Part VIII, Chapter 36

Crouzeix–Raviart approximation

In Part VIII, composed of Chapters 36 to 41, we study various nonconforming approximations of an elliptic model problem. We first study the Poisson equation with a homogeneous Dirichlet condition and then address a diffusion PDE with contrasted coefficients. Nonconformity means that the discrete trial and test spaces are not subspaces of $H^1(D)$. Nonconformity has many sources. It may be that the discrete shape functions have nonzero jumps across the mesh interfaces. It may be that the Dirichlet conditions are enforced weakly. Another possible reason is that the approximation involves discrete unknowns associated with the mesh faces as in hybrid methods. All of these situations are studied in the following chapters. The objective of the present chapter is to study the nonconforming approximation of the Poisson equation by Crouzeix–Raviart finite elements. Another objective is to illustrate the abstract error analysis of Chapter 27.

36.1 Model problem

Let D be a Lipschitz domain in $\mathbb{R}^d$. We assume for simplicity that D is a polyhedron. We focus on the Poisson equation with homogeneous Dirichlet boundary conditions:

$$-\Delta u = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{36.1}$$

with source term $f \in L^2(D)$. The weak formulation is as follows:

$$\begin{cases} \text{Find } u \in V := H_0^1(D) \text{ such that} \\ a(u, w) = \ell(w), \quad \forall w \in V, \end{cases} \tag{36.2}$$

with

$$a(v, w) := \int_D \nabla v \cdot \nabla w \, \mathrm{d}x, \qquad \ell(w) := \int_D f w \, \mathrm{d}x. \tag{36.3}$$

Owing to the Poincaré–Steklov inequality (see (3.11) with $p := 2$), there is $C_{\mathrm{PS}} > 0$ such that $C_{\mathrm{PS}}\|v\|_{L^2(D)} \le \ell_D \|\nabla v\|_{\boldsymbol{L}^2(D)}$ for all $v \in V$, where ℓ_D is a length scale associated with D, e.g., $\ell_D := \mathrm{diam}(D)$. Hence, V equipped with the norm $\|v\|_V := \|\nabla v\|_{\boldsymbol{L}^2(D)} = |v|_{H^1(D)}$ is a Hilbert space, and the bilinear form a coincides with the inner product in V. Owing to the Lax–Milgram lemma, (36.2) is well-posed. We refer the reader to §41.2 for the more general PDE $-\nabla\cdot(\lambda \nabla u) = f$ with contrasted diffusivity λ.

36.2 Crouzeix–Raviart discretization

In this section, we recall Crouzeix–Raviart finite element, we define the corresponding approximation space, we formulate the discrete problem, and we establish its well-posedness. We also derive some important stability estimates for Crouzeix–Raviart finite elements.

36.2.1 Crouzeix–Raviart finite elements

The Crouzeix–Raviart finite element is introduced in §7.5; see [151] for the original work to approximate the Stokes equations. Let $\widehat{K}$ be the unit simplex in $\mathbb{R}^d$ with vertices $\{\widehat{\boldsymbol{z}}_i\}_{i\in\{0:d\}}$. Let $\widehat{F}_i$ be the face of $\widehat{K}$ opposite to $\widehat{\boldsymbol{z}}_i$. The *Crouzeix–Raviart finite element* is defined by setting $\widehat{P} := \mathbb{P}_{1,d}$ and by using the following degrees of freedom (dofs) on $\widehat{P}$:

$$\widehat{\sigma}_i^{\mathrm{CR}}(\widehat{p}) := \frac{1}{|\widehat{F}_i|}\int_{\widehat{F}_i} \widehat{p}\,\mathrm{d}s, \qquad \forall i \in \{0:d\}. \tag{36.4}$$

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular matching mesh sequence composed of affine simplices so that each mesh covers D exactly. Let $\mathcal{T}_h$ be a mesh and let K be a cell in $\mathcal{T}_h$. Using the Crouzeix–Raviart element as reference finite element and letting the transformation ψ_K be the pullback by the geometric mapping, i.e., $\psi_K(v) := v \circ \boldsymbol{T}_K$, Proposition 9.2 allows us to generate a Crouzeix–Raviart finite element in K. We have $P_K := \psi_K^{-1}(\widehat{P}) = \mathbb{P}_{1,d}\circ \boldsymbol{T}_K^{-1} = \mathbb{P}_{1,d}$ since $\boldsymbol{T}_K$ is affine, and the local dofs in K are for all $p \in P_K$,

$$\sigma_{K,i}^{\mathrm{CR}}(p) := \widehat{\sigma}_i^{\mathrm{CR}}(\psi_K(p)) = \frac{1}{|\widehat{F}_i|}\int_{\widehat{F}_i} p\circ \boldsymbol{T}_K\,\mathrm{d}\widehat{s} = \frac{1}{|F_{K,i}|}\int_{F_{K,i}} p\,\mathrm{d}s, \tag{36.5}$$

for all $i \in \{1:d\}$, where $\{F_{K,i} := \boldsymbol{T}_K(\widehat{F}_i)\}_{i\in\{0:d\}}$ are the faces of K. The local interpolation operator $\mathcal{I}_K^{\mathrm{CR}} : V(K) := W^{1,1}(K) \to P_K$ is such that $\mathcal{I}_K^{\mathrm{CR}}(v) := \sum_{i\in\{0:d\}} \sigma_{K,i}^{\mathrm{CR}}(v)\theta_{K,i}^{\mathrm{CR}}$ for all $v \in V(K)$, where $\{\theta_{K,i}\}_{i\in\{0:d\}}$ are the local shape functions in K s.t. $\sigma_{K,i}^{\mathrm{CR}}(\theta_{K,j}^{\mathrm{CR}}) = \delta_{ij}$ for all $i,j \in \{0:d\}$. Recall that $\theta_i^{\mathrm{CR}} := 1 - d\lambda_i$, where $\{\lambda_i\}_{i\in\{0:d\}}$ are the barycentric coordinates in K.

Lemma 36.1 (Local interpolation). *There is c s.t. for all $r \in [0,1]$, all $p \in [1,\infty]$, all $v \in W^{1+r,p}(K)$, all $K \in \mathcal{T}_h$, and all $h \in \mathcal{H}$,*

$$\|v - \mathcal{I}_K^{\mathrm{CR}}(v)\|_{L^p(K)} + h_K|v - \mathcal{I}_K^{\mathrm{CR}}(v)|_{W^{1,p}(K)} \le c\, h_K^{1+r}|v|_{W^{1+r,p}(K)}. \tag{36.6}$$

Proof. Let $v \in W^{1+r,p}(K)$. The error estimates for $r \in \{0,1\}$ follow from Theorem 11.13 with $k := 1$ and $l := 1$ since $V(K) := W^{1,1}(K)$. For $r \in (0,1)$, we use Corollary 12.13, the $W^{1,p}$-stability of $\mathcal{I}_K^{\mathrm{CR}}$, and the fact that $P_K := \mathbb{P}_{1,d}$ is pointwise invariant under $\mathcal{I}_K^{\mathrm{CR}}$ to infer that

$$\begin{aligned} |v - \mathcal{I}_K^{\mathrm{CR}}(v))|_{W^{1,p}(K)} &\le \inf_{p\in\mathbb{P}_{1,d}} |v - p - \mathcal{I}_K^{\mathrm{CR}}(v-p))|_{W^{1,p}(K)} \\ &\le c \inf_{p\in\mathbb{P}_{1,d}} |v-p|_{W^{1,p}(K)} \le c'\, h_K^r |v|_{W^{1+r,p}(K)}. \end{aligned}$$

The bound on $\|v - \mathcal{I}_K^{\mathrm{CR}}(v)\|_{L^p(K)}$ follows by proceeding similarly and using that $\|\mathcal{I}_K^{\mathrm{CR}}(v)\|_{L^p(K)} \le \|v\|_{L^p(K)} + ch_K|v|_{W^{1,p}(K)}$. □

36.2.2 Crouzeix–Raviart finite element space

Consider the broken finite element space defined in (18.4) with $k := 1$,

$$P_1^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^\infty(D) \mid v_{h|K} \in \mathbb{P}_{1,d},\, \forall K \in \mathcal{T}_h\}.$$

