Part IX, Chapter 43

Maxwell's equations: $\boldsymbol{H}(\mathrm{curl})$-approximation

The objective of this chapter is to introduce some model problems derived from Maxwell's equations that all fit the Lax-Milgram formalism in $\boldsymbol{H}(\mathrm{curl})$. The approximation is performed using $\boldsymbol{H}(\mathrm{curl})$-conforming edge (Nédélec) finite elements. The analysis relies on a coercivity argument in $\boldsymbol{H}(\mathrm{curl})$ that exploits the presence of a uniformly positive zero-order term in the formulation. A more robust technique controlling the divergence of the approximated field is presented in Chapter 44. The space dimension is 3 in the entire chapter ($d = 3$), and D is a Lipschitz domain in $\mathbb{R}^3$.

43.1 Maxwell's equations

We start by recalling some basic facts about Maxwell's equations. The reader is referred to Bossavit [74, Chap. 1], Monk [303, Chap. 1], Assous et al. [27, Chap. 1] for a detailed discussion on this model. Maxwell's equations are partial differential equations providing a macroscopic description of electromagnetic phenomena. These equations describe how the electric field $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, the electric displacement field $\boldsymbol{D}$, and the magnetic induction $\boldsymbol{B}$ (sometimes called magnetic flux density) interact through the action of currents $\boldsymbol{j}$ and charges ρ:

$$\partial_t \boldsymbol{D} - \nabla{\times}\boldsymbol{H} = -\boldsymbol{j} \quad \text{(Ampère's law)}, \tag{43.1a}$$
$$\partial_t \boldsymbol{B} + \nabla{\times}\boldsymbol{E} = \boldsymbol{0} \quad \text{(Faraday's law of induction)}, \tag{43.1b}$$
$$\nabla{\cdot}\boldsymbol{D} = \rho \quad \text{(Gauss's law for electricity)}, \tag{43.1c}$$
$$\nabla{\cdot}\boldsymbol{B} = 0 \quad \text{(Gauss's law for magnetism)}. \tag{43.1d}$$

Notice that if $(\nabla{\cdot}\boldsymbol{B})_{|t=0} = 0$, taking the divergence of (43.1b) implies that (43.1d) is satisfied at all times. Similarly, assuming $(\nabla{\cdot}\boldsymbol{D})_{|t=0} = \rho_{|t=0}$ and that the charge conservation equation $\partial_t \rho + \nabla{\cdot}\boldsymbol{j} = 0$ is satisfied at all times implies that (43.1c) is satisfied at all times. This shows that if the data ρ,

$\boldsymbol{j}$, $\boldsymbol{B}_{|t=0}$, and $\boldsymbol{D}_{|t=0}$ satisfy the proper constraints, Gauss's laws are just consequences of Ampère's law and Faraday's law.

The system (43.1) is closed by relating the fields through constitutive laws describing microscopic mechanisms of polarization and magnetization:

$$\boldsymbol{D} - \varepsilon_0 \boldsymbol{E} = \boldsymbol{P}, \qquad \boldsymbol{B} = \mu_0(\boldsymbol{H} + \boldsymbol{M}), \tag{43.2}$$

where ε_0 and μ_0 are the electric permittivity and the magnetic permeability of vacuum, and $\boldsymbol{P}$ and $\boldsymbol{M}$ are the polarization and the magnetization fields, respectively. These quantities are the average representatives at the macroscopic scale of complex microscopic interactions that must be modeled. The models in question always involve parameters that need to be identified by measurements or other techniques like homogenization or multiscale models. We have $\boldsymbol{P} := \boldsymbol{0}$ and $\boldsymbol{M} := \boldsymbol{0}$ in vacuum, and it is common to use $\boldsymbol{P} := \varepsilon_0\varepsilon_r\boldsymbol{E}$ and $\boldsymbol{M} := \mu_r\boldsymbol{H}$ to model isotropic homogeneous dielectric and magnetic materials, where ε_r is the electric susceptibility and μ_r is the magnetic susceptibility. In the rest of the chapter, we assume that

$$\boldsymbol{D} := \epsilon\boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} := \mu\boldsymbol{H}, \tag{43.3}$$

where ϵ and μ are given coefficients that may be space-dependent. The current $\boldsymbol{j}$ and charge density ρ are a priori given, but it is also possible to make these quantities depend on the other fields through phenomenological mechanisms. For instance, it is possible to further decompose the current into one component that depends on the material and another one that is a source. The simplest model doing that is Ohm's law, $\boldsymbol{j} = \boldsymbol{j}_{\mathrm{s}} + \sigma\boldsymbol{E}$, where σ is the electrical conductivity and $\boldsymbol{j}_{\mathrm{s}}$ an imposed current.

We now formulate Maxwell's equations in two different regimes: the time-harmonic regime and the eddy current limit.

