \tag{14.6b}$$

Remark 14.11 (dofs). The unit of all the dofs is a surface times the dimension of $\boldsymbol{v}$. We could also have written $\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) := \ell_K^{-1}\int_K(\boldsymbol{v}\cdot\boldsymbol{e}_j)\boldsymbol{\psi}_m\,\mathrm{d}x$ for the cell dofs, where ℓ_K is a length scale of K and $\{\boldsymbol{e}_j\}_{j\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. We will see that the definition (14.5b) is more natural when using the contravariant Piola transformation to generate other finite elements. The dofs are defined here on $\mathbb{RT}_{k,d}$. Their extension to some larger space $\boldsymbol{V}(K)$ is addressed in Chapters 16 and 17. □

Lemma 14.12 (Invariance w.r.t. $\boldsymbol{T}_F$). *Assume that every affine bijective mapping* $\boldsymbol{S}:\widehat{S}^{d-1}\to\widehat{S}^{d-1}$ *leaves the basis* $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ *globally invariant, i.e.,* $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}} = \{\zeta_m\circ\boldsymbol{S}\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$. *Then for all* $F\in\mathcal{F}_K$, *the set* Σ^{f}_F *is independent of the affine bijective mapping* $\boldsymbol{T}_F$.

Proof. Let $\boldsymbol{T}_F, \tilde{\boldsymbol{T}}_F$ be two affine bijective mappings from $\widehat{S}^{d-1}$ to F. Then $\boldsymbol{S} := \boldsymbol{T}_F^{-1}\circ\tilde{\boldsymbol{T}}_F$ is an affine bijective mapping from $\widehat{S}^{d-1}$ to $\widehat{S}^{d-1}$. Let $m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}$. The invariance assumption implies that there exists ζ_n, $n\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}$, s.t. $\zeta_m\circ\boldsymbol{S} = \zeta_n$. Hence, with obvious notation we have

$$\begin{aligned}
&|F|\sigma^{\mathrm{f}}_{F,m}(\boldsymbol{v}) = \int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)(\zeta_m\circ\boldsymbol{T}_F^{-1})\,\mathrm{d}s \\
&= \int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)((\zeta_m\circ\boldsymbol{S})\circ\tilde{\boldsymbol{T}}_F^{-1})\,\mathrm{d}s = \int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)(\zeta_n\circ\tilde{\boldsymbol{T}}_F^{-1})\,\mathrm{d}s = |F|\tilde{\sigma}^{\mathrm{f}}_{F,n}(\boldsymbol{v}).
\end{aligned}$$
□

Example 14.13 (Vertex permutation). For every affine bijective mapping $\boldsymbol{S}:\widehat{S}^{d-1}\to\widehat{S}^{d-1}$, there is a unique permutation σ of the set $\{0:d-1\}$ s.t. $\boldsymbol{S}(\widehat{\boldsymbol{z}}_i) := \widehat{\boldsymbol{z}}_{\sigma(i)}$ for all $i\in\{0:d-1\}$, where $\{\widehat{\boldsymbol{z}}_i\}_{i\in\{0:d-1\}}$ are the vertices of $\widehat{S}^{d-1}$. Then the above invariance holds true holds true iff all the vertices of $\widehat{S}^{d-1}$ play symmetric roles when defining the basis functions $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$. For instance, for $d:=2$, $\widehat{S}^1 := [0,1]$, and $k:=1$, the basis $\{1,s\}$ of $\mathbb{P}_{1,1}$ is not invariant w.r.t. vertex permutation, but the basis $\{1-s,s\}$ is. □

A graphic representation of the dofs is shown in Figure 14.3. The number of arrows on a face counts the number of moments of the normal component considered over the face. The number of pairs of gray circles inside the triangle counts the number of moments inside the cell (one circle for the component along $\boldsymbol{\nu}_{K,1}$ and one for the component along $\boldsymbol{\nu}_{K,2}$).

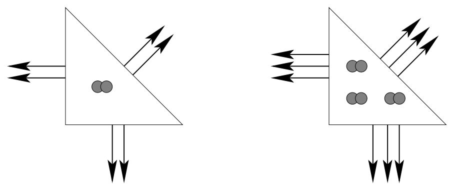

Fig. 14.3 Degrees of freedom of $\mathbb{RT}_{k,d}$ finite elements for $d = 2$ and $k = 1$ (left) or $k = 2$ (right) (assuming that all the normals point outward).

