Part IV, Chapter 18

From broken to conforming spaces

In Parts II and III, we have introduced many examples of finite elements and devised techniques to generate finite elements in each cell of a mesh. In Part IV, composed of Chapters 18 to 23, we show how these methods can be used to build finite-dimensional spaces composed of piecewise smooth functions whose gradient, curl, or divergence is integrable. We also devise quasi-interpolation operators enjoying fundamental stability, approximation, and commutation properties. These spaces and operators will be used repeatedly in Volumes II and III to approximate various PDEs and estimate the approximation error. In the present chapter, we introduce broken Sobolev spaces and broken finite element spaces based on a mesh from a family of meshes $(\mathcal{T}_h)_{h\in\mathcal{H}}$ covering exactly a domain $D \subsetneq \mathbb{R}^d$. Then we identify jump conditions across the mesh interfaces that are necessary and sufficient for every function in some broken Sobolev space to have an integrable gradient, curl, or divergence. These conditions lead to the notion of conforming finite element spaces. Finally, we show how to construct L^1-stable (local) interpolation operators in the broken finite element space with optimal local approximation properties.

18.1 Broken spaces and jumps

In this section, we are only concerned with broken Sobolev spaces and with broken finite element spaces. Membership to broken spaces is defined by requiring that some property be satisfied in each mesh cell without requiring any continuity across the mesh interfaces.

18.1.1 Broken Sobolev spaces and jumps

The notions introduced hereafter will be used repeatedly in this book. We consider $\mathbb{R}^q$-valued functions for some integer $q \geq 1$.

Definition 18.1 (Broken Sobolev space). *Let $p \in [1, \infty]$ and $s > 0$ be a positive real number. The space defined by*

$$W^{s,p}(\mathcal{T}_h; \mathbb{R}^q) := \{v \in L^p(D; \mathbb{R}^q) \mid v_{|K} \in W^{s,p}(K; \mathbb{R}^q), \forall K \in \mathcal{T}_h\}, \tag{18.1}$$

is called broken Sobolev space*. This space is equipped with the norm*

$$\|v\|^p_{W^{s,p}(\mathcal{T}_h;\mathbb{R}^q)} := \sum_{K \in \mathcal{T}_h} \|v\|^p_{W^{s,p}(K;\mathbb{R}^q)}, \tag{18.2}$$

if $p \in [1, \infty)$ and $\|v\|_{W^{s,\infty}(\mathcal{T}_h;\mathbb{R}^q)} := \max_{K \in \mathcal{T}_h} \|v\|_{W^{s,\infty}(K;\mathbb{R}^q)}$ if $p = \infty$. We write $W^{s,p}(\mathcal{T}_h) := W^{s,p}(\mathcal{T}_h; \mathbb{R})$ when $q = 1$.

An important notion in broken Sobolev spaces is the jump of functions across mesh interfaces (see Figure 18.1). Recall from the Definition 8.10 that the collection of the mesh interfaces is denoted by $\mathcal{F}_h^\circ$ and that for all $F \in \mathcal{F}_h^\circ$, there are two distinct mesh cells $K_l, K_r \in \mathcal{T}_h$ such that $F = \partial K_l \cap \partial K_r$. The interface F is oriented by means of the unit normal vector $\boldsymbol{n}_F$ pointing from K_l to K_r.

Definition 18.2 (Jump). *Let $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ be a mesh interface. Let $v \in W^{s,p}(\mathcal{T}_h; \mathbb{R}^q)$ with $s > \frac{1}{p}$ if $p \in (1, \infty)$ or $s \geq 1$ if $p = 1$ (notice that $(v_{|K_l})_{|F} \in L^1(F)$ and $(v_{|K_r})_{|F} \in L^1(F)$). The* jump *of v across F is defined as follows a.e. in F:*

$$[\![v]\!]_F := v_{|K_l} - v_{|K_r}. \tag{18.3}$$

The subscript F is dropped when the context is unambiguous.

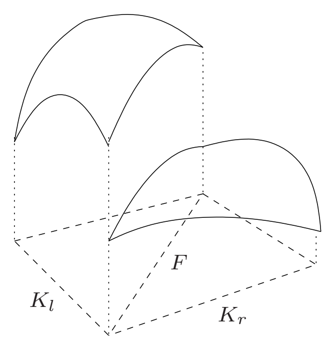

Fig. 18.1 Jump of a piecewise smooth function across the interface $F := \partial K_l \cap \partial K_r$.

Remark 18.3 (Alternative definition). Another definition of the jump where K_l, K_r play symmetric roles consists of setting $[\![v]\!]^*_F := v_{|K_l} \otimes \boldsymbol{n}_{K_l|F} + v_{|K_r} \otimes \boldsymbol{n}_{K_r|F}$, where $\boldsymbol{n}_{K_i|F}$, $i \in \{l, r\}$, is the unit normal to F pointing away from K_i, i.e., $[\![v]\!]_F \otimes \boldsymbol{n}_F = [\![v]\!]^*_F$. The advantage of (18.3) over this definition is that the jump $[\![v]\!]_F$ is $\mathbb{R}^q$-valued instead of being $\mathbb{R}^{q \times d}$-valued. Both definitions are commonly used in the literature. □

Remark 18.4 (Zero-jumps in $W^{s,p}$). Let $p \in (1,\infty)$ and $s > \frac{1}{p}$, or $p = 1$ and $s \geq \frac{1}{p}$. Owing to Theorem 2.21, smooth functions are dense in $W^{s,p}(D)$. Let $v \in W^{s,p}(D)$ and let $(v_n)_{n\in\mathbb{N}}$ be a sequence in $C^\infty(D)\cap W^{s,p}(D)$ converging to v in $W^{s,p}(D)$. Let $F \in \mathcal{F}_h^\circ$ be a mesh interface. Then $0 = [\![v_n]\!]_F \to [\![v]\!]_F$ as $n \to \infty$ since the trace map is bounded on $W^{s,p}(D)$. Hence, $0 = [\![v]\!]_F$ for all $F \in \mathcal{F}_h^\circ$. This shows that functions in $W^{s,p}(D)$ have a single-valued trace in $L^1(F)$ for all $F \in \mathcal{F}_h^\circ$. □

18.1.2 Broken finite element spaces

Let $(\widehat{K}, \widehat{P}, \widehat{\Sigma})$ be the reference finite element of degree $k \geq 0$, where $\widehat{P}$ is composed of $\mathbb{R}^q$-valued functions for some integer $q \geq 1$. We assume that $\widehat{P} \subset L^\infty(\widehat{K};\mathbb{R}^q)$ (this is a mild assumption since in general $\widehat{P}$ is composed of polynomial functions). Consider a $\mathcal{T}_h$-based family of finite elements $\{(K, P_K, \Sigma_K)\}_{K\in\mathcal{T}_h}$ constructed as in Proposition 9.2 by using the geometric mappings $\boldsymbol{T}_K : \widehat{K} \to K$ and the transformations $\psi_K : V(K) \to V(\widehat{K})$ for all $K \in \mathcal{T}_h$. We assume henceforth that $\psi_K \in \mathcal{L}(L^\infty(K;\mathbb{R}^q), L^\infty(\widehat{K};\mathbb{R}^q))$. Recall that we denote by $\{\theta_{K,i}\}_{i\in\mathcal{N}}$ the local shape functions in K and by $\{\sigma_{K,i}\}_{i\in\mathcal{N}}$ the local degrees of freedom (dofs).