43.1.1 The time-harmonic regime

We first consider Maxwell's equations in the *time-harmonic regime* where the time-dependence is assumed to be of the form $e^{\mathrm{i}\omega t}$ with $\mathrm{i}^2 = -1$ and ω is a given angular frequency. The time-harmonic version of (43.1a)-(43.1b) is

$$\mathrm{i}\omega\epsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{j}_{\mathrm{s}}, \qquad \text{in } D, \tag{43.4a}$$

$$\mathrm{i}\omega\mu\boldsymbol{H} + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \qquad \text{in } D, \tag{43.4b}$$

$$\boldsymbol{H}_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{d}}, \quad \boldsymbol{E}_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}, \quad \text{on } \partial D, \tag{43.4c}$$

where $\{\partial D_{\mathrm{d}}, \partial D_{\mathrm{n}}\}$ forms a partition of the boundary ∂D of D. The dependent variables are the electric field $\boldsymbol{E}$ and the magnetic field $\boldsymbol{H}$. The data are the conductivity σ, the permittivity ϵ, the permeability μ, the current $\boldsymbol{j}_{\mathrm{s}}$, and the boundary data $\boldsymbol{a}_{\mathrm{d}}$ and $\boldsymbol{a}_{\mathrm{n}}$. The material coefficients ϵ and μ can be complex-valued. The system (43.4) models for instance a microwave oven; see e.g., [74, Chap. 9]. The conditions $\boldsymbol{H}_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{E}_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n} = \boldsymbol{0}$ are usually called perfect magnetic conductor and perfect electric conductor boundary conditions, respectively.

Let us assume that the modulus of the magnetic permeability μ is bounded away from zero uniformly in D. It is then possible to eliminate $\boldsymbol{H}$ by using $\boldsymbol{H} = \mathrm{i}(\omega\mu)^{-1}\nabla\times\boldsymbol{E}$. The system then takes the following form:

$$(-\omega^2\epsilon + \mathrm{i}\omega\sigma)\boldsymbol{E} + \nabla\times(\mu^{-1}\nabla\times\boldsymbol{E}) = -\mathrm{i}\omega\boldsymbol{j}_{\mathrm{s}}, \qquad \text{in } D, \tag{43.5a}$$

$$(\nabla\times\boldsymbol{E})_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = -\mathrm{i}\omega\mu\boldsymbol{a}_{\mathrm{d}}, \quad \boldsymbol{E}_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}, \quad \text{on } \partial D. \tag{43.5b}$$

Notice that Gauss's law for electricity is contained in (43.5a) since taking the divergence of the equation yields $\nabla\cdot((-\omega^2\epsilon + \mathrm{i}\omega\sigma)\boldsymbol{E}) = \nabla\cdot(-\mathrm{i}\omega\boldsymbol{j}_{\mathrm{s}})$, which is the time-harmonic counterpart of (43.1c) combined with (43.1a). The system (43.5) is often used to model the propagation of electromagnetic waves through various media.

43.1.2 The eddy current problem

When the time scale of interest, say τ, is such that the ratio $\epsilon/(\tau\sigma) \ll 1$, it is legitimate to neglect the displacement current in Ampère's law (i.e., Maxwell's correction $\partial_t\boldsymbol{D}$). This situation occurs in particular in systems with moving parts (either solid or fluids) whose characteristic speed is much slower than the speed of light. The resulting system, called *eddy current problem*, is as follows:

$$\sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{j}_{\mathrm{s}}, \qquad \text{in } D, \tag{43.6a}$$

$$\partial_t(\mu\boldsymbol{H}) + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \qquad \text{in } D, \tag{43.6b}$$

$$\boldsymbol{H}_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{d}}, \quad \boldsymbol{E}_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{n}}, \quad \text{on } \partial D, \tag{43.6c}$$

where $\{\partial D_{\mathrm{d}}, \partial D_{\mathrm{n}}\}$ forms a partition of the boundary ∂D of D. The system (43.6) arises in magneto-hydrodynamics (MHD). In this case, $\boldsymbol{j}_{\mathrm{s}}$ is further decomposed into $\boldsymbol{j}_{\mathrm{s}} = \boldsymbol{j}'_{\mathrm{s}} + \sigma\boldsymbol{u}\times\boldsymbol{B}$, where $\boldsymbol{u}$ is the velocity of the fluid occupying the domain D, i.e., the actual current is decomposed into $\boldsymbol{j} = \boldsymbol{j}'_{\mathrm{s}} + \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$.