Recall that the set $\mathcal{F}_h^\circ$ is the collection of the interior faces (interfaces) in the mesh, and the faces are oriented by the unit normal vector $\boldsymbol{n}_F$ (see Chapter 10 on mesh orientation). For all $F \in \mathcal{F}_h^\circ$, there are two cells K_l, K_r s.t. $F := \partial K_l \cap \partial K_r$ and $\boldsymbol{n}_F$ points from K_l to K_r, i.e., $\boldsymbol{n}_F := \boldsymbol{n}_{K_l} = -\boldsymbol{n}_{K_r}$. The notion of jump across F is defined by setting $[\![v]\!]_F := v_{|K_l} - v_{|K_r}$. It is convenient to use a common notation for interfaces and boundary faces by writing $[\![v]\!]_F := v_{|K_l}$ for every boundary face $F := \partial K_l \cap \partial D \in \mathcal{F}_h^\partial$. The Crouzeix–Raviart finite element space is defined as

$$P_1^{\mathrm{CR}}(\mathcal{T}_h) := \{v_h \in P_1^{\mathrm{b}}(\mathcal{T}_h) \mid \int_F [\![v_h]\!]_F \,\mathrm{d}s = 0,\, \forall F \in \mathcal{F}_h^\circ\}. \tag{36.7}$$

The condition $\int_F [\![v_h]\!]_F \,\mathrm{d}s = 0$ is equivalent to the continuity of v_h at the barycenter $\boldsymbol{x}_F$ of F. Note that $P_1^{\mathrm{CR}}(\mathcal{T}_h)$ is not H^1-conforming since membership in $H^1(D)$ requires having zero-jumps pointwise (see Theorem 18.8).

Let $F \in \mathcal{F}_h$ be a mesh face. Let us denote by $\mathcal{T}_F := \{K \in \mathcal{T}_h \mid F \in \mathcal{F}_K\}$ the collection of the mesh cells having F as face ($\mathcal{T}_F$ contains two cells for $F \in \mathcal{F}_h^\circ$ and one cell for $F \in \mathcal{F}_h^\partial$). Let φ_F^{CR} be the function such that $\varphi_{F|K}^{\mathrm{CR}}$ is the local shape function in K associated with F if $K \in \mathcal{T}_F$ and $\varphi_{F|K}^{\mathrm{CR}} := 0$ otherwise; see Figure 36.1 for $d = 2$. Note that $\mathrm{supp}(\varphi_F^{\mathrm{CR}}) = D_F := \mathrm{int}(\bigcup_{K\in\mathcal{T}_F} K)$, i.e., D_F is the collection of all the points in the (one or two) mesh cells containing F. Let γ_F^{CR} be the linear form on $P_1^{\mathrm{CR}}(\mathcal{T}_h)$ such that $\gamma_F^{\mathrm{CR}}(v_h) := |F|^{-1}\int_F v_h \,\mathrm{d}s$

for all $v_h \in P_1^{\mathrm{CR}}(\mathcal{T}_h)$. Although v_h may be multivalued at F, the quantity $\gamma_F^{\mathrm{CR}}(v_h)$ is well defined since $\int_F [\![v_h]\!]_F \,\mathrm{d}s = 0$.

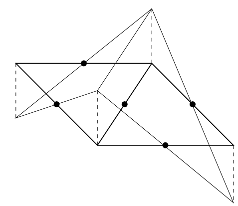

Fig. 36.1 Global shape function for the Crouzeix–Raviart finite element. The support is materialized by thick lines and the graph by thin lines. Bullets indicate the barycenter of the edges.

Proposition 36.2 (Global dofs). *$\{\varphi_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h}$ is a basis of $P_1^{\mathrm{CR}}(\mathcal{T}_h)$, and $\{\gamma_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h}$ is a basis of $\mathcal{L}(P_1^{\mathrm{CR}}(\mathcal{T}_h);\mathbb{R})$.*

Proof. φ_F^{CR} is a member of $P_1^{\mathrm{CR}}(\mathcal{T}_h)$ since φ_F^{CR} is piecewise affine by construction and its mean value on a mesh face is 0 or 1. Consider now real numbers $\{\alpha_F\}_{F\in\mathcal{F}_h}$ s.t. the function $w := \sum_{F\in\mathcal{F}_h} \alpha_F \varphi_F^{\mathrm{CR}}$ vanishes identically. Observing that $\gamma_{F'}^{\mathrm{CR}}(\varphi_F^{\mathrm{CR}}) = \delta_{FF'}$ for all $F, F' \in \mathcal{F}_h$, where $\delta_{FF'}$ denotes the Kronecker symbol, we infer that $\alpha_{F'} = \gamma_{F'}^{\mathrm{CR}}(w) = 0$ for all $F' \in \mathcal{F}_h$. Hence, the functions $\{\varphi_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h}$ are linearly independent. Finally, let $v_h \in P_1^{\mathrm{CR}}(\mathcal{T}_h)$ and set $w_h := \sum_{F\in\mathcal{F}_h} \gamma_F^{\mathrm{CR}}(v_h)\varphi_F^{\mathrm{CR}}$. Then, $v_{h|K}$ and $w_{h|K}$ are in P_K for all $K \in \mathcal{T}_h$, and $\sigma_{K,i}(w_{h|K}) = \sigma_{K,i}(v_{h|K})$ for all $i \in \{0{:}d\}$. Unisolvence implies that $v_{h|K} = w_{h|K}$, so that $v_h = w_h$ since $K \in \mathcal{T}_h$ is arbitrary. This shows that $\{\varphi_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h}$ is a basis of $P_1^{\mathrm{CR}}(\mathcal{T}_h)$. By using similar arguments, it follows that $\{\gamma_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h}$ is a basis of $\mathcal{L}(P_1^{\mathrm{CR}}(\mathcal{T}_h);\mathbb{R})$. □

Proposition 36.2 implies that the dimension of $P_1^{\mathrm{CR}}(\mathcal{T}_h)$ is equal to the number of faces (edges in dimension two) in the mesh. Moreover, the global Crouzeix–Raviart interpolation operator acts on every function v in $W^{1,1}(D)$ as follows: For all $\boldsymbol{x} \in D$,

$$\mathcal{I}_h^{\mathrm{CR}}(v)(\boldsymbol{x}) := \sum_{F\in\mathcal{F}_h} \gamma_F^{\mathrm{CR}}(v)\varphi_F^{\mathrm{CR}}(\boldsymbol{x}) = \sum_{F\in\mathcal{F}_h} \left(\frac{1}{|F|}\int_F v \,\mathrm{d}s\right) \varphi_F^{\mathrm{CR}}(\boldsymbol{x}).$$

Since $\mathcal{I}_h^{\mathrm{CR}}(v)_{|K} = \mathcal{I}_K^{\mathrm{CR}}(v_{|K})$ for all $K \in \mathcal{T}_h$, the approximation results of Lemma 36.1 can be rephrased in terms of $\mathcal{I}_h^{\mathrm{CR}}$.

36.2.3 Discrete problem and well-posedness

We account for the homogeneous Dirichlet boundary condition by considering the following subspace of $P_1^{\mathrm{CR}}(\mathcal{T}_h)$:

$$P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h) := \left\{ v_h \in P_1^{\mathrm{CR}}(\mathcal{T}_h) \mid \int_F v_h \,\mathrm{d}s = 0, \forall F \in \mathcal{F}_h^{\partial} \right\}, \tag{36.8}$$

where $\mathcal{F}_h^\partial$ is the collection of the mesh faces located at the boundary. By proceeding as in Proposition 36.2, one can verify that $\{\varphi_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h^\circ}$ is a basis of $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$, and $\{\gamma_F^{\mathrm{CR}}\}_{F\in\mathcal{F}_h^\circ}$ is a basis of $\mathcal{L}(P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h);\mathbb{R})$. The dimension of $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ is the number of internal faces (edges if $d=2$) in the mesh.

The bilinear form a introduced in (36.3) is not well defined on $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ since this space is not H^1-conforming. Since functions in $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ are piecewise smooth, we can localize their gradient to the mesh cells. To this purpose, we introduce the notion of broken gradient on the broken Sobolev space $W^{1,p}(\mathcal{T}_h)$ with $p\in[1,\infty]$. Recall from Definition 18.1 that a function $v\in W^{1,p}(\mathcal{T}_h)$ is s.t. $\nabla(v_{|K})\in \boldsymbol{L}^p(K)$ for all $K\in\mathcal{T}_h$.

