Part VI, Chapter 27

Error analysis with variational crimes

We have shown in the previous chapter how the Galerkin method can be used to approximate the solution to the model problem (26.1), and we have derived an error estimate in the simple setting where $V_h \subset V$, $W_h \subset W$, $a_h := a_{|V_h \times W_h}$, and $\ell_h := \ell_{|W_h}$. Departures from this setting are often called *variational crimes* in the literature. In this chapter, we perform the error analysis when variational crimes are committed. The main results, Lemma 27.5 and Lemma 27.8, will be invoked frequently in this book. They give an upper bound on the approximation error in terms of the best-approximation error of the exact solution by members of the discrete trial space. These error estimates are based on the notions of stability and consistency/boundedness. Combined with an approximability property, they allow us to conclude that the approximation method is convergent. Two simple examples illustrate the theory: a first-order PDE approximated by the Galerkin/least-squares technique and a second-order PDE approximated by a boundary penalty method.

27.1 Setting

In the entire chapter, we suppose that the assumptions of the BNB theorem (Theorem 25.9 or its variant Theorem 25.15) are satisfied, so that the exact problem (26.1) is well-posed. The inf-sup and boundedness constants on $V \times W$ of the exact sesquilinear form a are denoted by α and $\|a\|$; see (26.2). The exact solution is denoted by $u \in V$.

Recall that the Galerkin approximation (26.3) relies on the discrete trial space V_h and the discrete test space W_h. These spaces are equipped with the norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{W_h}$, respectively. The discrete problem uses a discrete sesquilinear form a_h defined on $V_h \times W_h$ and a discrete antilinear form ℓ_h defined on W_h. The sesquilinear form a_h and the antilinear form ℓ_h must be viewed, respectively, as some approximations to a and ℓ. The solution to the discrete problem (26.3) is denoted by $u_h \in V_h$. We always assume that

$\dim(V_h) = \dim(W_h)$, so that the well-posedness of the discrete problem is equivalent to the following inf-sup condition:

$$\inf_{v_h \in V_h} \sup_{w_h \in W_h} \frac{|a_h(v_h, w_h)|}{\|v_h\|_{V_h} \|w_h\|_{W_h}} =: \alpha_h > 0. \tag{27.1}$$

We say that the approximation (26.3) is *stable* whenever (27.1) holds true, i.e., $\alpha_h > 0$.

The goal of this chapter is to bound the error, i.e., we want to estimate how far the discrete solution $u_h \in V_h$ lies from the exact solution $u \in V$. We say that the method *converges* if the error tends to zero as the approximation capacity of the discrete trial space V_h increases. The approximation capacity of V_h increases by refining an underlying mesh. We will see that there are three key properties to establish convergence: (i) stability, (ii) consistency/boundedness, and (iii) approximability. Stability and approximability have already emerged as important notions in the error analysis presented in §26.3. The notion of consistency was present in the simple form of the Galerkin orthogonality property, and the boundedness of the sesquilinear form a on $V \times W$ was also invoked.

Remark 27.1 (Lax principle). A loose principle in numerical analysis, known as *Lax Principle*, is that stability and consistency imply convergence. The fact that boundedness and approximability are not mentioned does not mean that these properties should be taken for granted. We refer the reader to the upcoming chapters for numerous examples. □

Remark 27.2 (Norms). Since all the norms are equivalent in finite-dimensional vector spaces, if (27.1) holds true for one choice of norms in V_h and W_h, it holds true also for every other choice. The goal is to select norms s.t. (i) a_h is *uniformly stable*, i.e., $\alpha_h \geq \alpha_0 > 0$ for all $h \in \mathcal{H}$, and (ii) a_h is *uniformly bounded* on $V_h \times W_h$ with respect to $h \in \mathcal{H}$. □

27.2 Main results

This section contains our two main abstract error estimates.

27.2.1 The spaces V_{s} and $V_\sharp$

In a nonconforming approximation setting where $V_h \not\subset V$, the exact solution u and the discrete solution u_h may be objects of different nature. This poses the question of how to measure the approximation error. For instance, does the expression $(u - u_h)$ make sense? We are going to assume that it is possible to define a common ground between u and u_h to evaluate the error. A simple way to do this is to assume that it is meaningful to define the linear space $(V + V_h)$. If it is indeed the case, then the error belongs to this space.

However, we will see in numerous examples that the error analysis often requires to assume that the exact solution has slightly more smoothness than just being a member of V. We formalize this assumption by introducing a functional space $V_{\rm S}$ such that $u \in V_{\rm S} \subseteq V$. Our setting for the error analysis is therefore as follows:

$$u \in V_{\rm S} \subseteq V, \qquad u - u_h \in V_\sharp := V_{\rm S} + V_h. \tag{27.2}$$

Note that this setting allows for $V_{\rm S} := V$, and in the conforming setting, where $V_h \subset V$, this then implies that $V_\sharp := V$.

27.2.2 Consistency/boundedness

A crucial notion in the error analysis is that of consistency/boundedness. Loosely speaking the idea behind consistency is to insert the exact solution into the discrete equations and to verify that the discrepancy is small. This may not be possible in a nonconforming approximation setting because it may turn out that the discrete sesquilinear form a_h is not meaningful when its first argument is the exact solution. To stay general, we are going to define a consistency error for every discrete trial function $v_h \in V_h$ with the expectation that this error is small if the difference $(u - v_h) \in V_\sharp$ is small. Let us now formalize this idea. Recall that the norm of any antilinear form $\phi_h \in W_h' := \mathcal{L}(W_h;\mathbb{C})$ is defined by $\|\phi_h\|_{W_h'} := \sup_{w_h \in W_h} \frac{|\phi_h(w_h)|}{\|w_h\|_{W_h}}$.