Let us assume that σ is bounded from below away from zero uniformly in D. It is then possible to eliminate the electric field from (43.6) by using $\boldsymbol{E} = \sigma^{-1}(\nabla\times\boldsymbol{H} - \boldsymbol{j}_{\mathrm{s}})$. The new system to be solved is rewritten as follows:

$$\partial_t(\mu\boldsymbol{H}) + \nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H} - \boldsymbol{u}\times(\mu\boldsymbol{H})) = \nabla\times(\sigma^{-1}\boldsymbol{j}'_{\mathrm{s}}), \qquad \text{in } D, \tag{43.7a}$$

$$\boldsymbol{H}_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = \boldsymbol{a}_{\mathrm{d}}, \quad (\sigma^{-1}\nabla\times\boldsymbol{H} - \boldsymbol{u}\times(\mu\boldsymbol{H}))_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n} = \boldsymbol{c}_{\mathrm{n}}, \quad \text{on } \partial D, \tag{43.7b}$$

where $\boldsymbol{c}_{\mathrm{n}} := \boldsymbol{a}_{\mathrm{n}} + (\sigma^{-1}\boldsymbol{j}'_{\mathrm{s}})_{|\partial D_{\mathrm{n}}}\times\boldsymbol{n}$. At this point, it is possible to further simplify the problem by assuming that either the time evolution is harmonic, i.e., $\boldsymbol{H}(\boldsymbol{x},t) := \boldsymbol{H}_{\mathrm{sp}}(\boldsymbol{x})e^{\mathrm{i}\omega t}$, or the time derivative is approximated as $\partial_t\boldsymbol{H}(\boldsymbol{x},t) \approx \tau^{-1}(\boldsymbol{H}(\boldsymbol{x},t) - \boldsymbol{H}(\boldsymbol{x},t-\tau))$, where τ is the time step of the time discretization. After appropriately renaming the dependent variable and the data, say either $\widetilde{\mu} := \mathrm{i}\omega\mu$ and $\boldsymbol{f} := \nabla\times(\sigma^{-1}\boldsymbol{j}_\mathrm{s}')$, or $\widetilde{\mu} := \mu\tau^{-1}$ and $\boldsymbol{f} := \nabla\times(\sigma^{-1}\boldsymbol{j}_\mathrm{s}') + \widetilde{\mu}\boldsymbol{H}(\boldsymbol{x}, t-\tau)$, the above system reduces to solving the following problem:

$$\widetilde{\mu}\boldsymbol{H} + \nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H} - \boldsymbol{u}\times(\mu\boldsymbol{H})) = \boldsymbol{f}, \qquad \text{in } D, \qquad (43.8\text{a})$$

$$\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{d}, \quad (\sigma^{-1}\nabla\times\boldsymbol{H} - \boldsymbol{u}\times(\mu\boldsymbol{H}))_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{c}_\mathrm{n}, \quad \text{on } \partial D. \quad (43.8\text{b})$$

Notice that $\nabla\cdot\boldsymbol{f} = 0$ in both cases. Hence, Gauss's law for magnetism is contained in (43.8a) since taking the divergence of the equation yields $\nabla\cdot(\mu\boldsymbol{H}) = 0$ whether $\widetilde{\mu} := \mathrm{i}\omega\mu$ or $\widetilde{\mu} := \mu\tau^{-1}$.

43.2 Weak formulation

The time-harmonic problem and the eddy current problem have a very similar structure. After lifting the boundary condition (either on ∂D_n for the time-harmonic problem or on ∂D_d for the eddy current problem) and making appropriate changes of notation, the above two problems (43.5) and (43.8) can be reformulated as follows: Find $\boldsymbol{A} : D \to \mathbb{C}^3$ such that

$$\nu\boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}, \quad \boldsymbol{A}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{0}, \quad (\kappa\nabla\times\boldsymbol{A})_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{0}, \quad (43.9)$$

where ν, κ, and $\boldsymbol{f}$ are complex-valued. We have taken $\boldsymbol{u} := \boldsymbol{0}$ in the MHD problem for simplicity. We have also assumed that the Neumann data is zero to avoid unnecessary technicalities. We have $\nu := -\omega^2\epsilon + \mathrm{i}\omega\sigma$ and $\kappa := \mu^{-1}$ for the time-harmonic problem, and $\nu := \mathrm{i}\omega\mu$ or $\nu := \mu\tau^{-1}$ and $\kappa := \sigma^{-1}$ for the eddy current problem.

43.2.1 Functional setting

Let us assume that $\boldsymbol{f} \in \boldsymbol{L}^2(D) := L^2(D;\mathbb{C}^3)$ and $\nu, \kappa \in L^\infty(D;\mathbb{C})$. A weak formulation of (43.9) is obtained by multiplying the PDE by the complex conjugate of a smooth test function $\boldsymbol{b}$ with zero tangential component over ∂D_d and integrating by parts. Recalling (4.11), we obtain

$$\int_D (\nu\boldsymbol{A}\cdot\overline{\boldsymbol{b}} + \kappa\nabla\times\boldsymbol{A}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x = \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x.$$

The integral on the left-hand side makes sense if $\boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D)$. To be dimensionally coherent, we equip $\boldsymbol{H}(\mathrm{curl}; D)$ with the norm $\|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)} := (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})^{\frac{1}{2}}$, where ℓ_D is some characteristic length of D, e.g., $\ell_D := \mathrm{diam}(D)$.