Definition 36.3 (Broken gradient). *Let $p\in[1,\infty]$. The* broken gradient *operator $\nabla_h : W^{1,p}(\mathcal{T}_h)\to \boldsymbol{L}^p(D)$ is defined by setting $(\nabla_h v)_{|K} := \nabla(v_{|K})$ for all $K\in\mathcal{T}_h$.*

A crucial consequence of Lemma 18.9 is that $\nabla_h v = \nabla v$ whenever $v\in W^{1,p}(D)$. This property will be often used for the solution to the model problem (36.2) since $u\in H_0^1(D)$. We define the following discrete bilinear and linear forms on $V_h\times V_h$ and on V_h, respectively:

$$a_h(v_h,w_h) := \int_D \nabla_h v_h{\cdot}\nabla_h w_h \,\mathrm{d}x, \qquad \ell_h(w_h) := \int_D f w_h \,\mathrm{d}x, \tag{36.9}$$

and we consider the following discrete problem:

$$\begin{cases} \text{Find } u_h\in V_h := P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h) \text{ such that} \\ a_h(u_h,w_h) = \ell_h(w_h), \quad \forall w_h\in V_h. \end{cases} \tag{36.10}$$

Lemma 36.4 (Coercivity, well-posedness). (i) *The map*

$$v_h \mapsto \|v_h\|_{V_h} := a_h(v_h,v_h)^{\frac12} = \|\nabla_h v_h\|_{\boldsymbol{L}^2(D)} \tag{36.11}$$

is a norm on $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$. (ii) *Equipping V_h with this norm, the bilinear form a_h is coercive on V_h with $\alpha_h := 1$.* (iii) *The discrete problem* (36.10) *is well-posed.*

Proof. (i) The only nontrivial property is to prove that $\|v_h\|_{V_h}=0$ implies that $v_h=0$ for all $v_h\in V_h$. If $\|v_h\|_{V_h}=0$, then v_h is piecewise constant. The additional property $\int_F [\![v_h]\!]_F \,\mathrm{d}s = 0$ for all $F\in\mathcal{F}_h^\circ$ implies that v_h is globally constant on D. That $v_h=0$ follows from $\int_F v_h\,\mathrm{d}s=0$ for all $F\in\mathcal{F}_h^\partial$.
(ii)-(iii) Since $\|{\cdot}\|_{V_h}$ is a norm on V_h, coercivity follows from the definition of $\|{\cdot}\|_{V_h}$, and well-posedness follows from the Lax–Milgram lemma. □

Remark 36.5 (Nonsmooth right-hand side). We observe that it is not clear how one should account for a source term f in $H^{-1}(D)$ in (36.10), since it is not clear how f would act on (discrete) functions that are not in $H_0^1(D)$. One possibility is to consider the discrete linear form $\ell_h(w_h) :=$

$\langle f, \mathcal{J}^{\mathrm{av}}_{h,0}(w_h)\rangle_{H^{-1}(D),H^1_0(D)}$ where $\mathcal{J}^{\mathrm{av}}_{h,0} : P^{\mathrm{b}}_1(\mathcal{T}_h) \to P^{\mathrm{g}}_{1,0}(\mathcal{T}_h)$ is the averaging operator with boundary conditions introduced in §22.4.1. A general theory addressing this type of difficulty is developed in Veeser and Zanotti [373]. □

36.2.4 Discrete Poincaré–Steklov inequality

On the H^1_0-conforming subspace $P^{\mathrm{g}}_{1,0}(\mathcal{T}_h) := P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h) \cap H^1_0(D)$, the norm $\|\cdot\|_{V_h}$ defined in (36.11) coincides with the H^1-seminorm. Owing to the Poincaré–Steklov inequality, we know that there is $C_{\mathrm{PS}} > 0$ s.t. $C_{\mathrm{PS}}\|v_h\|_{L^2(D)} \leq \ell_D\|\nabla v_h\|_{\boldsymbol{L}^2(D)} = \ell_D\|v_h\|_{V_h}$ for all $v_h \in P^{\mathrm{g}}_{1,0}(\mathcal{T}_h)$. We now prove that a similar inequality is available on the larger space $P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$.

Lemma 36.6 (Discrete Poincaré–Steklov inequality). *There is $C^{\mathrm{CR}}_{\mathrm{PS}} > 0$ s.t. for all $v_h \in P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$ and all $h \in \mathcal{H}$,*

$$C^{\mathrm{CR}}_{\mathrm{PS}}\|v_h\|_{L^2(D)} \leq \ell_D\|\nabla_h v_h\|_{\boldsymbol{L}^2(D)}. \tag{36.12}$$

Proof. Let $v_h \in P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$. Let $\phi \in H^1_0(D)$ solve $\Delta\phi = v_h$ and let $\boldsymbol{\sigma} := \nabla\phi$. Then $\nabla\cdot\boldsymbol{\sigma} = v_h$. Elliptic regularity implies that there is $s > \frac{1}{2}$ such that $\phi \in H^{1+s}(D)$ (see Theorem 31.33) so that $\boldsymbol{\sigma} \in \boldsymbol{H}^s(D)$. Moreover, there is $\gamma_D > 0$ such that $\gamma_D(\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)} + \ell^s_D|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(D)}) \leq \ell_D\|v_h\|_{L^2(D)}$. Integrating by parts cellwise, we infer that

$$\begin{aligned}
\|v_h\|^2_{L^2(D)} &= \int_D v_h\nabla\cdot\boldsymbol{\sigma}\,\mathrm{d}x = \sum_{K\in\mathcal{T}_h}\int_K v_{h|K}\nabla\cdot\boldsymbol{\sigma}\,\mathrm{d}x \\
&= -\sum_{K\in\mathcal{T}_h}\int_K \boldsymbol{\sigma}\cdot\nabla(v_{h|K})\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K}\int_F \boldsymbol{\sigma}\cdot\boldsymbol{n}_K v_{h|K}\,\mathrm{d}s \\
&= -\int_D \boldsymbol{\sigma}\cdot\nabla_h v_h\,\mathrm{d}x + \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K}\int_F \boldsymbol{\sigma}\cdot\boldsymbol{n}_K v_{h|K}\,\mathrm{d}s =: \mathfrak{T}_1 + \mathfrak{T}_2,
\end{aligned}$$

where $\mathcal{F}_K$ is the collection of the faces of K and $\boldsymbol{n}_K$ the outward unit normal to K (observe that $\boldsymbol{\sigma}$ is single-valued on F since $\boldsymbol{\sigma} \in \boldsymbol{H}^s(D)$ with $s > \frac{1}{2}$). The Cauchy–Schwarz inequality implies that

$$|\mathfrak{T}_1| \leq \|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)}\|\nabla_h v_h\|_{\boldsymbol{L}^2(D)}.$$

Consider now $\mathfrak{T}_2$. If $F := \partial K_l \cap \partial K_r$ is an interface, the integral over F appears twice in the sum. Since $\int_F v_{h|K_l}\,\mathrm{d}s = \int_F v_{h|K_r}\,\mathrm{d}s$ by definition of $P^{\mathrm{CR}}_1(\mathcal{T}_h)$ and since $\boldsymbol{n}_{K_l} = -\boldsymbol{n}_{K_r}$, we can subtract from $\boldsymbol{\sigma}$ a constant function on F that we take equal to $\underline{\boldsymbol{\sigma}}_F := \frac{1}{|F|}\int_F \boldsymbol{\sigma}\,\mathrm{d}s$. The same conclusion is valid for the boundary faces since $\int_F v_h\,\mathrm{d}s = 0$ on such faces by definition of $P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$. This leads to

$$\begin{aligned}\mathfrak{T}_2 &= \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K}\int_F(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}}_F)\cdot\boldsymbol{n}_K v_{h|K}\,\mathrm{d}s\\ &= \sum_{K\in\mathcal{T}_h}\sum_{F\in\mathcal{F}_K}\int_F(\boldsymbol{\sigma}-\underline{\boldsymbol{\sigma}}_F)\cdot\boldsymbol{n}_K (v_{h|K}-\underline{v}_F)\,\mathrm{d}s,\end{aligned}$$

where the subtraction of the single-valued quantity $\underline{v}_F := \frac{1}{|F|}\int_F v_h\,\mathrm{d}s$ is justified as above. Applying Lemma 36.8 below to $\sigma_{|K}$ and to $v_{h|K}$, using $h_K \le \ell_D$ for all $K\in\mathcal{T}_h$, and invoking the Cauchy–Schwarz inequality yields

$$\begin{aligned}|\mathfrak{T}_2| &\le c\sum_{K\in\mathcal{T}_h} h_K^{s-\frac12}|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(K)} h_K^{\frac12}\|\nabla(v_{h|K})\|_{\boldsymbol{L}^2(K)}\\ &\le c\,\ell_D^s\sum_{K\in\mathcal{T}_h}|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(K)}\|\nabla(v_{h|K})\|_{\boldsymbol{L}^2(K)} \le c\,\ell_D^s|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(D)}\|\nabla_h v_h\|_{\boldsymbol{L}^2(D)},\end{aligned}$$

since $\sum_{K\in\mathcal{T}_h}|\boldsymbol{\sigma}|^2_{\boldsymbol{H}^s(K)} \le |\boldsymbol{\sigma}|^2_{\boldsymbol{H}^s(D)}$. Combining the above bounds on $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we infer that

$$\|v_h\|^2_{L^2(D)} \le \big(\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)} + c\,\ell_D^s|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(D)}\big)\|\nabla_h v_h\|_{\boldsymbol{L}^2(D)},$$

and (36.12) follows from $\gamma_D(\|\boldsymbol{\sigma}\|_{\boldsymbol{L}^2(D)} + \ell_D^s|\boldsymbol{\sigma}|_{\boldsymbol{H}^s(D)}) \le \ell_D\|v_h\|_{L^2(D)}$. □