Definition 18.5 (Broken finite element space). *The* broken finite element space *is defined as follows:*

$$P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q) := \{v_h \in L^\infty(D;\mathbb{R}^q) \mid \psi_K(v_{h|K}) \in \widehat{P}, \forall K \in \mathcal{T}_h\}. \tag{18.4}$$

We simply write $P_k^{\mathrm{b}}(\mathcal{T}_h)$ whenever $q = 1$.

Recalling that $P_K := \psi_K^{-1}(\widehat{P})$ (see (9.4a)), we have $v_h \in P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ iff $v_{h|K} \in P_K$ for all $K \in \mathcal{T}_h$. The above assumptions on $\widehat{P}$ and ψ_K imply that $P_K \subset L^\infty(K;\mathbb{R}^q)$, which in turn means that $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ is indeed a subspace of $L^\infty(D;\mathbb{R}^q)$. Moreover, since functions in $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ can be defined independently in each mesh cell, we have

$$\dim(P^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)) = \mathrm{card}(\mathcal{N}) \times \mathrm{card}(\mathcal{T}_h) =: n_{\mathrm{sh}} \times N_{\mathrm{c}}, \tag{18.5}$$

where n_{sh} is the number of dofs in $\widehat{\Sigma}$ (i.e., the cardinality of the set $\mathcal{N}$), and N_{c} is the number of mesh cells in $\mathcal{T}_h$. Then the set $\{\tilde{\theta}_{K,i}\}_{(K,i)\in\mathcal{T}_h\times\mathcal{N}}$, where $\tilde{\theta}_{K,i}$ is the zero-extension of $\theta_{K,i}$ to D, is a basis of $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$. The functions $\tilde{\theta}_{K,i}$ are called *global shape functions* in $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$.

Example 18.6 (Piecewise polynomials). On affine meshes the choice $\widehat{P} := \mathbb{P}_{k,d}$ (resp., $\widehat{P} := \mathbb{Q}_{k,d}$) together with $\psi_K(v) := v \circ \boldsymbol{T}_K$ and $q := 1$ (i.e., scalar-valued functions) leads to $P_k^{\mathrm{b}}(\mathcal{T}_h) = \{v_h \in L^\infty(D) \mid v_{h|K} \in \mathbb{P}_{k,d}, \forall K \in \mathcal{T}_h\}$ (resp., $\{v_h \in L^\infty(D) \mid v_{h|K} \in \mathbb{Q}_{k,d}, \forall K \in \mathcal{T}_h\}$) since $v_{h|K} \in \mathbb{P}_{k,d}$ iff $v_h \circ \boldsymbol{T}_K \in \mathbb{P}_{k,d}$ (resp., $\mathbb{Q}_{k,d}$). □

Remark 18.7 (Connectivity array). In practice, the global shape functions are enumerated, say from 1 to I. For the broken finite element space, we have $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q) = \mathrm{span}\{\varphi_1,\ldots,\varphi_I\}$ with $I = n_{\mathrm{sh}}N_{\mathrm{c}}$. The connection between the local and the global shape functions is materialized by a connectivity array `j_dof` : $\{1{:}N_{\mathrm{c}}\}\times\mathcal{N} \to \{1{:}I\}$ defined such that $\varphi_{\mathtt{j\_dof}(m,n)|K_m} := \theta_{K_m,i}$ for all $m \in \{1{:}N_{\mathrm{c}}\}$ and all $n \in \mathcal{N}$. The most common approach to define `j_dof` consists of enumerating first the dofs in the first cell, then in the second cell, and so on, leading to $\mathtt{j\_dof}(m,n) := (m-1)n_{\mathrm{sh}} + n$. $\square$

18.2 Conforming finite element subspaces

Given a piecewise smooth function on the mesh $\mathcal{T}_h$, either scalar- or vector-valued, depending on the context, we want to find necessary and sufficient conditions for this function to be in $H^1(D)$, $\boldsymbol{H}(\mathrm{curl};D)$, or $\boldsymbol{H}(\mathrm{div};D)$. It turns out that the answer to this question hinges on the continuity properties of the function, its normal component, or its tangential component across the mesh interfaces.

18.2.1 Membership in H^1

The global integrability of the gradient of a piecewise smooth function is characterized by the following result.

Theorem 18.8 (Integrability of ∇). *Let $v \in W^{1,p}(\mathcal{T}_h;\mathbb{R}^q)$ with $p \in [1,\infty]$. Then $\nabla v \in \boldsymbol{L}^p(D)$ iff $[\![v]\!]_F = 0$ a.e. on all $F \in \mathcal{F}_h^{\circ}$.*

Proof. We prove the assertion for $q = 1$. The general case is treated by working componentwise. Let $v \in W^{1,p}(\mathcal{T}_h)$ and let $C_0^\infty(D)$ be the set of the smooth functions compactly supported in D. For all $\boldsymbol{\Phi} \in \boldsymbol{C}_0^\infty(D)$, we have

$$\begin{aligned}
\int_D v\nabla{\cdot}\boldsymbol{\Phi}\,\mathrm{d}x &= \sum_{K\in\mathcal{T}_h}\int_K v_{|K}\nabla{\cdot}\boldsymbol{\Phi}\,\mathrm{d}x \\
&= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(v_{|K}){\cdot}\boldsymbol{\Phi}\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\int_{\partial K} v_{|K}\boldsymbol{n}_K{\cdot}\boldsymbol{\Phi}\,\mathrm{d}s \\
&= -\sum_{K\in\mathcal{T}_h}\int_K \nabla(v_{|K}){\cdot}\boldsymbol{\Phi}\,\mathrm{d}x + \sum_{F\in\mathcal{F}_h^{\circ}}\int_F [\![v]\!]_F\boldsymbol{n}_F{\cdot}\boldsymbol{\Phi}\,\mathrm{d}s,
\end{aligned}$$

where $\boldsymbol{n}_K$ is the outward unit normal to K and $\boldsymbol{n}_F$ is the unit vector defining the orientation of F.
(i) If $[\![v]\!]_F = 0$ a.e. on all $F \in \mathcal{F}_h^{\circ}$, we infer from the above identity that $\int_D v\nabla{\cdot}\boldsymbol{\Phi}\,\mathrm{d}x = -\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\Phi}{\cdot}\nabla(v_{|K})\,\mathrm{d}x$, which shows that v has a weak gradient in $\boldsymbol{L}^p(D)$ s.t. $(\nabla v)_{|K} = \nabla(v_{|K})$ for all $K \in \mathcal{T}_h$. Hence, $v \in W^{1,p}(D)$.
(ii) Conversely let $v \in W^{1,p}(D)$. We can conclude by invoking Remark 18.4.