Definition 27.3 (Consistency/boundedness). *Let $\delta_h : V_h \to W_h'$ be defined by setting*

$$\langle \delta_h(v_h), w_h \rangle_{W_h',W_h} := \ell_h(w_h) - a_h(v_h, w_h) = a_h(u_h - v_h, w_h). \tag{27.3}$$

The quantity $\|\delta_h(v_h)\|_{W_h'}$ is called consistency error *for the discrete trial function $v_h \in V_h$. We say that* consistency/boundedness *holds true if the space $V_\sharp$ can be equipped with a norm $\|\cdot\|_{V_\sharp}$ such that there is a real number $\omega_{\sharp h}$, uniform w.r.t. $u \in V_{\rm S}$, such that for all $v_h \in V_h$ and all $h \in \mathcal{H}$,*

$$\|\delta_h(v_h)\|_{W_h'} \le \omega_{\sharp h} \|u - v_h\|_{V_\sharp}. \tag{27.4}$$

Example 27.4 (Simple setting). Assume conformity (i.e., $V_h \subset V$ and $W_h \subset W$), $a_h := a_{|V_h \times W_h}$, and $\ell_h := \ell_{|W_h}$. Take $V_{\rm S} := V$, so that $V_\sharp := V$, and take $\|\cdot\|_{V_\sharp} := \|\cdot\|_V$. The consistency error (27.3) is such that

$$\langle \delta_h(v_h), w_h \rangle_{W_h',W_h} = \ell(w_h) - a(v_h, w_h) = a(u - v_h, w_h),$$

where we used that $\ell(w_h) = a(u, w_h)$ (i.e., the Galerkin orthogonality property). Since a is bounded on $V \times W$, (27.4) holds true with $\omega_{\sharp h} := \|a\|$. □

27.2.3 Error estimate using one norm

We can now establish our first abstract error estimate. This estimate will be applied to various nonconforming approximation settings of elliptic PDEs. It hinges on the assumption that there is a real number $c_\sharp$, uniform w.r.t. $h \in \mathcal{H}$, s.t.

$$\|v_h\|_{V_\sharp} \le c_\sharp \|v_h\|_{V_h}, \qquad \forall v_h \in V_h. \tag{27.5}$$

Recall that $\|\cdot\|_{V_h}$ is the stability norm on V_h used in (27.1) and $\|\cdot\|_{V_\sharp}$ is the consistency/boundedness norm on $V_\sharp$ used in (27.4).

Lemma 27.5 (Quasi-optimal error estimate). *Assume the following:* (i) Stability, *i.e.,* (27.1) *holds true;* (ii) Consistency/boundedness, *i.e.,* $u \in V_{\text{S}}$ *and* (27.4) *holds true. Assume that* (27.5) *holds true. Then we have*

$$\|u - u_h\|_{V_\sharp} \le \left(1 + c_\sharp \frac{\omega_{\sharp h}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}. \tag{27.6}$$

Proof. Owing to the assumptions, we infer that for all $v_h \in V_h$,

$$\begin{aligned}
\|u - u_h\|_{V_\sharp} &\le \|u - v_h\|_{V_\sharp} + \|v_h - u_h\|_{V_\sharp} \\
&\le \|u - v_h\|_{V_\sharp} + c_\sharp \|v_h - u_h\|_{V_h} \\
&\le \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \sup_{w_h \in W_h} \frac{|a_h(u_h - v_h, w_h)|}{\|w_h\|_{W_h}} \\
&= \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \|\delta_h(v_h)\|_{W_h'} \\
&\le \|u - v_h\|_{V_\sharp} + \frac{c_\sharp \omega_{\sharp h}}{\alpha_h} \|u - v_h\|_{V_\sharp}.
\end{aligned}$$

Taking the infimum over $v_h \in V_h$ yields (27.6). □

Example 27.6 (Simple setting). In the setting of Example 27.4, we can equip V_h and $V_\sharp$ with the norm $\|\cdot\|_V$, so that $c_\sharp = 1$. Since $\omega_{\sharp h} = \|a\|$, the error estimate (27.6) coincides with the error estimate in Lemma 26.14. □

Remark 27.7 (Literature). A general framework for the error analysis of nonconforming methods for elliptic PDEs can be found in Veeser and Zanotti [373]. This framework introduces a different notion of consistency and leads to quasi-optimal error estimates in the $\|\cdot\|_V$-norm without any smoothness assumption on the exact solution $u \in V$ (or equivalently for all data $\ell \in V'$), i.e., the space V_{S} and the norm $\|\cdot\|_{V_\sharp}$ are not invoked. This remarkable result is achieved at the expense of a specific design of the discrete form ℓ_h. We also refer the reader to the gradient discretization method discussed in Droniou et al. [172] which can be used to analyze nonconforming methods. □

27.2.4 Error estimate using two norms

It turns out that the assumption (27.5) on the $\|\cdot\|_{V_\sharp}$-norm cannot be satisfied when one considers the approximation of first-order PDEs using stabilization techniques. A more general setting consists of introducing a second norm on $V_\sharp$, say $\|\cdot\|_{V_\flat}$, and assuming that there exists a real number $c_\flat$ s.t.

$$\|v_h\|_{V_\flat} \le c_\flat \|v_h\|_{V_h}, \ \forall v_h \in V_h, \qquad \|v\|_{V_\flat} \le c_\flat \|v\|_{V_\sharp}, \ \forall v \in V_\sharp, \tag{27.7}$$

where $\|\cdot\|_{V_h}$ is the stability norm on V_h used in (27.1) and $\|\cdot\|_{V_\sharp}$ is the consistency/boundedness norm on $V_\sharp$ used in (27.4).

