# Part X, Chapter 46

## Symmetric elliptic eigenvalue problems

The three chapters composing Part X deal with the finite element approximation of the spectrum of elliptic differential operators. Ellipticity is crucial here to provide a compactness property that guarantees that the spectrum of the operators in question is well structured. We start by recalling fundamental results on compact operators and symmetric operators in Hilbert spaces. Then, we study the finite element approximation of the spectrum of compact operators. We first focus on the $H^1$-conforming approximation of symmetric operators, then we treat the (possibly nonconforming) approximation of nonsymmetric operators.

The present chapter contains a brief introduction to the spectral theory of compact operators together with illustrative examples. Eigenvalue problems occur when analyzing the response of devices, buildings, or vehicles to vibrations, or when performing the linear stability analysis of dynamical systems.

## 46.1 Spectral theory

We briefly recall in this section some essential facts regarding the spectral theory of linear operators. Most of the proofs are omitted since the material is classical and can be found in Brezis [89, Chap. 6], Chatelin [116, pp. 95-120], Dunford and Schwartz [179, Part I, pp. 577-580], Lax [278, Chap. 21&32], Kreyszig [271, pp. 365-521]. In the entire section, $L$ is a complex Banach space, we use the shorthand notation $\mathcal{L}(L) := \mathcal{L}(L; L)$, and $I_L$ denotes the identity operator in $L$.

### 46.1.1 Basic notions and examples

**Definition 46.1 (Resolvent, spectrum, eigenvalues, eigenvectors).** *Let $T \in \mathcal{L}(L)$. The* resolvent set *of $T$, $\rho(T)$, and the* spectrum *of $T$, $\sigma(T)$, are subsets of $\mathbb{C}$ defined as follows:*

$$\rho(T) := \{\mu \in \mathbb{C} \mid \mu I_L - T \textit{ is bijective}\}, \tag{46.1a}$$
$$\sigma(T) := \mathbb{C}\backslash\rho(T) = \{\mu \in \mathbb{C} \mid \mu I_L - T \textit{ is not bijective}\}. \tag{46.1b}$$

*(Since $L$ is a Banach space, $\mu \in \rho(T)$ iff $(\mu I_L - T)^{-1} \in \mathcal{L}(L)$.) The spectrum of $T$ is decomposed into the following disjoint union:*

$$\sigma(T) = \sigma_{\rm p}(T) \cup \sigma_{\rm c}(T) \cup \sigma_{\rm r}(T), \tag{46.2}$$

*where the* point spectrum, *$\sigma_{\rm p}(T)$, the* continuous spectrum, *$\sigma_{\rm c}(T)$, and the* residual spectrum, *$\sigma_{\rm r}(T)$, are defined as follows:*

$$\sigma_{\rm p}(T) := \{\mu \in \mathbb{C} \mid \mu I_L - T \textit{ is not injective}\},$$
$$\sigma_{\rm c}(T) := \{\mu \in \mathbb{C} \mid \mu I_L - T \textit{ is injective, not surjective, } \overline{\mathrm{im}(\mu I_L - T)} = L\},$$
$$\sigma_{\rm r}(T) := \{\mu \in \mathbb{C} \mid \mu I_L - T \textit{ is injective, not surjective, } \overline{\mathrm{im}(\mu I_L - T)} \neq L\}.$$

*Whenever $\sigma_{\rm p}(T)$ is nonempty, members of $\sigma_{\rm p}(T)$ are called* eigenvalues, *and the nonzero vectors in $\ker(\mu I_L - T)$ are called* eigenvectors *associated with $\mu$, i.e., $0 \neq z \in L$ is an eigenvector associated with $\mu$ iff $T(z) = \mu z$.*

**Example 46.2 (Finite dimension).** If $L$ is finite-dimensional, $\ker(\mu I_L - T) \neq \{0\}$ iff $(\mu I_L - T)$ is not invertible. In this case, the spectrum of $T$ only consists of eigenvalues, i.e., $\sigma(T) = \sigma_{\rm p}(T)$ and $\sigma_{\rm c}(T) = \sigma_{\rm r}(T) = \emptyset$. □

**Example 46.3 (Volterra operator).** Let $L := L^2((0,1);\mathbb{C})$ and let us identify $L$ and $L'$ by setting $\langle l', l\rangle_{L',L} := \int_0^1 l'(x)\overline{l}(x)\,\mathrm{d}x$. Let $T : L \to L$ be s.t. $T(f)(x) := \int_0^x f(t)\,\mathrm{d}t$ for a.e. $x \in (0,1)$. We have $\rho(T) = \mathbb{C}\backslash\{0\}$, $\sigma_{\rm p}(T) = \emptyset$, $\sigma_{\rm c}(T) = \{0\}$, and $\sigma_{\rm r}(T) = \emptyset$; see Exercise 46.4. □

**Theorem 46.4 (Spectral radius).** *Let $T \in \mathcal{L}(L)$.* (i) *The subsets $\rho(T)$ and $\sigma(T)$ are both nonempty.* (ii) *$\sigma(T)$ is a compact subset of $\mathbb{C}$.* (iii) *Let*

$$r(T) := \max_{\mu\in\sigma(T)} |\mu| \tag{46.3}$$

*be the* spectral radius *of $T$. Then*

$$r(T) = \lim_{n\to\infty} \|T^n\|_{\mathcal{L}(L)}^{\frac{1}{n}} \leq \|T\|_{\mathcal{L}(L)}. \tag{46.4}$$

*Proof.* See Kreyszig [271], Thm. 7.5.4 for (i), Thm. 7.3.4 for (ii), and Thm. 7.5.5 for (iii). □