Let us give a more direct proof. Owing to Lemma 18.9 below, we infer that $(\nabla v)_{|K} = \nabla(v_{|K})$ for all $K \in \mathcal{T}_h$. Hence, the above identity implies that $\sum_{F\in\mathcal{F}_h^\circ} \int_F [\![v]\!]_F \boldsymbol{n}_F{\cdot}\boldsymbol{\Phi}\,\mathrm{d}s = 0$ for all $\boldsymbol{\Phi} \in \boldsymbol{C}_0^\infty(D)$. Let $F \in \mathcal{F}_h^\circ$ be an arbitrary interface. After localizing the support of $\boldsymbol{\Phi}$ in such a way that it intersects F and no other interface in $\mathcal{F}_h^\circ$, it follows from the vanishing integral theorem (Theorem 1.32) that $[\![v]\!]_F = 0$, since $\boldsymbol{\Phi}_{|F}{\cdot}\boldsymbol{n}_F$ can be arbitrarily chosen, and $[\![v]\!]_F \in L^1(F)$ because the trace map is bounded on $W^{1,p}(D)$. □

Lemma 18.9 (Local weak derivative). *Let $p \in [1,\infty]$ and let $v \in W^{1,p}(D)$. Then $\nabla(v_{|K}) = (\nabla v)_{|K}$ a.e. in K for all $K \in \mathcal{T}_h$.*

Proof. Let $K \in \mathcal{T}_h$ and let $\boldsymbol{\phi} \in \boldsymbol{C}_0^\infty(K)$. Let $\widetilde{\boldsymbol{\phi}} \in \boldsymbol{C}_0^\infty(D)$ be the zero-extension of $\boldsymbol{\phi}$ to D. For all $v \in W^{1,p}(D)$, we infer that

$$\begin{aligned}\int_K \nabla(v_{|K}){\cdot}\boldsymbol{\phi}\,\mathrm{d}x &= -\int_K v_{|K}\nabla{\cdot}\boldsymbol{\phi}\,\mathrm{d}x \\ &= -\int_D v\nabla{\cdot}\widetilde{\boldsymbol{\phi}}\,\mathrm{d}x = \int_D \nabla v{\cdot}\widetilde{\boldsymbol{\phi}}\,\mathrm{d}x = \int_K (\nabla v)_{|K}{\cdot}\boldsymbol{\phi}\,\mathrm{d}x.\end{aligned}$$

The assertion follows from Theorem 1.32 since $\boldsymbol{\phi}$ is arbitrary in $\boldsymbol{C}_0^\infty(K)$. □

Figure 18.2 illustrates Theorem 18.8 in dimension one.

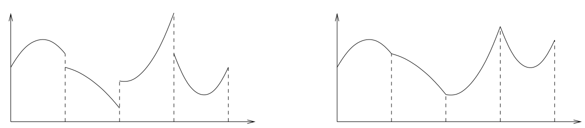

Fig. 18.2 One-dimensional example with two piecewise quadratic functions. The one on the left is not in H^1, the one on the right is.

18.2.2 Membership in $\boldsymbol{H}(\mathrm{curl})$ and $\boldsymbol{H}(\mathrm{div})$

Let us now consider the integrability of the curl or the divergence of vector-valued piecewise smooth functions. Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\mathcal{T}_h) := W^{1,p}(\mathcal{T}_h;\mathbb{R}^d)$, $p \in [1,\infty)$. We also use the notation $\boldsymbol{W}^{s,p}(\mathcal{T}_h) := W^{s,p}(\mathcal{T}_h;\mathbb{R}^d)$, $s > 0$. The jump of the tangential component of $\boldsymbol{v}$ (if $d = 3$) and the jump of its normal component across a mesh interface $F \in \mathcal{F}_h^\circ$, with $F := \partial K_l \cap \partial K_r$, are defined as follows a.e. in F:

$$[\![\boldsymbol{v}{\times}\boldsymbol{n}]\!]_F := (\boldsymbol{v}_{|K_l}{\times}\boldsymbol{n}_F) - (\boldsymbol{v}_{|K_r}{\times}\boldsymbol{n}_F) = [\![\boldsymbol{v}]\!]_F{\times}\boldsymbol{n}_F, \tag{18.6a}$$

$$[\![\boldsymbol{v}{\cdot}\boldsymbol{n}]\!]_F := (\boldsymbol{v}_{|K_l}{\cdot}\boldsymbol{n}_F) - (\boldsymbol{v}_{|K_r}{\cdot}\boldsymbol{n}_F) = [\![\boldsymbol{v}]\!]_F{\cdot}\boldsymbol{n}_F, \tag{18.6b}$$

where $[\![\boldsymbol{v}]\!]_F$ is the componentwise jump of $\boldsymbol{v}$ across F from Definition 18.2. The subscript F is dropped when the context is unambiguous.

Theorem 18.10 (Integrability of $\nabla\times$ and $\nabla\cdot$). *Let $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(\mathcal{T}_h)$ with $p \in [1,\infty]$.* (i) *If $d = 3$, $\nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)$ if and only if $[\![\boldsymbol{v}\times\boldsymbol{n}]\!]_F = 0$ a.e. on all $F \in \mathcal{F}_h^\circ$.* (ii) *$\nabla\cdot\boldsymbol{v} \in L^p(D)$ if and only if $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_F = 0$ a.e. on all $F \in \mathcal{F}_h^\circ$.*

Proof. Proceed as in the proof of Theorem 18.8. See Exercise 18.1. □

Remark 18.11 (Extension). The statement of Theorem 18.10 can be extended to functions $\boldsymbol{v} \in \boldsymbol{W}^{s,p}(\mathcal{T}_h)$ with $s > \frac{1}{p}$ if $p \in (1,\infty)$ or $s \geq 1$ if $p = 1$. The following holds true: (i) If $d = 3$ and $\nabla\times(\boldsymbol{v}_{|K}) \in \boldsymbol{L}^p(K)$ for all $K \in \mathcal{T}_h$, then $\nabla\times\boldsymbol{v} \in \boldsymbol{L}^p(D)$ iff $[\![\boldsymbol{v}\times\boldsymbol{n}]\!]_F = 0$ for all $F \in \mathcal{F}_h^\circ$. (ii) If $\nabla\cdot(\boldsymbol{v}_{|K}) \in L^p(K)$ for all $K \in \mathcal{T}_h$, then $\nabla\cdot\boldsymbol{v} \in L^p(D)$ iff $[\![\boldsymbol{v}\cdot\boldsymbol{n}]\!]_F = 0$ for all $F \in \mathcal{F}_h^\circ$. □

18.2.3 Unified notation for conforming subspaces

To allow for a unified treatment of H^1-, $\boldsymbol{H}(\mathrm{curl})$-, and $\boldsymbol{H}(\mathrm{div})$-conformity, we use the superscript $\mathrm{x} \in \{\mathrm{g},\mathrm{c},\mathrm{d}\}$ (referring to the gradient, curl, and divergence operators), and we consider $\mathbb{R}^q$-valued functions with $q := 1$ if $\mathrm{x} = \mathrm{g}$, $q = d = 3$ if $\mathrm{x} = \mathrm{c}$, and $q = d$ if $\mathrm{x} = \mathrm{d}$. Let $p \in [1,\infty)$ and let $s > \frac{1}{p}$ if $p > 1$ or $s \geq 1$ if $p = 1$. Let $K \in \mathcal{T}_h$ be a mesh cell and let $F \in \mathcal{F}_K$ be a face of K. We define the local trace operators $\gamma^{\mathrm{x}}_{K,F} : W^{s,p}(K;\mathbb{R}^q) \to L^1(F;\mathbb{R}^t)$ s.t.