Lemma 14.14 (Face unisolvence). *For all $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ and all $F \in \mathcal{F}_K$,*

$$[\,\sigma(\boldsymbol{v}) = 0,\ \forall \sigma \in \Sigma_F^{\mathrm{f}}\,] \iff [\,\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0\,]. \tag{14.7}$$

Proof. The condition $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma_F^{\mathrm{f}}$ means that $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F$ is orthogonal to $\mathbb{P}_{k,d-1} \circ \boldsymbol{T}_F^{-1}$. Since Lemma 14.7 implies that $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F \in \mathbb{P}_{k,d-1} \circ \boldsymbol{T}_F^{-1}$, we infer that $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0$. □

Proposition 14.15 (Finite element). $(K, \mathbb{RT}_{k,d}, \Sigma)$ *is a finite element.*

Proof. We have already established the assertion for $k = 0$. Let us consider $k \geq 1$. Observe first that the cardinality of Σ can be evaluated as follows:

$$\begin{aligned}\mathrm{card}(\Sigma) &= d\, n_{\mathrm{sh}}^{\mathrm{c}} + (d+1) n_{\mathrm{sh}}^{\mathrm{f}} = d\binom{d+k-1}{k-1} + (d+1)\binom{d+k-1}{k} \\ &= \frac{(d+k-1)!}{(d-1)!(k-1)!}\left(1 + \frac{d+1}{k}\right) = \dim(\mathbb{RT}_{k,d}).\end{aligned}$$

Hence, the statement will be proved once it is established that zero is the only function in $\mathbb{RT}_{k,d}$ that annihilates the dofs in Σ. Let $\boldsymbol{v} \in \mathbb{RT}_{k,d}$ be such that $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma$. Owing to Lemma 14.14, we infer that $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0$ for all $F \in \mathcal{F}_K$. This in turn implies that $\int_K \boldsymbol{v}\cdot(\nabla\nabla\cdot\boldsymbol{v})\,\mathrm{d}x = -\int_K (\nabla\cdot\boldsymbol{v})^2\,\mathrm{d}x$. Observing that $\nabla\nabla\cdot\boldsymbol{v}$ is in $\boldsymbol{\mathbb{P}}_{k-1,d}$ (recall that $\nabla\cdot\boldsymbol{v} \in \mathbb{P}_{k,d}$ from Lemma 14.9), the assumption that $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma^{\mathrm{c}}$ (i.e., $\boldsymbol{v}$ is orthogonal to $\boldsymbol{\mathbb{P}}_{k-1,d}$), together with the above identity imply that $\nabla\cdot\boldsymbol{v} = 0$. Using Lemma 14.9, we conclude that $\boldsymbol{v} \in \boldsymbol{\mathbb{P}}_{k,d}$ and $\boldsymbol{v}_{|F}\cdot\boldsymbol{\nu}_F = 0$ for all $F \in \mathcal{F}_K$. Let $j \in \{1{:}d\}$. Since $\boldsymbol{\nu}_{K,j} = \boldsymbol{\nu}_{F_j} = |F_j|\boldsymbol{n}_{F_j}$ for some face $F_j \in \mathcal{F}_K$, we infer that $\boldsymbol{v}(\boldsymbol{x})\cdot\boldsymbol{\nu}_{K,j} = \lambda_j(\boldsymbol{x}) r_j(\boldsymbol{x})$ for all $\boldsymbol{x} \in K$, where λ_j is the barycentric coordinate of K associated with the vertex opposite to F_j (i.e., λ_j vanishes on F_j) and $r_j \in \mathbb{P}_{k-1,d}$; see Exercise 7.4(iv). The condition $\sigma(\boldsymbol{v}) = 0$ for all $\sigma \in \Sigma^{\mathrm{c}}$ implies that $\int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j}) r_j \,\mathrm{d}x = 0$, which in turn means that $0 = \int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j}) r_j \,\mathrm{d}x = \int_K \lambda_j r_j^2 \,\mathrm{d}x$, thereby proving that $r_j = 0$ since λ_j is positive in the interior of K. Hence, $\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j}$ vanishes identically for all $j \in \{1{:}d\}$. This proves that $\boldsymbol{v} = \boldsymbol{0}$ since $\{\boldsymbol{\nu}_{K,j}\}_{j\in\{1:d\}}$ is a basis of $\mathbb{R}^d$. □

