Part V, Chapter 25

Main results on well-posedness

The starting point of this chapter is the model problem derived in §24.4. Our goal is to specify conditions under which this problem is well-posed. Two important results are presented: the Lax–Milgram lemma and the more fundamental Banach–Nečas–Babuška theorem. The former provides a *sufficient* condition for well-posedness, whereas the latter, relying on slightly more sophisticated assumptions, provides *necessary and sufficient* conditions. The reader is invited to review the material of Appendix C on bijective operators in Banach spaces before reading this chapter.

25.1 Mathematical setting

To stay general, we consider complex vector spaces. The case of real vector spaces is recovered by replacing the field $\mathbb{C}$ by $\mathbb{R}$, by removing the real part symbol $\Re(\cdot)$ and the complex conjugate symbol $\bar{\cdot}$, and by interpreting the symbol $|\cdot|$ as the absolute value instead of the modulus.

We consider the following model problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, w) = \ell(w), \quad \forall w \in W. \end{cases} \tag{25.1}$$

The spaces V and W are complex Banach spaces equipped with norms denoted by $\|\cdot\|_V$ and $\|\cdot\|_W$, respectively. In many applications, V and W are Hilbert spaces. The map $a : V \times W \to \mathbb{C}$ is a sesquilinear form (bilinear in the real case). We assume that a is *bounded*, which means that

$$\|a\|_{V\times W} := \sup_{v\in V} \sup_{w\in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} < \infty. \tag{25.2}$$

It is henceforth implicitly understood that this type of supremum is taken over nonzero arguments (notice that the order in which the suprema are taken in (25.2) does not matter). Furthermore, the map $\ell : W \to \mathbb{C}$ is an antilinear form (linear in the real case). We assume that ℓ is *bounded*, and we write $\ell \in W'$. The boundedness of ℓ means that

$$\|\ell\|_{W'} := \sup_{w\in W} \frac{|\ell(w)|}{\|w\|_W} < \infty. \tag{25.3}$$

Notice that it is possible to replace the modulus by the real part in (25.2) and (25.3) (replace w by ξw with a unitary complex number ξ), and in the real case, the absolute value is not needed (replace w by $\pm w$).

Definition 25.1 (Well-posedness, Hadamard [236]). *We say that the problem* (25.1) *is* well-posed *if it admits one and only one solution for all $\ell \in W'$, and there is c, uniform with respect to ℓ, s.t. the a priori estimate $\|u\|_V \le c\,\|\ell\|_{W'}$ holds true.*

The goal of this chapter is to study the well-posedness of (25.1). The key idea is to introduce the bounded linear operator $A \in \mathcal{L}(V;W')$ that is naturally associated with the bilinear form a on $V\times W$ by setting

$$\langle A(v), w\rangle_{W',W} := a(v,w), \qquad \forall (v,w) \in V\times W. \tag{25.4}$$

This definition implies that A is linear and bounded with norm $\|A\|_{\mathcal{L}(V;W')} = \|a\|_{V\times W}$. The problem (25.1) can be reformulated as follows: Find $u \in V$ such that $A(u) = \ell$ in W'. Hence, proving the existence and uniqueness of the solution to (25.1) amounts to proving that the operator A is bijective. Letting $A^*\colon W''\to V'$ be the adjoint of A, the way to do this is to prove the following three conditions:

$$\underbrace{\text{(i) } A \text{ is injective},\quad \overbrace{\text{(ii) } \mathrm{im}(A) \text{ is closed}}^{}}_{\iff \exists\alpha>0,\ \|A(v)\|_{W'}\ge\alpha\|v\|_V,\ \forall v\in V}\!\!\overbrace{,\quad \text{(iii) } A^* \text{ is injective}}^{\iff A \text{ is surjective}}. \tag{25.5}$$

The conditions (ii)-(iii) in (25.5) are equivalent to A being surjective since the closure of $\mathrm{im}(A)$ is $(\ker(A^*))^\perp \subset W'$ owing to Lemma C.34 (see also (C.14b)). That the conditions (i)-(ii) are equivalent to the existence of some $\alpha > 0$ s.t. $\|A(v)\|_{W'} \ge \alpha\|v\|_V$, for all $v \in V$, is established in Lemma C.39 (these two conditions are also equivalent to the surjectivity of A^*).

25.2 Lax–Milgram lemma

The Lax–Milgram lemma is applicable only if the solution and the test spaces are *identical*. Assuming $W = V$, the model problem (25.1) becomes

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u,w) = \ell(w), \quad \forall w \in V. \end{cases} \tag{25.6}$$

Lemma 25.2 (Lax–Milgram). *Let V be a Hilbert space, let a be a bounded sesquilinear form on $V \times V$, and let $\ell \in V'$. Assume the following* coercivity *property: There is a real number $\alpha > 0$ and a complex number ξ with $|\xi| = 1$ such that*

$$\Re\left(\xi a(v,v)\right) \geq \alpha \|v\|_V^2, \qquad \forall v \in V. \tag{25.7}$$

Then (25.6) *is well-posed with the a priori estimate $\|u\|_V \leq \frac{1}{\alpha}\|\ell\|_{V'}$.*

