Part VII, Chapter 35

The Helmholtz problem

The objective of this chapter is to give a brief overview of the analysis of the Helmholtz problem and its approximation using H^1-conforming finite elements. The Helmholtz problem arises when modeling electromagnetic or acoustic scattering problems in the frequency domain. One specificity of this elliptic problem is that one cannot apply the Lax-Milgram lemma to establish well-posedness. The correct way to tackle the Helmholtz problem is to invoke the BNB theorem (Theorem 25.9). In the entire chapter, D is a Lipschitz domain in $\mathbb{R}^d$ with $d \geq 1$, i.e., a nonempty open bounded and connected subset of $\mathbb{R}^d$ with a Lipschitz boundary.

35.1 Robin boundary conditions

We investigate in this section the Helmholtz problem with Robin boundary conditions. Given $f \in L^2(D)$, $g \in L^2(\partial D)$, and $\kappa \in \mathbb{R}$, our goal is to find a function $u : D \to \mathbb{C}$ such that

$$-\Delta u - \kappa^2 u = f \quad \text{in } D, \qquad \partial_n u - \mathrm{i}\kappa u = g \quad \text{on } \partial D, \tag{35.1}$$

with $\mathrm{i}^2 = -1$. Notice that the Robin boundary condition couples the real and imaginary parts of u. The sign of the parameter κ is irrelevant in what follows, but to simplify some expressions, we henceforth assume that $\kappa > 0$. All that is said below remains valid when $\kappa < 0$ by replacing κ by $|\kappa|$ in the definitions of the norms and in the upper bounds. Note that κ^{-1} is a length scale. The problem (35.1) can be reformulated as follows in weak form:

$$\begin{cases} \text{Find } u \in V := H^1(D) \text{ such that} \\ a(u, w) = \ell(w), \quad \forall w \in V, \end{cases} \tag{35.2}$$

with the sesquilinear form

$$a(v,w) := \int_D (\nabla v{\cdot}\nabla\overline{w} - \kappa^2 v\overline{w})\,\mathrm{d}x - \mathrm{i}\kappa \int_{\partial D} \gamma^{\mathrm{g}}(v)\gamma^{\mathrm{g}}(\overline{w})\,\mathrm{d}s, \tag{35.3}$$

and the antilinear form $\ell(w) := \int_D f\overline{w}\,\mathrm{d}x + \int_{\partial D} g\gamma^{\mathrm{g}}(\overline{w})\,\mathrm{d}s$, where $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is the trace map.

Remark 35.1 (Sommerfeld radiation condition). The Helmholtz problem is in general posed on unbounded domains, and the proper "boundary condition to set at infinity" is the Sommerfeld radiation condition $\lim_{r\to\infty} r^{\frac{d-1}{2}}(\boldsymbol{e}{\cdot}\nabla u(r\boldsymbol{e}) - \mathrm{i}\kappa u(r\boldsymbol{e})) = 0$ for every unit vector $\boldsymbol{e} \in \mathbb{R}^d$ and the convergence must be uniform with respect to $\boldsymbol{n}$. One usually simplifies this problem by truncating the domain and replacing the Sommerfeld radiation condition by a Robin boundary condition as in (35.1). □

Remark 35.2 (Wave equation). The Helmholtz problem can be derived by considering the wave equation $\partial_{tt}v - c^2\Delta v = g(\boldsymbol{x})\cos(\omega t)$ in $D\times(0,T)$ with appropriate initial data and boundary conditions; see §46.2.1 and §46.2.2. Here, c is the wave speed and g is some forcing. Assuming that the solution is of the form $v(\boldsymbol{x},t) = \Re(u(\boldsymbol{x})e^{\mathrm{i}\omega t})$, the complex amplitude u solves $\omega^2 u - c^2\Delta u = g$. We then recover (35.1) by setting $\kappa := \frac{\omega}{c}$. □

35.1.1 Well-posedness

Contrary to what was done in the previous chapters, we cannot apply the Lax–Milgram lemma to establish that the weak formulation (35.2) is well-posed since the sesquilinear form a is not coercive. We are going to invoke instead the BNB theorem (Theorem 25.9), and with this goal in mind, we first establish an abstract result.

Lemma 35.3 (Gårding). *Let $V \hookrightarrow L$ be two Banach spaces with compact embedding. Let $a : V\times V \to \mathbb{C}$ be a bounded sesquilinear form. Assume that there exist two real numbers $\beta,\gamma > 0$ such that the following holds true:*

$$|a(v,v)| + \beta\|v\|_L^2 \ge \gamma\|v\|_V^2, \quad \forall v \in V, \tag{35.4a}$$

$$[\,a(v,w) = 0,\ \forall w \in V\,] \implies [\,v = 0\,]. \tag{35.4b}$$

Then there is $\alpha > 0$ such that $\inf_{v\in V}\sup_{w\in V} \frac{|a(v,w)|}{\|v\|_V\|w\|_V} \ge \alpha$.

Proof. Let us argue by contradiction like in the proof of the Peetre–Tartar lemma (Lemma A.20). Assume that for every integer $n \ge 1$, there is $v_n \in V$ with $\|v_n\|_V = 1$ and $\sup_{w\in V} |a(v_n,w)|/\|w\|_V \le \frac{1}{n}$. Since the embedding $V \hookrightarrow L$ is compact, there is a subsequence $(v_l)_{l\in\mathcal{S}}$, $\mathcal{S} \subset \mathbb{N}$, such that $(v_l)_{l\in\mathcal{S}}$ converges strongly to some v in L. The assumption (35.4a) implies that

$$\begin{aligned}\gamma\|v_m - v_n\|_V^2 &\le \beta\|v_m - v_n\|_L^2 + |a(v_m - v_n, v_m - v_n)|\\ &\le \beta\|v_m - v_n\|_L^2 + |a(v_m,v_m)| + |a(v_m,v_n)| + |a(v_n,v_m)| + |a(v_n,v_n)|.\end{aligned}$$

Since $|a(v_l, v_{l'})| = |a(v_l, v_{l'})|/\|v_{l'}\|_V \leq \frac{1}{l}$, for all $l, l' \in \{m, n\}$, we infer that $\gamma\|v_m - v_n\|_V^2 \leq \beta\|v_m - v_n\|_L^2 + 2(m^{-1} + n^{-1})$, which in turn implies that $(v_l)_{l\in\mathcal{S}}$ is a Cauchy sequence in V. As a result, $v \in V$ and $\sup_{w\in V} |a(v, w)|/\|w\|_V = 0$, which means that $a(v, w) = 0$ for all $w \in V$. The assumption (35.4b) implies that $v = 0$, which contradicts $1 = \lim_{\mathcal{S}\ni l\to\infty} \|v_l\|_V = \|v\|_V$. □

Remark 35.4 (Gårding's inequality). Inequalities like (35.4a) are called *Gårding's inequality* in the literature. □

Theorem 35.5 (BNB, Robin BCs). *Let $V := H^1(D)$ be equipped with the norm $\|v\|_V := \{\|\nabla v\|^2_{\boldsymbol{L}^2(D)} + \kappa\|v\|^2_{L^2(\partial D)}\}^{\frac{1}{2}}$. The sesquilinear form a defined in* (35.3) *satisfies the conditions of the BNB theorem.*

Proof. We are going to verify (35.4a) and (35.4b) from Lemma 35.3.
(1) Let $v \in V$. The real and imaginary parts of $a(v, v)$ are

$$\Re(a(v, v)) = \|\nabla v\|^2_{\boldsymbol{L}^2(D)} - \kappa^2\|v\|^2_{L^2(D)}, \tag{35.5a}$$

$$\Im(a(v, v)) = -\kappa\|v\|^2_{L^2(\partial D)}. \tag{35.5b}$$

Using that $\sqrt{2}(x^2 + y^2)^{\frac{1}{2}} \geq x - y$ for all $x, y \in \mathbb{R}$, this implies that