The shape functions $\{\theta_i\}_{i\in\mathcal{N}}$ associated with the dofs $\{\sigma_i\}_{i\in\mathcal{N}}$ defined in (14.5) can be constructed by choosing a basis $\{\phi_i\}_{i\in\mathcal{N}}$ of the polynomial space $\mathbb{RT}_{k,d}$ and by inverting the corresponding generalized Vandermonde matrix $\mathcal{V}$ as explained in Proposition 5.5. Recall that this matrix has entries $\mathcal{V}_{ij} = \sigma_j(\phi_i)$ and that the i-th line of $\mathcal{V}^{-1}$ gives the components of the shape function θ_i in the basis $\{\phi_i\}_{i\in\mathcal{N}}$. The basis $\{\phi_i\}_{i\in\mathcal{N}}$ chosen in Bonazzoli and Rapetti [31] (built by dividing the simplex into smaller sub-simplices following the ideas in Rapetti and Bossavit [163], Christiansen and Rapetti [70]) is particularly interesting since the entries of $\mathcal{V}^{-1}$ are integers. One could also choose $\{\phi_i\}_{i\in\mathcal{N}}$ to be the hierarchical basis of $\mathbb{RT}_{k,d}$ constructed in Fuentes et al. [103, §7.3]. This basis can be organized into functions attached to the faces of K and to K itself in such a way that the generalized Vandermonde matrix $\mathcal{V}$ is block-triangular (notice though that this matrix is not block-diagonal).

Remark 14.16 (Dof independence). As in Remark 7.20, we infer from Exercise 5.2 that the interpolation operator $\mathcal{I}_K^{\mathrm{d}}$ associated with the $\mathbb{RT}_{k,d}$ element is independent of the bases $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$ and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ used to define the dofs in (14.5). This operator is also independent of the mappings $\boldsymbol{T}_F$ and of the orientation vectors $\{\boldsymbol{\nu}_F\}_{F\in\mathcal{F}_K}$ and $\{\boldsymbol{\nu}_{K,j}\}_{j\in\{1:d\}}$. □

Remark 14.17 (Literature). The $\mathbb{RT}_{k,d}$ finite element has been introduced in Raviart and Thomas [164, 165] for $d = 2$; see also Weil [198, p. 127], Whitney [199, Eq. (12), p. 139] for $k = 0$. The generalization to $d \geq 3$ is due to Nédélec [151]. The reading of [151] is highly recommended; see also Boffi et al. [29, §2.3.1], Hiptmair [117], Monk [145, pp. 118-126]. The name Raviart–Thomas seems to be an accepted practice in the literature. □

14.4 Generation of Raviart–Thomas elements

Let $\widehat{K}$ be the reference simplex in $\mathbb{R}^d$. Let $\mathcal{T}_h$ be an affine simplicial mesh. Let $K = \boldsymbol{T}_K(\widehat{K})$ be a mesh cell, where $\boldsymbol{T}_K : \widehat{K} \to K$ is the geometric mapping, and let $\mathbb{J}_K$ be the Jacobian matrix of $\boldsymbol{T}_K$. Let $F \in \mathcal{F}_K$ be a face of K. We have $F = \boldsymbol{T}_K(\widehat{F})$ for some face $\widehat{F} \in \mathcal{F}_{\widehat{K}}$. Owing to Theorem 10.8, it is possible (using the increasing vertex-index enumeration) to orient the faces F and $\widehat{F}$ in a way that is compatible with the geometric mapping $\boldsymbol{T}_K$. This means that the unit normal vectors $\boldsymbol{n}_F$ and $\widehat{\boldsymbol{n}}_{\widehat{F}}$ satisfy (10.6b), i.e., $\boldsymbol{n}_F = \boldsymbol{\Phi}_K^{\mathrm{d}}(\widehat{\boldsymbol{n}}_{\widehat{F}})$ with $\boldsymbol{\Phi}_K^{\mathrm{d}}$ defined in (9.14a). In other words, we have

$$\boldsymbol{n}_F \circ \boldsymbol{T}_{K|\widehat{F}} = \epsilon_K \frac{1}{\|\mathbb{J}_K^{-\mathsf{T}} \widehat{\boldsymbol{n}}_{\widehat{F}}\|_{\ell^2}} \mathbb{J}_K^{-\mathsf{T}} \widehat{\boldsymbol{n}}_{\widehat{F}}, \tag{14.8}$$

where $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|} = \pm 1$. Recalling that $\boldsymbol{\nu}_F := |F|\boldsymbol{n}_F$, $\widehat{\boldsymbol{\nu}}_{\widehat{F}} := |\widehat{F}|\widehat{\boldsymbol{n}}_{\widehat{F}}$ and that $|F| = |\det(\mathbb{J}_K)| \|\mathbb{J}_K^{-\mathsf{T}} \widehat{\boldsymbol{n}}_{\widehat{F}}\|_{\ell^2} |\widehat{F}|$ owing to Lemma 9.12, we infer that

$$\boldsymbol{\nu}_F \circ \boldsymbol{T}_{K|\widehat{F}} = \det(\mathbb{J}_K) \mathbb{J}_K^{-\mathsf{T}} \widehat{\boldsymbol{\nu}}_{\widehat{F}}. \tag{14.9}$$