Remark 36.7 (Literature). The above proof is adapted from Temam [363, Prop. 4.13]; see also Croisille and Greff [150]. □

Lemma 36.8 (Poincaré–Steklov on faces). *Let $s\in(\frac12,1]$. There is c s.t.*

$$\|\psi-\underline{\psi}_F\|_{L^2(F)} \le c\,h_K^{s-\frac12}|\psi|_{H^s(K)}, \tag{36.13}$$

for all $\psi\in H^s(K)$ with $\underline{\psi}_F := \frac{1}{|F|}\int_F\psi\,\mathrm{d}s$; all $K\in\mathcal{T}_h$, all $F\in\mathcal{F}_K$, and all $h\in\mathcal{H}$ (the constant c grows unboundedly as $s\downarrow\frac12$).

Proof. Let $\tilde\psi := \psi - \frac{1}{|K|}\int_K\psi\,\mathrm{d}x$. With obvious notation, we have $\psi-\underline{\psi}_F = \tilde\psi-\underline{\tilde\psi}_F$. The triangle inequality and the Cauchy–Schwarz inequality imply that $\|\psi-\underline{\psi}_F\|_{L^2(F)} \le 2\|\tilde\psi\|_{L^2(F)}$. Using the trace inequality (12.17) yields

$$\|\psi-\underline{\psi}_F\|_{L^2(F)} \le c(h_K^{-\frac12}\|\tilde\psi\|_{L^2(K)} + h_K^{s-\frac12}|\tilde\psi|_{H^s(K)}).$$

The expected bound follows from $|\tilde\psi|_{H^s(K)} = |\psi|_{H^s(K)}$ and the Poincaré–Steklov inequality ((12.13) if $s=1$ or (12.14) if $s\in(\frac12,1)$) on K, which gives $\|\tilde\psi\|_{L^2(K)} \le ch_K^s|\psi|_{H^s(K)}$. □

36.2.5 Bound on the jumps

Bounding the jumps of functions in $P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$ is useful in many situations. The following result will be invoked in the next section.

Lemma 36.9 (Bound on the jumps). *There is c s.t. for all $v_h \in P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$ and all $h \in \mathcal{H}$,*

$$c^{-1} \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]\|^2_{L^2(F)} \le \inf_{v\in H^1_0(D)} \|\nabla_h(v-v_h)\|^2_{\boldsymbol{L}^2(D)} \le c \sum_{F\in\mathcal{F}_h} h_F^{-1} \|[\![v_h]\!]\|^2_{L^2(F)}. \tag{36.14}$$

Proof. Let $v_h \in P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$. For all $K \in \mathcal{T}_h$, let us set $H^1_*(K) := \{\phi \in H^1(K) \mid \int_K \phi \,\mathrm{d}x = 0\}$ and let $\mathcal{F}_K$ be the collection of the faces of K. For all $F \in \mathcal{F}_K$, let $\psi_{K,F} \in H^1_*(K)$ solve the local Neumann problem:

$$\int_K \nabla\psi_{K,F}{\cdot}\nabla\phi \,\mathrm{d}x = \epsilon_{K,F} \int_F [\![v_h]\!]_F \phi \,\mathrm{d}s, \quad \forall \phi \in H^1_*(K), \tag{36.15}$$

where $\epsilon_{K,F} := \boldsymbol{n}_K{\cdot}\boldsymbol{n}_F = \pm 1$. This problem is well-posed since $\int_F [\![v_h]\!]_F \,\mathrm{d}s = 0$ for all $F \in \mathcal{F}_h$. Since $\psi_{K,F} \in H^1_*(K)$, the multiplicative trace inequality (12.17) (with $s := 1$ and $p := 2$) together with the Poincaré–Steklov inequality (12.13) implies that $\|\psi_{K,F}\|_{L^2(F)} \le c h_K^{\frac12} \|\nabla\psi_{K,F}\|_{\boldsymbol{L}^2(K)}$. Taking $\phi := \psi_{K,F}$ as a test function in (36.15), we infer that

$$\begin{aligned}\|\nabla\psi_{K,F}\|^2_{\boldsymbol{L}^2(K)} &= \epsilon_{K,F} \int_F [\![v_h]\!]_F \psi_{K,F} \,\mathrm{d}s \le \|[\![v_h]\!]\|_{L^2(F)} \|\psi_{K,F}\|_{L^2(F)} \\ &\le c\, h_K^{\frac12} \|[\![v_h]\!]_F\|_{L^2(F)} \|\nabla\psi_{K,F}\|_{\boldsymbol{L}^2(K)}.\end{aligned}$$

Owing to the regularity of the mesh sequence, we infer that

$$\|\nabla\psi_{K,F}\|_{\boldsymbol{L}^2(K)} \le c\, h_F^{\frac12} \|[\![v_h]\!]_F\|_{L^2(F)}.$$

(1) Let us prove the first bound in (36.14). Let $v \in H^1_0(D)$. Let c_K be the mean value of the function $(v_h - v)$ over K. The restriction of $(v_h - v - c_K)$ to K is in $H^1_*(K)$. Let $F \in \mathcal{F}_h$. Taking $\phi_K := (v_h - v)_{|K} - c_K$ as a test function in (36.15) and summing over $K \in \mathcal{T}_F$, we infer that

$$\begin{aligned}&\sum_{K\in\mathcal{T}_F} \int_K \nabla\psi_{K,F}{\cdot}\nabla(v_h - v)_{|K} \,\mathrm{d}x = \sum_{K\in\mathcal{T}_F} \int_K \nabla\psi_{K,F}{\cdot}\nabla\phi_K \,\mathrm{d}x \\ &= \sum_{K\in\mathcal{T}_F} \epsilon_{K,F} \int_F [\![v_h]\!]_F \phi_K \,\mathrm{d}s = \sum_{K\in\mathcal{T}_F} \epsilon_{K,F} \int_F [\![v_h]\!]_F (v_{h|K} - v - c_K) \,\mathrm{d}s \\ &= \int_F [\![v_h]\!]_F [\![v_h - v - c_K]\!]_F \,\mathrm{d}s = \int_F [\![v_h]\!]_F [\![v_h - v]\!]_F \,\mathrm{d}s = \int_F [\![v_h]\!]^2_F \,\mathrm{d}s,\end{aligned}$$

where we used that $\int_F [\![v_h]\!]_F \,\mathrm{d}s = 0$ to eliminate c_K and the fact that $v \in H^1_0(D)$ to eliminate $[\![v]\!]_F$. Using the Cauchy–Schwarz inequality and the above bound on $\|\nabla\psi_{K,F}\|_{\boldsymbol{L}^2(K)}$, we obtain

$$h_F^{-1}\|[\![v_h]\!]_F\|^2_{L^2(F)} \le c \sum_{K\in\mathcal{T}_F} \|\nabla(v - v_{h|K})\|^2_{\boldsymbol{L}^2(K)}. \tag{36.16}$$

Summing over $F \in \mathcal{F}_h$ leads to the first bound in (36.14).
(2) To prove the second bound in (36.14), we estimate the infimum over $v \in H^1_0(D)$ by taking $v := \mathcal{J}^{\mathrm{g,av}}_{h,0}(v_h)$ where $\mathcal{J}^{\mathrm{g,av}}_{h,0} : P^{\mathrm{b}}_1(\mathcal{T}_h) \to P^{\mathrm{g}}_{1,0}(\mathcal{T}_h) \subset H^1_0(D)$ is the averaging operator with zero trace introduced in §22.4.1. Then the second bound in (36.14) follows from Lemma 22.12 and the regularity of the mesh sequence. □

The bound (36.14) can be adapted to the case where $v_h \in P^{\mathrm{CR}}_1(\mathcal{T}_h)$, i.e., without any boundary prescription. The summations over the mesh faces are then restricted to the mesh interfaces, and the infimum is taken over the functions v in $H^1(D)$. The idea of introducing the local Neumann problem (36.15) has been considered in Achdou et al. [4].