**Remark 46.5 ((46.4)).** The identity $r(T) = \lim_{n\to\infty} \|T^n\|_{\mathcal{L}(L)}^{\frac{1}{n}}$ is often called Gelfand's formula (see [213, p. 11]). The inequality $\lim_{n\to\infty} \|T^n\|_{\mathcal{L}(L)}^{\frac{1}{n}} \leq \|T\|_{\mathcal{L}(L)}$ may sometimes be strict. For instance, $r(T) = 0$ if $\sigma(T) = \{0\}$, but it can happen in that case that $\|T\|_{\mathcal{L}(L)} > 0$. A simple example is the operator $T : \mathbb{R}^2 \to \mathbb{R}^2$ s.t. $T(X) := \mathbb{A}X$ with $\mathbb{A} := \left(\begin{smallmatrix} 0 & 1 \\ 0 & 0 \end{smallmatrix}\right)$. □

Let us consider more specifically the eigenvalues of $T$. Assume that $\sigma_{\mathrm{p}}(T) \neq \emptyset$ and let $\mu \in \sigma_{\mathrm{p}}(T)$. Let us set $K_i := \ker(\mu I_L - T)^i$ for all $i \in \mathbb{N}\setminus\{0\}$. One readily verifies that the spaces $K_i$ are invariant under $T$. Moreover, $K_1 \subset K_2 \ldots$, and if there is an integer $j \geq 1$ such that $K_j = K_{j+1}$, then $K_j = K_{j'}$ for all $j' > j$.

**Definition 46.6 (Ascent, algebraic and geometric multiplicity).** *Assume that $\sigma_{\mathrm{p}}(T) \neq \emptyset$ and let $\mu \in \sigma_{\mathrm{p}}(T)$. We say that $\mu$ has finite ascent if there is $j \in \mathbb{N} \setminus \{0\}$ such that $K_j = K_{j+1}$, and the smallest integer satisfying this property is called* ascent *of $\mu$ and is denoted by $\alpha(\mu)$ (or simply $\alpha$). Moreover, if $K_\alpha$ is finite-dimensional, then the* algebraic multiplicity *of $\mu$, say $m$, and the* geometric multiplicity *of $\mu$, say $g$, are defined as follows:*

$$m := \dim(K_\alpha) \geq \dim(K_1) =: g. \tag{46.5}$$

*Whenever $\alpha \geq 2$, nonzero vectors in $K_\alpha$ are called* generalized eigenvectors.

If the eigenvalue $\mu$ has finite ascent $\alpha$ and if $K_\alpha$ is finite-dimensional, then elementary arguments from linear algebra show that $\alpha+g-1 \leq m \leq \alpha g$ (note that $\alpha = 1$ iff $m = g$). This inequalities are proved by showing that $g_1+i-1 \leq g_i \leq g_{i-1} + g_1$ for all $i \in \{1{:}\alpha\}$ with $g_i := \dim(K_i)$; see Exercise 46.2. All the eigenvalues have a finite ascent and a finite multiplicity if $L$ is finite-dimensional, or if the operator $T$ is compact (see Theorem 46.14(iv)), but this may not be the case in general.

**Example 46.7 (Ascent, algebraic and geometric multiplicity).** To illustrate Definition 46.6 in a finite-dimensional setting, we consider the operator $T : \mathbb{R}^4 \to \mathbb{R}^4$ defined by $T(X) := \mathbb{A}X$ for all $X \in L := \mathbb{R}^4$, where

$$\mathbb{A} := \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

Then $\mu = 1$ is the only eigenvalue of $T$. Since

$$\mathbb{I}_4 - \mathbb{A} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbb{I}_4 - \mathbb{A})^2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad (\mathbb{I}_4 - \mathbb{A})^3 = \mathbb{O}_4,$$

we have $\ker(I_L - T) = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_4\}$, $\ker(I_L - T)^2 = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_4\}$, and $\ker(I_L - T)^3 = \ker(I_L - T)^4 = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$, where $\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}$ is the canonical Cartesian basis of $\mathbb{R}^4$. Thus, the ascent of $\mu = 1$ is $\alpha = 3$, its algebraic multiplicity is $m = \dim(\ker(I_L - T)^3) = 4$, and its geometric multiplicity is $g = \dim(\ker(I_L - T)) = 2$. Notice that $\alpha + g - 1 = 4 = m \leq 6 = \alpha g$. $\square$

Let us finally explore the relation between the spectrum of $T$ and that of its adjoint $T^* : L' \to L'$ s.t. $\langle T^*(l'), l\rangle_{L',L} := \langle l', T(l)\rangle_{L',L}$ for all $l \in L$ and all $l' \in L'$ (see Definition C.29). Recall that we have adopted the convention that dual spaces are composed of antilinear forms (see Definition A.11 and §C.4), so that $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{C}$. (The reader should be aware that a usual convention in the mathematical physics literature is that dual spaces are composed of linear forms, in which case $(\lambda T)^* = \lambda T^*$.) Moreover, for any subset $A \subset \mathbb{C}$, we denote $\mathrm{conj}(A) := \{\mu \in \mathbb{C} \mid \overline{\mu} \in A\}$.