Lemma 27.8 (Error estimate). *Assume the following:* (i) Stability*, i.e.,* (27.1) *holds true;* (ii) Consistency/boundedness*, i.e.,* $u \in V_{\mathrm{S}}$ *and* (27.4) *holds true. Assume that* (27.7) *holds true. Then we have*

$$\|u - u_h\|_{V_\flat} \le c_\flat \left(1 + \frac{\omega_{\sharp h}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}. \tag{27.8}$$

Proof. The proof is similar to that of Lemma 27.5. Owing to the assumptions, we infer that for all $v_h \in V_h$,

$$\begin{aligned}
\|u - u_h\|_{V_\flat} &\le \|u - v_h\|_{V_\flat} + \|v_h - u_h\|_{V_\flat} \\
&\le c_\flat \|u - v_h\|_{V_\sharp} + c_\flat \|v_h - u_h\|_{V_h} \\
&\le c_\flat \|u - v_h\|_{V_\sharp} + \frac{c_\flat}{\alpha_h} \sup_{w_h \in W_h} \frac{|a_h(u_h - v_h, w_h)|}{\|w_h\|_{W_h}} \\
&= c_\flat \|u - v_h\|_{V_\sharp} + \frac{c_\flat}{\alpha_h} \|\delta_h(v_h)\|_{W_h'} \\
&\le c_\flat \|u - v_h\|_{V_\sharp} + \frac{c_\flat \omega_{\sharp h}}{\alpha_h} \|u - v_h\|_{V_\sharp}.
\end{aligned}$$

Taking the infimum over $v_h \in V_h$ yields (27.8). □

Remark 27.9 (Lemma 27.5 vs. Lemma 27.8). Lemma 27.5 estimates the approximation error by the best-approximation error using the same norm $\|\cdot\|_{V_\sharp}$. We say that this estimate is quasi-optimal *over the whole computational range*. In contrast, Lemma 27.8 estimates the approximation error in the $\|\cdot\|_{V_\flat}$-norm by the best-approximation error in the stronger $\|\cdot\|_{V_\sharp}$-norm. We will see numerous examples where the best-approximation errors in both norms actually exhibit the same decay rate in terms of the meshsize $h \in \mathcal{H}$ for smooth solutions. In this situation, we say that the error estimate from Lemma 27.8 is quasi-optimal *in the asymptotic range*. □

27.2.5 Convergence

We are now ready to state a convergence result. The last missing ingredient that we introduce now is approximability.

Corollary 27.10 (Convergence). *We have* $\lim_{h\to 0}\|u-u_h\|_{V_\sharp}=0$ *in the setting of Lemma* 27.5 *and* $\lim_{h\to 0}\|u-u_h\|_{V_\flat}=0$ *in the setting of Lemma* 27.8*, provided the following properties hold true:*

(i) Uniform stability: $\alpha_h \geq \alpha_0 > 0$ *for all* $h\in\mathcal{H}$;
(ii) Uniform consistency/boundedness: $\omega_{\sharp h}\leq \omega_{\sharp 0}<\infty$ *for all* $h\in\mathcal{H}$;
(iii) Approximability: $\lim_{h\to 0}(\inf_{v_h\in V_h}\|v-v_h\|_{V_\sharp})=0$ *for all* $v\in V_{\rm S}$.

Proof. Direct consequence of the assumptions. □

27.3 Two simple examples

This section presents two one-dimensional examples illustrating how to use the above error estimates: (i) a boundary penalty method applied to an elliptic PDE where Lemma 27.5 is applied; (ii) a stabilized approximation applied to a first-order PDE where Lemma 27.8 is applied.

27.3.1 Boundary penalty method for an elliptic PDE

Consider the PDE $-u''=f$ in $D:=(0,1)$ with $u(0)=u(1)=0$, $f\in L^2(D)$. The trial and test spaces are $V=W:=H^1_0(D)$. The corresponding bilinear and linear forms are $a(v,w):=\int_0^1 v'w'\,\mathrm{d}t$ and $\ell(w):=\int_0^1 fw\,\mathrm{d}t$. Consider the standard Galerkin approximation using as discrete trial and test spaces the spaces $V_h=W_h$ built using continuous $\mathbb{P}_1$ Lagrange finite elements on a uniform mesh $\mathcal{T}_h$ of step $h\in\mathcal{H}$. We do not enforce any boundary condition on V_h. As a result, the approximation setting is *nonconforming*. Let us define the discrete forms

$$\begin{aligned} a_h(v_h,w_h) &:= \int_0^1 v_h'w_h'\,\mathrm{d}t - (v_h'(1)w_h(1)-v_h'(0)w_h(0)) \\ &\quad + h^{-1}(v_h(1)w_h(1)+v_h(0)w_h(0)), \\ \ell_h(w_h) &:= \int_0^1 fw_h\,\mathrm{d}t. \end{aligned}$$

One can show that coercivity holds true with the stability norm

$$\|v_h\|^2_{V_h} := \|v_h'\|^2_{L^2(D)} + h^{-1}|v_h(0)|^2 + h^{-1}|v_h(1)|^2,$$

i.e., $a_h(v_h,v_h)\geq \alpha_0\|v_h\|^2_{V_h}$ with $\alpha_0:=\frac{3}{8}$ for all $v_h\in V_h$; see Exercise 27.2 and Chapter 37.