Proof. Although this lemma is a consequence of the more abstract BNB theorem (Theorem 25.9), we present a direct proof for completeness. Let $A : V \to V'$ be the bounded linear operator defined in (25.4) and let us prove the three conditions (i)-(ii)-(iii) in (25.5). Since $\xi a(v,v) = a(v, \overline{\xi} v)$, the coercivity property (25.7) implies that

$$\alpha \|v\|_V \leq \frac{\Re(a(v,\overline{\xi}v))}{\|v\|_V} \leq \sup_{w \in V} \frac{\Re(a(v,\overline{\xi}w))}{\|w\|_V} = \sup_{w \in V} \frac{|a(v,w)|}{\|w\|_V} = \|A(v)\|_{V'},$$

so that the conditions (i)-(ii) hold true. Since V is reflexive, we identify V and V'', so that the adjoint operator $A^* : V \to V'$ is such that $\langle A^*(v), w\rangle_{V',V} = \overline{\langle A(w), v\rangle_{V',V}}$ for all $v, w \in V$. Let $v \in V$ and assume that $A^*(v) = 0$. Then $0 = \overline{0} = \overline{\langle A^*(v), \xi v\rangle_{V',V}} = \xi a(v,v)$. We then infer from (25.7) that $\alpha\|v\|_V^2 \leq \Re\left(\xi a(v,v)\right) = 0$, i.e., $v = 0$. This proves that A^* is injective. Hence, the condition (iii) also holds true. Finally, the a priori estimate follows from $\alpha\|u\|_V \leq \frac{\Re(a(u,\overline{\xi}u))}{\|u\|_V} = \frac{\Re(\ell(\overline{\xi}u))}{\|u\|_V} \leq \|\ell\|_{V'}$. □

Remark 25.3 (Hilbertian setting). An important observation is that the Lax–Milgram lemma relies on the notion of coercivity which is applicable only in Hilbertian settings; see Proposition C.59. □

Example 25.4 (Laplacian). Consider the weak formulation (24.7) of the Poisson equation with homogeneous Dirichlet condition. The functional setting is $V = W := H_0^1(D)$ equipped with the norm $\|\cdot\|_{H^1(D)}$, the bilinear form is $a(v,w) := \int_D \nabla v \cdot \nabla w \,\mathrm{d}x$, and the linear form is $\ell(w) := \int_D f w \,\mathrm{d}x$. Owing to the Cauchy–Schwarz inequality, the forms a and ℓ are bounded on $V \times V$ and V, respectively. Moreover, the Poincaré–Steklov inequality (3.11) (with $p := 2$) implies that (see Remark 3.29)

$$a(v,v) = \|\nabla v\|_{\boldsymbol{L}^2(D)}^2 = |v|_{H^1(D)}^2 \geq \ell_D^{-2} \frac{C_{\text{PS}}^2}{1 + C_{\text{PS}}^2} \|v\|_{H^1(D)}^2,$$

for all $v \in V$. Hence, (25.7) holds true with $\alpha := \ell_D^{-2} \frac{C_{\text{PS}}^2}{1+C_{\text{PS}}^2}$ and $\xi := 1$, and by the Lax–Milgram lemma, the problem (24.7) is well-posed. Alternatively one can equip V with the norm $\|v\|_V := \ell_D^{-1}\|\nabla v\|_{\boldsymbol{L}^2(D)}$ which is equivalent to the norm $\|\cdot\|_{H^1(D)}$ owing to the Poincaré–Steklov inequality. The coercivity constant of a is then $\alpha := \ell_D^{-2}$. $\square$

Example 25.5 (Complex case). Consider the PDE $\mathrm{i}u - \nu\Delta u = f$ in D with $\mathrm{i}^2 = -1$, a real number $\nu > 0$, a source term $f \in L^2(D;\mathbb{C})$, and a homogeneous Dirichlet condition. The functional setting is $V = W := H_0^1(D;\mathbb{C})$ equipped with the norm $\|\cdot\|_{H^1(D;\mathbb{C})}$, the sesquilinear form is $a(v,w) := \int_D \mathrm{i}v\overline{w}\,\mathrm{d}x + \nu\int_D \nabla v{\cdot}\nabla\overline{w}\,\mathrm{d}x$, and the antilinear form is $\ell(w) := \int_D f\overline{w}\,\mathrm{d}x$. Then (24.25) shows that the coercivity property (25.7) holds true with $\xi := e^{-\mathrm{i}\frac{\pi}{4}}$ and $\alpha := \frac{1}{\sqrt{2}}\min(1,\nu\ell_D^{-2})$. $\square$

Remark 25.6 (Definition of coercivity). The coercivity property can also be defined in the following way: There is a real number $\alpha > 0$ such that $|a(v,v)| \geq \alpha\|v\|_V^2$ for all $v \in V$. It is shown in Lemma C.58 that this definition and (25.7) are equivalent. $\square$

Definition 25.7 (Hermitian/symmetric form). *Let V be a Hilbert space. In the complex case, we say that a sesquilinear form $a : V\times V \to \mathbb{C}$ is* Hermitian *whenever $a(v,w) = \overline{a(w,v)}$ for all $v,w \in V$. In the real case, we say that a bilinear form a is* symmetric *whenever $a(v,w) = a(w,v)$ for all $v,w \in V$.*

Whenever the sesquilinear form a is Hermitian and coercive (with $\xi := 1$ for simplicity), setting $(\!(\cdot,\cdot)\!)_V := a(\cdot,\cdot)$ one defines an inner product in V, and the induced norm is equivalent to $\|\cdot\|_V$ owing to the coercivity and the boundedness of a. Then solving the problem (25.6) amounts to finding the representative $u \in V$ of the linear form $\ell \in V'$, i.e., $(\!(u,w)\!)_V = \ell(w)$ for all $w \in V$. This problem is well-posed by the Riesz–Fréchet theorem (Theorem C.24). Thus, the Lax–Milgram lemma can be viewed as an extension of the Riesz–Fréchet theorem to non-Hermitian forms.