36.3 Error analysis

In this section, we first establish an error estimate by using the coercivity norm and the abstract error estimate from Lemma 27.5. Then we derive an improved L^2-error estimate by adapting the duality argument from §32.3.

36.3.1 Energy error estimate

We perform the error analysis under the assumption that the solution to the model problem (36.2) is in $H^{1+r}(D)$ with $r > \frac{1}{2}$, i.e., we set

$$V_{\mathrm{S}} := H^{1+r}(D) \cap H^1_0(D), \quad r > \frac{1}{2}. \tag{36.17}$$

The assumption $u \in V_{\mathrm{S}}$ is reasonable in the setting of the Poisson equation with Dirichlet conditions in a Lipschitz polyhedron since it is consistent with the elliptic regularity theory (see Theorem 31.33). The important property of a function $v \in V_{\mathrm{S}}$ that we use here is that its normal derivative $\boldsymbol{n}_K{\cdot}\nabla v$ is meaningful in $L^2(\partial K)$ for all $K \in \mathcal{T}_h$. Actually, the full trace of ∇v on ∂K is meaningful on $\boldsymbol{L}^2(\partial K)$, and this trace is single-valued on any interface $F \in \mathcal{F}^\circ_h$ (see Remark 18.4). Therefore, we have $[\![\nabla v]\!]_F = \boldsymbol{0}$ for all $v \in V_{\mathrm{S}}$ and all $F \in \mathcal{F}^\circ_h$.

The discrete space $V_h := P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$ is equipped with the norm $\|{\cdot}\|_{V_h}$ defined in (36.11), and we introduce the space $V_\sharp := V_{\mathrm{S}} + V_h$ equipped with the norm $\|{\cdot}\|_{V_\sharp}$ defined by

$$\|v\|^2_{V_\sharp} := \sum_{K\in\mathcal{T}_h} \Big(\|\nabla v\|^2_{\boldsymbol{L}^2(K)} + h_K \|\boldsymbol{n}_K{\cdot}\nabla v_{|K}\|^2_{L^2(\partial K)} \Big). \tag{36.18}$$

A discrete trace inequality shows that there is $c_\sharp$ s.t. $\|v_h\|_{V_\sharp} \le c_\sharp \|v_h\|_{V_h}$ for all $v_h \in V_h$ and all $h \in \mathcal{H}$, i.e., (27.5) holds true. Using the forms a_h and ℓ_h defined in (36.9), the consistency error is s.t.

$$\langle \delta_h(v_h), w_h \rangle_{V_h',V_h} := \ell_h(w_h) - a_h(v_h, w_h), \qquad \forall v_h, w_h \in V_h. \tag{36.19}$$

Lemma 36.10 (Consistency/boundedness). *Assume* (36.17)*. There is $\omega_\sharp$, uniform w.r.t. $u \in V_{\mathrm{S}}$, s.t. for all $v_h \in V_h$ and all $h \in \mathcal{H}$,*

$$\|\delta_h(v_h)\|_{V_h'} \le \omega_\sharp \, \|u - v_h\|_{V_\sharp}. \tag{36.20}$$

Proof. Let $v_h, w_h \in V_h$. Since the normal derivative $\boldsymbol{n}_K{\cdot}\nabla u$ is meaningful in $L^2(\partial K)$ for all $K \in \mathcal{T}_h$, we have

$$\begin{aligned}
\ell_h(w_h) &= \sum_{K\in\mathcal{T}_h} \int_K f w_{h|K} \,\mathrm{d}x = \sum_{K\in\mathcal{T}_h} \int_K -(\Delta u) w_{h|K} \,\mathrm{d}x \\
&= \sum_{K\in\mathcal{T}_h} \int_K \nabla u{\cdot}\nabla w_{h|K} \,\mathrm{d}x - \sum_{K\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_K} \int_F (\boldsymbol{n}_K{\cdot}\nabla u) w_{h|K} \,\mathrm{d}s \\
&= \int_D \nabla u{\cdot}\nabla_h w_h \,\mathrm{d}x - \sum_{K\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_K} \int_F (\boldsymbol{n}_K{\cdot}\nabla u) w_{h|K} \,\mathrm{d}s.
\end{aligned}$$

Note that we write $\boldsymbol{n}_K{\cdot}\nabla u$ instead of $\boldsymbol{n}_K{\cdot}\nabla u_{|K}$ since ∇u is single-valued on F because $u \in V_{\mathrm{S}}$. We want to exchange the order of the summations on the right-hand side. Recalling that for every interface $F := \partial K_l \cap \partial K_r \in \mathcal{F}_h^\circ$ with $\boldsymbol{n}_F$ pointing from K_l to K_r, i.e., $\boldsymbol{n}_F := \boldsymbol{n}_{K_l} = -\boldsymbol{n}_{K_r}$, we have

$$(\boldsymbol{n}_{K_l}{\cdot}\nabla u) w_{h|K_l} + (\boldsymbol{n}_{K_r}{\cdot}\nabla u) w_{h|K_r} = (\boldsymbol{n}_{K_l}{\cdot}\nabla u) [\![w_h]\!]_F.$$

For every boundary face $F := \partial K_l \cap \partial D \in \mathcal{F}_h^\partial$, recall that we have conventionally set $[\![w_h]\!]_F := w_{h|K_l}$. Thus, we infer that

$$\sum_{K\in\mathcal{T}_h} \sum_{F\in\mathcal{F}_K} \int_F (\boldsymbol{n}_K{\cdot}\nabla u) w_{h|K} \,\mathrm{d}s = \sum_{F\in\mathcal{F}_h} \int_F (\boldsymbol{n}_{K_l}{\cdot}\nabla u) [\![w_h]\!]_F \,\mathrm{d}s.$$

Setting $\eta := u - v_h$, we can write the consistency error as follows:

$$\begin{aligned}
\langle \delta_h(v_h), w_h \rangle_{V_h',V_h} &= \int_D \nabla_h \eta{\cdot}\nabla_h w_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F (\boldsymbol{n}_{K_l}{\cdot}\nabla u) [\![w_h]\!]_F \,\mathrm{d}s \\
&= \int_D \nabla_h \eta{\cdot}\nabla_h w_h \,\mathrm{d}x - \sum_{F\in\mathcal{F}_h} \int_F (\boldsymbol{n}_{K_l}{\cdot}\nabla \eta_{|K_l}) [\![w_h]\!]_F \,\mathrm{d}s,
\end{aligned}$$

where we used that $\int_F (\boldsymbol{n}_{K_l}{\cdot}\nabla v_{h|K_l}) [\![w_h]\!]_F \,\mathrm{d}s = 0$ for all $F \in \mathcal{F}_h$ by definition of the Crouzeix–Raviart space $V_h = P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$. We conclude by invoking the Cauchy–Schwarz inequality, the first bound on the jumps in (36.14) which implies that $\sum_{F\in\mathcal{F}_h} h_F^{-1}\|[\![w_h]\!]_F\|^2_{L^2(F)} \le c\|w_h\|^2_{V_h}$ (bound the infimum by taking $v := 0$), and the regularity of the mesh sequence. □

Theorem 36.11 (Convergence). *Let u solve* (36.2) *and let u_h solve* (36.10). *Assume* (36.17). (i) *There is c s.t. the following quasi-optimal error estimate holds true for all $h \in \mathcal{H}$,*

$$\|u - u_h\|_{V_\sharp} \le c \inf_{v_h\in V_h} \|u - v_h\|_{V_\sharp}. \tag{36.21}$$

(ii) *Letting $t := \min(1, r)$, we have*

$$\|u - u_h\|_{V_\sharp} \le c\left(\sum_{K\in\mathcal{T}_h} h_K^{2t}|u|^2_{H^{1+t}(K)}\right)^{\frac{1}{2}}. \tag{36.22}$$