Let us perform the error analysis using Lemma 27.5. The assumption $u\in V_{\rm S}:=H^2(D)\cap H^1_0(D)$ is natural here since $f\in L^2(D)$ and $-u''=f$. We equip the space $V_\sharp:=V_{\rm S}+V_h$ with the norm

$$\|v\|^2_{V_\sharp} := \|v'\|^2_{L^2(D)} + h^{-1}|v(0)|^2 + h^{-1}|v(1)|^2 + h|v'(0)|^2 + h|v'(1)|^2.$$

(Recall that $H^2(D) \hookrightarrow C^1(\overline{D})$ in one dimension.) Using a discrete trace inequality shows that the norms $\|\cdot\|_{V_h}$ and $\|\cdot\|_{V_\sharp}$ are equivalent on V_h uniformly w.r.t. $h \in \mathcal{H}$. Hence, (27.5) holds true. It remains to establish consistency/boundedness. Since $u \in H^2(D)$, integrating by parts leads to

$$\ell_h(w_h) = -\int_0^1 u'' w_h \,\mathrm{d}t = \int_0^1 u' w_h' \,\mathrm{d}t - (u'(1)w_h(1) - u'(0)w_h(0)),$$

so that letting $\eta := u - v_h$ and since $u(0) = u(1) = 0$, we obtain

$$\begin{aligned}\langle \delta_h(v_h), w_h \rangle_{V_h', V_h} &= \ell_h(w_h) - a_h(v_h, w_h) \\ &= \int_0^1 \eta' w_h' \,\mathrm{d}t - (\eta'(1)w_h(1) - \eta'(0)w_h(0)) \\ &\quad + h^{-1}(\eta(1)w_h(1) + \eta(0)w_h(0)).\end{aligned}$$

Using the Cauchy–Schwarz inequality, we conclude that (27.4) holds true with $\omega_{\sharp h} = 1$. In conclusion, Lemma 27.5 implies that

$$\|u - u_h\|_{V_\sharp} \le c \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}. \tag{27.9}$$

Since $u \in H^2(D)$, we use the approximation properties of finite elements to obtain $\inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp} \le ch|u|_{H^2(D)}$, so that

$$\|u - u_h\|_{V_\sharp} \le c\, h|u|_{H^2(D)}. \tag{27.10}$$

This shows that the error in the $\|\cdot\|_{V_\sharp}$-norm tends to zero at rate h.

27.3.2 Stabilized approximation of a first-order PDE

Consider the PDE $u' = f$ in $D := (0,1)$ with $u(0) = 0$ and $f \in L^2(D)$. Following §24.2.2, we consider the L^2-based weak formulation with the trial space $V := \{v \in H^1(D) \mid v(0) = 0\}$ and the test space $W := L^2(D)$. The exact forms are $a(v,w) := \int_0^1 v'w \,\mathrm{d}t$ and $\ell(w) := \int_0^1 fw \,\mathrm{d}t$. The model problem consists of seeking $u \in V$ such that $a(u,w) = \ell(w)$ for all $w \in W$. This problem is well-posed; see Exercise 25.9.

Consider the standard Galerkin approximation using as discrete trial and test spaces the space V_h built by using continuous $\mathbb{P}_1$ Lagrange finite elements on a uniform mesh $\mathcal{T}_h$ of step $h \in \mathcal{H}$ and by enforcing the boundary condition $v_h(0) = 0$. The discrete problem consists of seeking $u_h \in V_h$ such that $a(u_h, w_h) = \ell(w_h)$ for all $w_h \in V_h$. (The reader is invited to verify that the resulting linear system is identical to that obtained with centered finite differences.) The approximation setting is *conforming* since $V_h \subset V$ and $W_h = V_h \subset W$. Unfortunately, it turns out that the bilinear form a is not uniformly stable on $V_h \times V_h$. Indeed, one can show (see Exercise 27.3) that there are $0 < c_1 \le c_2$ s.t. for all $h \in \mathcal{H}$,

$$c_1\, h \leq \inf_{v_h\in V_h} \sup_{w_h\in V_h} \frac{|a(v_h,w_h)|}{\|v_h\|_{H^1(D)}\|w_h\|_{L^2(D)}} =: \alpha_h \leq c_2\, h. \tag{27.11}$$

This result shows that the above naive Galerkin approximation of first-order PDEs cannot produce optimal error estimates, even though it yields an invertible linear system ($c_1 \neq 0$). In practice, this problem manifests itself through the presence of spurious wiggles in the approximate solution. To circumvent this difficulty, let us define the discrete bilinear and linear forms

$$a_h(v_h,w_h) := \int_0^1 (v_h' w_h + h v_h' w_h')\,\mathrm{d}t, \qquad \ell_h(w_h) := \int_0^1 f(w_h + h w_h')\,\mathrm{d}t,$$

for all $v_h, w_h \in V_h$. Referring to Exercise 27.4 (see also §57.3 and §61.4), one can establish the uniform inf-sup condition

$$\inf_{v_h\in V_h} \sup_{w_h\in V_h} \frac{|a_h(v_h,w_h)|}{\|v_h\|_{V_h}\|w_h\|_{V_h}} \geq \alpha_0 > 0, \tag{27.12}$$

with the stability norm

$$\|v_h\|_{V_h}^2 := \ell_D^{-1}\|v_h\|_{L^2(D)}^2 + |v_h(1)|^2 + h\|v_h'\|_{L^2(D)}^2,$$

where we introduced the length scale $\ell_D := 1$ to be dimensionally consistent.