**Lemma 46.8 (Spectrum of $T^*$).** *Let $T \in (\mathcal{L})$. The following holds true:*

$$\sigma(T^*) = \mathrm{conj}(\sigma(T)), \qquad \sigma_{\mathrm{r}}(T) \subset \mathrm{conj}(\sigma_{\mathrm{p}}(T^*)) \subset \sigma_{\mathrm{r}}(T) \cup \sigma_{\mathrm{p}}(T). \tag{46.6}$$

*Proof.* Corollary C.52 implies that $\mu I_L - T$ is not bijective iff $(\mu I_L - T)^* = \overline{\mu} I_{L'} - T^*$ is not bijective. This proves the first equality. See Exercise 46.1 for the proof of the other two inclusions. □

**Example 46.9 (Left and right shifts).** Let $p \in (1, \infty)$ and let $\ell^p$ be the Banach space composed of the complex-valued sequences $x := (x_n)_{n\in\mathbb{N}}$ s.t. $\sum_{n\in\mathbb{N}} |x_n|^p < \infty$. We can identify the dual space of $\ell^p$ with $\ell^{p'}$, where $\frac{1}{p} + \frac{1}{p'} = 1$, by setting $\langle x, y\rangle_{\ell^{p'},\ell^p} := \sum_{n\in\mathbb{N}} x_n \overline{y_n}$ with $x := (x_n)_{n\in\mathbb{N}}$ and $y := (y_n)_{n\in\mathbb{N}}$. Consider the left shift operator $\mathsf{L} : \ell^{p'} \to \ell^{p'}$ defined by $\mathsf{L}(x) := (x_1, x_2, \ldots)$ and the right shift operator $\mathsf{R} : \ell^p \to \ell^p$ defined by $\mathsf{R}(x) := (0, x_0, x_1, \ldots)$. Then $\langle x, \mathsf{R}(y)\rangle_{\ell^{p'},\ell^p} := \sum_{n\geq 1} x_n \overline{y_{n-1}} = \sum_{n\geq 0} x_{n+1}\overline{y_n} = \langle \mathsf{L}(x), y\rangle_{\ell^{p'},\ell^p}$. This shows that $\mathsf{L} = \mathsf{R}^*$. Similarly, $\mathsf{R} = \mathsf{L}^*$ once the dual of $\ell^{p'}$ is identified with $\ell^p$. Observe that $\|\mathsf{R}\|_{\mathcal{L}(\ell^p;\ell^p)} = \|\mathsf{L}\|_{\mathcal{L}(\ell^{p'};\ell^{p'})} = 1$, so that both $\sigma(\mathsf{R})$ and $\sigma(\mathsf{L})$ are contained in the unit disk $\{\lambda \in \mathbb{C} \mid |\lambda| \leq 1\}$ owing to Theorem 46.4(iii). Notice that $0 \notin \sigma_{\mathrm{p}}(\mathsf{R})$ since $\mathsf{R}$ is injective. Assume that there exists $\mu \in \sigma_{\mathrm{p}}(\mathsf{R})$, i.e., there is a nonzero $x \in \ell^p$ s.t. $(\mu x_0, \mu x_1 - x_0, \mu x_2 - x_1, \ldots) = 0$. Then $x_n = 0$ for all $n \in \mathbb{N}$, i.e., $x = 0$, which is absurd (recall that $\mu \neq 0$). Hence, $\sigma_{\mathrm{p}}(\mathsf{R}) = \emptyset$. Lemma 46.8 in turn implies that $\sigma_{\mathrm{r}}(\mathsf{L}) = \emptyset$ because $\mathsf{L}^* = \mathsf{R}$. Similarly, Lemma 46.8 implies that $\sigma_{\mathrm{r}}(\mathsf{R}) \subset \mathrm{conj}(\sigma_{\mathrm{p}}(\mathsf{L})) \subset \sigma_{\mathrm{r}}(\mathsf{R})$, i.e., $\sigma_{\mathrm{r}}(\mathsf{R}) = \mathrm{conj}(\sigma_{\mathrm{p}}(\mathsf{L}))$. Assuming that $\mu \in \sigma_{\mathrm{p}}(\mathsf{L})$, there is a nonzero vector $x \in \ell^{p'}$ s.t. $\mathsf{L}(x) = \mu x$, which means that $x = x_0(1, \mu, \mu^2, \ldots)$. This vector is in $\ell^{p'}$ iff $|\mu| < 1$. Hence, $\sigma_{\mathrm{p}}(\mathsf{L}) = \{\mu \in \mathbb{C} \mid |\mu| < 1\}$. Since $\sigma_{\mathrm{p}}(\mathsf{L})$ is invariant under complex conjugation, we conclude that $\sigma_{\mathrm{r}}(\mathsf{R}) = \sigma_{\mathrm{p}}(\mathsf{L})$. Finally, since $\sigma(\mathsf{L})$ is closed (see Theorem 46.4(ii)) and $\|\mathsf{L}\|_{\mathcal{L}(\ell^{p'};\ell^{p'})} = 1$, we have $\sigma(\mathsf{L}) \subset \{\mu \in \mathbb{C} \mid |\mu| \leq 1\}$. But $\sigma(\mathsf{L})$ must also contain the closure in $\mathbb{C}$ of $\sigma_{\mathrm{p}}(\mathsf{L}) = \{\mu \in \mathbb{C} \mid |\mu| < 1\}$. Hence, $\sigma(\mathsf{L}) = \{\mu \in \mathbb{C} \mid |\mu| \leq 1\}$. This, in turn, implies that $\sigma_{\mathrm{c}}(\mathsf{L}) = \{\mu \in \mathbb{C} \mid |\mu| = 1\}$. In conclusion, we have established that

$$\begin{aligned}
\sigma_{\mathrm{p}}(\mathsf{L}) &= \{\mu \in \mathbb{C} \mid |\mu| < 1\} = \sigma_{\mathrm{r}}(\mathsf{R}),\\
\sigma_{\mathrm{c}}(\mathsf{L}) &= \{\mu \in \mathbb{C} \mid |\mu| = 1\} = \sigma_{\mathrm{c}}(\mathsf{R}),\\
\sigma_{\mathrm{r}}(\mathsf{L}) &= \emptyset = \sigma_{\mathrm{p}}(\mathsf{R}).
\end{aligned}$$ □