$$\sqrt{2}|a(v, v)| \geq \|v\|_V^2 - \kappa^2\|v\|^2_{L^2(D)}.$$

Hence, (35.4a) holds true with $\beta := \frac{1}{\sqrt{2}}\kappa^2$ and $\gamma := \frac{1}{\sqrt{2}}$.
(2) Let us now assume that $a(v, w) = 0$ for all $w \in V$. We are going to prove that $v = 0$ by arguing by contradiction. The inequality $|a(v, v)| \geq -\Im(a(v, v)) = \kappa\|v\|^2_{L^2(\partial D)}$ implies that $\gamma^{\mathrm{g}}(v) = 0$. Hence, $v \in H^1_0(D)$. Let us embed D into a ball of radius R large enough, say $R > R_0 := \mathrm{diam}(D)$, and without loss of generality, we assume that this ball is centered at $\mathbf{0}$. Let B_R be the ball in question and let us set $D^{\mathrm{c}}_R := D^{\mathrm{c}} \cap B_R$, where D^{c} denotes the complement of D in $\mathbb{R}^d$. Since $v_{|\partial D} = 0$, we can extend v by zero over D^{c}_R, and we denote by $\widetilde{v}_R$ the extension in question. We have $\widetilde{v}_R \in H^1_0(B_R)$, $(\nabla\widetilde{v}_R)_{|D} \in \boldsymbol{H}(\mathrm{div}; D)$, and $(\nabla\widetilde{v}_R)_{|D^{\mathrm{c}}_R} \in \boldsymbol{H}(\mathrm{div}; D^{\mathrm{c}}_R)$. Since the Robin boundary condition implies that $\partial_n v_{|\partial D} = 0$, we infer that the normal component of $\nabla\widetilde{v}_R$ is continuous across ∂D. Reasoning as in the proof of Theorem 18.10, we conclude that $\nabla\widetilde{v}_R$ is a member of $\boldsymbol{H}(\mathrm{div}; B_R)$. This means that $\Delta\widetilde{v}_R \in L^2(B_R)$. Since $\widetilde{v}_R \in H^1_0(B_R)$ and $\widetilde{v}_R$ vanishes on an open subset of B_R, we can invoke the unique continuation principle (see Theorem 31.4) to infer that $\widetilde{v}_R = 0$ in B_R. Hence, $v = 0$ in D and the property (35.4b) holds true. □

Remark 35.6 (Alternative proof). Instead of invoking the unique continuation principle in the above proof, one can use the spectral theorem for symmetric compact operators (see Theorem 46.21). The above reasoning shows that $\widetilde{v}_R \in H^1_0(B_R)$ and $-\Delta\widetilde{v}_R = \kappa^2\widetilde{v}_R$ in B_R. Hence, if $\widetilde{v}_R$ is not zero, then κ^2 is an eigenvalue of the Laplace operator equipped with homogeneous

Dirichlet boundary conditions on every ball centered at $\mathbf{0}$ in $\mathbb{R}^d$ with radius larger than R_0. However, Theorem 46.21 says that the eigenvalues of the Laplace operator in $H_0^1(B_R)$ are countable with no accumulation point and are of the form $(R^{-2}\lambda_n)_{n\in\mathbb{N}}$ for every $R>0$, where $(\lambda_n)_{n\in\mathbb{N}}$ are the eigenvalues of the Laplace operator in $H_0^1(B_1)$. Assuming that the eigenvalues are ordered in increasing order, let $R_0' > R_0$ be large enough so that there is some $n\in\mathbb{N}$ such that $\kappa^2(R_0')^2 = \lambda_n$ with $\lambda_n < \lambda_{n+1}$. Let δ be defined by $\kappa^2(R_0'+\delta)^2 := \frac{1}{2}(\lambda_n+\lambda_{n+1})$. Then $\kappa^2(R_0'+\delta)^2$ cannot be in the set $\{\lambda_n\}_{n\in\mathbb{N}}$, but this is a contradiction since the above reasoning with $R := R_0'+\delta$ shows that $\kappa^2R^2 = \kappa^2(R_0'+\delta)^2$ is a member of the sequence $(\lambda_n)_{n\in\mathbb{N}}$ if $\widetilde{v}_R$ is not zero. This proves that $\widetilde{v}_R = 0$. □

35.1.2 A priori estimates on the solution

In this section, we derive a priori estimates on the weak solution of (35.2). We are particularly interested in estimating the possible dependence of the upper bound on the (nondimensional) quantity $\kappa\ell_D$ with $\ell_D := \mathrm{diam}(D)$. The following result, established in Melenk [299, Prop. 8.1.4] and Hetmaniuk [243], delivers a sharp upper bound on the V-norm of the weak solution that relies on the relatively strong assumption that the domain D is star-shaped with respect to some point in D which we take to be $\mathbf{0}$.

Lemma 35.7 (A priori estimate). *Assume that D is a bounded Lipschitz domain and star-shaped w.r.t. $\mathbf{0}$, i.e., there exists $r>0$ s.t. $\boldsymbol{x}\cdot\boldsymbol{n} > r\ell_D$ for all $\boldsymbol{x}\in\partial D$. Let $V := H^1(D)$ be equipped with the norm $\|v\|_V := \{\|\nabla v\|^2_{\boldsymbol{L}^2(D)} + \kappa\|v\|^2_{L^2(\partial D)}\}^{\frac{1}{2}}$. There is a constant c that depends only on D (i.e., it is independent of $\kappa\ell_D$) such that the weak solution of* (35.2) *satisfies*

$$\kappa\|u\|_{L^2(D)} + \|u\|_V \le c\,(\ell_D\|f\|_{L^2(D)} + \ell_D^{\frac{1}{2}}\|g\|_{L^2(\partial D)}). \tag{35.6}$$

Proof. We only give the proof when κ is bounded away from zero, say $\kappa\ell_D \ge 1$ since the proof in the other case is similar; see [299, 243]. Since we assume that $\mathbf{0}\in D$, we have $\|\boldsymbol{x}\|_{\ell^2} \le \ell_D$ for all $\boldsymbol{x}\in D$. We write $C(f,g) := c(\ell_D\|f\|_{L^2(D)} + \ell_D^{\frac{1}{2}}\|g\|_{L^2(\partial D)})$, where as usual the value of the constant c can change at each occurrence as long as it is independent of κ.

(1) In the first step of the proof, we assume that $\nabla u_{|\partial D} \in \boldsymbol{L}^2(\partial D)$ (we establish this smoothness property in the second step). Let us multiply the PDE $-\Delta u - \kappa^2 u = f$ with $\boldsymbol{x}\cdot\nabla\overline{u}$ and integrate over D. The identity (35.11) from Lemma 35.8 with $\boldsymbol{m} := \boldsymbol{x}$ implies that

$$\begin{aligned} -\Re\left(\int_D \Delta u\,\boldsymbol{x}\cdot\nabla\overline{u}\,\mathrm{d}x\right) &= \left(1-\frac{d}{2}\right)\|\nabla u\|^2_{\boldsymbol{L}^2(D)} \\ &\quad + \frac{1}{2}\int_{\partial D}(\boldsymbol{x}\cdot\boldsymbol{n})\|\nabla u\|^2_{\ell^2}\,\mathrm{d}s - \Re\left(\int_{\partial D}(\partial_n u)(\boldsymbol{x}\cdot\nabla\overline{u})\,\mathrm{d}s\right), \end{aligned}$$

since $\nabla \boldsymbol{x} = (\nabla \boldsymbol{x})^{\mathsf{T}} = \mathbb{I}_d$ and $\nabla{\cdot}\boldsymbol{x} = d$ so that $\mathbb{e}(\boldsymbol{x}) = (1 - \frac{d}{2})\mathbb{I}_d$ (see Lemma 35.8). This identity is often called *Rellich's identity* in the literature. Using the PDE $-\Delta u - \kappa^2 u = f$, the Robin boundary condition $\partial_n u = \mathrm{i}\kappa u + g$, and the assumption $\boldsymbol{x}{\cdot}\boldsymbol{n} > r\ell_D$ on ∂D, we obtain

$$\begin{aligned}\frac{r\ell_D}{2}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} \le \Big(\frac{d}{2} - 1\Big)\|\nabla u\|^2_{\boldsymbol{L}^2(D)} + \Re\left(\int_D \kappa^2 u(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}x\right) \\ + \Re\left(\int_D f(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}x\right) + \Re\left(\int_{\partial D} (\mathrm{i}\kappa u + g)(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}s\right).\end{aligned}$$

Since $\Re(\int_D u(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}x) = -\frac{d}{2}\|u\|^2_{L^2(D)} + \frac{1}{2}\int_{\partial D}(\boldsymbol{x}{\cdot}\boldsymbol{n})|u|^2\,\mathrm{d}s$, this leads to

$$\begin{aligned}\frac{r\ell_D}{2}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{d\kappa^2}{2}\|u\|^2_{L^2(D)} \le \Big(\frac{d}{2} - 1\Big)\|\nabla u\|^2_{\boldsymbol{L}^2(D)} + \frac{\kappa^2\ell_D}{2}\|u\|^2_{L^2(\partial D)} \\ + \Re\left(\int_D f(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}x\right) + \Re\left(\int_{\partial D} (\mathrm{i}\kappa u + g)(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}s\right).\end{aligned}$$