Proof. (i) The estimate (36.21) follows from Lemma 27.5 combined with stability (Lemma 36.4) and consistency/boundedness (Lemma 36.10).
(ii) The bound (36.22) follows from (36.21) by taking $v_h := \mathcal{I}_h^{\mathrm{CR}}(u)$. Letting $\eta := u - \mathcal{I}_h^{\mathrm{CR}}(u)$, we indeed have $\|\nabla\eta_{|K}\|_{\boldsymbol{L}^2(K)} \le ch_K^t|u|_{H^{1+t}(K)}$ for all $K \in \mathcal{T}_h$ owing to Lemma 36.1. Moreover, invoking the multiplicative trace inequality (12.17), we obtain

$$h_K^{\frac{1}{2}}\|\boldsymbol{n}_K{\cdot}\nabla\eta_{|K}\|_{L^2(\partial K)} \le \|\nabla\eta_{|K}\|_{\boldsymbol{L}^2(K)} + h_K^t|\eta_{|K}|_{H^{1+t}(K)},$$

and we have $|\eta_{|K}|_{H^{1+t}(K)} = |u|_{H^{1+r}(K)}$ since $\mathcal{I}_h^{\mathrm{CR}}(u)$ is affine in K. □

Remark 36.12 (Strang 2). The analysis can also be done by invoking Strang's second lemma (Lemma 27.15). Let us set $V_\sharp := H_0^1(D)+P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ and let us equip this space with the norm $\|{\cdot}\|_{V_\sharp}$ defined in (36.18). The discrete bilinear form a_h can be extended to a bilinear form $a_\sharp$ having boundedness constant equal to 1 on $V_\sharp{\times}V_h$. Lemma 27.15 leads to the error bound

$$\|u - v_h\|_{V_\sharp} \le c\left(\inf_{v_h\in V_h}\|u - v_h\|_{V_\sharp} + \|\delta_h^{\mathrm{St2}}(u)\|_{V_h'}\right),$$

with the consistency error s.t. for all $w_h \in P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$,

$$\begin{aligned}\langle\delta_h^{\mathrm{St2}}(u), w_h\rangle_{V_h',V_h} := \ell_h(w_h) - a_h(u, w_h) &= \sum_{K\in\mathcal{T}_h}\int_K (fw_h - \nabla u{\cdot}\nabla w_{h|K})\,\mathrm{d}x \\ &= -\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{n}_K{\cdot}\nabla u)w_{h|K}\,\mathrm{d}s.\end{aligned}$$

Thus, the consistency error does not vanish identically, i.e., the Crouzeix–Raviart finite element method is not strongly consistent in the sense defined in Remark 27.16. Since we have

$$\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{n}_K\cdot\nabla u)w_{h|K}\,\mathrm{d}s=\sum_{K\in\mathcal{T}_h}\int_{\partial K}(\boldsymbol{n}_K\cdot\nabla(u-v_h))w_{h|K}\,\mathrm{d}s,$$

for all $v_h \in P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h)$, by proceeding as in the proof of Theorem 36.11, we infer again that the quasi-optimal error estimate (36.21) holds true. □

36.3.2 L^2-error estimate

The goal of this section is to derive an improved L^2-error estimate of the form $\|u-u_h\|_{L^2(D)} \le ch^\gamma \ell_D^{1-\gamma}\|u-u_h\|_{V_\sharp}$ for some real number $\gamma > 0$, where ℓ_D is a length scale associated with D, e.g., $\ell_D := \mathrm{diam}(D)$.

Proceeding as in §32.3, we invoke a *duality argument*. We consider for all $g \in L^2(D)$ the adjoint solution $\zeta_g \in V := H^1_0(D)$ such that

$$a(v,\zeta_g)=(v,g)_{L^2(D)}, \qquad \forall v\in V. \tag{36.23}$$

Notice that $-\Delta\zeta_g = g$ in D and $\gamma^{\mathrm{g}}(\zeta_g) = 0$. Owing to the elliptic regularity theory (see §31.4), there is $s \in (0, 1]$ and a constant c_{smo} such that $\|\zeta_g\|_{H^{1+s}(D)} \le c_{\mathrm{smo}}\ell_D^2\|g\|_{L^2(D)}$ for all $g \in L^2(D)$. In the present setting of the Poisson equation with Dirichlet conditions in a Lipschitz polyhedron, it is reasonable to assume that $s \in (\frac{1}{2}, 1]$.

Theorem 36.13 (L^2-estimate). *Let u solve* (36.2) *and let u_h solve* (36.10). *Assume that the elliptic regularity index satisfies $s \in (\frac{1}{2}, 1]$. There is c s.t. for all $h \in \mathcal{H}$,*

$$\|u-u_h\|_{L^2(D)} \le c\,h^s\ell_D^{1-s}\|u-u_h\|_{V_\sharp}. \tag{36.24}$$

Proof. Let $e := u - u_h$ and set $Y_h := P^{\mathrm{g}}_{1,0}(\mathcal{T}_h) := P^{\mathrm{CR}}_{1,0}(\mathcal{T}_h) \cap H^1_0(D)$. Then $(\nabla_h e, \nabla y_h)_{\boldsymbol{L}^2(D)} = (\nabla u, \nabla y_h)_{\boldsymbol{L}^2(D)} - (\nabla_h u_h, \nabla y_h)_{\boldsymbol{L}^2(D)} = 0$ for all $y_h \in Y_h$. Since $\|e\|^2_{L^2(D)} = -(e, \Delta\zeta_e)_{\boldsymbol{L}^2(D)}$, we have

$$\begin{aligned}\|e\|^2_{L^2(D)} &= (\nabla_h e,\nabla\zeta_e)_{\boldsymbol{L}^2(D)} - \big((e,\Delta\zeta_e)_{L^2(D)} + (\nabla_h e,\nabla\zeta_e)_{\boldsymbol{L}^2(D)}\big)\\ &= (\nabla_h e,\nabla(\zeta_e-y_h))_{\boldsymbol{L}^2(D)} - \langle\delta^{\mathrm{adj}}(\zeta_e),e\rangle_{V'_\sharp,V_\sharp},\end{aligned}$$

where we introduced $\delta^{\mathrm{adj}}(\zeta_e) \in V'_\sharp$ s.t. $\langle\delta^{\mathrm{adj}}(\zeta_e), v\rangle_{V'_\sharp,V_\sharp} := (v, \Delta\zeta_e)_{L^2(D)} + (\nabla_h v, \nabla\zeta_e)_{\boldsymbol{L}^2(D)}$ and used that $(\nabla_h e, \nabla y_h)_{\boldsymbol{L}^2(D)} = 0$ for all $y_h \in V_h$. Let us set $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V'_\sharp} := \sup_{v\in V_\sharp}\frac{|\langle\delta^{\mathrm{adj}}(\zeta_e),v\rangle_{V'_\sharp,V_\sharp}|}{\|v\|_{V_\sharp}}$. The Cauchy–Schwarz inequality and the definition of the $\|\cdot\|_{V_\sharp}$- and $\|\cdot\|_{V'_\sharp}$-norms imply that

$$\|e\|^2_{L^2(D)} \le \left(\inf_{y_h\in Y_h}\|\nabla(\zeta_e-y_h)\|_{\boldsymbol{L}^2(D)} + \|\delta^{\mathrm{adj}}(\zeta_e)\|_{V'_\sharp}\right)\|e\|_{V_\sharp}.$$