Due to the role played by the normal component of vector fields on the faces of K, we are going to use in Proposition 9.2 the contravariant Piola transformation

$$\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \det(\mathbb{J}_K) \mathbb{J}_K^{-1} (\boldsymbol{v} \circ \boldsymbol{T}_K) \tag{14.10}$$

to define Raviart–Thomas elements on K from a reference Raviart–Thomas element defined on $\widehat{K}$. For scalar fields, we consider the pullback by the geometric mapping, i.e., $\psi_K^{\mathrm{g}}(q) := q \circ \boldsymbol{T}_K$. Finally, we orient K and $\widehat{K}$ with the d vectors $\{\boldsymbol{\nu}_{K,j} := |F_j|\boldsymbol{n}_{F_j}\}_{j\in\{1:d\}}$ and $\{\widehat{\boldsymbol{\nu}}_{\widehat{K},j} := |\widehat{F}_j|\widehat{\boldsymbol{n}}_{F_j}\}_{j\in\{1:d\}}$ associated with the d faces of K and $\widehat{K}$ that share the vertex with the lowest index, i.e., we have $F_j = \boldsymbol{T}_K(\widehat{F}_j)$ for all $j \in \{1:d\}$. The above considerations show that $\boldsymbol{\nu}_{K,j} \circ \boldsymbol{T}_K = \det(\mathbb{J}_K) \mathbb{J}_K^{-\mathsf{T}} \widehat{\boldsymbol{\nu}}_{\widehat{K},j}$ for all $j \in \{1:d\}$.

Lemma 14.18 (Transformation of dofs). *Let $\boldsymbol{v} \in \boldsymbol{C}^0(K)$ and let $q \in C^0(K)$. The following holds true:*

$$\frac{1}{|F|} \int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F) q \,\mathrm{d}s = \frac{1}{|\widehat{F}|} \int_{\widehat{F}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{\nu}}_{\widehat{F}}) \psi_K^{\mathrm{g}}(q) \,\mathrm{d}\widehat{s}, \qquad \forall F \in \mathcal{F}_K, \tag{14.11a}$$

$$\frac{1}{|K|} \int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j}) q \,\mathrm{d}x = \frac{1}{|\widehat{K}|} \int_{\widehat{K}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{\nu}}_{\widehat{K},j}) \psi_K^{\mathrm{g}}(q) \,\mathrm{d}\widehat{x}, \ \forall j \in \{1:d\}. \tag{14.11b}$$

Proof. The identity (14.11a) is nothing but (10.7a) from Lemma 10.4, which itself is a reformulation of (9.15a) from Lemma 9.13 (the fact that $\boldsymbol{T}_K$ is affine is not used here). The proof of (14.11b) is similar since

$$\begin{aligned} \int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j}) q \,\mathrm{d}x &= \int_{\widehat{K}} (\boldsymbol{v} \circ \boldsymbol{T}_K)\cdot(\boldsymbol{\nu}_{K,j} \circ \boldsymbol{T}_K) \psi_K^{\mathrm{g}}(q) |\det(\mathbb{J}_K)| \,\mathrm{d}\widehat{x} \\ &= \int_{\widehat{K}} (\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\widehat{\boldsymbol{\nu}}_{\widehat{K},j}) \psi_K^{\mathrm{g}}(q) |\det(\mathbb{J}_K)| \,\mathrm{d}\widehat{x}, \end{aligned}$$

and since $\boldsymbol{T}_K$ is affine, we have $|K| = |\det(\mathbb{J}_K)|\,|\widehat{K}|$. □