### 46.1.2 Compact operators in Banach spaces

Since we are going to focus later our attention on the approximation of the eigenvalues and eigenspaces of compact operators, we now recall important facts about such operators. Given two Banach spaces $V$, $W$, we say that $T \in \mathcal{L}(V;W)$ is compact if $T$ maps the unit ball of $V$ into a relatively compact set in $W$ (see Definition A.17). Let us also recall (see Theorem A.21) that if there exists a sequence $(T_n)_{n\in\mathbb{N}}$ of operators in $\mathcal{L}(V;W)$ of finite rank s.t. $\lim_{n\to\infty} \|T - T_n\|_{\mathcal{L}(V;W)} = 0$, then $T$ is compact. Conversely, if $W$ is a Hilbert space and $T \in \mathcal{L}(V;W)$ is a compact operator, then there exists a sequence of operators in $\mathcal{L}(V;W)$ of finite rank, $(T_n)_{n\in\mathbb{N}}$, such that $\lim_{n\to\infty} \|T - T_n\|_{\mathcal{L}(V;W)} = 0$.

**Example 46.10 (Rellich–Kondrachov).** For every bounded Lipschitz domain $D$, the Rellich–Kondrachov theorem states that the injection $W^{s,p}(D) \hookrightarrow L^q(D)$ is compact for all $q \in [1, \frac{pd}{d-sp})$ if $sp \le d$ (see Theorem 2.35). □

**Example 46.11 (Hilbert–Schmidt operators).** Let $K \in L^2(D\times D;\mathbb{C})$, where $D$ is a bounded set in $\mathbb{R}^d$. Then the Hilbert–Schmidt operator $T : L^2(D;\mathbb{C}) \to L^2(D;\mathbb{C})$ s.t. $T(f)(x) := \int_D f(y)K(x,y)\,\mathrm{d}y$ a.e. in $D$ is compact (see Brezis [89, Thm. 6.12]). Note that $T^*(f)(x) := \int_D f(y)\overline{K(y,x)}\,\mathrm{d}y$. □

**Example 46.12 (Identity).** The identity $I_{\ell^p} : \ell^p \to \ell^p$, $p \in [1,\infty]$, is not compact. Indeed, consider the sequence $e_n := (\delta_{mn})_{m\in\mathbb{N}}$. For all $N \ge 0$ and $n, m \ge N$, $n \ne m$, we have $\|e_n - e_m\|_{\ell^p} = 2^{\frac{1}{p}}$ for all $p \in [1,\infty)$, and $\|e_n - e_m\|_{\ell^\infty} = 1$. Hence, one cannot extract any Cauchy subsequence in $\ell^p$ from $(e_n)_{n\in\mathbb{N}}$. □

Let us now state some important results on compact operators.

**Theorem 46.13 (Fredholm alternative).** *Let $T \in \mathcal{L}(L)$ be a compact operator. The following properties hold true for all $\mu \in \mathbb{C}\backslash\{0\}$:*

(i) $\mu I_L - T$ *is injective iff* $\mu I_L - T$ *is surjective.*
(ii) $\ker(\mu I_L - T)$ *is finite-dimensional.*
(iii) $\mathrm{im}(\mu I_L - T)$ *is closed, i.e.,* $\mathrm{im}(\mu I_L - T) = \ker(\overline{\mu} I_{L'} - T^*)^\perp$.
(iv) $\dim\ker(\mu I_L - T) = \dim\ker(\overline{\mu} I_{L'} - T^*)$.

*Proof.* See Brezis [89, Thm. 6.6]. □

The Fredholm alternative usually refers to Item (i), which implies that every nonzero member of the spectrum of $T$ is an eigenvalue when $T$ is compact. The key result for compact operators is the following theorem.

**Theorem 46.14 (Spectrum of compact operators).** *Let $T \in \mathcal{L}(L)$ be a compact operator with $\dim(L) = \infty$. The following holds true:*

(i) $0 \in \sigma(T)$.
(ii) $\sigma(T)\backslash\{0\} = \sigma_{\mathrm{p}}(T)\backslash\{0\}$.

(iii) *One of the following three cases holds:* (1) $\sigma(T) = \{0\}$; (2) $\sigma(T)\backslash\{0\}$ *is a finite set;* (3) $\sigma(T)\backslash\{0\}$ *is a sequence converging to* 0.
(iv) *Any* $\mu \in \sigma(T)\backslash\{0\}$ *has a finite ascent* $\alpha$*, and the space* $\ker(\mu I_L - T)^\alpha$ *is finite-dimensional, i.e.,* $\mu$ *has finite algebraic and geometric multiplicity.*
(v) $\mu \in \sigma(T)$ *iff* $\overline{\mu} \in \sigma(T^*)$*, i.e.,* $\sigma(T^*) = \mathrm{conj}(\sigma(T))$*. The ascent, algebraic and geometric multiplicities of* $\mu \in \sigma(T)\backslash\{0\}$ *and of* $\overline{\mu}$ *are equal.*

*Proof.* See Brezis [89, Thm. 6.8], Lax [278, p. 238], or Kreyszig [271, Thm. 8.3.1 & 8.4.4] for (i)-(iii) and [271, Thm. 8.4.3] for (iv)-(v). □

The first two items in Theorem 46.14 imply that either $T$ is not injective (i.e., $0 \in \sigma_{\mathrm{p}}(T)$) and then $\sigma(T) = \sigma_{\mathrm{p}}(T)$ (and $\sigma_{\mathrm{c}}(T) = \sigma_{\mathrm{r}}(T) = \emptyset$), or $T$ is injective (i.e., $0 \notin \sigma_{\mathrm{p}}(T)$) and then $\sigma(T) = \sigma_{\mathrm{p}}(T) \cup \{0\}$ (and $\sigma_{\mathrm{c}}(T) = \{0\}$, $\sigma_{\mathrm{r}}(T) = \emptyset$ or $\sigma_{\mathrm{r}}(T) = \{0\}$, $\sigma_{\mathrm{c}}(T) = \emptyset$).