$$\gamma^{\mathrm{g}}_{K,F}(v) := v_{|F} \qquad (q = t = 1), \tag{18.7a}$$

$$\gamma^{\mathrm{c}}_{K,F}(\boldsymbol{v}) := \boldsymbol{v}_{|F}\times\boldsymbol{n}_F \qquad (q = t = d = 3), \tag{18.7b}$$

$$\gamma^{\mathrm{d}}_{K,F}(\boldsymbol{v}) := \boldsymbol{v}_{|F}\cdot\boldsymbol{n}_F \qquad (q = d,\ t = 1). \tag{18.7c}$$

This leads to the following notion of γ-jump: For all $v \in W^{s,p}(\mathcal{T}_h;\mathbb{R}^q)$,

$$[\![v]\!]^{\mathrm{x}}_F(\boldsymbol{x}) := \gamma^{\mathrm{x}}_{K_l,F}(v_{|K_l})(\boldsymbol{x}) - \gamma^{\mathrm{x}}_{K_r,F}(v_{|K_r})(\boldsymbol{x}) \qquad \text{a.e. on } F. \tag{18.8}$$

Let $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \Sigma^{\mathrm{g}})$ be one of the Lagrange elements or the canonical hybrid element introduced in Chapters 6 and 7. Let $k \geq 1$ be the degree of the finite element. The corresponding broken finite element space is

$$P^{\mathrm{g,b}}_k(\mathcal{T}_h) := \{v_h \in L^\infty(D) \mid \psi^{\mathrm{g}}_K(v_{h|K}) \in \widehat{P}^{\mathrm{g}}, \forall K \in \mathcal{T}_h\}, \tag{18.9}$$

where $\psi^{\mathrm{g}}_K(v) := v \circ \boldsymbol{T}_K$ is the pullback by the geometric mapping $\boldsymbol{T}_K$. The H^1-conforming finite element subspace is defined as follows:

$$P^{\mathrm{g}}_k(\mathcal{T}_h) := P^{\mathrm{g,b}}_k(\mathcal{T}_h) \cap H^1(D). \tag{18.10}$$

Similarly, let $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{c}}, \Sigma^{\mathrm{c}})$ be one of the Nédélec elements introduced in Chapter 15, and let $(\widehat{K}, \widehat{\boldsymbol{P}}^{\mathrm{d}}, \Sigma^{\mathrm{d}})$ be one of the Raviart–Thomas elements introduced in Chapter 14. Let $k \geq 0$ be the degree of the finite element. The corresponding broken finite element spaces are

$$\boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^\infty(D) \mid \boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{c}},\ \forall K \in \mathcal{T}_h\}, \tag{18.11a}$$
$$\boldsymbol{P}_k^{\mathrm{d,b}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^\infty(D) \mid \boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{d}},\ \forall K \in \mathcal{T}_h\}, \tag{18.11b}$$

where $\boldsymbol{\psi}_K^{\mathrm{c}}(\boldsymbol{v}) := \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K)$ is the covariant Piola transformation and $\boldsymbol{\psi}_K^{\mathrm{d}}(\boldsymbol{v}) := \det(\mathbb{J}_K)\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K)$ is the contravariant Piola transformation. The corresponding $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element subspaces are defined as follows:

$$\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{curl}; D), \tag{18.12a}$$
$$\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) := \boldsymbol{P}_k^{\mathrm{d,b}}(\mathcal{T}_h) \cap \boldsymbol{H}(\mathrm{div}; D). \tag{18.12b}$$

The zero-jump conditions from Theorem 18.8 and Theorem 18.10 imply that

$$P_k^{\mathrm{g}}(\mathcal{T}_h) = \{v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h) \mid [\![v_h]\!]_F^{\mathrm{g}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}, \tag{18.13a}$$
$$\boldsymbol{P}_k^{\mathrm{c}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{c,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = \boldsymbol{0},\ \forall F \in \mathcal{F}_h^\circ\}, \tag{18.13b}$$
$$\boldsymbol{P}_k^{\mathrm{d}}(\mathcal{T}_h) = \{\boldsymbol{v}_h \in \boldsymbol{P}_k^{\mathrm{d,b}}(\mathcal{T}_h) \mid [\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}. \tag{18.13c}$$

In the next chapters, we study the construction and the interpolation properties of the above conforming finite element subspaces. To stay general, we employ the following unified notation with $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}\}$:

$$P_k^{\mathrm{x}}(\mathcal{T}_h; \mathbb{R}^q) := \{v_h \in \boldsymbol{P}_k^{\mathrm{x,b}}(\mathcal{T}_h; \mathbb{R}^q) \mid [\![v_h]\!]_F^{\mathrm{x}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}, \tag{18.14}$$

where $\boldsymbol{P}_k^{\mathrm{x,b}}(\mathcal{T}_h; \mathbb{R}^q)$ is one of the broken finite element spaces defined above.

Remark 18.12 (2D discrete Sobolev inequality). We have $P_k^{\mathrm{g}}(\mathcal{T}_h) \subset L^\infty(D) \cap H^1(D)$ by construction, but as shown in Example 2.33, if $d \geq 2$, there exist functions in $H^1(D)$ that are unbounded. It turns out that in dimension two, it is possible to derive a bound on the $\|\cdot\|_{L^\infty}$-norm of functions in $P_k^{\mathrm{g}}(\mathcal{T}_h)$ that blows up very mildly w.r.t. the meshsize. This bound involves a global length scale associated with D, say δ_D. More precisely, since D is Lipschitz, one can show that there exist a length scale $\delta_D > 0$ and an angle $\omega \in (0, 2\pi)$ such that any point $\boldsymbol{x} \in D$ is the vertex of a cone $\mathfrak{C}(\boldsymbol{x}) \subset D$, where $\mathfrak{C}(\boldsymbol{x})$ is the image by a translation and rotation of the cone $\mathfrak{C} := \{(r, \theta) \mid r \in (0, \delta_D),\ \theta \in (0, \omega)\}$ defined in polar coordinates; see Lemma 3.4. Then assuming $d := 2$, one can show (see Exercise 18.2 and Bramble et al. [42]) the following inverse inequality, called *discrete Sobolev inequality*: There is $c > 0$ s.t.

$$c\,\delta_D^{-\frac{1}{2}} \|v_h\|_{L^\infty(K)} \leq \delta_D^{-1} \|v_h\|_{L^2(D)} + \ln\left(\frac{\delta_D}{h_K}\right)^{\frac{1}{2}} \|\nabla v_h\|_{\boldsymbol{L}^2(D)}, \tag{18.15}$$

for all $v_h \in P_k^{\mathrm{g}}(\mathcal{T}_h)$, all $K \in \mathcal{T}_h$ such that $h_K \leq \frac{1}{2}\delta_D$, and all $h \in \mathcal{H}$. $\square$

18.3 L^1-stable local interpolation

In this section, we devise a local interpolation operator that is L^1-stable and maps $L^1(D)$ onto the broken finite element space $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ defined in (18.4). The construction is local in each mesh cell. The key idea is to extend the dofs of the reference finite element so as to be able to interpolate boundedly all the functions that are in $L^1(D)$.