We now bound the last two terms on the right-hand side by using Young's inequality, which yields

$$\begin{aligned}&\Re\left(\int_D f(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}x\right) + \Re\left(\int_{\partial D} (\mathrm{i}\kappa u + g)(\boldsymbol{x}{\cdot}\nabla\overline{u})\,\mathrm{d}s\right) \le \gamma_1\|\nabla u\|^2_{\boldsymbol{L}^2(D)} \\ &+ \frac{1}{4\gamma_1}\ell_D^2\|f\|^2_{L^2(D)} + \frac{r\ell_D}{4}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{2\ell_D}{r}\big(\kappa^2\|u\|^2_{L^2(\partial D)} + \|g\|^2_{L^2(\partial D)}\big),\end{aligned}$$

where $\gamma_1 > 0$ can be chosen as small as needed. Rearranging the terms gives

$$\begin{aligned}\frac{r\ell_D}{4}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{d\kappa^2}{2}\|u\|^2_{L^2(D)} \le \Big(\frac{d}{2} - 1 + \gamma_1\Big)\|\nabla u\|^2_{\boldsymbol{L}^2(D)} \\ + \frac{r+4}{2r}\kappa^2\ell_D\|u\|^2_{L^2(\partial D)} + C(f,g)^2. \quad (35.7)\end{aligned}$$

Let us now bound the norms $\|\nabla u\|^2_{\boldsymbol{L}^2(D)}$ and $\|u\|^2_{L^2(\partial D)}$ appearing on the right-hand side. Owing to (35.5a) and Young's inequality, we infer that

$$\begin{aligned}&\|\nabla u\|^2_{\boldsymbol{L}^2(D)} = \kappa^2\|u\|^2_{L^2(D)} + \Re\Big((f,u)_{L^2(D)} + (g,\gamma^{\mathrm{g}}(u))_{L^2(\partial D)}\Big) \\ &\le (1+\gamma_2)\kappa^2\|u\|^2_{L^2(D)} + \frac{1}{4\gamma_2\kappa^2}\|f\|^2_{L^2(D)} + \frac{1}{2\kappa}\|g\|^2_{L^2(\partial D)} + \frac{1}{2}\kappa\|u\|^2_{L^2(\partial D)},\end{aligned}$$

where $\gamma_2 > 0$ can be chosen as small as needed. Since we assumed above that $\kappa\ell_D \ge 1$, we obtain

$$\|\nabla u\|^2_{\boldsymbol{L}^2(D)} \le (1+\gamma_2)\kappa^2\|u\|^2_{L^2(D)} + \frac{1}{2}\kappa\|u\|^2_{L^2(\partial D)} + C(f,g)^2. \quad (35.8)$$

Owing to (35.5b), we infer that

$$\kappa\|u\|^2_{L^2(\partial D)} = -\Im\Big((f,u)_{L^2(D)} + (g,\gamma^{\mathrm{g}}(u))_{L^2(\partial D)}\Big),$$

and applying Young's inequality with a positive real number θ gives

$$\frac{1}{2}\kappa\|u\|^2_{L^2(\partial D)} \le \theta\kappa\|u\|^2_{L^2(D)} + \frac{1}{4\theta\kappa}\|f\|^2_{L^2(D)} + \frac{1}{2\kappa}\|g\|^2_{L^2(\partial D)}.$$

Taking $\theta := \gamma_3\kappa$ with $\gamma_3 > 0$ as small as needed leads to (recall that $\kappa\ell_D \ge 1$)

$$\frac{1}{2}\kappa\|u\|^2_{L^2(\partial D)} \le \gamma_3\kappa^2\|u\|^2_{L^2(D)} + C(f,g)^2. \tag{35.9}$$

In addition, taking $\theta := \frac{1}{2\ell_D}\frac{r}{r+4}$ and multiplying by $\frac{r+4}{r}\kappa\ell_D$ yields

$$\frac{r+4}{2r}\kappa^2\ell_D\|u\|^2_{L^2(\partial D)} \le \frac{1}{2}\kappa^2\|u\|^2_{L^2(D)} + C(f,g)^2. \tag{35.10}$$

Inserting (35.9) into (35.8) gives $\|\nabla u\|^2_{\boldsymbol{L}^2(D)} \le (1+\gamma_2+\gamma_3)\kappa^2\|u\|^2_{L^2(D)} + C(f,g)^2$, and inserting this bound into (35.7), we obtain

$$\begin{aligned}\frac{r\ell_D}{4}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{d\kappa^2}{2}\|u\|^2_{L^2(D)} \le \Big(\frac{d}{2} - 1 + \gamma_1\Big)&(1+\gamma_2+\gamma_3)\kappa^2\|u\|^2_{L^2(D)} \\ &+ \frac{r+4}{2r}\kappa^2\ell_D\|u\|^2_{L^2(\partial D)} + C(f,g)^2.\end{aligned}$$

Using now the bound on $\|u\|^2_{L^2(\partial D)}$ from (35.10), we infer that

$$\begin{aligned}&\frac{r\ell_D}{4}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{d}{2}\kappa^2\|u\|^2_{L^2(D)} \le \\ &\qquad\Big(\Big(\frac{d}{2} - 1 + \gamma_1\Big)(1+\gamma_2+\gamma_3) + \frac{1}{2}\Big)\kappa^2\|u\|^2_{L^2(D)} + C(f,g)^2.\end{aligned}$$

Letting $\gamma_1 := \frac{1}{4d}, \gamma_2 = \gamma_3 := \frac{1}{8d}$, we observe that $(\frac{d}{2} - 1 + \gamma_1)(1+\gamma_2+\gamma_3) = \frac{d}{2} - \frac{7}{8} + \frac{1}{16d^2} \le \frac{d}{2} - \frac{1}{4}$ for all $d \ge 1$. We conclude that

$$\frac{r\ell_D}{4}\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} + \frac{\kappa^2}{4}\|u\|^2_{L^2(D)} \le C(f,g)^2.$$

Invoking once again the bounds (35.8) and (35.9), we infer that

$$\kappa^2\|u\|^2_{L^2(D)} + \kappa\|u\|^2_{L^2(\partial D)} + \|\nabla u\|^2_{\boldsymbol{L}^2(D)} + \ell_D\|\nabla u\|^2_{\boldsymbol{L}^2(\partial D)} \le C(f,g)^2,$$

which shows that the a priori estimate (35.6) holds true.
(2) It remains to prove that indeed $\nabla u_{|\partial D} \in \boldsymbol{L}^2(\partial D)$. Recall that u is in the functional space $Y := \{y \in H^1(D) \mid \Delta y \in L^2(D),\, \partial_n y \in L^2(\partial D)\}$

owing to (35.1) and our assumption that $f \in L^2(D)$ and $g \in L^2(\partial D)$. We are going to show by means of a density argument that any function $y \in Y$ is such that $\nabla y_{|\partial D} \in \boldsymbol{L}^2(\partial D)$. Let $(\varphi_m)_{m\in\mathbb{N}}$ be a sequence in $C^\infty(\overline{D})$ converging to y in Y (such a sequence can be constructed by using mollifying operators, as in §23.1). Let us set $f_m := -\Delta\varphi_m - \varphi_m$ and $g_m := \partial_n\varphi_m - \mathrm{i}\kappa\varphi_m$. Then $(f_m)_{m\in\mathbb{N}}$ and $(g_m)_{m\in\mathbb{N}}$ are Cauchy sequences in $L^2(D)$ and $L^2(\partial D)$, respectively. Moreover, the bound from Step (1) implies that $\|\nabla(\varphi_m - \varphi_p)\|_{\boldsymbol{L}^2(\partial D)} \le c(\ell_D^{\frac12}\|f_m - f_p\|_{L^2(D)} + \|g_m - g_p\|_{L^2(\partial D)})$ for all $m,p \in \mathbb{N}$, which shows that $(\nabla\varphi_m)_{m\in\mathbb{N}}$ is a Cauchy sequence in $\boldsymbol{L}^2(\partial D)$. The uniqueness of the limit in the distribution sense finally shows that $\nabla y_{|\partial D} \in \boldsymbol{L}^2(\partial D)$. □

Lemma 35.8 (Special identity). *For all $q \in \{v \in H^1(D;\mathbb{C}) \mid \Delta v \in L^2(D;\mathbb{C}),\ \nabla v \in L^2(\partial D;\mathbb{C}^d)\}$ and all $\boldsymbol{m} \in W^{1,\infty}(D;\mathbb{R}^d)$, letting $\mathbb{e}(\boldsymbol{m}) := \frac12(\nabla\boldsymbol{m} + (\nabla\boldsymbol{m})^{\mathsf{T}} - (\nabla{\cdot}\boldsymbol{m})\mathbb{I}_d)$, we have*

$$-\Re\left(\int_D \Delta q(\boldsymbol{m}{\cdot}\nabla\overline{q})\,\mathrm{d}x\right) = \Re\left(\int_D \nabla q{\cdot}(\mathbb{e}(\boldsymbol{m})\nabla\overline{q})\,\mathrm{d}x\right) + \frac12\int_{\partial D}(\boldsymbol{m}{\cdot}\boldsymbol{n})\|\nabla q\|_{\ell^2}^2\,\mathrm{d}s - \Re\left(\int_{\partial D}(\boldsymbol{n}{\cdot}\nabla q)(\boldsymbol{m}{\cdot}\nabla\overline{q})\,\mathrm{d}s\right). \quad (35.11)$$

Proof. See Exercise 35.4 and Hetmaniuk [243, Lem. 3.2]. □

A detailed analysis of the Helmholtz problem (35.2) using integral representations is done in Esterhazy and Melenk [195, §2]. The following result is established therein.