Let $\gamma^\mathrm{c} : \boldsymbol{H}(\mathrm{curl}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ denote the tangential trace operator introduced in (4.11) and let $\langle\cdot,\cdot\rangle_{\partial D}$ denote the duality pairing between

$\boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ and $\boldsymbol{H}^{\frac{1}{2}}(\partial D)$. Since the Dirichlet condition $\gamma^{\mathrm{c}}(\boldsymbol{A}) = \boldsymbol{0}$ is enforced on ∂D_{d} only, we must consider the restriction of the linear forms in $\boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ to functions that are only defined on ∂D_{d}. Let $\widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{d}})$ be composed of the functions $\boldsymbol{\theta}$ defined on ∂D_{d} whose zero-extension to ∂D, say $\tilde{\boldsymbol{\theta}}$, is in $\boldsymbol{H}^{\frac{1}{2}}(\partial D)$. Then for all $\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D)$, the restriction $\gamma^{\mathrm{c}}(\boldsymbol{b})_{|\partial D_{\mathrm{d}}}$ is defined in $\widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{d}})'$ by using the duality product $\langle \gamma^{\mathrm{c}}(\boldsymbol{b})_{|\partial D_{\mathrm{d}}}, \boldsymbol{\theta}\rangle_{\partial D_{\mathrm{d}}} := \langle \gamma^{\mathrm{c}}(\boldsymbol{b}), \tilde{\boldsymbol{\theta}}\rangle_{\partial D}$ for all $\boldsymbol{\theta} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{d}})$. A weak formulation of (43.9) is the following:

$$\begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{V}_{\mathrm{d}} := \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \gamma^{\mathrm{c}}(\boldsymbol{b})_{|\partial D_{\mathrm{d}}} = \boldsymbol{0}\} \text{ such that} \\ a_{\nu,\kappa}(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b}), \quad \forall \boldsymbol{b} \in \boldsymbol{V}_{\mathrm{d}}, \end{cases} \tag{43.10}$$

with the following sesquilinear and antilinear forms:

$$a_{\nu,\kappa}(\boldsymbol{a}, \boldsymbol{b}) := \int_D (\nu \boldsymbol{a}\cdot\overline{\boldsymbol{b}} + \kappa \nabla\times\boldsymbol{a}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x, \qquad \ell(\boldsymbol{b}) := \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x. \tag{43.11}$$

43.2.2 Well-posedness

We assume that there are real numbers θ, $\nu_\flat > 0$, and $\kappa_\flat > 0$ s.t.

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{i\theta}\nu(\boldsymbol{x})\big) \geq \nu_\flat \quad \text{and} \quad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re\big(e^{i\theta}\kappa(\boldsymbol{x})\big) \geq \kappa_\flat. \tag{43.12}$$

Let us set $\nu_\sharp := \|\nu\|_{L^\infty(D;\mathbb{C})}$ and $\kappa_\sharp := \|\kappa\|_{L^\infty(D;\mathbb{C})}$.

Theorem 43.1 (Coercivity, well-posedness). (i) *Assume* $\boldsymbol{f} \in \boldsymbol{L}^2(D)$, $\nu, \kappa \in L^\infty(D;\mathbb{C})$, *and* (43.12). *Then the sesquilinear form* $a_{\nu,\kappa}$ *is coercive and bounded:*

$$\Re\big(e^{i\theta}a_{\nu,\kappa}(\boldsymbol{b}, \boldsymbol{b})\big) \geq \min(\nu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}, \tag{43.13a}$$

$$|a_{\nu,\kappa}(\boldsymbol{a}, \boldsymbol{b})| \leq \max(\nu_\sharp, \ell_D^{-2}\kappa_\sharp)\|\boldsymbol{a}\|_{\boldsymbol{H}(\mathrm{curl};D)}\|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)}, \tag{43.13b}$$

for all $\boldsymbol{a}, \boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D)$. (ii) *The problem* (43.10) *is well-posed.*

Proof. Let us first verify that $\boldsymbol{V}_{\mathrm{d}}$ is a closed subspace of $\boldsymbol{H}(\mathrm{curl}; D)$. Let $(\boldsymbol{b}_n)_{n\in\mathbb{N}}$ be a Cauchy sequence in $\boldsymbol{V}_{\mathrm{d}}$. Then $\boldsymbol{b}_n \to \boldsymbol{b}$ in $\boldsymbol{H}(\mathrm{curl}; D)$, and for all $\boldsymbol{\theta} \in \widetilde{\boldsymbol{H}}^{\frac{1}{2}}(\partial D_{\mathrm{d}})$, we have

$$0 = \langle \gamma^{\mathrm{c}}(\boldsymbol{b}_n)_{|\partial D_{\mathrm{d}}}, \boldsymbol{\theta}\rangle_{\partial D_{\mathrm{d}}} := \langle \gamma^{\mathrm{c}}(\boldsymbol{b}_n), \tilde{\boldsymbol{\theta}}\rangle_{\partial D} \to \langle \gamma^{\mathrm{c}}(\boldsymbol{b}), \tilde{\boldsymbol{\theta}}\rangle_{\partial D} =: \langle \gamma^{\mathrm{c}}(\boldsymbol{b}), \boldsymbol{\theta}\rangle_{\partial D_{\mathrm{d}}},$$