Proposition 14.19 (Generation). *Let $(\widehat{K}, \widehat{\boldsymbol{P}}, \widehat{\Sigma})$ be a simplicial $\mathbb{RT}_{k,d}$ element with face and cell dofs defined using the polynomial bases $\{\zeta_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}}$ and $\{\psi_m\}_{m\in\{1:n_{\mathrm{sh}}^{\mathrm{c}}\}}$ (if $k \geq 1$) of $\mathbb{P}_{k,d-1}$ and $\mathbb{P}_{k-1,d}$, respectively, as in* (14.5)*. Assume that the geometric mapping $\boldsymbol{T}_K$ is affine and that* (14.9) *holds true. Then the finite element $(K, \boldsymbol{P}_K, \Sigma_K)$ generated using Proposition* 9.2 *with* *the contravariant Piola transformation* (14.10) *is a simplicial* $\mathbb{RT}_{k,d}$ *finite element with dofs*

$$\sigma^{\mathrm{f}}_{F,m}(\boldsymbol{v}) = \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)(\zeta_m \circ \boldsymbol{T}^{-1}_{K,F})\,\mathrm{d}s, \qquad \forall F \in \mathcal{F}_K, \tag{14.12a}$$

$$\sigma^{\mathrm{c}}_{j,m}(\boldsymbol{v}) = \frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j})(\psi_m \circ \boldsymbol{T}^{-1}_{K})\,\mathrm{d}x, \quad \forall j \in \{1{:}d\}, \tag{14.12b}$$

where $\boldsymbol{T}_{K,F} := \boldsymbol{T}_{K|\widehat{F}} \circ \boldsymbol{T}_{\widehat{F}}$ *is the affine bijective mapping from* $\widehat{S}^{d-1}$ *to* F *that maps the* d *vertices of* $\widehat{S}^{d-1}$ *to the* d *vertices of* F *with increasing indices.*

Proof. See Exercise 14.4 for the proof that $\boldsymbol{P}_K = \mathbb{RT}_{k,d}$. Use Lemma 14.18 to prove (14.12a)-(14.12b). □

Remark 14.20 (Unit). Given some length unit L, the shape functions scale as L^{1-d} since the unit of all the dofs is L^{d-1}. □

Remark 14.21 (Nonaffine meshes). Proposition 9.2 together with the map $\boldsymbol{\psi}^{\mathrm{d}}_K$ defined in (14.10) can still be used to generate a finite element $(K, \boldsymbol{P}_K, \Sigma_K)$ if the geometric mapping $\boldsymbol{T}_K$ is nonaffine. The function space $\boldsymbol{P}_K$ and the dofs in Σ_K then differ from those of the $\mathbb{RT}_{k,d}$ element. □

14.5 Other $\boldsymbol{H}(\mathrm{div})$ finite elements

14.5.1 Brezzi–Douglas–Marini elements

Brezzi–Douglas–Marini (BDM) elements [49, 50] offer an interesting alternative to Raviart–Thomas elements since in this case the polynomial space is $\boldsymbol{P} := \mathbb{P}_{k,d} \subsetneq \mathbb{RT}_{k,d}$, $k \geq 1$. This space is optimal from the approximation viewpoint. The price to pay for this simplification is that the divergence operator $\nabla\cdot$ is surjective from $\mathbb{P}_{k,d}$ onto $\mathbb{P}_{k-1,d}$ only. This is not a limitation if the functions one wants to interpolate are divergence-free (or have a divergence that belongs to $\mathbb{P}_{k-1,d}$).

Let K be a simplex in $\mathbb{R}^d$. The dofs of BDM elements are attached to the $(d+1)$ faces of K and to K itself (for $k \geq 2$). The face dofs are the same as for Raviart–Thomas elements, i.e., the linear forms $\sigma^{\mathrm{f}}_{F,m}$ defined in (14.5a) for all $F \in \mathcal{F}_K$ and every $m \in \{1{:}n^{\mathrm{f}}_{\mathrm{sh}}\}$ with $n^{\mathrm{f}}_{\mathrm{sh}} := \dim(\mathbb{P}_{k,d-1})$. Note that the cell dofs for Raviart–Thomas elements are moments against a set of basis functions of $\mathbb{P}_{k-1,d}$, whereas those for BDM elements are moments against a set of basis functions of the Nédélec polynomial space $\mathbb{N}_{k-2,d}$ introduced in the next chapter (see §15.2). At this stage, it is sufficient to know that $\mathbb{P}_{k-2,d} \subsetneq \mathbb{N}_{k-2,d} \subsetneq \mathbb{P}_{k-1,d}$ and that $\dim(\mathbb{N}_{k-2,2}) = (k-1)(k+1)$ and $\dim(\mathbb{N}_{k-2,3}) = \frac{1}{2}(k-1)(k+1)(k+2)$ (see Lemma 15.7). We define

$$\tilde{\sigma}_m^{\mathrm{c}}(\boldsymbol{v}) := \int_K \boldsymbol{v}\cdot\tilde{\boldsymbol{\psi}}_m \,\mathrm{d}x, \qquad \forall m \in \{1{:}\tilde{n}_{\mathrm{sh}}^{\mathrm{c}}\}, \tag{14.13}$$