Theorem 35.9 (BNB, Robin BCs). *Let D be a Lipschitz domain in $\mathbb{R}^d$, $d \in \{2,3\}$. Let $V := H^1(D)$ be equipped with the norm $\|v\|_V := \kappa\|v\|_{L^2(D)} + \|\nabla v\|_{\boldsymbol{L}^2(D)}$. Let $k_0 > 0$ be a fixed number and set $\kappa_0 := k_0\ell_D^{-1}$. Then there is $c > 0$, depending on D and k_0, such that the following holds true for all $\kappa \ge \kappa_0$:*

$$\inf_{v\in V}\sup_{w\in V}\frac{|a(v,w)|}{\|v\|_V\|w\|_V} \ge c\,(\kappa\ell_D)^{-s}, \quad (35.12)$$

with $s := \frac72$ in general, and $s := 1$ if D is convex or if D is star-shaped or if ∂D is smooth.

This theorem implies, in particular, that for every $f \in V' := (H^1(D))'$ and $g \in H^{-\frac12}(\partial D) = (H^{\frac12}(\partial D))'$, the problem (35.2) is uniquely solvable in V, and its solution satisfies the a priori bound $\|u\|_V \le c(\kappa\ell_D)^{\frac72}(\|f\|_{V'} + \|g\|_{H^{-\frac12}(\partial D)})$. If $f \in L^2(D)$ and $g \in L^2(\partial D)$, this estimate can be improved to $\|u\|_V \le c(\kappa\ell_D)^{\frac52}(\ell_D\|f\|_{L^2(D)} + \kappa^{-\frac12}\|g\|_{L^2(\partial D)})$; see [195, Thm. 2.5].

35.2 Mixed boundary conditions

We consider in this section the Helmholtz problem with mixed Dirichlet and Robin boundary conditions. The problem is formulated as follows: For $f \in L^2(D)$, $g \in L^2(\partial D_\mathrm{r})$, and $\kappa \in \mathbb{R}$, find a complex-valued function u such that

$$-\Delta u - \kappa^2 u = f \text{ in } D, \quad u = 0 \text{ on } \partial D_\mathrm{d}, \quad \partial_n u - \mathrm{i}\kappa u = g \text{ on } \partial D_\mathrm{r}, \quad (35.13)$$

where $\{\partial D_\mathrm{d}, \partial D_\mathrm{r}\}$ is a partition of ∂D. We assume that the subsets ∂D_d and ∂D_r have a Lipschitz boundary and have positive (surface) measure. As before, we assume that $\kappa > 0$ for simplicity. The above problem is reformulated as follows:

$$\begin{cases} \text{Find } u \in V := \{v \in H^1(D) \mid \gamma^{\mathrm{g}}(v)_{|\partial D_\mathrm{d}} = 0\} \text{ such that} \\ a(u,w) = \ell(w), \quad \forall w \in V, \end{cases} \quad (35.14)$$

with the sesquilinear form

$$a(v,w) := \int_D (\nabla v{\cdot}\nabla \overline{w} - \kappa^2 v\overline{w})\,\mathrm{d}x - \mathrm{i}\kappa \int_{\partial D_\mathrm{r}} v\overline{w}\,\mathrm{d}s, \quad (35.15)$$

and the antilinear form $\ell(w) := \int_D f\overline{w}\,\mathrm{d}x + \int_{\partial D_\mathrm{r}} g\overline{w}\,\mathrm{d}s$. Here again, we cannot apply the Lax–Milgram lemma since a is not coercive on V. We are going to invoke instead the BNB theorem.

Theorem 35.10 (BNB, mixed BCs). *Let the space V defined in* (35.14) *be equipped with the norm $\|v\|_V := \|\nabla v\|_{\boldsymbol{L}^2(D)}$. The sesquilinear form a defined in* (35.15) *satisfies the conditions of the BNB theorem.*

Proof. We are going to invoke Lemma 35.3. We can proceed as in the proof of Theorem 35.5 to prove the Gårding inequality (35.4a), but we proceed slightly differently to prove (35.4b). Let us assume that $a(v,w) = 0$ for all $w \in V$. The inequality $|a(v,v)| \geq \kappa \|v\|^2_{L^2(\partial D_\mathrm{r})}$ implies that $v_{|\partial D_\mathrm{r}} = 0$. Since $|\partial D_\mathrm{r}| > 0$, there exists a point $\boldsymbol{x}_0 \in \partial D_\mathrm{r}$ and there is $r_0 > 0$ such that $B(\boldsymbol{x}_0, r_0) \cap \partial D \subset \partial D_\mathrm{r}$. Let $D^\mathrm{c}_{r_0} := D^\mathrm{c} \cap B(\boldsymbol{x}_0, r_0)$. We extend v by zero over $D^\mathrm{c}_{r_0}$, denote the extension in question by $\widetilde{v}_{r_0}$ and set $\widetilde{D}_{r_0} := \mathrm{int}(\overline{D} \cup \overline{D}^\mathrm{c}_{r_0})$. We have $\widetilde{v}_{r_0} \in H^1_0(\widetilde{D}_{r_0})$, $(\nabla \widetilde{v}_{r_0})_{|D} \in \boldsymbol{H}(\mathrm{div}; D)$, and $(\nabla \widetilde{v}_{r_0})_{|D^\mathrm{c}_{r_0}} \in \boldsymbol{H}(\mathrm{div}; D^\mathrm{c}_{r_0})$. Since the Robin boundary condition implies that $(\partial_n v)_{|\partial D_\mathrm{r}} = 0$, we infer that the normal component of $\nabla \widetilde{v}_{r_0}$ is continuous across $\partial D_\mathrm{r} \cap B(\boldsymbol{x}_0, r_0)$. Reasoning as in the proof of Theorem 18.10, we conclude that $\nabla \widetilde{v}_{r_0}$ is a member of $\boldsymbol{H}(\mathrm{div}; \widetilde{D}_{r_0})$, i.e., $\Delta \widetilde{v}_{r_0} \in L^2(\widetilde{D}_{r_0})$. In conclusion, we have $\widetilde{v}_{r_0} \in H^1_0(\widetilde{D}_{r_0})$, $-\Delta \widetilde{v}_{r_0} = \kappa^2 \widetilde{v}_{r_0}$ in $\widetilde{D}_{r_0}$, and $\widetilde{v}_{r_0|D^\mathrm{c}_{r_0}} = 0$. The unique continuation principle (Theorem 31.4) implies that $\widetilde{v}_{r_0} = 0$. Hence, $v = 0$. □

Following Ihlenburg and Babuška [251], we now set $D := (0, \ell_D)$ and investigate the one-dimensional version of the problem (35.13). A homogeneous Dirichlet boundary condition is enforced at $\{x = 0\}$, and a homogeneous Robin condition is enforced at $\{x = \ell_D\}$. The space V becomes $V := \{v \in H^1(D) \mid v(0) = 0\}$.