so that $\boldsymbol{b} \in \boldsymbol{V}_{\mathrm{d}}$. (Recall that (4.11) implies that $\gamma^{\mathrm{c}} : \boldsymbol{H}(\mathrm{curl}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ is continuous.) Moreover, coercivity follows from (43.12) since we have

$$\begin{aligned} \Re(e^{\mathrm{i}\theta}a_{\nu,\kappa}(\boldsymbol{b}, \boldsymbol{b})) &= \int_D \Big(\Re(e^{i\theta}\nu)|\boldsymbol{b}|^2 + \Re(e^{i\theta}\kappa)|\nabla\times\boldsymbol{b}|^2\Big)\,\mathrm{d}x \\ &\geq \int_D \Big(\nu_\flat|\boldsymbol{b}|^2 + \kappa_\flat|\nabla\times\boldsymbol{b}|^2\Big)\,\mathrm{d}x \geq \min(\nu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}. \end{aligned}$$

Similarly, the boundedness of $a_{\nu,\kappa}$ follows from $\nu, \kappa \in L^\infty(D;\mathbb{C})$, and the boundedness of ℓ follows from $\boldsymbol{f} \in \boldsymbol{L}^2(D)$. Finally, well-posedness follows from the complex version of the Lax–Milgram lemma. □

Example 43.2 (Property (43.12)**).** Assume to fix the ideas that κ is real and uniformly positive. If ν is also real and uniformly positive, (43.12) is satisfied with $\theta := 0$, $\kappa_\flat := \operatorname{ess\,inf}_{\boldsymbol{x}\in D} \kappa(\boldsymbol{x})$, and $\nu_\flat := \operatorname{ess\,inf}_{\boldsymbol{x}\in D} \nu(\boldsymbol{x})$. If instead ν is purely imaginary with a uniformly positive imaginary part, (43.12) is satisfied with $\theta := -\frac{\pi}{4}$, $\kappa_\flat := \frac{\sqrt{2}}{2} \operatorname{ess\,inf}_{\boldsymbol{x}\in D} \kappa(\boldsymbol{x})$, and $\nu_\flat := \frac{\sqrt{2}}{2} \operatorname{ess\,inf}_{\boldsymbol{x}\in D} \Im(\nu(\boldsymbol{x}))$. More generally, if $\nu := \rho_\nu e^{i\theta_\nu}$ with $\operatorname{ess\,inf}_{\boldsymbol{x}\in D} \rho_\nu(\boldsymbol{x}) =: \rho_\flat > 0$ and $\theta_\nu(\boldsymbol{x}) \in [\theta_{\min}, \theta_{\max}] \subset (-\pi, \pi)$ a.e. in D, then setting $\delta := \theta_{\max} - \theta_{\min}$ and assuming that $\delta < \pi$, (43.12) is satisfied with $\theta := -\frac{1}{2}(\theta_{\min} + \theta_{\max})\frac{\pi}{2\pi-\delta}$, $\nu_\flat := \min(\cos(\theta_{\min} + \theta), \cos(\theta_{\max} + \theta))\rho_\flat$ and $\kappa_\flat := \cos(\theta) \operatorname{ess\,inf}_{\boldsymbol{x}\in D} \kappa(\boldsymbol{x})$ (see Exercise 43.3). An important example where the condition (43.12) fails is when the two complex numbers ν and κ are collinear and point in opposite directions. In this case, resonances may occur and (43.10) has to be replaced by an eigenvalue problem. □

43.2.3 Regularity

In the case of constant or smooth coefficients, a smoothness property on the solution to (43.10) can be inferred from the following important result.

Lemma 43.3 (Regularity). *Let D be a Lipschitz domain in $\mathbb{R}^3$.* (i) *There is $c > 0$ s.t. the following holds true:*

$$c\,\ell_D^s |\boldsymbol{v}|_{\boldsymbol{H}^s(D)} \le \|\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla\cdot\boldsymbol{v}\|_{L^2(D)}, \tag{43.14}$$

with $s := \frac{1}{2}$, for all vector fields $\boldsymbol{v} \in \boldsymbol{H}(\mathrm{curl}; D) \cap \boldsymbol{H}(\mathrm{div}; D)$ with either zero normal trace or zero tangential trace over ∂D. (ii) *The estimate remains valid with $s \in (\frac{1}{2}, 1]$ if D is a Lipschitz polyhedron, and with $s := 1$ if D is convex.*

Proof. (i) For the proof of (43.14), see Birman and Solomyak [57, Thm. 3.1] and Costabel [142, Thm. 2]. (ii) See Amrouche et al. [10, Prop. 3.7] when D is a Lipschitz polyhedron and [10, Thm. 2.17] when D is convex. □