Whenever V is a real Hilbert space and the bilinear form a is symmetric and coercive with $\xi := 1$, the problem (25.6) can be interpreted as a minimization problem (or a maximization problem if $\xi := -1$). In this context, (25.6) is called *variational formulation.*

Proposition 25.8 (Variational formulation). *Let V be a real Hilbert space, let a be a bounded bilinear form on $V\times V$, and let $\ell \in V'$. Assume that a is coercive with $\xi := 1$. Assume that a is* symmetric*, i.e.,*

$$a(v,w) = a(w,v), \qquad \forall v,w \in V. \tag{25.8}$$

Then introducing the energy functional $\mathfrak{E} : V \to \mathbb{R}$ *such that*

$$\mathfrak{E}(v) := \frac{1}{2}a(v,v) - \ell(v), \tag{25.9}$$

u solves (25.6) *iff u minimizes $\mathfrak{E}$ over V.*

Proof. The proof relies on the fact that for all $u,w \in V$ and all $t \in \mathbb{R}$,

$$\mathfrak{E}(u + tw) = \mathfrak{E}(u) + t(a(u, w) - \ell(w)) + \frac{1}{2}t^2 a(w, w), \tag{25.10}$$

which results from the symmetry of a. (i) Assume that u solves (25.6). Since $a(w, w) \geq 0$ owing to the coercivity of a with $\xi := 1$, (25.10) implies that u minimizes $\mathfrak{E}$ over V. (ii) Conversely, assume that u minimizes $\mathfrak{E}$ over V. The right-hand side of (25.10) is a quadratic polynomial in t reaching its minimum value at $t = 0$. Hence, the derivative of this polynomial vanishes at $t = 0$, which amounts to $a(u, w) - \ell(w) = 0$. Since w is arbitrary in V, we conclude that u solves (25.6). □

25.3 Banach–Nečas–Babuška (BNB) theorem

The BNB theorem plays a fundamental role in this book. We use this terminology since, to our knowledge, the BNB theorem was stated by Nečas in 1962 [310] and Babuška in 1970 in the context of finite element methods [33]. From a functional analysis point of view, the BNB theorem is a rephrasing of two fundamental results by Banach: the closed range theorem and the open mapping theorem. We present two settings for the BNB theorem depending on whether the test functions in the model problem belong to a reflexive Banach space or to the dual of a Banach space. Recall from Definition C.18 that a Banach space W is said to be reflexive if the canonical isometry $J_W : W \to W''$ is an isomorphism. This is always the case if W is a Hilbert space.

25.3.1 Test functions in reflexive Banach space

Theorem 25.9 (Banach–Nečas–Babuška (BNB)). *Let V be a Banach space and let W be a reflexive Banach space. Let a be a bounded sesquilinear form on $V \times W$ and let $\ell \in W'$. Then the problem (25.1) is well-posed iff:*

$$(\textsc{bnb}1) \qquad \inf_{v\in V} \sup_{w\in W} \frac{|a(v, w)|}{\|v\|_V \|w\|_W} =: \alpha > 0, \tag{25.11a}$$

$$(\textsc{bnb}2) \qquad \forall w \in W, \quad [\,\forall v \in V,\ a(v, w) = 0\,] \implies [\,w = 0\,]. \tag{25.11b}$$

(It is implicitly understood that the argument is nonzero in the above infimum and supremum.) Moreover, we have the a priori estimate $\|u\|_V \leq \frac{1}{\alpha}\|\ell\|_{W'}$.

Proof. Let $A \in \mathcal{L}(V; W')$ be defined by (25.4) and let us prove that the three conditions (i)-(ii)-(iii) in (25.5) are equivalent to (BNB1)-(BNB2). The conditions (i)-(ii) are equivalent to (BNB1) since for all $v \in V$,

$$\|Av\|_{W'} = \sup_{w\in W} \frac{|\langle A(v), w\rangle_{W',W}|}{\|w\|_W} = \sup_{w\in W} \frac{|a(v, w)|}{\|w\|_W}.$$

Since $\langle A^*(J_W(w)), v\rangle_{V',V} = \langle J_W(w), A(v)\rangle_{W'',W'} = \overline{\langle A(v), w\rangle_{W',W}} = \overline{a(v,w)}$ for all $(v,w) \in V\times W$, stating that $a(v,w) = 0$ for all $v \in V$ is equivalent to stating that $(A^*\circ J_W)(w) = 0$. Hence, (BNB2) is equivalent to stating that $A^*\circ J_W$ is injective. Furthermore, since W is reflexive, the canonical isometry $J_W : W \to W''$ from Proposition C.17 is an isomorphism. Hence, (BNB2) is equivalent to stating that $A^* : W'' \to V'$ is injective, which is the condition (iii) in (25.5). Finally, the a priori estimate follows from the inequalities $\alpha\|u\|_V \le \sup_{w\in W} \frac{|a(u,w)|}{\|w\|_W} = \sup_{w\in W} \frac{|\ell(w)|}{\|w\|_W} = \|\ell\|_{W'}$. □