We assume that the geometric mappings $\boldsymbol{T}_K$ are affine for all $K \in \mathcal{T}_h$, and that all the transformations ψ_K are of the form $\psi_K(v) := \mathbb{A}_K(v \circ \boldsymbol{T}_K)$ (see (11.1)) where $\mathbb{A}_K \in \mathbb{R}^{q\times q}$ satisfies (see (11.12))

$$\|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2} \le c\,\|\mathbb{J}_K\|_{\ell^2}\|\mathbb{J}_K^{-1}\|_{\ell^2}, \tag{18.16}$$

with c uniform w.r.t. $K \in \mathcal{T}_h$ and $h \in \mathcal{H}$, where $\mathbb{J}_K$ is the Jacobian matrix of $\boldsymbol{T}_K$. Let us define the adjoint transformation $\phi_K(w) := \mathbb{B}_K(w \circ \boldsymbol{T}_K)$ where $\mathbb{B}_K := |\det(\mathbb{J}_K)|\mathbb{A}_K^{-\mathsf{T}}$. The terminology is motivated by the following identity:

$$(w,v)_{L^2(K;\mathbb{R}^q)} = (\phi_K(w),\psi_K(v))_{L^2(\widehat{K};\mathbb{R}^q)}, \tag{18.17}$$

for all $v \in L^p(K;\mathbb{R}^q)$, all $w \in L^{p'}(K;\mathbb{R}^q)$, and all $p \in [1,\infty]$ with $\frac{1}{p}+\frac{1}{p'}=1$. Indeed, we have

$$\begin{aligned}(\phi_K(w),\psi_K(v))_{L^2(\widehat{K};\mathbb{R}^q)} &= \int_{\widehat{K}} |\det(\mathbb{J}_K)|(\mathbb{A}_K^{-\mathsf{T}}(w\circ \boldsymbol{T}_K),\mathbb{A}_K(v\circ \boldsymbol{T}_K))_{\ell^2(\mathbb{R}^q)}\,\mathrm{d}\widehat{x}\\ &= \int_K (w\circ \boldsymbol{T}_K, v\circ \boldsymbol{T}_K)_{\ell^2(\mathbb{R}^q)}\,\mathrm{d}x = (w,v)_{L^2(K;\mathbb{R}^q)}.\end{aligned}$$

Moreover, we have $\|\mathbb{B}_K\|_{\ell^2}\|\mathbb{B}_K^{-1}\|_{\ell^2} = \|\mathbb{A}_K\|_{\ell^2}\|\mathbb{A}_K^{-1}\|_{\ell^2}$ since $\|\mathbb{A}_K^{\mathsf{T}}\|_{\ell^2} = \|\mathbb{A}_K\|_{\ell^2}$.

We first extend the dofs of the reference finite element. Let $\widehat{\rho}_i \in \widehat{P}$ for all $i \in \mathcal{N}$ be such that

$$\frac{1}{|\widehat{K}|}(\widehat{\rho}_i,\widehat{p})_{L^2(\widehat{K};\mathbb{R}^q)} := \widehat{\sigma}_i(\widehat{p}), \qquad \forall \widehat{p} \in \widehat{P}. \tag{18.18}$$

The function $\widehat{\rho}_i$ is well defined owing to the Riesz–Fréchet theorem (see either Exercise 5.9 or Theorem A.16 applied here in the finite-dimensional space $\widehat{P}$ equipped with the L^2-inner product weighted by $|\widehat{K}|^{-1}$). This leads us to define the extended dofs as follows:

$$\widehat{\sigma}_i^\sharp(\widehat{v}) := \frac{1}{|\widehat{K}|}(\widehat{\rho}_i,\widehat{v})_{L^2(\widehat{K};\mathbb{R}^q)}, \qquad \forall \widehat{v} \in L^1(\widehat{K};\mathbb{R}^q). \tag{18.19}$$

We then define the interpolation operator s.t. for all $\widehat{\boldsymbol{x}} \in \widehat{K}$,

$$\mathcal{I}^\sharp_{\widehat{K}}(\widehat{v})(\widehat{\boldsymbol{x}}) := \sum_{i\in\mathcal{N}} \widehat{\sigma}^\sharp_i(\widehat{v})\widehat{\theta}_i(\widehat{\boldsymbol{x}}), \qquad \forall \widehat{v} \in L^1(\widehat{K};\mathbb{R}^q). \tag{18.20}$$

We can take $V(\widehat{K}) := L^1(\widehat{K};\mathbb{R}^q)$ for the domain of $\mathcal{I}^\sharp_{\widehat{K}}$. One can show that $\mathcal{I}^\sharp_{\widehat{K}}$ is actually the L^2-orthogonal projection onto $\widehat{P}$; see Exercise 18.3.

Lemma 18.13 (Invariance and stability). *Let $\mathcal{I}^\sharp_{\widehat{K}}$ be defined in* (18.20). (i) $\widehat{P}$ *is pointwise invariant under $\mathcal{I}^\sharp_{\widehat{K}}$.* (ii) $\mathcal{I}^\sharp_{\widehat{K}}$ *is L^p-stable for all $p \in [1,\infty]$, i.e., there is $\widehat{c}$ s.t.*

$$\|\mathcal{I}^\sharp_{\widehat{K}}(\widehat{v})\|_{L^p(\widehat{K};\mathbb{R}^q)} \le \widehat{c}\,\|\widehat{v}\|_{L^p(\widehat{K};\mathbb{R}^q)}, \qquad \forall \widehat{v} \in L^p(\widehat{K};\mathbb{R}^q). \tag{18.21}$$

Proof. (i) Since $\widehat{\sigma}^\sharp_i(\widehat{p}) = \widehat{\sigma}_i(\widehat{p})$ for all $\widehat{p} \in \widehat{P}$ and all $i \in \mathcal{N}$, we obtain $\mathcal{I}^\sharp_{\widehat{K}}(\widehat{p}) = \sum_{i\in\mathcal{N}} \widehat{\sigma}_i(\widehat{p})\widehat{\theta}_i = \widehat{p}$. (ii) Since $\widehat{P} \subset L^\infty(\widehat{K};\mathbb{R}^q)$, we have $\widehat{\rho}_i \in L^\infty(\widehat{K};\mathbb{R}^q)$. Hölder's inequality implies that

$$|\widehat{\sigma}^\sharp_i(\widehat{v})| \le |\widehat{K}|^{-\frac{1}{p}}\|\widehat{\rho}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}\|\widehat{v}\|_{L^p(\widehat{K};\mathbb{R}^q)},$$

for all $\widehat{v} \in L^p(\widehat{K};\mathbb{R}^q)$. We conclude that (18.21) holds true with $\widehat{c} := \sum_{i\in\mathcal{N}} |\widehat{K}|^{-\frac{1}{p}}\|\widehat{\rho}_i\|_{L^\infty(\widehat{K};\mathbb{R}^q)}\|\widehat{\theta}_i\|_{L^p(\widehat{K};\mathbb{R}^q)}$. $\square$