Let us consider the problem (43.10) and assume that $\boldsymbol{f} \in \boldsymbol{H}(\mathrm{div}; D)$ and ν is constant (or smooth) over D. Then the unique solution $\boldsymbol{A}$ is such that $\nabla\cdot\boldsymbol{A} = \nu^{-1}\nabla\cdot\boldsymbol{f} \in L^2(D)$. Hence, $\boldsymbol{A} \in \boldsymbol{H}(\mathrm{curl}; D) \cap \boldsymbol{H}(\mathrm{div}; D)$. Moreover, (43.9) implies that $\nabla\times(\kappa\nabla\times\boldsymbol{A}) \in \boldsymbol{L}^2(D)$ so that, assuming that κ is constant (or smooth) over D, we infer that $\nabla\times\boldsymbol{A} \in \boldsymbol{H}(\mathrm{curl}; D)\cap\boldsymbol{H}(\mathrm{div}; D)$. In addition to the above assumptions on ν and κ, let us also assume that $\partial D_{\mathrm{n}} = \emptyset$ (i.e., $\boldsymbol{A}$ has a zero tangential trace, which implies that $\nabla\times\boldsymbol{A}$ has a zero normal trace a.e. on ∂D). Lemma 43.3 implies that there exists $r > 0$ so that

$$\boldsymbol{A} \in \boldsymbol{H}^r(D), \qquad \nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D), \tag{43.15}$$

with $r := \frac{1}{2}$ in general, $r \in (\frac{1}{2}, 1]$ if D is a Lipschitz polyhedron, and $r := 1$ if D is convex. In the more general case of heterogeneous coefficients, we will see in the next chapter (see Lemma 44.2) that the smoothness assumption (43.15) is still valid with a smoothness index $r > 0$ under appropriate assumptions on ν. In the rest of this chapter, we are going to assume that (43.15) holds true with $r > 0$.

43.3 Approximation using edge elements

We assume that the hypotheses of Theorem 43.1 are satisfied so that the boundary-value problem (43.10) is well-posed.

43.3.1 Discrete setting

We consider a shape-regular sequence of affine meshes $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of D. We assume that D is a Lipschitz polyhedron so that each mesh covers D exactly. We also assume that the meshes are compatible with the partition of the boundary into $\{\partial D_{\rm d}, \partial D_{\rm n}\}$. We consider the Nédélec (or edge) finite elements of some order $k \geq 0$ from Chapter 15 and the corresponding $\boldsymbol{H}(\mathrm{curl})$-conforming finite element space $\boldsymbol{P}_k^{\rm c}(\mathcal{T}_h)$ built in Chapter 19. Let $\boldsymbol{V}_{h\rm d}$ be the subspace of $\boldsymbol{P}_k^{\rm c}(\mathcal{T}_h)$ defined by

$$\boldsymbol{V}_{h\rm d} := \{\boldsymbol{b}_h \in \boldsymbol{P}_k^{\rm c}(\mathcal{T}_h) \mid \boldsymbol{b}_{h|\partial D_{\rm d}}{\times}\boldsymbol{n} = \boldsymbol{0}\}. \tag{43.16}$$

Since the Dirichlet boundary condition is strongly enforced in $\boldsymbol{V}_{h\rm d}$, the approximation setting is conforming, i.e., $\boldsymbol{V}_{h\rm d} \subset \boldsymbol{V}_{\rm d}$. The discrete formulation of (43.10) is

$$\begin{cases} \text{Find } \boldsymbol{A}_h \in \boldsymbol{V}_{h\rm d} \text{ such that} \\ a_{\nu,\kappa}(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h), \quad \forall \boldsymbol{b}_h \in \boldsymbol{V}_{h\rm d}. \end{cases} \tag{43.17}$$

The Lax–Milgram lemma together with the conformity of the approximation setting implies that (43.17) has a unique solution.

43.3.2 $\boldsymbol{H}(\mathrm{curl})$-error estimate

Theorem 43.4 ($\boldsymbol{H}(\mathrm{curl})$-error estimate). (i) *Under the assumptions of Theorem 43.1, there is c s.t. for all $h \in \mathcal{H}$,*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)} \leq c \inf_{\boldsymbol{b}_h\in\boldsymbol{V}_{h\rm d}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}. \tag{43.18}$$

(ii) A*ssuming that either $\partial D_{\rm d} = \partial D$ or $\partial D_{\rm n} = \partial D$ and that there is $r \in (0, k+1]$ s.t. $\boldsymbol{A} \in \boldsymbol{H}^r(D)$ and $\nabla{\times}\boldsymbol{A} \in \boldsymbol{H}^r(D)$, where $k \geq 0$ is the degree of the finite element used to build $\boldsymbol{V}_{h\rm d}$, we have*