### 46.1.3 Symmetric operators in Hilbert spaces

In this section, $L$ denotes a complex Hilbert space. The reader is invited to review §C.3 for basic facts about Hilbert spaces. Let $T \in \mathcal{L}(L)$. The (Hermitian) transpose of $T$, say $T^{\mathsf{H}} \in \mathcal{L}(L)$, is defined by setting

$$(T^{\mathsf{H}}(w), v)_L := (w, T(v))_L, \qquad \forall v, w \in L. \tag{46.7}$$

Let $(J_L^{\mathrm{RF}})^{-1} : L' \to L$ be the Riesz–Fréchet representation operator (see Theorem C.24), that is, $((J_L^{\mathrm{RF}})^{-1}(l'), l)_L := \langle l', l\rangle_{L',L}$ for all $l' \in L'$ and $l \in L$. We recall that $J_L^{\mathrm{RF}}$ and $(J_L^{\mathrm{RF}})^{-1}$ are linear operators because we have chosen dual spaces to be composed of antilinear forms (see Exercise 46.5 and Remark C.26).

**Lemma 46.15 (Transpose and adjoint).** *Let* $T \in \mathcal{L}(L)$*. We have* $T^{\mathsf{H}} = (J_L^{\mathrm{RF}})^{-1} \circ T^* \circ J_L^{\mathrm{RF}}$*, and*

$$\sigma_{\mathrm{p}}(T^*) = \sigma_{\mathrm{p}}(T^{\mathsf{H}}), \quad \sigma_{\mathrm{c}}(T^*) = \sigma_{\mathrm{c}}(T^{\mathsf{H}}), \quad \sigma_{\mathrm{r}}(T^*) = \sigma_{\mathrm{r}}(T^{\mathsf{H}}). \tag{46.8}$$

*Finally, if the duality paring is identified with the inner product of* $L$*, i.e., if* $L$ *and* $L'$ *are identified, we have* $T^{\mathsf{H}} = T^*$.

*Proof.* The identities $((J_L^{\mathrm{RF}})^{-1}T^*(l'), l)_L = \langle T^*(l'), l\rangle_{L',L} = \langle l', T(l)\rangle_{L',L} = ((J_L^{\mathrm{RF}})^{-1}(l'), T(l))_L$ show that $T^{\mathsf{H}} = (J_L^{\mathrm{RF}})^{-1} \circ T^* \circ J_L^{\mathrm{RF}}$. This proves the first assertion. To prove (46.8), we observe that for all $\mu \in \mathbb{C}$, we have $\mu I_{L'} - T^* = \mu I_{L'} - J_L^{\mathrm{RF}} \circ T^{\mathsf{H}} \circ (J_L^{\mathrm{RF}})^{-1} = J_L^{\mathrm{RF}} \circ (\mu I_L - T^{\mathsf{H}}) \circ (J_L^{\mathrm{RF}})^{-1}$. The assertion (46.8) on the spectrum follows readily. Finally, if $L$ and $L'$ are identified, $J_L^{\mathrm{RF}}$ becomes the identity operator so that $T^{\mathsf{H}} = T^*$. □

**Definition 46.16 (Symmetric operator).** *Let* $T \in \mathcal{L}(L)$*. We say that* $T$ *is (Hermitian) symmetric if* $T = T^{\mathsf{H}}$.

**Theorem 46.17 (Spectrum, spectral radius, ascent).** *Let $T \in \mathcal{L}(L)$ be a symmetric operator. The following holds true:* (i) $\sigma(T) \subset \mathbb{R}$, $\sigma_{\rm r}(T) = \emptyset$, *and*

$$\{a, b\} \subset \sigma(T) \subset [a, b], \tag{46.9}$$

*with* $a := \inf_{v\in L, \|v\|_L=1}(T(v), v)_L$ *and* $b := \sup_{v\in L, \|v\|_L=1}(T(v), v)_L$. (ii) $\|T\|_{\mathcal{L}(L)} = r(T) = \max(|a|, |b|)$. (iii) *The ascent of any* $\mu \in \sigma_{\rm p}(T)$ *is equal to 1, i.e., every generalized eigenvector is an eigenvector, and if $T$ is compact, the algebraic multiplicity and the geometric multiplicity of $\mu$ are equal.*

*Proof.* See Lax [278, p. 356], Kreyszig [271, §9.2], and Exercise 46.6 for a proof of (i). See Exercise 46.6(iii) for a proof of (iii). □

**Corollary 46.18 (Characterization of $\sigma(T)$).** *Let $T \in \mathcal{L}(L)$ be a symmetric operator. Then $\mu \in \sigma(T)$ iff there is a sequence $(v_n)_{n\in\mathbb{N}}$ in $L$ such that $\|v_n\|_L = 1$ for all $n \in \mathbb{N}$ and $\|T(v_n) - \mu v_n\|_L \to 0$ as $n \to \infty$.*

*Proof.* Identifying $L$ and $L'$, we apply Corollary C.50 which says that $(\mu I_L - T)$ is not bijective iff there exists a sequence $(v_n)_{n\in\mathbb{N}}$ in $L$ such that $\|v_n\|_L = 1$ and $\|\mu v_n - T(v_n)\|_L \le \frac{1}{n+1}$. □

For the reader's convenience, we now recall the notion of Hilbert basis in a separable Hilbert space (separability is defined in Definition C.8).