Theorem 35.11 (BNB, mixed BCs, 1D). *Let $D := (0, \ell_D)$. Let the space V be equipped with the norm $\|v\|_V := \|\partial_x v\|_{L^2(D)}$. There are two constants $0 < c_\flat \le c_\sharp$, both uniform with respect to κ, such that*

$$\frac{c_\flat}{1+\kappa\ell_D} \le \inf_{v\in V}\sup_{w\in V}\frac{|a(v,w)|}{\|v\|_V\|w\|_V} \le \sup_{v\in V}\sup_{w\in V}\frac{|a(v,w)|}{\|v\|_V\|w\|_V} \le \frac{c_\sharp}{1+\kappa\ell_D}.$$

Proof. (1) Let us start with the lower bound. Let $v \in V$, $v \neq 0$, and let $z \in V$ solve $a(w, z) = (w, \kappa^2 v)_{L^2(D)}$ for all $w \in V$. It is shown in Exercise 35.1 that this problem has a unique solution in V, and it is shown in Exercise 35.2 that $\|z\|_V \le 4\kappa\ell_D\|v\|_V$. Then we have

$$\begin{aligned}
|a(v, v+z)| &\ge \Re(a(v, v+z)) = \Re(a(v,v)) + \kappa^2\|v\|^2_{L^2(D)} \\
&= \|v'\|^2_{L^2(D)} = \|v\|^2_V = \frac{1}{4\kappa\ell_D+1}\|v\|_V(\|v\|_V + 4\kappa\ell_D\|v\|_V) \\
&\ge \frac{1}{4\kappa\ell_D+1}\|v\|_V(\|v\|_V + \|z\|_V) \ge \frac{1}{4\kappa\ell_D+1}\|v\|_V\|v+z\|_V.
\end{aligned}$$

This shows that the lower bound holds true.
(2) Let us now prove the upper bound. Let $v \in V$.
(2.a) If $\kappa\ell_D \le 2$, then we can invoke the following Poincaré–Steklov inequality in V: there is a constant $\tilde{C}_{\text{PS}} > 0$ s.t. $\tilde{C}_{\text{PS}}(\ell_D^{-1}\|v\|_{L^2(D)} + \ell_D^{-\frac{1}{2}}|v(\ell_D)|) \le \|v\|_V$ (see the proof of Proposition 31.21). Using the Cauchy–Schwarz inequality in (35.3) implies that

$$\begin{aligned}
|a(v,w)| &\le \|v\|_V\|w\|_V + \kappa^2\|v\|_{L^2(D)}\|w\|_{L^2(D)} + \kappa|v(\ell_D)||w(\ell_D)| \\
&\le \max(1, \tilde{C}_{\text{PS}}^{-2})(1 + \kappa\ell_D + (\kappa\ell_D)^2)\|v\|_V\|w\|_V.
\end{aligned}$$

Since we assumed $\kappa\ell_D \le 2$, this leads to the bound $|a(v,w)| \le c(1 + \kappa\ell_D)^{-1}\|v\|_V\|w\|_V$ with $c := \max(1, \tilde{C}_{\text{PS}}^{-2})\max_{t\in[0,2]}(1+t+t^2)(1+t)$.
(2.b) Let us now assume that $\kappa\ell_D \ge 2$. Let φ be a smooth nonnegative function equal to 1 on $[0, \frac{1}{2}\ell_D]$ and such that $\varphi(\ell_D) = \partial_x\varphi(\ell_D) = 0$. Let us set $w(x) := \varphi(x)\sin(\kappa x)/\kappa$ so that $w \in V$, $w(0) = 0$, $w(\ell_D) = 0$, and $\partial_x w(\ell_D) = 0$. Let us set $\eta(x) := \partial_x w(x) - \partial_x w(0) + \kappa^2\int_0^x w(s)\,\mathrm{d}s$, and $c_\varphi := \max(2\ell_D\|\partial_x\varphi\|_{L^\infty(D)}, \ell_D^2\|\partial_{xx}\varphi\|_{L^\infty(D)})$. Since w is real-valued and vanishes at $x = \ell_D$ and $v(0) = 0$, we have

$$\begin{aligned}
a(v,w) &= \int_0^{\ell_D}\partial_x v\partial_x w\,\mathrm{d}x - \kappa^2\int_0^{\ell_D} vw\,\mathrm{d}x \\
&= \int_0^{\ell_D}(\partial_x v)\eta\,\mathrm{d}x + v(\ell_D)\partial_x w(0) - \kappa^2\int_0^{\ell_D}\left(vw + \partial_x v\int_0^x w(s)\,\mathrm{d}s\right)\mathrm{d}x.
\end{aligned}$$

The last term is equal to $-\kappa^2 v(\ell_D)\int_0^{\ell_D} w(s)\,\mathrm{d}s$ since $v(0)=0$. Since $\eta(\ell_D) = -\partial_x w(0) + \kappa^2\int_0^{\ell_D} w(s)\,\mathrm{d}s$ and $|v(\ell_D)| \le \ell_D^{\frac12}\|v\|_V$, we infer that

$$\begin{aligned}|a(v,w)| &= \int_0^{\ell_D} (\partial_x v)\eta\,\mathrm{d}x - v(\ell_D)\eta(\ell_D)\\ &\le \|v\|_V(\|\eta\|_{L^2(D)} + \ell_D^{\frac12}|\eta(\ell_D)|) \le 2\ell_D^{\frac12}\|v\|_V\|\eta\|_{L^\infty(D)}.\end{aligned}$$

Since $\eta(0)=0$, we have $\|\eta\|_{L^\infty(D)} \le \ell_D\|\partial_x\eta\|_{L^\infty(D)}$. After observing that $\partial_x\eta(x) = \partial_{xx}\varphi(x)\sin(\kappa x)/\kappa + 2\partial_x\varphi(x)\cos(\kappa x)$ and recalling the above bounds on the derivatives of φ, we deduce that $\|\eta\|_{L^\infty(D)} \le c_\varphi(1+(\kappa\ell_D)^{-1})$. Hence, we have $|a(v,w)| \le 2c_\varphi(1+(\kappa\ell_D)^{-1})\ell_D^{\frac12}\|v\|_V$. After observing that

$$\|w\|_V^2 \ge \int_0^{\frac12\ell_D} \cos(\kappa x)^2\,\mathrm{d}x \ge \frac{\ell_D}{4} - \frac{1}{4\kappa} \ge \frac{\ell_D}{8},$$

since $\kappa\ell_D \ge 2$, we conclude that $\|w\|_V \ge (\frac18\ell_D)^{\frac12}$. This proves that $|a(v,w)| \le c(1+\kappa\ell_D)^{-1}\|v\|_V\|w\|_V$, and the proof is complete. □

Remark 35.12 (Literature). Theorem 35.11 has been derived in Ihlenburg and Babuška [251, Thm. 1], and we refer the reader to this work for an exhaustive analysis of the continuous problem in one dimension with $g := 0$. Two- and three-dimensional versions of Lemma 35.7 for mixed boundary conditions are established in Hetmaniuk [243]. □

35.3 Dirichlet boundary conditions

We consider in this section the Helmholtz problem with Dirichlet boundary conditions: For $f \in L^2(D;\mathbb{R})$ and $\kappa \in \mathbb{R}$, find u such that

$$-\Delta u - \kappa^2 u = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D. \tag{35.16}$$

As before, we assume that $\kappa > 0$ for simplicity. Note that the solution is now real-valued. We reformulate the above problem as follows:

$$\begin{cases}\text{Find } u \in V := H_0^1(D) \text{ such that}\\ a(u,w) = \ell(w), \quad \forall w \in V,\end{cases} \tag{35.17}$$

with the bilinear form

$$a(v,w) := \int_D (\nabla v{\cdot}\nabla w - \kappa^2 vw)\,\mathrm{d}x, \tag{35.18}$$

and the linear form $\ell(v) := \int_D fv\,\mathrm{d}x$. As above, we are going to rely on the BNB theorem to establish the well-posedness (35.17) since a is not coercive.

But contrary to the case with Robin or mixed boundary conditions, the enforcement of Dirichlet conditions leads to a conditional stability depending on the value of κ. In other words, resonance phenomena can occur if κ takes values in some discrete subset of $\mathbb{R}_+$ associated with the spectrum of the Laplacian operator in D with Dirichlet conditions.

Since the embedding $H^1_0(D) \hookrightarrow L^2(D)$ is compact and the operator $(-\Delta)^{-1} : L^2(D) \to L^2(D)$ is self-adjoint, there exists a Hilbertian basis of $L^2(D)$ composed of eigenvectors of the Laplace operator (see Theorem 46.21). Let $(\psi_l)_{l\in\mathbb{N}}$ be the basis in question and let $(\lambda_l)_{l\in\mathbb{N}}$ be the corresponding eigenvalues with the normalization $\|\psi_l\|_{L^2(D)} = 1$. Then every function $v \in H^1_0(D)$ admits a unique expansion $v := \sum_{l\in\mathbb{N}} v_l\psi_l$ with $\|\nabla v\|^2_{\boldsymbol{L}^2(D)} = \sum_{l\in\mathbb{N}} \lambda_l v_l^2$, $\|v\|^2_{L^2(D)} = \sum_{l\in\mathbb{N}} v_l^2$. Notice that $a(v,w) = \sum_{l\in\mathbb{N}}(\lambda_l - \kappa^2)v_l w_l$ for all $v = \sum_{l\in\mathbb{N}} v_l\psi_l$, $w = \sum_{l\in\mathbb{N}} w_l\psi_l$ in $H^1_0(D)$. Let us denote by $l(\kappa)$ the largest integer such that $\lambda_{l(\kappa)} < \kappa^2$ with the convention that $l(\kappa) = -1$ if $\kappa^2 \le \lambda_0$. The well-posedness of the problem (35.17) follows from the following result.