Let us perform the error analysis using Lemma 27.8. We set $V_{\mathrm{S}} := V$ so that $V_\sharp = V + V_h = V$, and we equip $V_\sharp$ with the following norms (recall that $H^1(D) \hookrightarrow C^0(\overline{D})$ in one dimension):

$$\|v\|_{V_\flat} := \ell_D^{-1}\|v\|_{L^2(D)}^2 + |v(1)|^2 + h\|v'\|_{L^2(D)}^2, \tag{27.13}$$

$$\|v\|_{V_\sharp}^2 := h^{-1}\|v\|_{L^2(D)}^2 + |v(1)|^2 + h\|v'\|_{L^2(D)}^2, \tag{27.14}$$

so that (27.7) holds true with $c_\flat := 1$ since $h \leq \ell_D$. Notice that there is no uniform constant $c_\sharp$ s.t. (27.5) holds true, i.e., we cannot apply Lemma 27.5. To apply Lemma 27.8, it remains to establish consistency/boundedness. Since $u' = f$ in D, letting $\eta := u - v_h$, we infer that

$$\begin{aligned}\langle \delta_h(v_h), w_h\rangle_{V_h',V_h} &= \ell_h(w_h) - a_h(v_h,w_h) \\ &= \int_0^1 f(w_h + h w_h')\,\mathrm{d}t - \int_0^1 (v_h' w_h + h v_h' w_h')\,\mathrm{d}t \\ &= \int_0^1 (\eta' w_h + h\eta' w_h')\,\mathrm{d}t =: \mathfrak{T}_1 + \mathfrak{T}_2.\end{aligned}$$

Integrating by parts, we obtain

$$\mathfrak{T}_1 = \int_0^1 \eta' w_h\,\mathrm{d}t = -\int_0^1 \eta w_h'\,\mathrm{d}t + \eta(1)w_h(1),$$

since $\eta(0) = 0$. Using the Cauchy–Schwarz inequality, we infer that

$$|\mathfrak{T}_1| \leq h^{-\frac{1}{2}}\|\eta\|_{L^2(D)}h^{\frac{1}{2}}\|w_h'\|_{L^2(D)} + |\eta(1)|\,|w_h(1)| \leq \|\eta\|_{V_\sharp}\|w_h\|_{V_h},$$
$$|\mathfrak{T}_2| \leq h^{\frac{1}{2}}\|\eta'\|_{L^2(D)}h^{\frac{1}{2}}\|w_h'\|_{L^2(D)} \leq \|\eta\|_{V_\sharp}\|w_h\|_{V_h},$$

which shows that (27.4) holds true with $\omega_{\sharp h} := 2$. In conclusion, Lemma 27.8 implies that

$$\|u - u_h\|_{V_\flat} \leq c \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp}. \tag{27.15}$$

Assuming that $u \in H^{1+r}(D)$, $r \in [0,1]$, we use the approximation properties of finite elements to obtain $\inf_{v_h\in V_h} \|u - v_h\|_{V_\sharp} \leq ch^{\frac{1}{2}+r}|u|_{H^{1+r}(D)}$, so that

$$\|u - u_h\|_{V_\flat} \leq c\, h^{\frac{1}{2}+r}|u|_{H^{1+r}(D)}. \tag{27.16}$$

The error estimate (27.15) is quasi-optimal in the asymptotic range since the best-approximation errors in the $\|\cdot\|_{V_\flat}$- and $\|\cdot\|_{V_\sharp}$-norms converge to zero at the same rate (see Remark 27.9 for the terminology).

27.4 Strang's lemmas

We review in this section results due to Strang [358] and often called Strang's lemmas in the literature. These lemmas are historically important for the development of the analysis of finite element methods. In this book, we are going to use systematically Lemma 27.5 and Lemma 27.8 and only use Strang's lemmas at a few instances.

There are two Strang's lemmas: the first one is tailored to conforming approximations but allows for $a_h \neq a$ and $\ell_h \neq \ell$, and the second one can be applied to nonconforming approximations. Both lemmas can be seen as variants of Lemma 27.5 and Lemma 27.8, where the consistency error $\|\delta_h(v_h)\|_{W_h'}$ is further decomposed by adding/subtracting some terms so as to separate the approximation of a by a_h and the approximation of ℓ by ℓ_h (these contributions are sometimes called consistency error in the literature) from the best-approximation error of u by a function in V_h.

Remark 27.11 (Consistency). One should bear in mind that the notion of consistency in Strang's lemmas is somewhat arbitrary. This is illustrated in §27.4.3, where each lemma leads to a different notion of consistency for the same approximation method. We think that it is preferable to use the quantity $\|\delta_h(v_h)\|_{W_h'}$ defined in (27.3) as the only notion of consistency. This is the convention we are going to follow in the rest of the book. □

27.4.1 Strang's first lemma

Strang's first lemma is tailored to conforming approximations. It has been devised to estimate the error due to quadratures when approximating elliptic PDEs by H^1-conforming finite elements (see §33.3).

Lemma 27.12 (Strang 1). *Assume:* (i) Conformity: $V_h \subset V$ *and* $W_h \subset W$, *and set* $V_{\text{S}} := V$ *so that* $V_\sharp := V + V_h = V$; (ii) Stability: (27.1) *holds true;* (iii) Boundedness: *the sesquilinear form* a *is bounded on* $V \times W_h$, *and set*

$$\|a\|_{\sharp h} := \sup_{v \in V} \sup_{w_h \in W_h} \frac{|a(v, w_h)|}{\|v\|_{V_\sharp} \|w_h\|_{W_h}}, \tag{27.17}$$

where the norm $\|\cdot\|_{V_\sharp}$ *satisfies* (27.5). *Let* $\delta_h^{\text{St1}} : V_h \to W_h'$ *be defined by*

$$\langle \delta_h^{\text{St1}}(v_h), w_h \rangle_{W_h', W_h} := \ell_h(w_h) - \ell(w_h) + a(v_h, w_h) - a_h(v_h, w_h). \tag{27.18}$$

Then the following holds true:

$$\|u - u_h\|_{V_\sharp} \le \inf_{v_h \in V_h} \left[\left(1 + c_\sharp \frac{\|a\|_{\sharp h}}{\alpha_h}\right) \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \|\delta_h^{\text{St1}}(v_h)\|_{W_h'} \right]. \tag{27.19}$$