Remark 25.10 ((BNB1)). Condition (BNB1) is called *inf-sup condition* and it is equivalent to the following statement:

$$\exists\alpha > 0, \qquad \alpha\|v\|_V \le \sup_{w\in W} \frac{|a(v,w)|}{\|w\|_W}, \qquad \forall v \in V. \tag{25.12}$$

Establishing (25.12) is usually done by finding two positive real numbers c_1, c_2 s.t. for all $v \in V$, one can find a "partner" $w_v \in W$ s.t. $\|w_v\|_W \le c_1\|v\|_V$ and $|a(v,w_v)| \ge c_2\|v\|_V^2$. If this is indeed the case, then (25.12) holds true with $\alpha := \frac{c_2}{c_1}$. Establishing coercivity amounts to asserting that $w_v = \zeta v$ is a suitable partner for some $\zeta \in \mathbb{C}$ with $|\zeta| = 1$. □

Remark 25.11 ((BNB2)). The statement in (BNB2) is equivalent to asserting that for all w in W, either there exists v in V such that $a(v,w) \neq 0$ or $w = 0$. In view of the proof Theorem 25.9, (BNB2) says that for all w in W, either $A^*\circ J_W(w) \neq 0$ or $w = 0$. □

Remark 25.12 (Two-sided bound). Since $\|\ell\|_{W'} = \|A(u)\|_{W'} \le \omega\|u\|_V$ where $\omega := \|a\|_{V\times W} = \|A\|_{\mathcal{L}(V;W')}$ is the boundedness constant of the sesquilinear form a on $V\times W$, we infer the two-sided bound

$$\frac{1}{\|a\|_{V\times W}}\|\ell\|_{W'} \le \|u\|_V \le \frac{1}{\alpha}\|\ell\|_{W'}.$$

Since $\alpha^{-1} = \|A^{-1}\|_{\mathcal{L}(W';V)}$ owing to Lemma C.51, the quantity

$$\kappa(a) = \frac{\|a\|_{V\times W}}{\alpha} = \|A\|_{\mathcal{L}(V;W')}\|A^{-1}\|_{\mathcal{L}(W';V)} \ge 1$$

can be viewed as the *condition number* of the sesquilinear form a (or of the associated operator A). A similar notion of conditioning is developed for matrices in §28.2.1. □

Remark 25.13 (Link with Lax–Milgram). Let V be a Hilbert space and let a be a bounded and coercive bilinear form on $V\times V$. The proof of the Lax–Milgram lemma shows that a satisfies the conditions (BNB1) and (BNB2) (with $W = V$). The converse is false: the conditions (BNB1) and (BNB2) do not imply coercivity. Hence, (25.7) is *not necessary* for well-posedness, whereas (BNB1)-(BNB2) are *necessary and sufficient.* However, coercivity is both necessary and sufficient for well-posedness when the bilinear form a is Hermitian and positive semidefinite; see Exercise 25.7. □

Remark 25.14 (T-coercivity). Let V, W be Hilbert spaces. Then (BNB1)-(BNB2) are equivalent to the existence of a bijective operator $T \in \mathcal{L}(V;W)$ and a positive real number η such that

$$\Re(a(v, T(v))) \geq \eta \|v\|_V^2, \qquad \forall v \in V.$$

This property is called T-*coercivity* in Bonnet-Ben Dhia et al. [72, 73]; see Exercise 25.10. The advantage of this notion over coercivity is the possibility of treating different trial and test spaces and using a test function different from $v \in V$ to estimate $\|v\|_V^2$. Note that the bilinear form $(u, v) \mapsto a(u, T(v))$ is bounded and coercive on $V \times V$. Proposition C.59 then implies that V is necessarily a Hilbert space. This argument proves that T-coercivity is a notion relevant in Hilbert spaces only. The BNB theorem is more general than T-coercivity since it also applies to Banach spaces. □

25.3.2 Test functions in dual Banach space

The requirement on the reflexivity of the space W in the BNB theorem can be removed if the model problem is reformulated in such a way that the test functions act on the problem data instead of the data acting on the test functions. Assume that we are given a bounded operator $A \in \mathcal{L}(V;W)$ and some data $f \in W$, and we want to assert that there is a unique $u \in V$ s.t. $A(u) = f$. To recast this problem in the general setting of (25.1) using test functions, we define the bounded sesquilinear form on $V \times W'$ such that

$$a(v, w') := \overline{\langle w', A(v) \rangle_{W',W}}, \qquad \forall (v, w') \in V \times W', \tag{25.13}$$

and we consider the following model problem:

$$\begin{cases} \text{Find } u \in V \text{ such that} \\ a(u, w') = \overline{\langle w', f \rangle_{W',W}}, \quad \forall w' \in W'. \end{cases} \tag{25.14}$$

Then $u \in V$ solves (25.14) iff $\langle w', A(u) - f \rangle_{W',W} = 0$ for all $w' \in W'$, that is, iff $A(u) = f$. In (25.14), the data is f is in W and the test functions belong to W', whereas in the original model problem (25.1) the data is $\ell \in W'$ and the test functions belong to W. The functional setting of (25.14) is useful, e.g., when considering first-order PDEs; see §24.2.1.

Theorem 25.15 (Banach–Nečas–Babuška (BNB)). *Let V, W be Banach spaces. Let $A \in \mathcal{L}(V;W)$ and let $f \in W$. Let a be the bounded sesquilinear form on $V \times W'$ defined in* (25.13)*. The problem* (25.14) *is well-posed iff:*

$$(\text{BNB1'}) \qquad \inf_{v\in V}\sup_{w'\in W'} \frac{|a(v,w')|}{\|v\|_V\|w'\|_{W'}} := \alpha > 0, \tag{25.15}$$

$$(\text{BNB2'}) \qquad \forall w' \in W', \quad [\,\forall v \in V,\ a(v,w') = 0\,] \implies [\,w' = 0\,]. \tag{25.16}$$

Moreover, we have the a priori estimate $\|u\|_V \le \frac{1}{\alpha}\|f\|_W$.