Theorem 35.13 (BNB, Dirichlet BCs). *Let $V := H^1_0(D)$ be equipped with the norm $\|v\|_V := \|\nabla v\|_{\boldsymbol{L}^2(D)}$. Assume that $\kappa^2 \notin \{\lambda_l\}_{l\in\mathbb{N}}$. Then the bilinear form a satisfies the conditions of the BNB theorem with the constant $\alpha(\kappa) := \min_{l\in\mathbb{N}} |\lambda_l - \kappa^2|/\lambda_l > 0$.*

Proof. Let $v \in H^1_0(D)$ with $v := \sum_{l\in\mathbb{N}} v_l\psi_l$. Let us set $w := \sum_{l\le l(\kappa)} -v_l\psi_l + \sum_{l(\kappa)<l} v_l\psi_l$ with the convention that $l \in \mathbb{N}$ in the sums. Then we have

$$a(v,w) = \sum_{l\le l(\kappa)} (\kappa^2 - \lambda_l)v_l^2 + \sum_{l(\kappa)<l} (\lambda_l - \kappa^2)v_l^2 \ge \alpha(\kappa) \sum_{l\in\mathbb{N}} \lambda_l v_l^2 = \alpha(\kappa)\|v\|^2_V.$$

The assertion follows readily from $\|w\|_V = \|v\|_V$. The reader is referred to Ciarlet [120, §3.1] for more details on this problem. □

In general, $\alpha(\kappa)$ behaves like $\alpha_0\gamma(\kappa)(\kappa\ell_D)^{-1}$, where $\gamma(\kappa) \in (0,1]$ and α_0 only depends on D. For $D := (0,\ell_D)$, the eigenvalues of the Laplace operator are $\lambda_l := \pi l^2\ell_D^{-2}$. Let $\beta \in (0,1)$ and $L \in \mathbb{N}\setminus\{0\}$ be s.t. $\kappa^2 := \pi(L+\beta)^2\ell_D^{-2}$. Then $\alpha(\kappa) = \min(\beta(2L+\beta)/L^2, (1+\beta)(2L+1+\beta)/(L+1)^2)$, and the claim follows readily. Notice that $\gamma(\kappa)$ becomes arbitrarily small as κ approaches an eigenvalue of the Laplace operator, i.e., if β is close to 0.

35.4 H^1-conforming approximation

We now formulate an H^1-conforming approximation of the Helmholtz problem with one of the boundary conditions discussed in the previous sections (Robin, mixed or Dirichlet). At this stage, we do not specify the norm with which we equip the space V: we just assume that it is an H^1-like norm that can contain some lower-order terms depending on κ (see Example 35.18).

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular mesh sequence so that each mesh covers D exactly. In the case of mixed boundary conditions, we also assume that the meshes are compatible with the corresponding partition of the boundary ∂D. Let $k \geq 1$ be the degree of the underlying finite element. Let $P_k^{\mathrm{g}}(\mathcal{T}_h)$ be the H^1-conforming finite element space considered in §18.2.3 and §32.1. For the Robin problem, we set $V_h := P_k^{\mathrm{g}}(\mathcal{T}_h)$, and for the mixed and the Dirichlet problems we set

$$V_h := \{v_h \in P_k^{\mathrm{g}}(\mathcal{T}_h) \mid v_{h|\partial D_{\mathrm{d}}} = 0\}. \tag{35.19}$$

We construct an approximation of the Helmholtz problem as follows:

$$\begin{cases} \text{Find } u_h \in V_h \text{ such that} \\ a(u_h, w_h) = \ell(w_h), \quad \forall w_h \in V_h. \end{cases} \tag{35.20}$$

A first way to investigate the stability of the discrete problem (35.20) consists of reasoning by perturbation using the fact that the continuous problem is well-posed. Such a result can be obtained by invoking a variation of Fortin's lemma (a more abstract version of this variation is discussed in Exercise 35.3). Recall that the elliptic projection $\Pi_h^{\mathrm{E}} : V \to V_h$ is defined for all $v \in V$ s.t. $(\nabla(v - \Pi_h^{\mathrm{E}}(v)), \nabla w_h)_{\boldsymbol{L}^2(D)} = 0$ for all $w_h \in V_h$ (see §32.4).

Lemma 35.14 (Modified Fortin). *Assume that there are positive real numbers γ_{stb}, c_{app}, s such that the elliptic projection satisfies for all $v \in V$,*

$$\gamma_{\mathrm{stb}}\|\Pi_h^{\mathrm{E}}(v)\|_V \leq \|v\|_V, \qquad \|v - \Pi_h^{\mathrm{E}}(v)\|_{L^2(D)} \leq c_{\mathrm{app}} h^s \ell_D^{1-s}\|v\|_V. \tag{35.21}$$

Let α be the inf-sup constant of a on $V\times V$. Let $\iota_{L,V} > 0$ be such that

$$\|v\|_{L^2(D)} \leq \iota_{L,V}\ell_D\|v\|_V. \tag{35.22}$$

Assume that $h \in \mathcal{H} \cap (0, \ell_0(\kappa)]$ with $\ell_0(\kappa) := (\frac{1}{2}c_{\mathrm{app}}^{-1}\iota_{L,V}^{-1}\alpha\ell_D^{s-2}\kappa^{-2})^{\frac{1}{s}}$. Then the restriction of a to $V_h\times V_h$ satisfies the following inf-sup condition:

$$\inf_{v_h\in V_h}\sup_{w_h\in W_h}\frac{|a(v_h,w_h)|}{\|v_h\|_V\|w_h\|_V} \geq \alpha_0 := \frac{1}{2}\gamma_{\mathrm{stb}}\alpha > 0. \tag{35.23}$$

Proof. Using that $\Pi_h^{\mathrm{E}}(V) \subset V_h$ and the assumptions on Π_h^{E}, we have

$$\begin{aligned}
&\gamma_{\mathrm{stb}}^{-1}\sup_{w_h\in V_h}\frac{|a(v_h,w_h)|}{\|w_h\|_V} \geq \gamma_{\mathrm{stb}}^{-1}\sup_{w\in V}\frac{|a(v_h,\Pi_h^{\mathrm{E}}(w))|}{\|\Pi_h^{\mathrm{E}}(w)\|_V} \geq \sup_{w\in V}\frac{|a(v_h,\Pi_h^{\mathrm{E}}(w))|}{\|w\|_V} \\
&\geq \sup_{w\in V}\frac{|a(v_h,w) + \kappa^2(v_h, w - \Pi_h^{\mathrm{E}}(w))_{L^2(D)}|}{\|w\|_V} \\
&\geq \sup_{w\in V}\frac{|a(v_h,w)|}{\|w\|_V} - c_{\mathrm{app}}\iota_{L,V}h^s\ell_D^{2-s}\kappa^2\|v_h\|_V \geq (\alpha - c_{\mathrm{app}}\iota_{L,V}h^s\ell_D^{2-s}\kappa^2)\|v_h\|_V.
\end{aligned}$$

Since $h \le \ell_0(\kappa)$, using the definition of $\ell_0(\kappa)$ yields $\gamma_{\mathrm{stb}}^{-1} \sup_{w_h \in V_h} \frac{|a(v_h,w_h)|}{\|w_h\|_V} \ge \frac{1}{2}\alpha\|v_h\|_V$, i.e., (35.23) holds true with $\alpha_0 := \frac{1}{2}\gamma_{\mathrm{stb}}\alpha$. □

The above result can be applied with $s := 1$ when full elliptic regularity is available. One always has $s > \frac{1}{2}$ in polyhedra (see Theorem 31.31).

Remark 35.15 (Duality argument). A duality argument is implicitly present in the assumptions of Lemma 35.14 since duality has to be invoked to establish the approximation property $\|v - \Pi_h^{\mathrm{E}}(v)\|_{L^2(D)} \le c_{\mathrm{app}} h^s \ell_D^{1-s}\|v\|_V$ (see Theorem 32.15). □

A second way to investigate the stability of the discrete problem (35.20) is a technique introduced by Schatz [343] based on the Aubin–Nitsche duality argument.