Consider now a mesh cell $K \in \mathcal{T}_h$ from a shape-regular mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ and let (K, P_K, Σ_K) be the finite element generated in K using the transformation ψ_K (see Proposition 9.2). The assumption $\psi_K(v) = \mathbb{A}_K(v \circ \boldsymbol{T}_K)$ implies that $\psi_K^{-1}(L^1(\widehat{K};\mathbb{R}^q)) = L^1(K;\mathbb{R}^q)$. We extend the dofs in Σ_K to $L^1(K;\mathbb{R}^q)$ by setting $\sigma^\sharp_{K,i}(v) := \widehat{\sigma}^\sharp_i(\psi_K(v))$, i.e., owing to (18.17),

$$\sigma^\sharp_{K,i}(v) = \frac{1}{|\widehat{K}|}(\widehat{\rho}_i, \psi_K(v))_{L^2(\widehat{K};\mathbb{R}^q)} = \frac{1}{|\widehat{K}|}(\phi_K^{-1}(\widehat{\rho}_i), v)_{L^2(K;\mathbb{R}^q)}, \tag{18.22}$$

and we define the local interpolation operator in K s.t. for all $\boldsymbol{x} \in K$,

$$\mathcal{I}^\sharp_K(v)(\boldsymbol{x}) := \sum_{i\in\mathcal{N}} \sigma^\sharp_{K,i}(v)\theta_{K,i}(\boldsymbol{x}), \qquad \forall v \in V(K) := L^1(K;\mathbb{R}^q), \tag{18.23}$$

recalling that the local shape functions are given by $\theta_{K,i} := \psi_K^{-1}(\widehat{\theta}_i)$ for all $i \in \mathcal{N}$. The linearity of ψ_K implies that

$$\psi_K\left(\mathcal{I}^\sharp_K(v)\right) := \psi_K\left(\sum_{i\in\mathcal{N}} \sigma^\sharp_{K,i}(v)\psi_K^{-1}(\widehat{\theta}_i)\right) = \sum_{i\in\mathcal{N}} \widehat{\sigma}^\sharp_i(\psi_K(v))\widehat{\theta}_i = \mathcal{I}^\sharp_{\widehat{K}}(\psi_K(v)).$$

In other words, the following key relation holds true:

$$\mathcal{I}^\sharp_K = \psi_K^{-1} \circ \mathcal{I}^\sharp_{\widehat{K}} \circ \psi_K. \tag{18.24}$$

One can show that $\mathcal{I}_K^\sharp$ is the oblique projection onto $P_K = \psi_K^{-1}(\widehat{P})$ parallel to $Q_K^\perp$ with $Q_K := \Phi_K^{-1}(\widehat{P})$. Note that $\mathcal{I}_K^\sharp$ is L^2-orthogonal whenever the matrix $\mathbb{A}_K$ is unitary; see Exercise 18.3.

Theorem 18.14 (Local approximation). *Let $\mathcal{I}_K^\sharp$ be defined by* (18.23). *Let k be the degree of the finite element, i.e., $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)$.* (i) *$P_K$ is pointwise invariant under $\mathcal{I}_K^\sharp$.* (ii) *Assuming that the mesh sequence is shape-regular, there is c s.t. for all $r \in [0,k+1]$, all $p \in [1,\infty)$ if $r \notin \mathbb{N}$ or all $p \in [1,\infty]$ if $r \in \mathbb{N}$, every integer $m \in \{0{:}\lfloor r \rfloor\}$, all $v \in W^{r,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$|v - \mathcal{I}_K^\sharp(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K;\mathbb{R}^q)}. \tag{18.25}$$

Proof. The property (i) follows from (18.24). The property (ii) for $r \in \mathbb{N}$ follows from Theorem 11.13 with $l := 0$ since $\mathcal{I}_{\widehat{K}}^\sharp$ is stable in L^p owing to Lemma 18.13. Taking $m := r$ in (18.25) implies the $W^{m,p}$-stability of $\mathcal{I}_K^\sharp$ for every integer $m \in \{0{:}k+1\}$, i.e.,

$$|\mathcal{I}_K^\sharp(w)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, |w|_{W^{m,p}(K;\mathbb{R}^q)}, \qquad \forall w \in W^{m,p}(K;\mathbb{R}^q). \tag{18.26}$$

Since $\mathcal{I}_K^\sharp(g) = g$ for all $g \in P_K$, (18.26) and the triangle inequality yield

$$\begin{aligned} |v - \mathcal{I}_K^\sharp(v)|_{W^{m,p}(K;\mathbb{R}^q)} &= \inf_{q \in P_K} |v - g - \mathcal{I}_K^\sharp(v-g)|_{W^{m,p}(K;\mathbb{R}^q)} \\ &\le c \inf_{q \in P_K} |v - g|_{W^{m,p}(K;\mathbb{R}^q)}. \end{aligned}$$

Invoking the bound (12.18) on $\inf_{q\in P_K} |v-g|_{W^{m,p}(K;\mathbb{R}^q)}$, we infer that the property (ii) holds true for all $r \notin \mathbb{N}$ as well. □

Corollary 18.15 (Approximation on faces). (i) *Let $p \in [1,\infty)$ and $r \in (\frac{1}{p}, k+1]$ if $p > 1$ or $r \in [1,k+1]$ if $p = 1$. There is c s.t.*

$$\|v - \mathcal{I}_K^\sharp(v)\|_{L^p(F;\mathbb{R}^q)} \le c\, h_K^{r-\frac{1}{p}} |v|_{W^{r,p}(K;\mathbb{R}^q)}, \tag{18.27}$$

for all $v \in W^{r,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, all $F \in \mathcal{F}_K$, and all $h \in \mathcal{H}$, where the constant c grows unboundedly as $rp \downarrow 1$ if $p > 1$. (ii) *Assume $k \ge 1$. Let $p \in [1,\infty)$ and $r \in (\frac{1}{p}, k]$ if $p > 1$ or $r \in [1,k]$ if $p = 1$. There is c s.t.*

$$\|\nabla(v - \mathcal{I}_K^\sharp(v))\|_{\boldsymbol{L}^p(F;\mathbb{R}^q)} \le c\, h_K^{r-\frac{1}{p}} |v|_{W^{1+r,p}(K;\mathbb{R}^q)}, \tag{18.28}$$

for all $v \in W^{1+r,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$, where the constant c grows unboundedly as $rp \downarrow 1$ if $p > 1$.