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} \le c\, h^r(|\boldsymbol{A}|_{\boldsymbol{H}^r(D)} + \ell_D|\nabla\times\boldsymbol{A}|_{\boldsymbol{H}^r(D)}). \tag{43.19}$$

Proof. (i) The estimate (43.18) is a direct consequence of Céa's lemma.
(ii) We prove the estimate (43.19) when $\partial D_{\rm d} = \partial D$, that is, when $\boldsymbol{V}_{h\rm d} := \boldsymbol{P}^{\rm c}_{k,0}(\mathcal{T}_h) := \{\boldsymbol{b}_h \in \boldsymbol{P}^{\rm c}_k(\mathcal{T}_h) \mid \boldsymbol{b}_{h|\partial D}\times\boldsymbol{n} = \boldsymbol{0}\}$. We estimate the infimum in (43.18) by taking $\boldsymbol{b}_h := \mathcal{J}^{\rm c}_{h0}(\boldsymbol{A})$, where $\mathcal{J}^{\rm c}_{h0} : \boldsymbol{L}^1(D) \to \boldsymbol{P}^{\rm c}_{k,0}(\mathcal{T}_h)$ is the commuting quasi-interpolation operator with zero tangential trace introduced in §23.3.3. Owing to the items (ii) and (iii) in Theorem 23.12, we infer that

$$\begin{aligned}
\|\boldsymbol{A}-\mathcal{J}^{\rm c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{H}(\text{curl};D)} &\le \|\boldsymbol{A}-\mathcal{J}^{\rm c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \ell_D\|\nabla\times(\boldsymbol{A}-\mathcal{J}^{\rm c}_{h0}(\boldsymbol{A}))\|_{\boldsymbol{L}^2(D)}\\
&= \|\boldsymbol{A}-\mathcal{J}^{\rm c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \ell_D\|\nabla\times\boldsymbol{A}-\mathcal{J}^{\rm d}_{h0}(\nabla\times\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}\\
&\le c \inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{\rm c}_{k,0}(\mathcal{T}_h)} \|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} + c'\ell_D \inf_{\boldsymbol{d}_h\in\boldsymbol{P}^{\rm d}_{k,0}(\mathcal{T}_h)} \|\nabla\times\boldsymbol{A}-\boldsymbol{d}_h\|_{\boldsymbol{L}^2(D)}\\
&\le c''\, h^r(|\boldsymbol{A}|_{\boldsymbol{H}^r(D)} + \ell_D|\nabla\times\boldsymbol{A}|_{\boldsymbol{H}^r(D)}),
\end{aligned}$$

where the last step follows from Corollary 22.16. The proof for $\partial D_{\rm n} = \partial D$ is similar if one uses $\mathcal{J}^{\rm c}_h$, $\mathcal{J}^{\rm d}_h$ instead of $\mathcal{J}^{\rm c}_{h0}$, $\mathcal{J}^{\rm d}_{h0}$. □

Remark 43.5 ($\nu_\flat$-dependency). The coercivity and boundedness properties in (43.13) show that the constant in the error estimate (43.18) is $c = \frac{\max(\nu_\sharp,\ell_D^{-2}\kappa_\sharp)}{\min(\nu_\flat,\ell_D^{-2}\kappa_\flat)}$, which becomes unbounded when $\nu_\flat$ is very small. This difficulty is addressed in Chapter 44. □

Remark 43.6 (Variants). It is possible to localize (43.19) by using Theorem 22.14 instead of Corollary 22.16 when $\partial D_{\rm d} = \partial D$, and using Theorem 22.6 instead of Corollary 22.9 when $\partial D_{\rm n} = \partial D$. Using that $\boldsymbol{A} \in \boldsymbol{H}_0(\text{curl};D)$, $\nabla\times\boldsymbol{A} \in \boldsymbol{H}_0(\text{div};D)$, and the regularity of the mesh sequence, Theorem 22.14 and Theorem 22.6 imply that

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} \le c\left(\sum_{K\in\mathcal{T}_h} h_K^{2r}(|\boldsymbol{A}|_{\boldsymbol{H}^r(K)} + \ell_D|\nabla\times\boldsymbol{A}|_{\boldsymbol{H}^r(K)})^2\right)^{\frac{1}{2}},$$

when $r > \frac{1}{2}$. The seminorm $|\cdot|_{\boldsymbol{H}^r(K)}$ has to be replaced by $|\cdot|_{\boldsymbol{H}^r(D_K)}$ whenever $r \le \frac{1}{2}$, where D_K is the set of the points composing the mesh cells sharing a degree of freedom with K. One can also extend the estimate (43.19) to the case of mixed boundary conditions by adapting the construction of the quasi-interpolation operator and of the commuting projection from Chapters 22 and 23. Finally, we refer the reader to Ciarlet [121, Prop. 4] for an alternative proof of (43.19). □