**Definition 46.19 (Hilbert basis).** *Let $L$ be a separable Hilbert space. A sequence $(e_n)_{n\in\mathbb{N}}$ in $L$ is said to be a Hilbert basis of $L$ if it satisfies the following two properties:*

(i) $(e_m, e_n)_L = \delta_{mn}$ *for all* $m, n \in \mathbb{N}$.
(ii) *The linear space composed of all the finite linear combinations of the vectors in $(e_n)_{n\in\mathbb{N}}$ is dense in $L$.*

Not every Hilbert space is separable, but all the Hilbert spaces encountered in this book are separable (or by default are always assumed to be separable).

**Lemma 46.20 (Pareseval).** *Let $L$ be a separable Hilbert space and let $(e_n)_{n\in\mathbb{N}}$ be a Hilbert basis of $L$. For all $u \in L$, set $u_n := \sum_{k\in\{0:n\}}(u, e_k)_L e_k$. The following holds true:*

$$\lim_{n\to\infty} \|u - u_n\|_L = 0 \quad \textit{and} \quad \|u\|_L^2 = \sum_{k\in\mathbb{N}} |(u, e_k)_L|^2. \tag{46.10}$$

*Conversely, let $(\alpha_n)_{n\in\mathbb{N}}$ be a sequence in $\ell^2(\mathbb{C})$ and set $u_{\alpha,n} := \sum_{k\in\{0:n\}} \alpha_k e_k$. Then the sequence $(u_{\alpha,n})_{n\in\mathbb{N}}$ converges to some $u_\alpha$ in $L$ such that $(u_\alpha, e_n)_V = \alpha_n$ for all $n \in \mathbb{N}$, and we have $\|u_\alpha\|_L^2 = \lim_{n\to\infty} \sum_{k\in\{0:n\}} \alpha_k^2$.*

*Proof.* See Brezis [89, Cor. 5.10]. □

**Theorem 46.21 (Symmetric compact operator).** *Let $L$ be a separable Hilbert space and let $T \in \mathcal{L}(L)$ be a symmetric compact operator. Then there exists a Hilbert basis of $L$ composed of eigenvectors of $T$.*

*Proof.* See [89, Thm. 6.11]. □

The above results mean that the eigenvectors of a symmetric compact operator $T$ form a sequence $(v_n)_{n\in\mathbb{N}}$ s.t. $(v_m, v_n)_L = \delta_{mn}$ for all $m, n \in \mathbb{N}$. Moreover, for all $u \in L$, letting $\alpha_n := (u, v_n)_L$ and $u_n := \sum_{k\in\{0:n\}} \alpha_k v_k$, the sequence $(u_n)_{n\in\mathbb{N}}$ converges to $u$ in $L$ and we have $\|u\|_L^2 = \sum_{k\in\mathbb{N}} |\alpha_k|^2$.

## 46.2 Introductory examples

We review in this section some typical examples that give rise to an eigenvalue problem, and we illustrate some of the concepts introduced in §46.1.

### 46.2.1 Example 1: Vibrating string

Consider a vibrating string of linear density $\rho$, length $\ell$, attached at $x = 0$ and $x = \ell$, and maintained under tension with the force $\tau$. Let us set $D := (0, \ell)$, $J := (0, T_{\max})$, $T_{\max} > 0$, and denote by $u : D\times J \to \mathbb{R}$ the displacement of the string in the direction orthogonal to the $x$-axis. Denoting by $u_0(x)$ and $u_1(x)$ the initial position and the initial velocity (i.e., the time derivative of the displacement), the displacement of the string can be modeled by the linear wave equation

$$
\begin{aligned}
&\partial_{tt} u(x,t) - c^2 \partial_{xx} u(x,t) = 0 && \text{in } D\times J, && (46.11a)\\
&u(0,t) = 0,\ u(\ell,t) = 0 && \text{in } J, && (46.11b)\\
&u(x,0) = u_0(x),\ \partial_t u(x,0) = u_1(x) && \text{in } D, && (46.11c)
\end{aligned}
$$

where the wave speed is $c := (\frac{\tau}{\rho})^{\frac{1}{2}}$. The method of the separation of variables gives the following representation of the solution:

$$
u(x,t) = \sum_{n\geq 1} (\alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t))\, \psi_n(x), \tag{46.12}
$$

with $\omega_n := c\lambda_n^{\frac{1}{2}}$, $\lambda_n := \frac{n^2\pi^2}{\ell^2}$, $\psi_n(x) := \sin(n\pi\frac{x}{\ell})$, $\alpha_n := \frac{2}{\ell}\int_0^\ell u_0(x)\psi_n(x)\,\mathrm{d}x$, and $\beta_n := \frac{2}{cn\pi}\int_0^\ell u_1(x)\psi_n(x)\,\mathrm{d}x$. A remarkable fact is that for all $n \geq 1$, $(\lambda_n, \psi_n)$ is an eigenpair for the Laplace eigenvalue problem

$$
-\partial_{xx}\psi_n(x) = \lambda_n \psi_n(x), \quad \psi_n(0) = 0,\ \psi_n(\ell) = 0. \tag{46.13}
$$

The eigenfunctions $\psi_n$ are called normal modes. In musical language, they are called harmonics of the string. Note that $\alpha_n = \int_0^\ell u_0(x)\psi_n(x)\,\mathrm{d}x / \int_0^\ell \psi_n^2(x)\,\mathrm{d}x$and $\omega_n\beta_n = \int_0^\ell u_1(x)\psi_n(x)\,\mathrm{d}x / \int_0^\ell \psi_n^2(x)\,\mathrm{d}x$.