It remains to bound the two terms between parentheses on the right-hand side. Using the quasi-interpolation operator $\mathcal{I}^{\mathrm{g,av}}_{h0}$ from §22.4, we infer that

$$\inf_{y_h\in Y_h}\|\nabla(\zeta_e - y_h)\|_{\boldsymbol{L}^2(D)} \le \|\nabla(\zeta_e - \mathcal{I}^{\mathrm{g,av}}_{h0}(\zeta_e))\|_{\boldsymbol{L}^2(D)}$$
$$\le c\,h^s|\zeta_e|_{H^{1+s}(D)} \le c\,h^s\ell_D^{-1-s}\|\zeta_e\|_{H^{1+s}(D)} \le c\,c_{\mathrm{smo}}\,h^s\ell_D^{1-s}\|e\|_{L^2(D)},$$

where we used the approximation properties of $\mathcal{I}^{\mathrm{g,av}}_{h0}$ from Theorem 22.14 and the elliptic regularity theory to bound $\|\zeta_e\|_{H^{1+s}(D)}$ by $\|e\|_{L^2(D)}$. Let us now estimate $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'}$. By proceeding as in the proof of Lemma 36.10 (observe that $[\![\nabla\zeta_e]\!]_F = \mathbf{0}$ for all $F\in\mathcal{F}_h^\circ$), we infer that we have, for all $v := v_{\mathrm{s}} + v_h \in V_\sharp := V_{\mathrm{s}} + V_h$ with $v_{\mathrm{s}}\in V_{\mathrm{s}}$ and $v_h\in V_h$, and all $z_h\in V_h$,

$$\begin{aligned}\langle\delta^{\mathrm{adj}}(\zeta_e), v\rangle_{V_\sharp',V_\sharp} &= \sum_{F\in\mathcal{F}_h}\int_F \boldsymbol{n}_{K_l}\cdot\nabla\zeta_e[\![v_h]\!]_F\,\mathrm{d}s\\ &= \sum_{F\in\mathcal{F}_h}\int_F \boldsymbol{n}_{K_l}\cdot\nabla(\zeta_e - z_h)_{|K_l}[\![v_h]\!]_F\,\mathrm{d}s\\ &\le c\,\|\zeta_e - z_h\|_{V_\sharp}\left(\sum_{F\in\mathcal{F}_h} h_F^{-1}\|[\![v_h]\!]_F\|^2_{L^2(F)}\right)^{\frac12},\end{aligned}$$

where we used that $\boldsymbol{n}_{K_l}\cdot\nabla z_{h|K_l}$ is constant on F. Using the leftmost inequality in (36.14) with $\inf_{w\in H_0^1(D)}\|\nabla_h(w-v_h)\|^2_{\boldsymbol{L}^2(D)} \le \|\nabla_h(v_{\mathrm{s}}+v_h)\|^2_{\boldsymbol{L}^2(D)}$, we infer that $\sum_{F\in\mathcal{F}_h} h_F^{-1}\|[\![v_h]\!]_F\|^2_{L^2(F)} \le c\|v_{\mathrm{s}} + v_h\|^2_{V_\sharp} = c\|v\|^2_{V_\sharp}$. Thus, $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'} \le c'\inf_{z_h\in V_h}\|\zeta_e - z_h\|_{V_\sharp}$. Using the approximation properties of V_h, we conclude that $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'} \le ch^s|\zeta_e|_{H^{1+s}(D)}$, and reasoning as above yields $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'} \le ch^s\ell_D^{1-s}\|e\|_{L^2(D)}$. $\square$

36.3.3 Abstract nonconforming duality argument

Let us finish with an abstract formulation of the above duality argument that can be applied in the context of nonconforming approximation techniques. Let V a Banach space, L be a Hilbert space, and assume that V embeds continuously into L (i.e., $V\hookrightarrow L$) and V is dense in L. Identifying L with L', we are in the situation where

$$V\hookrightarrow L\equiv L'\hookrightarrow V', \tag{36.25}$$

with continuous and dense embeddings. Let $a : V\times V\to\mathbb{C}$ be a bounded sesquilinear form satisfying the assumptions of the BNB theorem (Theorem 25.9). For all $f\in L$ we denote by ξ_f the unique solution to the problem

$$a(\xi_f, v) = (f, v)_L, \qquad \forall v\in V. \tag{36.26}$$

Similarly, for all $g\in L$ we denote by $\zeta_g\in V$ the unique solution to the adjoint problem

$$a(v,\zeta_g) = (v, g)_L, \qquad \forall v\in V. \tag{36.27}$$

These two problems are well-posed since a satisfies the assumptions of the BNB theorem. Let $A^{\mathrm{adj}} \in \mathcal{L}(V;V')$ be s.t. $\langle A^{\mathrm{adj}}(w), v\rangle_{V',V} = \overline{a(v,w)}$ for all $(v,w) \in V\times V$. Owing to (36.25) and (36.27), we have $A^{\mathrm{adj}}(\zeta_g) = g$ in L.

We assume that we have at hand two subspaces $V_{\mathrm{S}} \subset V$ and $Z_{\mathrm{S}} \subset V$ s.t. the maps $V' \ni f \mapsto \xi_f \in V_{\mathrm{S}}$ and $V' \ni g \mapsto \zeta_g \in Z_{\mathrm{S}}$ are bounded. Let $V_h \subset L$ be a finite-dimensional subspace of L (but not necessarily of V). Let $Y_h \subseteq V_h$. We set $V_\sharp := V_{\mathrm{S}} + V_h$ and $Z_\sharp := Z_{\mathrm{S}} + Y_h$, and we equip these spaces with norms denoted by $\|\cdot\|_{V_\sharp}$ and $\|\cdot\|_{Z_\sharp}$.

Lemma 36.14 (L-norm estimate). *Let $a_\sharp$ be a bounded sesquilinear form on $V_\sharp\times Z_\sharp$. Let $\|a_\sharp\|$ be the norm of $a_\sharp$ on $V_\sharp\times Z_\sharp$. Let $u \in V_{\mathrm{S}}$ and $u_h \in V_h$. Assume that the following Galerkin orthogonality property holds true:*

$$a_\sharp(u - u_h, y_h) = 0, \qquad \forall y_h \in Y_h. \tag{36.28}$$

Let $e := u - u_h$ and let $\delta^{\mathrm{adj}}(\zeta_e) \in V_\sharp'$ be the adjoint consistency error:

$$\langle \delta^{\mathrm{adj}}(\zeta_e), v\rangle_{V_\sharp',V_\sharp} := (v, A^{\mathrm{adj}}(\zeta_e))_L - a_\sharp(v,\zeta_e), \qquad \forall v \in V_\sharp. \tag{36.29}$$

Then the following estimate holds true:

$$\|e\|_L \le \left(\frac{\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'}}{\|e\|_L} + \|a_\sharp\| \inf_{y_h\in Y_h} \frac{\|\zeta_e - y_h\|_{Z_\sharp}}{\|e\|_L} \right) \|e\|_{V_\sharp}, \tag{36.30}$$

Proof. Using the identity $A^{\mathrm{adj}}(\zeta_e) = e$ and the Galerkin orthogonality property (36.28), we infer that

$$\begin{aligned} \|e\|_L^2 &= (e, A^{\mathrm{adj}}(\zeta_e))_L = (e, A^{\mathrm{adj}}(\zeta_e))_L - a_\sharp(e,\zeta_e) + a_\sharp(e,\zeta_e) \\ &= \langle \delta^{\mathrm{adj}}(\zeta_e), e\rangle_{V_\sharp',V_\sharp} + a_\sharp(e, \zeta_e - y_h). \end{aligned}$$

The boundedness of $a_\sharp$ on $V_\sharp\times Z_\sharp$ and the definition of the dual norm $\|\delta^{\mathrm{adj}}(\zeta_e)\|_{V_\sharp'}$ imply that (36.30) holds true. ☐

Example 36.15 (Crouzeix–Raviart). Lemma 36.14 can be applied to the Crouzeix–Raviart approximation with $V_{\mathrm{S}} := H^{1+r}(D) \cap H^1_0(D)$, $Z_{\mathrm{S}} := H^{1+s}(D) \cap H^1_0(D)$, $a_\sharp(v,w) := (\nabla_h v, \nabla_h w)_{\boldsymbol{L}^2(D)}$, and equipping the spaces $V_\sharp := V_{\mathrm{S}} + V_h$, $Z_\sharp := Z_{\mathrm{S}} + Y_h$, $Y_h := V_h \cap H^1_0(D)$, with the broken energy norm. Note that the adjoint consistency error is nonzero, and that the proof of Theorem 36.13 shows that both terms on the right-hand side of (36.30) converge with the same rate w.r.t. $h \in \mathcal{H}$. ☐

Exercises

Exercise 36.1 (Commuting properties). Let K be a simplex in $\mathbb{R}^d$ and let $\mathit{\Pi}^0_K$ denote the L^2-orthogonal projection onto constants. Prove that

$\nabla(\mathcal{I}_K^{\mathrm{CR}}(p)) = \Pi_K^0(\nabla p)$ and $\nabla{\cdot}(\boldsymbol{\mathcal{I}}_K^{\mathrm{CR}}(\boldsymbol{\sigma})) = \Pi_K^0(\nabla{\cdot}\boldsymbol{\sigma})$ for all $p \in H^1(K)$ and all $\boldsymbol{\sigma} \in \boldsymbol{L}^2(K)$ with $\nabla{\cdot}\boldsymbol{\sigma} \in L^1(K)$ and $\boldsymbol{\mathcal{I}}_K^{\mathrm{CR}}$ defined componentwise using $\mathcal{I}_h^{\mathrm{CR}}$.