43.3.3 The duality argument

Recalling the material from §32.3, we would like to apply the Aubin–Nitsche duality argument to deduce an improved error estimate on $\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)}$. It is at this point that we realize that the approach we have taken so far is too simplistic. To better understand the problem, let us consider the case $\partial D_{\mathrm{d}} = \partial D$. In this context, we have $\boldsymbol{V}_{\mathrm{d}} := \boldsymbol{H}_0(\mathrm{curl}; D)$ and $\boldsymbol{L} := \boldsymbol{L}^2(D)$, and Theorem 32.8 tells us that the Aubin–Nitsche argument provides a better rate of convergence in the $\boldsymbol{L}^2$-norm if and only if the embedding $\boldsymbol{H}_0(\mathrm{curl}; D) \hookrightarrow \boldsymbol{L}^2(D)$ is compact, which is not the case as shown in Exercise 43.1. The conclusion of this argumentation is that the estimates we have derived so far cannot yield an improved error estimate on $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)}$. A way around this obstacle is to find a space smaller than $\boldsymbol{H}_0(\mathrm{curl}; D)$, where the weak solution $\boldsymbol{A}$ lives and that embeds compactly into $\boldsymbol{L}^2(D)$, and to show that $\boldsymbol{A}_h$ is a convergent nonconforming approximation of $\boldsymbol{A}$ in that space. We are going to see in Chapter 44 that a good candidate is $\boldsymbol{H}_0(\mathrm{curl}; D) \cap \boldsymbol{H}(\mathrm{div}; D)$, as pointed out in Weber [391, Thm. 2.1-2.3]. Recall that the unknown field $\boldsymbol{A}$ stands for $\boldsymbol{E}$ or $\boldsymbol{H}$, and that the Gauss laws (43.1c)-(43.1d) combined with (43.3) imply that $\nabla\cdot(\epsilon \boldsymbol{E}) = \nabla\cdot \boldsymbol{D} = \rho$ and that $\nabla\cdot(\mu \boldsymbol{H}) = \nabla\cdot\boldsymbol{B} = 0$. Thus, it is reasonable to expect some control on the divergence of $\boldsymbol{A}$ and, therefore, to hope for an improved estimate on $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)}$ provided $\nabla\cdot\boldsymbol{A}_h$ is controlled in some sense. This question is addressed in Chapter 44.

Exercises

Exercise 43.1 (Compactness). Let $D := (0,1)^3$ be the unit cube in $\mathbb{R}^3$. Show that the embedding $\boldsymbol{H}_0(\mathrm{curl}; D) \hookrightarrow \boldsymbol{L}^2(D)$ is not compact. (*Hint*: consider $\boldsymbol{v}_n := \nabla\phi_n$ with $\phi_n(x_1, x_2, x_3) := \frac{1}{n\pi}\sin(n\pi x_1)\sin(n\pi x_2)\sin(n\pi x_3)$, $n \ge 1$, and prove first that $(\boldsymbol{v}_n)_{n\ge 1}$ weakly converges to zero in $\boldsymbol{L}^2(D)$ (see Definition C.28), then compute $\|\boldsymbol{v}_n\|_{\boldsymbol{L}^2(D)}$ and argue by contradiction.)

Exercise 43.2 (Curl). (i) Let $\boldsymbol{v}$ be a smooth field. Show that $\|\nabla\times\boldsymbol{v}\|^2_{\ell^2} \le 2\nabla\boldsymbol{v}{:}\nabla\boldsymbol{v}$. (*Hint*: relate $\nabla\times\boldsymbol{v}$ to the components of $(\nabla\boldsymbol{v} - \nabla\boldsymbol{v}^{\mathsf{T}})$.) (ii) Show that $\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^2(D)} \le |\boldsymbol{v}|_{\boldsymbol{H}^1(D)}$ for all $\boldsymbol{v} \in \boldsymbol{H}^1_0(D)$. (*Hint*: use an integration by parts.)

Exercise 43.3 (Property (43.12)). Prove the claim in Example 43.2, i.e., for $[\theta_{\min}, \theta_{\max}] \subset (-\pi, \pi)$ with $\delta := \theta_{\max} - \theta_{\min} < \pi$, letting $\theta := -\frac{1}{2}(\theta_{\min} + \theta_{\max})\frac{\pi}{2\pi-\delta}$, prove that $\theta \in (-\frac{\pi}{2}, \frac{\pi}{2})$ and $[\theta_{\min} + \theta, \theta_{\max} + \theta] \subset (-\frac{\pi}{2}, \frac{\pi}{2})$.

Exercise 43.4 (Dirichlet/Neumann). Let $\boldsymbol{v}$ be a smooth vector field in D such that $\boldsymbol{v}_{|\partial D_{\mathrm{d}}}\times\boldsymbol{n} = \boldsymbol{0}$. Prove that $(\nabla\times\boldsymbol{v})_{|\partial D_{\mathrm{d}}}\cdot\boldsymbol{n} = \boldsymbol{0}$. (*Hint*: compute $\int_D (\nabla\times\boldsymbol{v})\cdot\nabla q \,\mathrm{d}x$ with q well chosen.)