Proof. For simplicity, we assume that $q = 1$. The general case is treated by reasoning componentwise. Let us prove (18.27). Assume first that $r \in$

$[1, k+1]$. Owing to the multiplicative trace inequality (12.16), we infer that, with $\eta := v - \mathcal{I}_K^\sharp(v)$,

$$\|\eta\|_{L^p(F)} \leq c\left(h_K^{-\frac{1}{p}}\|\eta\|_{L^p(K)} + \|\eta\|_{L^p(K)}^{1-\frac{1}{p}}\|\nabla\eta\|_{\boldsymbol{L}^p(K)}^{\frac{1}{p}}\right).$$

Invoking (18.25) with $m \in \{0,1\}$ (note that $m \leq \lfloor r \rfloor$) shows that (18.27) holds true in this case. Let us now assume that $r \in (\frac{1}{p}, 1)$ with $p > 1$. Let $q_0 \in \psi_K^{-1}(\mathbb{P}_{0,d}) = \mathbb{P}_{0,d}$ be arbitrary. We have

$$\begin{aligned}
h_K^{\frac{1}{p}}\|\eta\|_{L^p(F)} &\leq h_K^{\frac{1}{p}}\|v - q_0\|_{L^p(F)} + h_K^{\frac{1}{p}}\|\mathcal{I}_K^\sharp(v) - q_0\|_{L^p(F)} \\
&\leq c\left(\|v - q_0\|_{L^p(K)} + h_K^r|v|_{W^{r,p}(K)} + \|\mathcal{I}_K^\sharp(v) - q_0\|_{L^p(K)}\right) \\
&\leq c\left(\|v - q_0\|_{L^p(K)} + h_K^r|v|_{W^{r,p}(K)} + \|v - \mathcal{I}_K^\sharp(v)\|_{L^p(K)}\right),
\end{aligned}$$

where we used the triangle inequality in the first line, the fractional trace inequality (12.17), the discrete trace inequality (12.10) and $q_0 \in \mathbb{P}_{0,d}$ in the second line, and the triangle inequality in the third line. Invoking the best-approximation estimate (12.15) from Corollary 12.13 (observe that q_0 is arbitrary in $\mathbb{P}_{0,d}$) and (18.25) with $m = 0$ leads again to (18.27). Finally, the proof of (18.28) is similar and is left as an exercise. □

We define $\mathcal{I}_h^\sharp : L^1(D;\mathbb{R}^q) \to P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ s.t. for all $v \in L^1(D;\mathbb{R}^q)$,

$$\mathcal{I}_h^\sharp(v)_{|K} := \mathcal{I}_K^\sharp(v_{|K}), \qquad \forall K \in \mathcal{T}_h. \tag{18.29}$$

The approximation properties of $\mathcal{I}_h^\sharp$ readily follow from Theorem 18.14.

18.4 Broken L^2-orthogonal projection

Let $K \in \mathcal{T}_h$ be a mesh cell. The L^2-orthogonal projection $\mathcal{I}_K^{\mathrm{b}} : L^1(K;\mathbb{R}^q) \to P_K$ is defined s.t. for all $v \in L^1(K;\mathbb{R}^q)$,

$$(\mathcal{I}_K^{\mathrm{b}}(v) - v, q)_{L^2(K;\mathbb{R}^q)} = 0, \qquad \forall q \in P_K, \tag{18.30}$$

where $P_K := \psi_K^{-1}(\widehat{P})$ and $\psi_K(v) := \mathbb{A}_K(v \circ \boldsymbol{T}_K)$. Since (18.30) implies that

$$\|v - q\|_{L^2(K;\mathbb{R}^q)}^2 = \|v - \mathcal{I}_K^{\mathrm{b}}(v)\|_{L^2(K;\mathbb{R}^q)}^2 + \|\mathcal{I}_K^{\mathrm{b}}(v) - q\|_{L^2(K;\mathbb{R}^q)}^2, \tag{18.31}$$

we have the optimality property

$$\mathcal{I}_K^{\mathrm{b}}(v) = \arg\min_{q \in P_K}\|v - q\|_{L^2(K;\mathbb{R}^q)}. \tag{18.32}$$

The stability and approximation properties of $\mathcal{I}_K^{\mathrm{b}}$ can be analyzed by using the L^1-stable interpolation operator $\mathcal{I}_K^{\sharp}$ introduced in the previous section.

Theorem 18.16 (Stability and local approximation). *Let $\mathcal{I}_K^{\mathrm{b}}$ be defined by* (18.30)*. Let k be the degree of the finite element, i.e., $[\mathbb{P}_{k,d}]^q \subset \widehat{P} \subset W^{k+1,p}(\widehat{K};\mathbb{R}^q)$. Assume that the mesh sequence is shape-regular.* (i) *P_K is pointwise invariant under $\mathcal{I}_K^{\mathrm{b}}$.* (ii) *$\mathcal{I}_K^{\mathrm{b}}$ is L^p-stable for all $p \in [1,\infty]$, i.e., there is c s.t. $\|\mathcal{I}_K^{\mathrm{b}}(v)\|_{L^p(K;\mathbb{R}^q)} \le c\|v\|_{L^p(K;\mathbb{R}^q)}$ for all $v \in L^p(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.* (iii) *There is c s.t.*

$$|v - \mathcal{I}_K^{\mathrm{b}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} \le c\, h_K^{r-m} |v|_{W^{r,p}(K;\mathbb{R}^q)}, \tag{18.33}$$

for all $r \in [0,k+1]$, all $p \in [1,\infty)$ if $r \notin \mathbb{N}$ or all $p \in [1,\infty]$ if $r \in \mathbb{N}$, every integer $m \in \{0{:}\lfloor r \rfloor\}$, all $v \in W^{r,p}(K;\mathbb{R}^q)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$.

Proof. (i) The pointwise invariance of P_K under $\mathcal{I}_K^{\mathrm{b}}$ follows from (18.30).
(ii) Stability. Let $v \in L^p(K;\mathbb{R}^q)$. We observe that

$$\begin{aligned}
\|\mathcal{I}_K^{\mathrm{b}}(v)\|^2_{L^p(K;\mathbb{R}^q)} &\le c\, h_K^{d(\frac{2}{p}-1)} \|\mathcal{I}_K^{\mathrm{b}}(v)\|^2_{L^2(K;\mathbb{R}^q)} = c\, h_K^{d(\frac{2}{p}-1)} (v, \mathcal{I}_K^{\mathrm{b}}(v))_{L^2(K;\mathbb{R}^q)} \\
&\le c\, h_K^{d(\frac{2}{p}-1)} \|v\|_{L^p(K;\mathbb{R}^q)} \|\mathcal{I}_K^{\mathrm{b}}(v)\|_{L^{p'}(K;\mathbb{R}^q)} \\
&\le c'\, h_K^{d(\frac{2}{p}-1+\frac{1}{p'}-\frac{1}{p})} \|v\|_{L^p(K;\mathbb{R}^q)} \|\mathcal{I}_K^{\mathrm{b}}(v)\|_{L^p(K;\mathbb{R}^q)} \\
&= c'\, \|v\|_{L^p(K;\mathbb{R}^q)} \|\mathcal{I}_K^{\mathrm{b}}(v)\|_{L^p(K;\mathbb{R}^q)},
\end{aligned}$$

where we used the inverse inequality (12.3) (between L^p and L^2), (18.30) with $q := \mathcal{I}_K^{\mathrm{b}}(v)$, Hölder's inequality (with $\frac{1}{p} + \frac{1}{p'} = 1$), and again the inverse inequality (12.3) (between $L^{p'}$ and L^p). This proves the L^p-stability of $\mathcal{I}_K^{\mathrm{b}}$.
(iii) Local approximation. Since $\mathcal{I}_K^{\sharp}(v) \in P_K$ and P_K is left pointwise invariant by $\mathcal{I}_K^{\mathrm{b}}$, we have