Lemma 35.16 (Schatz). *Let V, W be two Banach spaces, W being reflexive. Let a be a bounded sesquilinear form on $V\times W$ satisfying the conditions of the BNB theorem with inf-sup and boundedness constants $0 < \alpha \le \|a\|$. Let L be a Hilbert space such that $\|v\|_L \le \iota_{L,V}\|v\|_V$ for all $v \in V$ (i.e., $V \hookrightarrow L$). Let $(V_h)_{h\in\mathcal{H}}$, $(W_h)_{h\in\mathcal{H}}$ be sequences of finite-dimensional subspaces equipped, respectively, with the norm of V and the norm of W. Assume the following:*

(i) *(Gårding's inequality) There are $c_V > 0$, $c_L \ge 0$ s.t. $c_V\|v_h\|_V - c_L\|v_h\|_L \le \sup_{w_h\in W_h} \frac{|a(v_h,w_h)|}{\|w_h\|_W}$ for all $v_h \in V_h$.*

(ii) *(Duality argument) There is a subspace $W_{\mathrm{s}} \hookrightarrow W$ and real numbers c_{smo}, c_{app}, and $s \in (0,1]$ s.t. $\inf_{w_h\in W_h} \|z - w_h\|_W \le c_{\mathrm{app}} h^s \|z\|_{W_{\mathrm{s}}}$ for all $z \in W_{\mathrm{s}}$ and all $h \in \mathcal{H}$. Moreover, for all $g \in L$, the unique solution $z \in W$ to the adjoint problem $a(v,z) = (v,g)_L$ for all $v \in V$, satisfies $\|z\|_{W_{\mathrm{s}}} \le c_{\mathrm{smo}}\|g\|_L$.*

Assume that $h \in \mathcal{H} \cap (0, \ell_0(\kappa)]$ with $\ell_0(\kappa) := (\frac{1}{2}c_V c_L^{-1}\|a\|^{-1}c_{\mathrm{app}}^{-1}c_{\mathrm{smo}}^{-1})^{\frac{1}{s}}$. Then the restriction of a to $V_h\times W_h$ satisfies the discrete inf-sup condition (35.23) *with $\alpha_0 \ge \frac{c_V}{2(\|a\|+c_L\iota_{L,V}+\frac{1}{2}c_V)}\alpha$.*

Proof. Let $v_h \ne 0$ be a member of V_h. Consider the antilinear form $\ell_h \in (W_h)'$ defined by $\ell_h(w_h) := a(v_h,w_h)$ for all $w_h \in W_h$. (Note that $\ell_h := A_h(v_h)$ with $A_h \in \mathcal{L}(V_h;W_h')$ s.t. $\langle A_h(y_h), w_h\rangle_{W_h',W_h} := a(y_h,w_h)$ for all $(y_h,w_h) \in V_h\times W_h$.) Owing to the Hahn–Banach theorem (Theorem C.13), we can extend ℓ_h to W. Let $\widetilde{\ell_h}$ be the extension in question with $\|\widetilde{\ell_h}\|_{W'} = \|\ell_h\|_{W_h'}$. Since a satisfies the conditions of the BNB theorem, there exists $u \in V$ such that $a(u,w) := \widetilde{\ell_h}(w)$ for all $w \in W$. (Notice that $u := A^{-1}(\widetilde{\ell_h})$ with $A \in \mathcal{L}(V;W')$ s.t. $\langle A(y), w\rangle_{W',W} := a(y,w)$ for all $(y,w) \in V\times W$.) Using the inf-sup condition satisfied by a on $V\times W$, we infer that

$$\sup_{w_h\in W_h} \frac{|a(v_h,w_h)|}{\|w_h\|_W} = \sup_{w_h\in W_h} \frac{|\ell_h(w_h)|}{\|w_h\|_W} = \|\ell_h\|_{W_h'} = \|\widetilde{\ell_h}\|_{W'}$$
$$= \sup_{w\in W} \frac{|a(u,w)|}{\|w\|_W} \ge \alpha\|u\|_V.$$

The rest of the proof consists of showing that there is c s.t. $\|u\|_V \geq c\|v_h\|_V$ for all $h \in \mathcal{H}$. Invoking Gårding's inequality on V_h gives

$$c_V\|v_h\|_V - c_L\|v_h\|_L \leq \sup_{w_h\in W_h}\frac{|a(v_h,w_h)|}{\|w_h\|_W} = \sup_{w_h\in W_h}\frac{|a(u,w_h)|}{\|w_h\|_W} \leq \|a\|\|u\|_V,$$

where we used that $a(u-v_h,w_h)=0$ for all $w_h \in W_h$ (Galerkin orthogonality property) and the boundedness of the sesquilinear form a on $V\times W$. Since $\|v\|_L \leq \iota_{L,V}\|v\|_V$ for all $v \in V$, we infer that

$$c_V\|v_h\|_V \leq c_L\|v_h-u\|_L + (c_L\iota_{L,V}+\|a\|)\|u\|_V.$$

We now establish an upper bound on $\|v_h-u\|_L$. Let $z \in W$ solve $a(v,z) = (v,u-v_h)_L$ for all v in V. The Galerkin orthogonality property implies that $\|u-v_h\|_L^2 = a(u-v_h,z) = a(u-v_h,z-z_h)$ for all $z_h \in W_h$. Hence, we have

$$\|u-v_h\|_L^2 \leq \|a\|\|u-v_h\|_V c_{\mathrm{a}}h^s\|z\|_{W_{\mathrm{s}}} \leq \|a\|\|u-v_h\|_V c_{\mathrm{app}}c_{\mathrm{smo}}h^s\|u-v_h\|_L,$$

so that $\|u-v_h\|_L \leq \|a\|c_{\mathrm{app}}c_{\mathrm{smo}}h^s\|u-v_h\|_V$. This in turn implies that

$$\begin{aligned} c_V\|v_h\|_V &\leq c_L\|v_h-u\|_L + (c_L\iota_{L,V}+\|a\|)\|u\|_V \\ &\leq c_L\|a\|c_{\mathrm{app}}c_{\mathrm{smo}}h^s\|u-v_h\|_V + (c_L\iota_{L,V}+\|a\|)\|u\|_V. \end{aligned}$$

Using the triangle inequality gives

$$(c_V - c_L\|a\|c_{\mathrm{app}}c_{\mathrm{smo}}h^s)\|v_h\|_V \leq \big(\|a\| + c_L\iota_{L,V} + c_L\|a\|c_{\mathrm{app}}c_{\mathrm{smo}}h^s\big)\|u\|_V.$$

Provided $h \leq \ell_0(\kappa)$ we obtain $c_L\|a\|c_{\mathrm{app}}c_{\mathrm{smo}}h^s \leq \frac{1}{2}c_V$, so that

$$\frac{c_V}{2(\|a\|+c_L\iota_{L,V}+\frac{1}{2}c_V)}\|v_h\|_V \leq \|u\|_V.$$

This concludes the proof. □

Both Lemma 35.14 and Lemma 35.16 imply that there is $\ell_0(\kappa)$ such that, if $h \in \mathcal{H}\cap(0,\ell_0(\kappa)]$, the discrete inf-sup condition (35.23) holds true with a constant that is uniform with respect to the meshsize but may depend on κ. To emphasize this dependency, let us write this constant as $\alpha_0(\kappa)$. We can now invoke Babuška's lemma (Lemma 26.14) to infer a quasi-optimal bound on the approximation error.

Corollary 35.17 (Error estimate). *There is $\ell_0(\kappa)$ s.t. the following quasi-optimal error estimate holds true for all $h \in \mathcal{H}\cap(0,\ell_0(\kappa)]$:*

$$\|u-u_h\|_V \leq \left(1+\frac{\|a\|}{\alpha_0(\kappa)}\right)\inf_{v_h\in V_h}\|u-v_h\|_V. \tag{35.24}$$

Example 35.18 (Dependence on κ). In order to illustrate the above results, let us assume that we impose Robin boundary conditions with the norm $\|v\|_V := \|\nabla v\|_{\boldsymbol{L}^2(D)} + \kappa\|v\|_{L^2(D)}$. Let us also assume that full elliptic regularity holds true, i.e., the conclusion of Theorem 35.9 is fulfilled with $s := 1$. Then $\alpha(\kappa) \sim (\kappa\ell_D)^{-1}$ for all $\kappa \geq \kappa_0$. Moreover, we have $c_{\text{app}} \sim 1$, $s := 1$, $\iota_{L,V} \sim \kappa\ell_D$ in Lemma 35.14, so that $\ell_0(\kappa) \sim \ell_D^{-1}\kappa^{-2}\kappa\ell_D(\ell_D\kappa)^{-1} = \kappa^{-2}\ell_D^{-1}$, and $\alpha_0(\kappa) \sim (\kappa\ell_D)^{-1}$. The error estimate (35.24) gives $\|u-u_h\|_V \leq (1+\kappa\ell_D)\inf_{v_h\in V_h}\|u-v_h\|_V$. Let us now use Lemma 35.16 with $\|a\| \sim 1$, $c_V := 1$, $c_L := \kappa$, $\iota_{L,V} := \kappa^{-1}$, $c_{\text{app}} \sim 1$, $s := 1$. In this case, it can be shown that $c_{\text{smo}} \sim \kappa\ell_D$. Then we have again $\ell_0(\kappa) \sim c_V c_L^{-1}\|a\|^{-1}c_{\text{a}}^{-1}c_{\text{smo}}^{-1} \sim \kappa^{-2}\ell_D^{-1}$ and $\alpha_0(\kappa) \sim (\kappa\ell_D)^{-1}$ leading to the same error estimate. □