Proof. Proceeding as in the proof of Lemma 27.5 leads to

$$\|u - u_h\|_{V_\sharp} \le \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \|\delta_h(v_h)\|_{W_h'}.$$

We write the consistency error as follows:

$$\begin{aligned}
\langle \delta_h(v_h), w_h \rangle_{W_h', W_h} &:= \ell_h(w_h) - a_h(v_h, w_h) \\
&= \ell_h(w_h) - \ell(w_h) + a(u, w_h) - a_h(v_h, w_h) \\
&= \ell_h(w_h) - \ell(w_h) + a(u, w_h) - a_h(v_h, w_h) + [a(v_h, w_h) - a(v_h, w_h)] \\
&= \langle \delta_h^{\text{St1}}(v_h), w_h \rangle_{W_h', W_h} + a(u - v_h, w_h),
\end{aligned}$$

where we used that $a(u, w_h) = \ell(w_h)$ since $W_h \subset W$. Using the triangle inequality and the boundedness property (27.17), we infer that

$$\|\delta_h(v_h)\|_{W_h'} \le \|\delta_h^{\text{St1}}(v_h)\|_{W_h'} + \|a\|_{\sharp h} \|u - v_h\|_{V_\sharp}.$$

Rearranging the terms leads to the expected estimate. □

Remark 27.13 (Comparison). In the original statement of Strang's first lemma, one takes $\|\cdot\|_{V_\sharp} := \|\cdot\|_V$, and one equips V_h with the $\|\cdot\|_V$-norm, so that the error estimate (27.19) holds true with $\|a\|_{\sharp h} := \|a\|$. Moreover, the terms $\ell_h(w_h) - \ell(w_h)$ and $a(v_h, w_h) - a_h(v_h, w_h)$ composing $\langle \delta_h^{\text{St1}}(v_h), w_h \rangle_{W_h', W_h}$ are separated, and the term $\|\ell_h - \ell\|_{W_h'}$ is taken out of the infimum over $v_h \in V_h$ in (27.19). The original statement is sufficient to analyze quadrature errors in the H^1-conforming approximation of elliptic PDEs, but as illustrated in §27.4.3, Strang's first lemma is not well adapted to analyze stabilized finite element approximations of first-order PDEs, since in this case one needs to invoke the two norms $\|\cdot\|_{V_\flat}$ and $\|\cdot\|_{V_\sharp}$ defined in (27.13)-(27.14). □

Remark 27.14 (Nonconforming setting). It is possible to derive an error estimate in the spirit of Strang's first lemma in some nonconforming settings. Following Gudi [226], the idea is to introduce an operator $T : W_h \to W$ acting on the discrete test functions. This operator can be built using the averaging operators analyzed in §22.2. We refer the reader to [226] and Exercise 27.5 for error estimates obtained with this technique. □

27.4.2 Strang's second lemma

Contrary to Strang's first lemma, the second lemma is applicable to nonconforming approximation settings.

Lemma 27.15 (Strang 2). *Let* $V_{\mathrm{S}} := V$ *so that* $V_\sharp := V + V_h$*. Assume:* (i) Stability: (27.1) *holds true;* (ii) Bounded extendibility*: There exists a bounded sesquilinear form* $a_\sharp$ *on* $V_\sharp \times W_h$ *that extends* a_h *originally defined on* $V_h \times W_h$*, i.e.,* $a_\sharp(v_h, w_h) = a_h(v_h, w_h)$ *for all* $(v_h, w_h) \in V_h \times W_h$ *and*

$$\|a_\sharp\|_{\sharp h} := \sup_{v \in V_\sharp} \sup_{w_h \in W_h} \frac{|a_\sharp(v, w_h)|}{\|v\|_{V_\sharp} \|w_h\|_{W_h}} < \infty, \tag{27.20}$$

with a norm $\|\cdot\|_{V_\sharp}$ *satisfying* (27.5)*. The following holds true:*

$$\|u - u_h\|_{V_\sharp} \le \left(1 + c_\sharp \frac{\|a_\sharp\|_{\sharp h}}{\alpha_h}\right) \inf_{v_h \in V_h} \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \|\delta_h^{\mathrm{St2}}(u)\|_{W_h'}, \tag{27.21}$$

with $\langle \delta_h^{\mathrm{St2}}(u), w_h \rangle_{W_h', W_h} := \ell_h(w_h) - a_\sharp(u, w_h)$*.*

Proof. The starting point is again the bound

$$\|u - u_h\|_{V_\sharp} \le \|u - v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h} \|\delta_h(v_h)\|_{W_h'}.$$

Now we write the consistency error as follows:

$$\begin{aligned} \langle \delta_h(v_h), w_h \rangle_{W_h', W_h} &:= \ell_h(w_h) - a_h(v_h, w_h) = \ell_h(w_h) - a_\sharp(v_h, w_h) \\ &= \ell_h(w_h) - a_\sharp(v_h, w_h) + [a_\sharp(u, w_h) - a_\sharp(u, w_h)] \\ &= a_\sharp(u - v_h, w_h) + \langle \delta_h^{\mathrm{St2}}(u), w_h \rangle_{W_h', W_h}. \end{aligned}$$

Using the triangle inequality and the boundedness property (27.20), we infer that

$$\|\delta_h(v_h)\|_{W_h'} \le \|\delta_h^{\mathrm{St2}}(u)\|_{W_h'} + \|a_\sharp\|_{\sharp h} \|u - v_h\|_{V_\sharp}.$$

Rearranging the terms leads to the expected estimate. □

Remark 27.16 (Strong consistency, quasi-optimality). Recalling the Galerkin orthogonality terminology introduced in the context of conforming approximations (see §26.3.1), we say that *strong consistency* holds true if $\delta_h^{\mathrm{St2}}(u)$ vanishes identically on W_h, i.e., if the exact solution satisfies the discrete equations rewritten using the extended sesquilinear form $a_\sharp$. In this case, (27.21) leads to a quasi-optimal error estimate. □