where $\{\tilde{\boldsymbol{\psi}}_m\}_{m\in\{1:\tilde{n}_{\mathrm{sh}}^{\mathrm{c}}\}}$ is a basis of $\mathbb{N}_{k-2,d}$ and $\tilde{n}_{\mathrm{sh}}^{\mathrm{c}} := \dim(\mathbb{N}_{k-2,d})$. Let us set $\Sigma := \{\sigma_{F,m}^{\mathrm{f}}\}_{F\in\mathcal{F}_K, m\in\{1:n_{\mathrm{sh}}^{\mathrm{f}}\}} \cup \{\tilde{\sigma}_m^{\mathrm{c}}\}_{m\in\{1:\tilde{n}_{\mathrm{sh}}^{\mathrm{c}}\}}$.

Proposition 14.22 (Finite element). *$(K, \mathbb{P}_{k,d}, \Sigma)$ is a finite element.*

Proof. See Boffi et al. [29, p. 88]. □

Hierarchical basis functions for the BDM element are constructed in Ainsworth and Coyle [6], Schöberl and Zaglmayr [176].

Remark 14.23 (Generation). Generating BDM elements also involves the covariant Piola transformation $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{w}) := \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{w}\circ\boldsymbol{T}_K)$ defined in (9.9b), so that $\int_K \boldsymbol{v}\cdot\tilde{\boldsymbol{\psi}}_m \,\mathrm{d}x = \epsilon_K \int_{\widehat{K}} \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\cdot\boldsymbol{\psi}_K^{\mathrm{c}}(\tilde{\boldsymbol{\psi}}_m)\,\mathrm{d}\widehat{x}$ with $\epsilon_K := \frac{\det(\mathbb{J}_K)}{|\det(\mathbb{J}_K)|} = \pm 1$. □

14.5.2 Cartesian Raviart–Thomas elements

Let us briefly review the Cartesian Raviart–Thomas finite elements. We refer the reader to Exercise 14.6 for the proofs. For a multi-index $\alpha \in \mathbb{N}^d$, we define the (anisotropic) polynomial space $\mathbb{Q}_{\alpha_1,\ldots,\alpha_d}$ composed of d-variate polynomials whose degree with respect to x_i is at most α_i for all $i \in \{1{:}d\}$. Let $k \in \mathbb{N}$ and define

$$\mathbb{RT}_{k,d}^{\square} := \mathbb{Q}_{k+1,k,\ldots,k}\times\ldots\times\mathbb{Q}_{k,\ldots,k,k+1}. \tag{14.14}$$

Since $\dim(\mathbb{Q}_{k+1,k,\ldots,k}) = \ldots = \dim(\mathbb{Q}_{k,\ldots,k,k+1}) = (k+2)(k+1)^{d-1}$, we have $\dim(\mathbb{RT}_{k,d}^{\square}) = d(k+2)(k+1)^{d-1}$. Moreover, one can verify that

$$\nabla\cdot\boldsymbol{v} \in \mathbb{Q}_{k,d}, \qquad \boldsymbol{v}_{|H}\cdot\boldsymbol{\nu}_H \in \mathbb{Q}_{k,d-1}\circ\boldsymbol{T}_H^{-1}, \tag{14.15}$$

for all $\boldsymbol{v} \in \mathbb{RT}_{k,d}^{\square}$ and every affine hyperplane H in $\mathbb{R}^d$ with normal vector $\boldsymbol{\nu}_H$ parallel to one of the vectors of the canonical basis of $\mathbb{R}^d$ and where $\boldsymbol{T}_H : \mathbb{R}^{d-1} \to H$ is any affine bijective mapping.

Let K be a cuboid in $\mathbb{R}^d$. Each face $F \in \mathcal{F}_K$ of K is oriented by the normal vector $\boldsymbol{\nu}_F$ with $\|\boldsymbol{\nu}_F\|_{\ell^2} = |F|$. Let $\boldsymbol{T}_F$ be an affine bijective mapping from $[0,1]^{d-1}$ onto F. Let us orient K using $\boldsymbol{\nu}_{K,j} := |F_j|\boldsymbol{e}_j$ for all $j \in \{1{:}d\}$, where $\{\boldsymbol{e}_j\}_{j\in\{1:d\}}$ is the canonical basis of $\mathbb{R}^d$ and $|F_j|$ is the measure of any of the two faces of K supported in a hyperplane perpendicular to $\boldsymbol{e}_j$. Let Σ be the set composed of the following linear forms:

$$\sigma_{F,m}^{\mathrm{f}}(\boldsymbol{v}) := \frac{1}{|F|}\int_F (\boldsymbol{v}\cdot\boldsymbol{\nu}_F)(\zeta_m \circ \boldsymbol{T}_F^{-1})\,\mathrm{d}s, \quad \forall F \in \mathcal{F}_K, \tag{14.16a}$$

$$\sigma_{j,m}^{\mathrm{c}}(\boldsymbol{v}) := \frac{1}{|K|}\int_K (\boldsymbol{v}\cdot\boldsymbol{\nu}_{K,j})\psi_{j,m}\,\mathrm{d}x, \qquad \forall j \in \{1{:}d\}, \tag{14.16b}$$

where $\{\zeta_m\}_{m\in\{1:n^{\mathrm{f}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{Q}_{k,d-1}$ with $n^{\mathrm{f}}_{\mathrm{sh}} := (k+1)^{d-1}$, and $\{\psi_{j,m}\}_{m\in\{1:n^{\mathrm{c}}_{\mathrm{sh}}\}}$ is a basis of $\mathbb{Q}_{k,\ldots,k,k-1,k,\ldots,k}$ with $n^{\mathrm{c}}_{\mathrm{sh}} := k(k+1)^{d-1}$ if $k \geq 1$, with the index $(k-1)$ at the j-th position for all $j \in \{1{:}d\}$.

Proposition 14.24 (Finite element). *$(K, \mathbb{RT}^{\square}_{k,d}, \Sigma)$ is a finite element.*

Cartesian Raviart–Thomas elements can be generated for all the mesh cells of an affine mesh composed of parallelotopes by using affine geometric mappings and the contravariant Piola transformation (recall, however, that orienting such meshes and making the orientation generation-compatible requires some care; see Theorem 10.10).

Example 14.25 (Shape functions and dofs for $\mathbb{RT}^{\square}_{0,d}$). Let $K := [0,1]^d$. Let F_i and F_{d+i} be the faces defined by $x_i = 0$ and $x_i = 1$, respectively, for all $i \in \{1{:}d\}$. Using the basis function $\zeta_1 := 1$ for $\mathbb{Q}_{0,d-1}$, the $2d$ dofs are the mean-value of the normal component over each face of K, and the shape functions are $\boldsymbol{\theta}^{\mathrm{f}}_i(\boldsymbol{x}) := (1-x_i)\boldsymbol{n}_{F_i}$ and $\boldsymbol{\theta}^{\mathrm{f}}_{d+i}(\boldsymbol{x}) := x_i\boldsymbol{n}_{F_i}$ for all $i \in \{1{:}d\}$. The dofs are illustrated in Figure 14.4. □

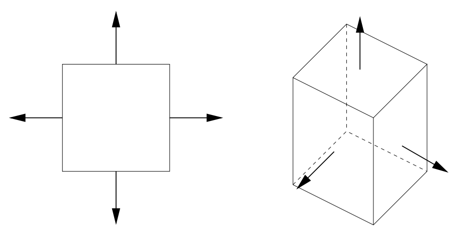

Fig. 14.4 Degrees of freedom of the lowest-order Cartesian Raviart–Thomas element $\mathbb{RT}^{\square}_{0,d}$ in dimensions two (left) and three (right, only visible dofs are shown).

Remark 14.26 (Other elements). Alternative elements are the Cartesian Brezzi–Douglas–Marini elements in dimension two, the Brezzi–Douglas–Durán–Fortin elements in dimension three (see [49, 50]), and their reduced versions by Brezzi–Douglas–Fortin–Marini [51]. □

Exercises

Exercise 14.1 ($\mathbb{RT}_{0,d}$). (i) Prove that $\int_K \iota_{F,K}\boldsymbol{\theta}^{\mathrm{f}}_F \,\mathrm{d}x = \boldsymbol{c}_F - \boldsymbol{c}_K$, where $\boldsymbol{\theta}^{\mathrm{f}}_F$ is defined in (14.3), and $\boldsymbol{c}_F$, $\boldsymbol{c}_K$ are the barycenters of F and K, respectively. (*Hint*: use (14.3) and $\int_F \boldsymbol{x}\,\mathrm{d}s = |F|\boldsymbol{c}_F$.) Provide a second proof without using (14.3). (*Hint*: fix $\boldsymbol{e} \in \mathbb{R}^d$, define $\phi(\boldsymbol{x}) = (\boldsymbol{x}-\boldsymbol{c}_F)\cdot\boldsymbol{e}$, observe that $\nabla\phi = \boldsymbol{e}$, and compute $\boldsymbol{e}\cdot\int_K \boldsymbol{\theta}^{\mathrm{f}}_F\,\mathrm{d}x$.) (ii) Prove that $\sum_{F\in\mathcal{F}_K} |F|\boldsymbol{\theta}^{\mathrm{f}}_F(\boldsymbol{x}) \otimes \boldsymbol{n}_F = \mathbb{I}_d$ for all $\boldsymbol{x} \in K$. (*Hint*: use (7.1).) (iii) Prove that $\boldsymbol{v}(\boldsymbol{x}) = \langle \boldsymbol{v} \rangle_K + \frac{1}{d}(\nabla{\cdot}\boldsymbol{v})(\boldsymbol{x} - \boldsymbol{c}_K)$ for all $\boldsymbol{v} \in \mathbb{RT}_{0,d}$, where $\langle \boldsymbol{v} \rangle_K := \frac{1}{|K|}\int_K \boldsymbol{v}\,\mathrm{d}x$ is the mean value of $\boldsymbol{v}$ on K.