Proof. The well-posedness of (25.14) is equivalent to the bijectivity of $A : V \to W$, and this property is equivalent to the three conditions (i)-(ii)-(iii) in (25.5) with W in lieu of W' and $A^* : W' \to V'$. Since $\|A(v)\|_W = \sup_{w'\in W'} \frac{|\langle w', A(v)\rangle_{W',W}|}{\|w'\|_{W'}}$ owing to Corollary C.14, the condition (BNB1') means that $\|A(v)\|_W \ge \alpha\|v\|_V$ for all $v \in V$. This condition is therefore equivalent to the conditions (i)-(ii). Moreover, since $a(v, w') = \overline{\langle w', A(v)\rangle_{W',W}} = \langle A^*(w'), v\rangle_{V',V}$, (BNB2') amounts to the condition (iii) (i.e., the injectivity of A^*). □

Remark 25.16 (A vs. a). In the first version of the BNB theorem (Theorem 25.9), it is equivalent to assume that we are given an operator $A \in \mathcal{L}(V; W')$ or a bounded sesquilinear form a on $V\times W$. But, in the second version of the BNB theorem (Theorem 25.15), we are given an operator $A \in \mathcal{L}(V; W)$, and the bounded sesquilinear form a on $V\times W'$ is defined *from* A. If we were given instead a bounded sesquilinear form a on $V\times W'$, proceeding as in (25.4) would be awkward since it would lead to an operator $\tilde{A} \in \mathcal{L}(V; W'')$ s.t. $\langle \tilde{A}(v), w'\rangle_{W'',W'} := a(v, w')$ for all $(v, w') \in V\times W'$. □

Remark 25.17 (Literature). Inf-sup conditions in nonreflexive Banach spaces are discussed in Amrouche and Ratsimahalo [9]. □

25.4 Two examples

In this section, we present two examples illustrating the above abstract results.

25.4.1 Darcy's equations

The weak formulation (24.12) fits the setting of the model problem (25.1) with $V := \boldsymbol{H}(\text{div}; D)\times H_0^1(D)$ and $W := \boldsymbol{L}^2(D)\times L^2(D)$, where $\|\boldsymbol{\sigma}\|_{\boldsymbol{H}(\text{div};D)} := (\|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\cdot\boldsymbol{\sigma}\|^2_{L^2(D)})^{\frac{1}{2}}$ (recall that ℓ_D is a characteristic length scale associated with D, e.g., $\ell_D := \text{diam}(D)$), and with the bilinear and linear forms

$$a(v, w) := \int_D \big(\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + \nabla p\cdot\boldsymbol{\tau} + (\nabla\cdot\boldsymbol{\sigma})q\big)\,\mathrm{d}x, \qquad \ell(w) := \int_D fq\,\mathrm{d}x, \tag{25.17}$$

with $v := (\boldsymbol{\sigma}, p) \in V$ and $w := (\boldsymbol{\tau}, q) \in W$.

Proposition 25.18. *Problem* (24.12) *is well-posed.*

Proof. We equip the Hilbert spaces V and W with the norms $\|v\|_V := \big(\|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathrm{div};D)} + |p|^2_{H^1(D)}\big)^{\frac12}$ and $\|w\|_W := \big(\|\boldsymbol{\tau}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^{-2}\|q\|^2_{L^2(D)}\big)^{\frac12}$ with $v := (\boldsymbol{\sigma}, p)$ and $w := (\boldsymbol{\tau}, q)$, respectively. That $\|\cdot\|_V$ is indeed a norm follows from the Poincaré–Steklov inequality (3.11) (see Remark 3.29). Since the bilinear form a and the linear form ℓ are obviously bounded, it remains to check the conditions (BNB1) and (BNB2).
(1) Proof of (BNB1). Let $(\boldsymbol{\sigma}, p) \in V$ and define $\mathbb{S} := \sup_{(\boldsymbol{\tau},q)\in W} \frac{|a((\boldsymbol{\sigma},p),(\boldsymbol{\tau},q))|}{\|(\boldsymbol{\tau},q)\|_W}$. Since $V \subset W$, we can take $(\boldsymbol{\sigma}, p)$ as the test function. Since p vanishes at the boundary, $a((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p)) = \|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)}$, whence we infer that

$$\|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)} = \frac{a((\boldsymbol{\sigma}, p), (\boldsymbol{\sigma}, p))}{\|(\boldsymbol{\sigma}, p)\|_W}\|(\boldsymbol{\sigma}, p)\|_W \le \mathbb{S}\,\|(\boldsymbol{\sigma}, p)\|_W.$$