Remark 27.17 (Bounded extendibility). Lemma 27.15 has been originally devised to analyze the Crouzeix–Raviart approximation of elliptic PDEs (see Chapter 36). In this context, the bounded extendibility assumption is indeed reasonable. However, it is no longer satisfied if a boundary penalty method or a discontinuous Galerkin method is used (see Chapters 37 and 38). For such methods, it is possible to recover the bounded extendibility assumption (and to prove strong consistency) provided the exact solution satisfies an additional smoothness assumption which is typically of the form $u \in H^{1+r}(D)$ with regularity pickup $r > \frac{1}{2}$. We will see that the error analysis based on Lemma 27.5 is more general since it only requires a regularity pickup $r > 0$ in the Sobolev scale. There are also other situations where the bounded extendibility assumption is simply not reasonable, e.g., when considering quadratures using point values or for stabilization techniques based on a two-scale hierarchical decomposition of the discrete spaces that is not meaningful for nondiscrete functions (see Chapter 59). □

27.4.3 Example: first-order PDE

Let us consider the first-order PDE and the discrete setting introduced in §27.3.2, and let us briefly illustrate how to estimate the error using Strang's lemmas in this context. Using Strang's first lemma, one finds that

$$\begin{aligned}\langle \delta_h^{\mathrm{St1}}(v_h), w_h\rangle_{V_h',V_h} &:= \ell_h(w_h) - \ell(w_h) + a(v_h, w_h) - a_h(v_h, w_h)\\ &= \int_0^1 h(f - v_h')w_h' \,\mathrm{d}t = \int_0^1 h\eta' w_h' \,\mathrm{d}t,\end{aligned}$$

since $f = u'$ and $\eta := u - v_h$, so that $\|\delta_h^{\mathrm{St1}}(v_h)\|_{V_h'} \le h^{\frac{1}{2}}\|\eta'\|_{L^2(D)} \le \|\eta\|_{V_\sharp}$, where $\|\cdot\|_{V_\sharp}$ is defined in (27.14). One also has $\|a\|_{\sharp h} \le \ell_D^{\frac{1}{2}} h^{-\frac{1}{2}}$. In conclusion, $\|u - u_h\|_{V_\sharp} \le (1 + \alpha_0^{-1}(\ell_D^{\frac{1}{2}} h^{-\frac{1}{2}} + 1)) \inf_{v_h\in V_h} \|u - v_h\|_{V_\sharp}$, which yields the suboptimal error estimate $\|u - u_h\|_{V_\sharp} \le ch^r \ell_D^{\frac{1}{2}} |u|_{H^{1+r}(D)}$ for all $r \in [0, 1]$ (compare with (27.16)). Using instead Strang's second lemma, one finds that $\langle \delta_h^{\mathrm{St2}}(u), w_h\rangle_{V_h',V_h} := \ell_h(w_h) - a_\sharp(u, w_h) = 0$ for all $w_h \in V_h$, i.e., strong consistency holds true, and one obtains again the suboptimal error estimate $\|u - u_h\|_{V_\sharp} \le ch^r \ell_D^{\frac{1}{2}} |u|_{H^{1+r}(D)}$. This example shows that the two Strang lemmas may lead to different notions of consistency, and, if applied blindly, they may yield suboptimal error estimates.

Exercises

Exercise 27.1 (Error identity). Assume stability, i.e., (27.1) holds true. Let $V_\sharp$ be defined in (27.2) and equip this space with a norm $\|\cdot\|_{V_\flat}$ s.t. there is $c_\flat$ s.t. $\|v_h\|_{V_\flat} \le c_\flat \|v_h\|_{V_h}$ for all $v_h \in V_h$. Prove that

$$\|u-u_h\|_{V_\flat} = \inf_{v_h\in V_h}\left[\|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\|\delta_h(v_h)\|_{W_h'}\right].$$

Exercise 27.2 (Boundary penalty). (i) Prove that $x^2 - 2\beta xy + \eta_0 y^2 \ge \frac{\eta_0-\beta^2}{1+\eta_0}(x^2+y^2)$ for all real numbers x, y, $\eta_0 \ge 0$ and $\beta \ge 0$. (ii) Using the notation of §27.3.1, prove that $a_h(v_h,v_h) \ge \frac{3}{8}\|v_h\|_{V_h}^2$ for all $v_h \in V_h$. (*Hint*: prove that $|v_h'(0)v_h(0)| \le \|v_h'\|_{L^2(0,h)} h^{-\frac12}|v_h(0)|$.)

Exercise 27.3 (First-order PDE). The goal is to prove (27.11). (i) Prove that $h^{-\frac12}\|G(v_h)\|_{\ell^2(\mathbb{R}^I)} \le \sup_{w_h\in V_h}\frac{|a(v_h,w_h)|}{\|w_h\|_{L^2(D)}} \le \sqrt{6}h^{-\frac12}\|G(v_h)\|_{\ell^2(\mathbb{R}^I)}$, where $G_i(v_h) := a(v_h,\varphi_i)$ for all $i \in \{1{:}I\}$ with $I := \dim(V_h)$. (*Hint*: use Simpson's rule to compare Euclidean norms of component vectors and L^2-norms of functions.) (ii) Assume that I is even (the odd case is treated similarly). Prove that $\alpha_h \le c_2 h$. (*Hint*: consider the oscillating function v_h s.t. $v_h(x_{2i}) := 2ih$ for all $i \in \{1{:}\frac{I}{2}\}$ and $v_h(x_{2i+1}) := 1$ for all $i \in \{0{:}\frac{I}{2}-1\}$.) (iii) Prove that $\alpha_h \ge c_1 h$. (*Hint*: prove that $\max_{i\in\{1:I\}}|v_h(x_i)| \le 2\sum_{k\in\{1:I\}}|G_k(v_h)|$.) (iv) Prove that $\inf_{v_h\in V_h}\sup_{w_h\in W_h}\frac{|a(v_h,w_h)|}{\|v_h\|_{W^{1,1}(D)}\|w_h\|_{L^\infty(D)}} \ge \alpha_0 > 0$ with $W_h := \{w_h \in L^\infty(D) \mid \forall i \in \{0{:}I-1\},\ w_{h|[x_i,x_{i+1}]} \in \mathbb{P}_0\}$. (*Hint*: see Proposition 25.19.)