$$\begin{aligned}
|v - \mathcal{I}_K^{\mathrm{b}}(v)|_{W^{m,p}(K;\mathbb{R}^q)} &\le |v - \mathcal{I}_K^{\sharp}(v)|_{W^{m,p}(K;\mathbb{R}^q)} + |\mathcal{I}_K^{\mathrm{b}}(v - \mathcal{I}_K^{\sharp}(v))|_{W^{m,p}(K;\mathbb{R}^q)} \\
&\le |v - \mathcal{I}_K^{\sharp}(v)|_{W^{m,p}(K;\mathbb{R}^q)} + c h_K^{-m} \|\mathcal{I}_K^{\mathrm{b}}(v - \mathcal{I}_K^{\sharp}(v))\|_{L^p(K;\mathbb{R}^q)} \\
&\le |v - \mathcal{I}_K^{\sharp}(v)|_{W^{m,p}(K;\mathbb{R}^q)} + c' h_K^{-m} \|v - \mathcal{I}_K^{\sharp}(v)\|_{L^p(K;\mathbb{R}^q)} \\
&\le c'' h_K^{r-m} |v|_{W^{r,p}(K;\mathbb{R}^q)},
\end{aligned}$$

where we used the triangle inequality, the inverse inequality from Lemma 12.1, the L^p-stability of $\mathcal{I}_K^{\mathrm{b}}$, and the approximation property (18.25) of $\mathcal{I}_K^{\sharp}$. □

We define $\mathcal{I}_h^{\mathrm{b}} : L^1(D;\mathbb{R}^q) \to P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$ s.t. for all $v \in L^1(D;\mathbb{R}^q)$, $\mathcal{I}_h^{\mathrm{b}}(v)_{|K} := \mathcal{I}_K^{\mathrm{b}}(v_{|K})$ for all $K \in \mathcal{T}_h$. One readily verifies that $\mathcal{I}_h^{\mathrm{b}}$ is the L^2-orthogonal projection onto $P_k^{\mathrm{b}}(\mathcal{T}_h;\mathbb{R}^q)$. The stability and approximation properties of $\mathcal{I}_h^{\mathrm{b}}$ follow from Theorem 18.16.

Remark 18.17 (Approximation on faces). A result similar to Corollary 18.15 holds true for $\mathcal{I}_K^{\mathrm{b}}$ on the mesh faces. □

Remark 18.18 (Pullback). One cannot investigate the approximation properties of $\mathcal{I}_K^{\mathrm{b}}$ by introducing the L^2-orthogonal projection onto $\widehat{P}$ (i.e., the operator $\mathcal{I}_{\widehat{K}}^{\sharp}$) and using Theorem 11.13, since we have seen that $\psi_K^{-1}\circ\mathcal{I}_{\widehat{K}}^{\sharp}\circ\psi_K$ is actually the oblique projection $\mathcal{I}_K^{\sharp}$ and not the L^2-orthogonal projection $\mathcal{I}_K^{\mathrm{b}}$. The two projections $\mathcal{I}_K^{\sharp}$ and $\mathcal{I}_K^{\mathrm{b}}$ coincide when the matrix $\mathbb{A}_K$ is unitary (see Exercise 18.3). This happens when ψ_K is the pullback by the geometric mapping $\boldsymbol{T}_K$, i.e., when $\mathbb{A}_K$ is the identity as is the case for scalar-valued elements. In this situation, Theorem 18.16 has already been established in Lemma 11.18 (at least for $r \in \{0{:}k+1\}$). $\square$

Remark 18.19 (Algebraic realization). To evaluate the L^2-orthogonal projection $\mathcal{I}_K^{\mathrm{b}}(v)$ of a function v, one has to solve the linear system $\mathcal{M}_K X = Y$, where the local mass matrix has entries $\mathcal{M}_{K,mn} := \int_K (\theta_{K,m}, \theta_{K,n})_{\ell^2(\mathbb{R}^q)}\,\mathrm{d}x$ for all $m, n \in \mathcal{N}$, and the right-hand side vector Y has components $Y_n := \int_K (v, \theta_{K,n})_{\ell^2(\mathbb{R}^q)}\,\mathrm{d}x$. Then we have $\mathcal{I}_K^{\mathrm{b}}(v) = \sum_{n\in\mathcal{N}} X_n \theta_{K,n}$; see §5.4.2. $\square$

Exercises

Exercise 18.1 ($\boldsymbol{H}$(div), $\boldsymbol{H}$(curl)). Prove Theorem 18.10. (*Hint*: use (4.8).)

Exercise 18.2 (Discrete Sobolev inequality). (i) Assume $d \geq 3$. Prove that $\|v_h\|_{L^\infty(K)} \leq c h_K^{1-\frac{d}{2}} \|\nabla v_h\|_{\boldsymbol{L}^2(K)}$ for all $v_h \in P_k^{\mathrm{g,b}}(\mathcal{T}_h)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$. (*Hint*: use Theorem 2.31.) (ii) Assume $d = 2$. Prove (18.15). (*Hint*: let $K \in \mathcal{T}_h$ with $h_K \leq \frac{\delta_D}{2}$, let $\boldsymbol{x} \in K$ and let $\boldsymbol{y}$ have polar coordinates (r,θ) with respect to $\boldsymbol{x}$ with $r \geq \frac{\delta_D}{2}$ and $\theta \in (0,\omega)$, use that $v_h(\boldsymbol{x}) = v_h(\boldsymbol{y}) - \int_0^r \partial_\rho v_h(\rho,\theta)\,\mathrm{d}\rho$, decompose the integral as $\int_0^r \cdot\,\mathrm{d}\rho = \int_0^{h_K} \cdot\,\mathrm{d}\rho + \int_{h_K}^r \cdot\,\mathrm{d}\rho$, and bound the two addends.)

Exercise 18.3 (Orthogonal and oblique projections). (i) Show that $\mathcal{I}_{\widehat{K}}^{\sharp}$ is the L^2-orthogonal projection onto $\widehat{P}$. (*Hint*: observe that $(\widehat{\rho}_i, \widehat{\theta}_j)_{L^2(\widehat{K};\mathbb{R}^q)} = |\widehat{K}|\delta_{ij}$ for all $i, j \in \mathcal{N}$.) (ii) Prove that $\mathcal{I}_K^{\sharp}$ is the oblique projection onto $P_K = \psi_K^{-1}(\widehat{P})$ parallel to $Q_K^{\perp}$ with $Q_K := \Phi_K^{-1}(\widehat{P})$. (*Hint*: use (18.17).) (iii) Show that $P_K = Q_K$ if the matrix $\mathbb{A}_K$ is unitary, i.e., $\mathbb{A}_K^{\mathsf{T}}\mathbb{A}_K = \mathbb{A}_K\mathbb{A}_K^{\mathsf{T}} = \mathbb{I}_q$.

Exercise 18.4 (Approximation on faces). Prove (18.28).