Since $\|\cdot\|_W \le \gamma\|\cdot\|_V$ on V with $\gamma := \max(1, C_{\mathrm{PS}}^{-1})$, we infer that $\|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)} \le \gamma\,\mathbb{S}\,\|(\boldsymbol{\sigma}, p)\|_V$. Moreover, we have

$$\begin{aligned}\Big(\|\nabla p\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\cdot\boldsymbol{\sigma}\|^2_{L^2(D)}\Big)^{\frac12} &= \sup_{(\boldsymbol{\tau},q)\in W} \frac{|\int_D \{\nabla p\cdot\boldsymbol{\tau} + (\nabla\cdot\boldsymbol{\sigma})q\}\,\mathrm{d}x|}{\|(\boldsymbol{\tau}, q)\|_W} \\ &\le \sup_{(\boldsymbol{\tau},q)\in W} \frac{|a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q))|}{\|(\boldsymbol{\tau}, q)\|_W} + \sup_{(\boldsymbol{\tau},q)\in W} \frac{|\int_D \boldsymbol{\sigma}\cdot\boldsymbol{\tau}\,\mathrm{d}x|}{\|(\boldsymbol{\tau}, q)\|_W}.\end{aligned}$$

Hence, $\big(\|\nabla p\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\cdot\boldsymbol{\sigma}\|^2_{L^2(D)}\big)^{\frac12} \le \mathbb{S} + \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)}$. Squaring this inequality and combining it with the above bound on $\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)}$, we infer that

$$\begin{aligned}\|(\boldsymbol{\sigma}, p)\|^2_V = \|\nabla p\|^2_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\sigma}\|^2_{\boldsymbol{H}(\mathrm{div};D)} &\le 2\mathbb{S}^2 + 3\|\boldsymbol{\sigma}\|^2_{\boldsymbol{L}^2(D)} \\ &\le 2\mathbb{S}^2 + 3\gamma\,\mathbb{S}\,\|(\boldsymbol{\sigma}, p)\|_V.\end{aligned}$$

Hence, the inf-sup condition (BNB1) holds true with $\alpha \ge (4 + 9\gamma^2)^{-\frac12}$.
(2) Proof of (BNB2). Let $(\boldsymbol{\tau}, q) \in W$ be such that $a((\boldsymbol{\sigma}, p), (\boldsymbol{\tau}, q)) = 0$ for all $(\boldsymbol{\sigma}, p) \in V$. This means on the one hand that $\int_D \nabla p\cdot\boldsymbol{\tau}\,\mathrm{d}x = 0$ for all $p \in H^1_0(D)$, so that $\nabla\cdot\boldsymbol{\tau} = 0$. On the other hand we obtain that $\int_D\{\boldsymbol{\sigma}\cdot\boldsymbol{\tau} + (\nabla\cdot\boldsymbol{\sigma})q\}\,\mathrm{d}x = 0$ for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}; D)$. Taking $\boldsymbol{\sigma} \in \boldsymbol{C}_0^\infty(D)$ we infer that $q \in H^1(D)$ and $\nabla q = \boldsymbol{\tau}$. Observing that $\boldsymbol{\tau} \in \boldsymbol{H}(\mathrm{div}; D)$ and taking $\boldsymbol{\sigma} := \boldsymbol{\tau}$, we infer that $0 = \int_D\{\boldsymbol{\tau}\cdot\boldsymbol{\tau} + (\nabla\cdot\boldsymbol{\tau})q\}\,\mathrm{d}x = \|\boldsymbol{\tau}\|^2_{\boldsymbol{L}^2(D)}$ since $\nabla\cdot\boldsymbol{\tau} = 0$. Hence, $\boldsymbol{\tau} = \boldsymbol{0}$. Finally, $\nabla q = \boldsymbol{\tau} = \boldsymbol{0}$, which implies that q is constant on D. Since $\int_D(\nabla\cdot\boldsymbol{\sigma})q\,\mathrm{d}x = 0$ for all $\boldsymbol{\sigma} \in \boldsymbol{H}(\mathrm{div}; D)$, q is identically zero in D (take for instance $\boldsymbol{\sigma}(\boldsymbol{x}) := \boldsymbol{x}$). □

25.4.2 First-order PDE

Consider the weak formulation (24.20) on $D := (0, 1)$. This formulation fits the setting of the model problem (25.14) with the spaces

$$V := \{v \in W^{1,1}(D) \mid v(0) = 0\}, \qquad W := L^1(D). \tag{25.18}$$

The data is $f \in W$ and we consider the bounded operator $A : V \to W$ s.t. $A(v) := \frac{\mathrm{d}v}{\mathrm{d}t}$ for all $v \in V$. (Here, we denote derivatives by $\frac{\mathrm{d}}{\mathrm{d}t}$ and reserve the primes to duality.) Recalling that $W' = L^\infty(D)$, the bilinear form a associated with the operator A is s.t.

$$a(v, w') := \int_0^1 \frac{\mathrm{d}v}{\mathrm{d}t} w' \,\mathrm{d}t, \qquad \forall (v, w') \in V \times W', \tag{25.19}$$

and the right-hand side is $\langle w', f\rangle_{W',W} := \int_0^1 w' f \,\mathrm{d}t$ with $f \in W$.