Exercise 27.4 (GaLS 1D). The goal is to prove (27.12). Let $v_h \in V_h$. (i) Compute $a_h(v_h,v_h)$. (ii) Let $\zeta(x) := -2x/\ell_D$, set $\zeta_h := \mathcal{I}_h^{\mathrm{b}}(\zeta)$, and show that $a_h(v_h,\mathcal{J}_h^{\mathrm{av}}(\zeta_h v_h)) \ge \frac12\ell_D^{-1}\|v_h\|_{L^2(D)}^2 - c_1 a(v_h,v_h)$ uniformly w.r.t. $h\in\mathcal{H}$, $\mathcal{J}_h^{\mathrm{av}}$ is the averaging operator defined in (22.9), and $\mathcal{I}_h^{\mathrm{b}}$ is the L^2-projection on the functions that are piecewise constant over the mesh. (iii) Prove (27.12). (*Hint*: use the test function $z_h := 2\mathcal{J}_h^{\mathrm{av}}(\zeta_h v_h) + 2(c_1+1)v_h$.)

Exercise 27.5 (Nonconforming Strang 1). Let $T : W_h \to W\cap W_h$. Let $V_{\mathrm{s}} := V$ so that $V_\sharp := V + V_h$, and assume that $V_\sharp$ is equipped with a norm $\|\cdot\|_{V_\sharp}$ satisfying (27.5). (i) Assume that a_h can be extended to $V_h\times(W+W_h)$. Assume that there is $\|a\|_{\sharp h}$ s.t. consistency/boundedness holds true in the form $|a(u,T(w_h)) - a_h(v_h,T(w_h))| \le \|a\|_{\sharp h}\|u-v_h\|_{V_\sharp}\|w_h\|_{W_h}$. Prove that

$$\|u-u_h\|_{V_\sharp} \le \inf_{v_h\in V_h}\left[\left(1+c_\sharp\frac{\|a\|_{\sharp h}}{\alpha_h}\right)\|u-v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h}\|\hat{\delta}_h^{\mathrm{St1}}(v_h)\|_{W_h'}\right],$$

with $\|\hat{\delta}_h^{\mathrm{St1}}(v_h)\|_{W_h'} := \|\ell_h - \ell\circ T + a_h(v_h,T(\cdot)) - a_h(v_h,\cdot)\|_{W_h'}$. (*Hint*: add/subtract $a_h(v_h,T(w_h))$.) (ii) We now derive another error estimate that avoids extending a_h but restricts the discrete trial functions to $V_h\cap V$ (this is reasonable provided the subspace $V_h\cap V$ has approximation properties that are similar to those of V_h). Assuming that there is $\|a\|_{V\times W_h}$ s.t. boundedness holds true in the form $|a(u-v_h, T(w_h))| \leq \|a\|_{V\times W_h}\|u-v_h\|_{V_\sharp}\|w_h\|_{W_h}$, prove that

$$\|u-u_h\|_{V_\sharp} \leq \inf_{v_h\in V_h\cap V}\left[\left(1+c_\sharp\frac{\|a\|_{V\times W_h}}{\alpha_h}\right)\|u-v_h\|_{V_\sharp}+\frac{c_\sharp}{\alpha_h}\|\check{\delta}_h^{\mathrm{St1}}(v_h)\|_{W_h'}\right],$$

with $\|\check{\delta}_h^{\mathrm{St1}}(v_h)\|_{W_h'} := \|\ell_h-\ell\circ T+a(v_h,T(\cdot))-a_h(v_h,\cdot)\|_{W_h'}$. (*Hint*: add/subtract $a(v_h,T(w_h))$.)

Exercise 27.6 (Orthogonal projection). Consider the setting of Exercise 25.4 with real vector spaces and coercivity with $\xi := 1$ for simplicity. Let u be the unique element in V such that $a(u,v-u) \geq \ell(v-u)$ for all $v\in U$. Let V_h be a finite-dimensional subspace of V, and let U_h be a nonempty, closed, and convex subset of V_h. We know from Exercise 25.4 that there is a unique u_h in V_h such that $a(u_h,v_h-u_h)\geq \ell(v_h-u_h)$ for all $v_h\in U_h$. (i) Show that there is $c_1(u)$ such that for all $(v,v_h)\in U\times V_h$,

$$\|u-u_h\|_V^2 \leq c_1(u)\big(\|u-v_h\|_V+\|u_h-v\|_V+\|u-u_h\|_V\|u-v_h\|_V\big).$$

(*Hint*: prove $\alpha\|u-u_h\|_V^2 \leq a(u,v-u_h)+\ell(u_h-v)+a(u_h,v_h-u)+\ell(u-v_h)$.) (ii) Show that there is $c_2(u)$ such that

$$\|u-u_h\|_V \leq c_2(u)\Big(\inf_{v_h\in U_h}\big(\|u-v_h\|_V+\|u-v_h\|_V^2\big)+\inf_{v\in U}\|u_h-v\|_V\Big)^{\frac{1}{2}}.$$