Exercise 14.2 ($\mathbb{RT}_{0,d}$ in 3D). Let $d = 3$. Let F_i, $i \in \{0{:}3\}$, be a face of K with vertices $\{\boldsymbol{a}_r, \boldsymbol{a}_p, \boldsymbol{a}_q\}$ s.t. $\big((\boldsymbol{z}_q - \boldsymbol{z}_r)\times(\boldsymbol{z}_p - \boldsymbol{z}_r)\big){\cdot}\boldsymbol{n}_{K|F_i} > 0$. (i) Prove that $\nabla\lambda_p \times \nabla\lambda_q = \frac{\boldsymbol{z}_r - \boldsymbol{z}_i}{6|K|}$ and prove similar formulas for $\nabla\lambda_q \times \nabla\lambda_r$ and $\nabla\lambda_r \times \nabla\lambda_p$. (*Hint*: prove the formula in the reference simplex, then use Exercise 9.5.) (ii) Prove that $\boldsymbol{\theta}_i^{\mathrm{f}} = -2(\lambda_p\nabla\lambda_q\times\nabla\lambda_r + \lambda_q\nabla\lambda_r\times\nabla\lambda_p + \lambda_r\nabla\lambda_p\times\nabla\lambda_q)$. Find the counterpart of this formula if $d = 2$.

Exercise 14.3 (Piola transformation). (i) Let $\boldsymbol{v} \in \boldsymbol{C}^1(K)$ and $q \in C^0(K)$. Prove that $\int_K q\,\nabla{\cdot}\boldsymbol{v}\,\mathrm{d}x = \int_{\widehat{K}} \psi_K^{\mathrm{g}}(q)\nabla{\cdot}\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v})\,\mathrm{d}\widehat{x}$. (ii) Show that $\int_K \boldsymbol{v}{\cdot}\boldsymbol{\theta}\,\mathrm{d}x = \epsilon_K \int_{\widehat{K}} \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}){\cdot}\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{\theta})\,\mathrm{d}\widehat{x}$ for all $\boldsymbol{\theta} \in \boldsymbol{C}^1(K)$.

Exercise 14.4 (Generating $\mathbb{RT}_{k,d}$). (i) Let $\boldsymbol{c} \in \mathbb{R}^d$, $q \in \mathbb{P}_{k,d}^{\mathrm{H}}$, and $\mathbb{A} \in \mathbb{R}^{d\times d'}$. Show that there is $r \in \mathbb{P}_{k-1,d'}$ such that $q(\mathbb{A}\boldsymbol{y} + \boldsymbol{c}) = q(\mathbb{A}\boldsymbol{y}) + r(\boldsymbol{y})$. (ii) Defining $s(\boldsymbol{y}) := q(\mathbb{A}\boldsymbol{y})$, show that $s \in \mathbb{P}_{k,d'}^{\mathrm{H}}$. (iii) Prove that $(\boldsymbol{\psi}_K^{\mathrm{d}})^{-1}(\mathbb{RT}_{k,d}) \subset \mathbb{RT}_{k,d}$. (iv) Prove the converse inclusion.

Exercise 14.5 (BDM). Verify that $\mathrm{card}(\Sigma) = \dim(\mathbb{P}_{k,d})$ for $d \in \{2,3\}$.

Exercise 14.6 (Cartesian Raviart–Thomas element). (i) Propose a basis for $\mathbb{RT}_{0,2}^{\square}$ and for $\mathbb{RT}_{0,3}^{\square}$ in $K := [0,1]^d$. (ii) Prove (14.15). (iii) Prove Proposition 14.24.