Exercise 36.2 (Best approximation). Let $v \in H^1(D)$. A global best-approximation of v in $P_1^{\mathrm{CR}}(\mathcal{T}_h)$ in the broken H^1-seminorm is a function $v_h^{\mathrm{CR}} \in P_1^{\mathrm{CR}}(\mathcal{T}_h)$ s.t.

$$\sum_{K\in\mathcal{T}_h} \|\nabla(v - v_h^{\mathrm{CR}})\|^2_{\boldsymbol{L}^2(K)} = \min_{v_h\in P_1^{\mathrm{CR}}(\mathcal{T}_h)} \sum_{K\in\mathcal{T}_h} \|\nabla(v - v_h)\|^2_{\boldsymbol{L}^2(K)}.$$

(i) Write a characterization of v_h^{CR} in weak form and show that v_h^{CR} is unique up to an additive constant. (*Hint*: adapt Proposition 25.8.) (ii) Let v_h^{b} be a global best-approximation of v in the broken finite element space $P_1^{\mathrm{b}}(\mathcal{T}_h)$; see §32.2. Prove that $\sum_{K\in\mathcal{T}_h} \|\nabla(v - v_h^{\mathrm{CR}})\|^2_{\boldsymbol{L}^2(K)} = \sum_{K\in\mathcal{T}_h} \|\nabla(v - v_h^{\mathrm{b}})\|^2_{\boldsymbol{L}^2(K)}$. (*Hint*: using Exercise 36.1, show that $v_h^{\mathrm{CR}} = \mathcal{I}_h^{\mathrm{CR}}(v)$ up to an additive constant.)

Exercise 36.3 ($\boldsymbol{H}$(div)-flux recovery). Let u_h solve (36.10). Assume that f is piecewise constant on $\mathcal{T}_h$. Set $\boldsymbol{\sigma}_{h|K} := -\nabla u_{h|K} + \frac{1}{d} f_{|K}(\boldsymbol{x} - \boldsymbol{x}_K)$, where $\boldsymbol{x}_K$ is the barycenter of K for all $K \in \mathcal{T}_h$. Prove that $\boldsymbol{\sigma}_h$ is in the lowest-order Raviart–Thomas finite element space $\boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h)$ and that $\nabla{\cdot}\boldsymbol{\sigma} = f$; see Marini [295] (*Hint*: evaluate $\int_F [\![\boldsymbol{\sigma}_h]\!]{\cdot}\boldsymbol{n}_F \varphi_F^{\mathrm{CR}}\,\mathrm{d}s$ for all $F \in \mathcal{F}_h^{\circ}$.)

Exercise 36.4 (Discrete Helmholtz). Let $D \subset \mathbb{R}^2$ be a simply connected polygon. Prove that $\boldsymbol{P}_0^{\mathrm{b}}(\mathcal{T}_h) = \nabla P_1^{\mathrm{g}}(\mathcal{T}_h) \oplus \nabla_h^{\perp} P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$, where

$$\nabla_h^{\perp} P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{P}_0^{\mathrm{b}}(\mathcal{T}_h) \mid \exists q_h \in P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h) \mid \boldsymbol{v}_{h|K} = \nabla^{\perp}(q_{h|K}), \forall K \in \mathcal{T}_h\},$$

and $\nabla^{\perp}$ is the two-dimensional curl operator defined in Remark 16.17. (*Hint*: prove that the decomposition is L^2-orthogonal and use a dimension argument based on Euler's relations.)

Exercise 36.5 (Rannacher–Turek). Let $K := [-1,1]^d$. For all $i \in \{1{:}d\}$ and $\alpha \in \{l, r\}$, let $F_{i,\alpha}$ be the face of K corresponding to $\{x_i = -1\}$ when $\alpha = l$ and to $\{x_i = 1\}$ when $\alpha = r$. Observe that there are $2d$ such faces, each of measure 2^{d-1}. Let P be spanned by the $2d$ functions $\{1, x_1, \ldots, x_d, x_1^2 - x_2^2, \ldots, x_{d-1}^2 - x_d^2\}$. Consider the linear forms $\sigma_{i,\alpha}(p) := 2^{1-d}\int_{F_{i,\alpha}} p\,\mathrm{d}s$ for all $i \in \{1{:}d\}$ and $\alpha \in \{l, r\}$. Setting $\Sigma := \{\sigma_{i,\alpha}\}_{i\in\{1:d\},\alpha\in\{l,r\}}$, prove that (K, P, Σ) is a finite element. *Note*: this element has been introduced by [330] for the mixed discretization of the Stokes equations on Cartesian grids.

Exercise 36.6 (Quadratic space). Let $\mathcal{T}_h$ be a triangulation of a simply connected domain $D \subset \mathbb{R}^2$ and let

$$P_2^{\mathrm{CR}}(\mathcal{T}_h) := \{v_h \in P_2^{\mathrm{b}}(\mathcal{T}_h) \mid \int_F [\![v_h]\!]_F (q \circ \boldsymbol{T}_F^{-1})\,\mathrm{d}s = 0, \forall F \in \mathcal{F}_h^{\circ}, \forall q \in \mathbb{P}_{1,1}\},$$

where $\boldsymbol{T}_F$ is an affine bijective mapping from the unit segment $\widehat{S}^1 = [-1,1]$ to F. Orient all the faces $F \in \mathcal{F}_h$ and define the two Gauss points $\boldsymbol{g}_F^{\pm}$ on F that are the image by $\boldsymbol{T}_F$ of $\widehat{g}^{\pm} := \pm\frac{\sqrt{3}}{3}$, in such a way that the orientation of F goes from $\boldsymbol{g}_F^-$ to $\boldsymbol{g}_F^+$. For all $K \in \mathcal{T}_h$, let $\{\lambda_{0,K}, \lambda_{1,K}, \lambda_{2,K}\}$ be the barycentric coordinates in K and set $b_K := 2 - 3(\lambda_{0,K}^2 + \lambda_{1,K}^2 + \lambda_{2,K}^2)$ (this function is usually called Fortin–Soulié bubble [204]). One can verify that a polynomial $p \in \mathbb{P}_{2,2}$ vanishes at the six points $\{\boldsymbol{g}_F^{\pm}\}_{F\in\mathcal{F}_K}$ if and only if $p = \alpha b_K$ for some $\alpha \in \mathbb{R}$. *Note*: this shows that these six points, which lie on an ellipse, cannot be taken as nodes of a $\mathbb{P}_{2,2}$ Lagrange element. (i) Extending b_K by zero outside K, verify that $b_K \in P_2^{\mathrm{CR}}(\mathcal{T}_h)$. (ii) Set $B := \mathrm{span}_{K\in\mathcal{T}_h}\{b_K\}$ and $B_* := \{v_h \in B \mid \int_D v_h \,\mathrm{d}x = 0\}$. Prove that $P_2^{\mathrm{g}}(\mathcal{T}_h) + B_* \subset P_2^{\mathrm{CR}}(\mathcal{T}_h)$ and that $P_2^{\mathrm{g}}(\mathcal{T}_h) \cap B_* = \{0\}$. (iii) Define $J : P_2^{\mathrm{CR}}(\mathcal{T}_h) \to \mathbb{R}^{2N_{\mathrm{f}}}$ s.t. $J(v_h) := (v_h(\boldsymbol{g}_F^-), v_h(\boldsymbol{g}_F^+))_{F\in\mathcal{F}_h}$ for all $v_h \in P_2^{\mathrm{CR}}(\mathcal{T}_h)$. Prove that $\dim(\ker(J)) = N_{\mathrm{c}}$ and $\dim(\mathrm{im}(J)) \le 2N_{\mathrm{f}} - N_{\mathrm{c}}$. (*Hint*: any polynomial $p \in \mathbb{P}_{2,2}$ satisfies $\sum_{F\in\mathcal{F}_K}(p(\boldsymbol{g}_F^+) - p(\boldsymbol{g}_F^-)) = 0$ for all $K \in \mathcal{T}_h$.) (iv) Prove that $P_2^{\mathrm{CR}}(\mathcal{T}_h) = P_2^{\mathrm{g}}(\mathcal{T}_h) \oplus B_*$; see Greff [222]. (*Hint*: use a dimensional argument and Euler's relation from Remark 8.13.)