Proposition 25.19. *Problem* (24.20) *is well-posed.*

Proof. We equip the Banach spaces V and W' with the norms $\|v\|_V := \|v\|_{L^1(D)} + \|\frac{\mathrm{d}v}{\mathrm{d}t}\|_{L^1(D)}$ and $\|w'\|_{W'} := \|w'\|_{L^\infty(D)}$, and we verify the conditions (BNB1') and (BNB2') from Theorem 25.15.
(1) Proof of (BNB1'). Let $v \in V$ and set $D^\pm := \{t \in D \mid \pm \frac{\mathrm{d}v}{\mathrm{d}t}(t) > 0\}$. Taking $w'_v := \mathbb{1}_{D^+} - \mathbb{1}_{D^-}$, where $\mathbb{1}_S$ denotes the indicator function of a measurable set S, we infer that

$$\sup_{w' \in W'} \frac{|a(v, w')|}{\|w'\|_{W'}} \geq \frac{|a(v, w'_v)|}{\|w'_v\|_{W'}} = \frac{|\int_0^1 \frac{\mathrm{d}v}{\mathrm{d}t} w'_v \,\mathrm{d}t|}{\|w'_v\|_{L^\infty(D)}} = \int_0^1 \left|\tfrac{\mathrm{d}v}{\mathrm{d}t}\right| \mathrm{d}t = \left\|\tfrac{\mathrm{d}v}{\mathrm{d}t}\right\|_{L^1(D)}.$$

Invoking the extended Poincaré–Steklov inequality on V (with $p := 1$ and the bounded linear form $v \mapsto v(0)$ in (3.13)) yields (BNB1').
(2) Proof of (BNB2'). Let $w' \in W'$ be such that $\int_0^1 \frac{\mathrm{d}v}{\mathrm{d}t} w' \,\mathrm{d}t = 0$ for all $v \in V$. Taking v in $C_0^\infty(D)$, we infer that the weak derivative of w' vanishes. Lemma 2.11 implies that w' is a constant. Choosing $v(t) := t$ as a test function leads to $\int_0^1 w' \,\mathrm{d}t = 0$. Hence, we have $w' = 0$. $\square$

Exercises

Exercise 25.1 (Riesz–Fréchet). The objective is to prove the Riesz–Fréchet theorem (Theorem C.24) by using the BNB theorem. Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$. (i) Show that for every $v \in V$, there is a unique $J_V^{\mathrm{RF}}(v) \in V'$ s.t. $\langle J_V^{\mathrm{RF}}(v), w\rangle_{V',V} := (v, w)_V$ for all $w \in V$. (ii) Show that $J_V^{\mathrm{RF}} : V' \to V$ is a linear isometry.

Exercise 25.2 (Reflexivity). Let V, W be two Banach spaces such that there is an isomorphism $A \in \mathcal{L}(V; W)$. Assume that V is reflexive. Prove that W is reflexive. (*Hint*: consider the map $A^{**} \circ J_V \circ A^{-1}$.)

Exercise 25.3 (Space $V_\mathbb{R}$). Let V be a set and assume that V has a vector space structure over the field $\mathbb{C}$. By restricting the scaling λv to $\lambda \in \mathbb{R}$ and

$v \in V$, V has also a vector space structure over the field $\mathbb{R}$, which we denote by $V_{\mathbb{R}}$ (V and $V_{\mathbb{R}}$ are the same sets, but they are equipped with different vector space structures); see Remark C.11. Let V' be the set of the bounded anti-linear forms on V and $V'_{\mathbb{R}}$ be the set of the bounded linear forms on $V_{\mathbb{R}}$. Prove that the map $I : V' \to V'_{\mathbb{R}}$ such that for all $\ell \in V'$, $I(\ell)(v) := \Re(\ell(v))$ for all $v \in V$, is a bijective isometry. (*Hint*: for $\psi \in V'_{\mathbb{R}}$, set $\ell(v) := \psi(v)+\mathrm{i}\psi(\mathrm{i}v)$ with $\mathrm{i}^2 = -1$.)

Exercise 25.4 (Orthogonal projection). Let V be a Hilbert space with inner product $(\cdot,\cdot)_V$ and induced norm $\|\cdot\|_V$. Let U be a nonempty, closed, and convex subset of V. Let $f \in V$. (i) Show that there is a unique u in U such that $\|f-u\|_V = \min_{v\in U}\|f-v\|_V$. (*Hint*: recall that $\frac{1}{4}(a-b)^2 = \frac{1}{2}(c-a)^2 + \frac{1}{2}(c-b)^2 - (c-\frac{1}{2}(a+b))^2$ and show that a minimizing sequence is a Cauchy sequence.) (ii) Show that $u \in U$ is the minimizer if and only if $\Re((f-u,v-u)_V) \le 0$ for all $v \in U$. (*Hint*: proceed as in the proof of Proposition 25.8.) (iii) Assuming that U is a (nontrivial) subspace of V, prove that the unique minimizer is characterized by $(f-u,v)_V = 0$ for all $v \in U$, and prove that the map $\Pi_U : V \ni f \mapsto u \in U$ is linear and $\|\Pi_U\|_{\mathcal{L}(V;U)} = 1$. (iv) Let a be a bounded, Hermitian, and coercive sesquilinear form (with $\xi := 1$ for simplicity). Let $\ell \in V'$. Set $\mathfrak{E}(v) := \frac{1}{2}a(v,v) - \ell(v)$. Show that there is a unique $u \in V$ such that $\mathfrak{E}(u) = \min_{v\in U}\mathfrak{E}(v)$ and that u is the minimizer if and only if $\Re(a(u,v-u) - \ell(v-u)) \ge 0$ for all $v \in U$.

Exercise 25.5 (Inf-sup constant). Let V be a Hilbert space, U a subset of V, and W a closed subspace of V. Let $\beta := \inf_{u\in U}\sup_{w\in W}\frac{|(u,w)_V|}{\|u\|_V\|w\|_W}$. (i) Prove that $\beta \in [0,1]$. (ii) Prove that $\beta = \inf_{u\in U}\frac{\|\Pi_W(u)\|_V}{\|u\|_V}$, where Π_W is the orthogonal projection onto W. (*Hint*: use Exercise 25.4.) (iii) Prove that $\|u - \Pi_W(u)\|_V \le (1-\beta^2)^{\frac{1}{2}}\|u\|_V$. (*Hint*: use the Pythagorean identity.)