We say that (46.13) is the *spectral problem* associated with the vibrating string. This problem can be reformulated in the following weak form:

$$\begin{cases} \text{Find } \psi \in H_0^1(D)\backslash\{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ \int_D \partial_x\psi\partial_x w\,\mathrm{d}x = \lambda \int_D \psi w\,\mathrm{d}x, \quad \forall w \in H_0^1(D). \end{cases} \tag{46.14}$$

Let $L := L^2(D)$ and let $T : L \to L$ be defined so that for all $f \in L$, $T(f) \in H_0^1(D)$ solves $\int_D \partial_x(T(f))\partial_x w\,\mathrm{d}x := \int_D fw\,\mathrm{d}x$ for all $w \in H_0^1(D)$. The operator $T$ is compact since the injection $H_0^1(D) \hookrightarrow L^2(D)$ is compact owing to the Rellich–Kondrachov theorem. This compactness property will be important for approximation purposes. Upon observing that $\int_D fT(g)\,\mathrm{d}x = \int_D \partial_x(T(f))\partial_x(T(g))\,\mathrm{d}x = \int_D T(f)g\,\mathrm{d}x$, we infer that $T$ is symmetric according to Definition 46.16. Owing to Theorem 46.17, all the eigenvalues of $T$ are real and $\sigma_{\mathrm{r}}(T) = 0$. According to Theorem 46.14, the eigenvalues of $T$ are well separated and form a sequence that goes to 0. Note that $T$ is injective, that is, 0 is not an eigenvalue. According to Theorem 46.14, this means that $\sigma_{\mathrm{c}}(T) = \{0\}$. Let $(\mu, \psi)$ be an eigenpair of $T$. Then $\mu \int_D \partial_x\psi\partial_x w\,\mathrm{d}x = \int_D \partial_x(T(\psi))\partial_x w\,\mathrm{d}x = \int_D \psi w\,\mathrm{d}x$. Hence, $(\mu^{-1}, \psi)$ solves (46.14). Conversely, one readily sees that if $(\lambda, \psi)$ solves (46.14), then $(\lambda^{-1}, \psi)$ is an eigenpair of $T$. Thus, we have established that $(\lambda, \psi)$ solves (46.14) iff $(\lambda^{-1}, \psi)$ is an eigenpair of $T$. Finally, Theorem 46.21 asserts that there exists a Hilbert basis of $L$ consisting of eigenvectors of $T$, and the basis in question is $((\frac{2}{\ell})^{\frac{1}{2}}\psi_n)_{n\geq 1}$.

## 46.2.2 Example 2: Vibrating drum

Consider a two-dimensional elastic homogeneous membrane occupying at rest the domain $D \subset \mathbb{R}^2$ and attached to a rigid frame on $\partial D$, as shown in Figure 46.1. We assume that $D$ is embedded in $\mathbb{R}^3$ and denote by $\boldsymbol{e}_z$ the

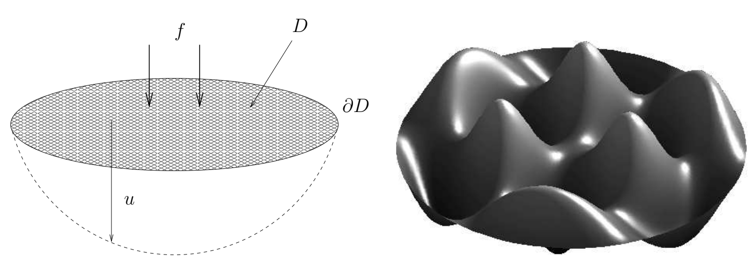


**Fig. 46.1** Vibrating membrane attached to a rigid frame. Left: reference configuration $D$, externally applied load $f$, and equilibrium displacement $u$. Right: one normal mode.

third direction. Assume that the membrane is of uniform thickness, has area density $\rho$, and that the tension tensor in the membrane, $\mathbb{t}$, is uniform and isotropic, i.e., it is of the form $\mathbb{t} = \tau \mathbb{I}_2$ for some positive real number $\tau$ (force per unit surface). Consider a time-dependent load $f(\boldsymbol{x}, t) := \rho g(\boldsymbol{x}) \cos(\omega t)$ with angular frequency $\omega$ for all $(\boldsymbol{x}, t) \in D \times J$ with $J := (0, T_{\max})$, $T_{\max} > 0$. Under the small strain assumption, the time-dependent displacement of the membrane in the $e_z$ direction, $u : D \times J \to \mathbb{R}$, is modeled by the two-dimensional wave equation

$$\begin{aligned} &\partial_{tt} u - c^2 \Delta u = g(\boldsymbol{x}) \cos(\omega t) && \text{in } D \times J, && (46.15\text{a}) \\ &u(\cdot, t)_{|\partial D} = 0 && \text{in } J, && (46.15\text{b}) \\ &u(\boldsymbol{x}, 0) = u_0(\boldsymbol{x}),\ \partial_t u(\boldsymbol{x}, 0) = u_1(\boldsymbol{x}) && \text{in } D, && (46.15\text{c}) \end{aligned}$$

where the wave speed is $c := (\frac{\tau}{\rho})^{\frac{1}{2}}$. As in §46.2.1, the solution to this problem can be expressed in terms of the normal modes (eigenmodes) of the membrane, $(\lambda_n, \psi_n)_{n \ge 1}$, which satisfy

$$-\Delta \psi_n = \lambda_n \psi_n \text{ in } D, \quad \psi_{n|\partial D} = 0. \tag{46.16}$$

Setting $\omega_n := c \lambda_n^{\frac{1}{2}}$, a straightforward calculation shows that if $\omega \notin \{\omega_n\}_{n \ge 1}$,

$$u(\boldsymbol{x}, t) = \sum_{n \ge 1} \left\{ \alpha_n \cos(\omega_n t) + \beta_n \sin(\omega_n t) + \frac{\gamma_n}{2} \frac{\sin\left(\frac{\omega - \omega_n}{2} t\right)}{\frac{\omega - \omega_n}{2}} \frac{\sin\left(\frac{\omega + \omega_n}{2} t\right)}{\frac{\omega + \omega_n}{2}} \right\} \psi_n(\boldsymbol{x}),$$

where $\alpha_n := (u_0, \psi_n)_{L^2(D)} / \|\psi_n\|^2_{L^2(D)}$, $\omega_n \beta_n := (u_1, \psi_n)_{L^2(D)} / \|\psi_n\|^2_{L^2(D)}$, $\gamma_n := (g, \psi_n)_{L^2(D)} / \|\psi_n\|^2_{L^2(D)}$. As the forcing angular frequency $\omega$ gets close to one of the $\omega_n$'s, a resonance phenomenon occurs. When $\omega = \omega_n$, $|u(\boldsymbol{x}, t)|$ grows linearly in time like $t|\sin(\omega_n t)|$.