Remark 35.19 (Literature). The reader is referred to Ihlenburg and Babuška [251] for an exhaustive analysis of the one-dimensional Helmholtz problem with mixed boundary conditions and its Galerkin approximation in one dimension with $g := 0$. In particular, the following statements are proved therein: (i) For piecewise linear continuous finite elements on a uniform mesh, α_h scales exactly like $(\kappa\ell_D)^{-1}$ uniformly in $h \in \mathcal{H}$, i.e., the discrete problem is well-posed for all $h \in \mathcal{H}$ (see [251, Thm. 4]); (ii) The $\mathbb{P}_1$ Galerkin method delivers a quasi-optimal error estimate in the H^1-seminorm with a constant proportional to $\kappa\ell_D$ if $\kappa h < 1 < \kappa\ell_D$ (see [251, Cor. 2]). □

Remark 35.20 (Dispersion error). It is shown in [251, Thm. 5] that $\|\nabla(u-u_h)\|_{\boldsymbol{L}^2(D)} \leq \ell_D(h\kappa/\pi)(1+ch\kappa^2\ell_D)\|f\|_{L^2}$, where c is independent of $h \in \mathcal{H}$ and $\kappa \geq 0$. The term proportional to $h\kappa^2\ell_D$ is usually called *pollution error* or *dispersion error*. This term grows unboundedly when κ grows even if $h\kappa < 1$. The question whether the pollution error could be reduced or eliminated by using stabilization techniques (i.e., discontinuous approximation techniques or methods similar to those presented in Chapters 57–60) has been extensively addressed in the literature. We refer the reader to Burman et al. [102], Feng and Wu [200], Melenk and Sauter [300], Peterseim [325], and the literature therein for more details. For instance, it is shown in [300, Thm. 5.8] that under some appropriate assumptions the pollution effect can be suppressed if one assumes that $\kappa h/k$ is sufficiently small and that the polynomial degree k is at least $\mathcal{O}(\ln(\kappa))$. It is shown in [102, Thm. 6] that the pollution error disappears in one dimension for some specific κ-dependent choices of the penalty parameter of the CIP method (see §58.3 for details on CIP). The pollution error is also shown to disappear in [325, Thm. 6.2] for a localized Petrov-Galerkin method where the global shape functions each have a support of size rh with the oversampling condition $r \gtrsim \ln(\kappa\ell_D)$. □

Exercises

Exercise 35.1 (1D Helmholtz, well-posedness). Let $D := (0,\ell_D)$, $\kappa > 0$, and consider the Helmholtz problem with mixed boundary conditions:

$-\partial_{xx}u - \kappa^2 u = f$ in D, $u(0) = 0$, and $\partial_x u(\ell_D) - \mathrm{i}\kappa u(\ell_D) = 0$. (i) Give a weak formulation in $V := \{v \in H^1(D) \mid v(0) = 0\}$. (ii) Show by invoking an ODE argument that if the weak formulation has a solution, then it is unique. (iii) Show that the weak problem is well-posed. (*Hint*: use Lemma 35.3.)

Exercise 35.2 (Green's function, 1D). Let $G : D\times D \to \mathbb{C}$ be the function defined by

$$G(x,s) := \kappa^{-1} \begin{cases} \sin(\kappa x)e^{\mathrm{i}\kappa s} & \text{if } x \in [0,s], \\ \sin(\kappa s)e^{\mathrm{i}\kappa x} & \text{if } x \in [s,1]. \end{cases}$$

(i) Prove that for all $x \in D$, the function $D \ni s \mapsto G(x,s) \in \mathbb{C}$ solves the PDE $-\partial_{ss}u - \kappa^2 u = \delta_{s=x}$ in D with the boundary conditions $u(0) = 0$ and $\partial_s u(\ell_D) - \mathrm{i}\kappa u(\ell_D) = 0$ (i.e., G is the Green's function of the Helmholtz problem from Exercise 35.1). (ii) Find $H(x,s)$ s.t. $\partial_s H(x,s) = \partial_x G(x,s)$. (iii) Let $u(x) := \int_0^{\ell_D} G(x,s)f(s)\,\mathrm{d}s$. Prove that $\|u\|_{L^2(D)} \le \kappa^{-1}\|f\|_{L^2(D)}$, $|u|_{H^1(D)} \le \|f\|_{L^2(D)}$, and $|u|_{H^2(D)} \le (\kappa+1)\|f\|_{L^2(D)}$. (iv) Let $v \in L^2(D)$ and let $\widetilde{z}(x) := \kappa^2 \int_0^{\ell_D} G(x,s)v(s)\,\mathrm{d}s$. What is the PDE solved by $\widetilde{z}$? Same question for $z(x) := \kappa^2 \int_0^{\ell_D} \overline{G}(x,s)v(s)\,\mathrm{d}s$. *Note*: The function z is invoked in Step (1) of the proof of Theorem 35.11. (v) Assume now that $v \in H^1(D)$ with $v(0) = 0$, and let z and $\widetilde{z}$ be defined as above. Prove that $\max(|z|_{H^1(D)}, |\widetilde{z}|_{H^1(D)}) \le 4\kappa\ell_D|v|_{H^1(D)}$. (*Hint*: see Ihlenburg and Babuška [251, p. 14] (up to the factor 4).)

Exercise 35.3 (Variation on Fortin's lemma). Let V, W be two Banach spaces and let a be a bounded sesquilinear form on $V\times W$ like in Fortin's Lemma 26.9. Let $(V_h)_{h\in\mathcal{H}}$, $(W_h)_{h\in\mathcal{H}}$ be sequences of subspaces of V and W equipped with the norm of V and W, respectively. Assume that there exists a map $\Pi_h : W \to W_h$ and constants $\gamma_{\Pi_h} > 0$, $c(h) > 0$ such that $|a(v_h, w - \Pi_h(w))| \le c(h)\|v_h\|_V\|w\|_W$, $\gamma_{\Pi_h}\|\Pi_h(w)\|_W \le \|w\|_W$ for all $v_h \in V_h$, all $w \in W$, and all $h \in \mathcal{H}$. Assume that $\lim_{h\to 0} c(h) = 0$. Prove that the discrete inf-sup condition (26.5a) holds true for $h \in \mathcal{H}$ small enough.

Exercise 35.4 (Lemma 35.8). (i) Prove that $\Re((\boldsymbol{m}\cdot\nabla v)\overline{v}) = \frac{1}{2}\boldsymbol{m}\cdot\nabla|v|^2$ for all $v \in H^1(D;\mathbb{C})$ and $\boldsymbol{m} \in \mathbb{R}^d$. (ii) Prove that $\Re(\boldsymbol{m}\cdot((\nabla\boldsymbol{v})^{\mathsf{T}}\overline{\boldsymbol{v}})) = \frac{1}{2}\boldsymbol{m}\cdot\nabla\|\boldsymbol{v}\|^2_{\ell^2(\mathbb{C}^d)}$ for all $\boldsymbol{v} \in H^1(D;\mathbb{C}^d)$ and $\boldsymbol{m} \in \mathbb{R}^d$. (iii) Let $q \in H^2(D;\mathbb{C})$ and let D^2q denote the Hessian matrix of q, i.e., $(D^2q)_{ij} = \partial^2_{x_ix_j}q$ for all $i,j \in \{1{:}d\}$. Show that $\Re(\boldsymbol{m}\cdot((D^2q)\nabla\overline{q})) = \frac{1}{2}\boldsymbol{m}\cdot\nabla\|\nabla q\|^2_{\ell^2(\mathbb{C}^d)}$. (iv) Prove that (35.11) holds true for all $q \in \{v \in H^1(D;\mathbb{C}) \mid \Delta v \in L^2(D;\mathbb{C}),\ \nabla v \in L^2(\partial D;\mathbb{C}^d)\}$ and all $\boldsymbol{m} \in W^{1,\infty}(D;\mathbb{R}^d)$. (*Hint*: assume first that $q \in H^2(D;\mathbb{C})$.)