Exercise 25.6 (Fixed-point argument). The goal of this exercise is to derive another proof of the Lax–Milgram lemma. Let $A \in \mathcal{L}(V;V)$ be defined by $(A(v),w)_V := a(v,w)$ for all $v,w \in V$ (note that we use an inner product to define A). Let L be the representative in V of the linear form $\ell \in V'$. Let λ be a positive real number. Consider the map $T_\lambda : V \to V$ s.t. $T_\lambda(v) := v - \lambda\xi(A(v) - L)$ for all $v \in V$. Prove that if λ is small enough, $\|T_\lambda(v) - T_\lambda(w)\|_V \le \rho_\lambda\|v-w\|_V$ for all $v,w \in V$ with $\rho_\lambda \in (0,1)$, and show that (25.6) is well-posed. (*Hint*: use Banach's fixed-point theorem.)

Exercise 25.7 (Coercivity as necessary condition). Let V be a reflexive Banach space and let $A \in \mathcal{L}(V;V')$ be a monotone self-adjoint operator; see Definition C.31. Prove that A is bijective if and only if A is coercive (with $\xi := 1$). (*Hint*: prove that $\Re(\langle A(v),w\rangle_{V',V}) \le \langle A(v),v\rangle_{V',V}^{\frac{1}{2}}\langle A(w),w\rangle_{V',V}^{\frac{1}{2}}$ for all $v,w \in V$.)

Exercise 25.8 (Darcy). Prove that the problem (24.14) is well-posed. (*Hint*: adapt the proof of Proposition 25.18.)

Exercise 25.9 (First-order PDE). Prove that the problem (24.21) is well-posed. (*Hint*: adapt the proof of Proposition 25.19.)

Exercise 25.10 (T-coercivity). Let V, W be Hilbert spaces. Prove that (BNB1)-(BNB2) are equivalent to the existence of a bijective operator $T \in \mathcal{L}(V;W)$ and a real number $\eta > 0$ such that $\Re(a(v,T(v))) \geq \eta\|v\|_V^2$ for all $v \in V$. (*Hint*: use J_W^{-1}, $(A^{-1})^*$, and the map J_V^{RF} from the Riesz–Fréchet theorem to construct T.)

Exercise 25.11 (Sign-changing diffusion). Let D be a Lipschitz domain D in $\mathbb{R}^d$ partitioned into two disjoint Lipschitz subdomains D_1 and D_2. Set $\Sigma := \partial D_1 \cap \partial D_2$, each having an intersection with ∂D of positive measure. Let κ_1, κ_2 be two real numbers s.t. $\kappa_1 > 0$ and $\kappa_2 < 0$. Set $\kappa(x) := \kappa_1 \mathbb{1}_{D_1}(x) + \kappa_2 \mathbb{1}_{D_2}(x)$ for all $x \in D$. Let $V := H_0^1(D)$ be equipped with the norm $\|\nabla v\|_{\boldsymbol{L}^2(D)}$. The goal is to show that the bilinear form $a(v,w) := \int_D \kappa \nabla v \cdot \nabla w$ satisfies conditions (BNB1)-(BNB2) on $V \times V$; see Chesnel and Ciarlet [118]. Set $V_m := \{v|_{D_m} \mid v \in V\}$ for all $m \in \{1,2\}$, equipped with the norm $\|\nabla v_m\|_{\boldsymbol{L}^2(D_m)}$ for all $v_m \in V_m$, and let $\gamma_{0,m}$ be the traces of functions in V_m on Σ. (i) Assume that there is $S_1 \in \mathcal{L}(V_1;V_2)$ s.t. $\gamma_{0,2}(S_1(v_1)) = \gamma_{0,1}(v_1)$. Define $T : V \to V$ s.t. for all $v \in V$, $T(v)(x) := v(x)$ if $x \in D_1$ and $T(v)(x) := -v(x) + 2S_1(v|_{D_1})(x)$ if $x \in D_2$. Prove that $T \in \mathcal{L}(V)$ and that T is an isomorphism. (*Hint*: verify that $T \circ T = I_V$, the identity in V.) (ii) Assume that $\frac{\kappa_1}{|\kappa_2|} > \|S_1\|^2_{\mathcal{L}(V_1;V_2)}$. Prove that the conditions (BNB1)-(BNB2) are satisfied. (*Hint*: use T-coercivity from Remark 25.14.) (iii) Let $D_1 := (-a,0)\times(0,1)$ and $D_2 := (0,b)\times(0,1)$ with $a > b > 0$. Show that if $\frac{\kappa_1}{|\kappa_2|} \notin [1, \frac{a}{b}]$, (BNB1)-(BNB2) are satisfied. (*Hint*: consider the map $S_1 \in \mathcal{L}(V_1;V_2)$ s.t. $S_1(v_1)(x,y) := v_1(-\frac{a}{b}x, y)$ for all $v_1 \in V_1$, and the map $S_2 \in \mathcal{L}(V_2;V_1)$ s.t. $S_2(v_2)(x,y) := v_2(-x,y)$ if $x \in (-b,0)$ and $S_2(v_2)(x,y) := 0$ otherwise, for all $v_2 \in V_2$.)