The *spectral problem* associated with the vibrating drum can be rewritten in weak form as follows:

$$\begin{cases} \text{Find } \psi \in H^1_0(D) \backslash \{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ \int_D \nabla \psi \cdot \nabla w \, \mathrm{d}x = \lambda \int_D \psi w \, \mathrm{d}x, \quad \forall w \in H^1_0(D). \end{cases} \tag{46.17}$$

If the tension tensor $\mathbb{t}$ in the membrane is not uniform and/or not isotropic (think of a membrane made of composite materials), and if the area density $\rho$ is not uniform, the above spectral problem takes the following form:

$$\begin{cases} \text{Find } \psi \in H^1_0(D) \backslash \{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ \int_D (\mathbb{t} \nabla \psi) \cdot \nabla w \, \mathrm{d}x = \lambda \int_D \rho \psi w \, \mathrm{d}x, \quad \forall w \in H^1_0(D). \end{cases} \tag{46.18}$$

By proceeding as in §46.2.1 and under reasonable assumptions on $\mathbb{t}$ and $\rho$, one can show that the solution operator associated with (46.18) is symmetric and compact from $L^2(D)$ to $L^2(D)$. Hence, the eigenvalues associated with the eigenvalue problem (46.18) are countable, isolated, and grow to infinity.

### 46.2.3 Example 3: Stability analysis of PDEs

It is common that one has to study the stability of physical systems modeled by PDEs. For instance, the following nonlinear reaction-diffusion equation (sometimes referred to as the Kolmogorov–Petrovsky–Piskounov equation):

$$\partial_t u - \Delta g(u) - f(u) = 0 \quad \text{in } D\times J, \tag{46.19}$$

models the spreading of biological populations when $f(u) := u(1-u)$, the Rayleigh–Benard convection when $f(u) := u(1-u^2)$, and combustion processes when $f(u) := u(1-u)(u-\alpha)$ with $\alpha \in (0,1)$. We assume here that $D := (0,1)^d$, periodic boundary conditions are enforced, $f$ and $g$ are smooth, and $g'$ is bounded from below by some positive constant. Assuming that this problem admits a particular time-independent solution (a standing wave), $u_{\mathrm{sw}}$, the natural question that follows is to determine whether this solution is stable under infinitesimal perturbations. Writing $u(\boldsymbol{x},t) := u_{\mathrm{sw}}(\boldsymbol{x}) + \psi(\boldsymbol{x})e^{-\lambda t}$, $\lambda \in \mathbb{C}$, where $\psi$ is assumed to be small compared to $u_{\mathrm{sw}}$, one obtains the following linearized form of the PDE:

$$-\lambda\psi - \Delta(g'(u_{\mathrm{sw}})\psi) - f'(u_{\mathrm{sw}})\psi = 0 \quad \text{in } D\times J. \tag{46.20}$$

Since $\nabla(g'(u_{\mathrm{sw}})\psi) = g'(u_{\mathrm{sw}})\nabla\psi + \psi g''(u_{\mathrm{sw}})\nabla u_{\mathrm{sw}}$, this problem leads to the following eigenvalue problem:

$$\begin{cases} \text{Find } \psi \in H^1_{\mathrm{per}}(D)\backslash\{0\} \text{ and } \lambda \in \mathbb{C} \text{ such that } \forall w \in H^1_{\mathrm{per}}(D), \\ \int_D \big((g'(u_{\mathrm{sw}})\nabla\psi + \psi\, g''(u_{\mathrm{sw}})\nabla u_{\mathrm{sw}})\cdot\nabla\overline{w} - f'(u_{\mathrm{sw}})\psi\overline{w}\big)\mathrm{d}x = \lambda \int_D \psi\overline{w}\,\mathrm{d}x, \end{cases} \tag{46.21}$$

where $H^1_{\mathrm{per}}(D)$ is composed of the functions in $H^1(D)$ that are periodic over $D$. The particular solution $u_{\mathrm{sw}}$ is said to be linearly stable if all the eigenvalues have a positive real part. Here again, it is the solution operator $T : L^2(D) \to L^2(D)$ that is of interest, where for all $s \in L^2(D)$, $T(s) \in H^1_{\mathrm{per}}(D) \subset L^2(D)$ solves $\int_D \big((g'(u_{\mathrm{sw}})\nabla T(s) + T(s)\, g''(u_{\mathrm{sw}})\nabla u_{\mathrm{sw}})\cdot\nabla\overline{w} - T(s)f'(u_{\mathrm{sw}})\overline{w}\big)\,\mathrm{d}x = \int_D s\overline{w}\,\mathrm{d}x$ for all $w \in H^1_{\mathrm{per}}(D)$. Under reasonable assumptions on $f, g, u_{\mathrm{sw}}$, the operator $T$ can be shown to be compact.

### 46.2.4 Example 4: Schrödinger equation and hydrogen atom

The vibrating string and the drum are typical examples where compactness directly results from the boundedness of the domain $D$. We now give an example where compactness results from an additional potential in the PDE.

An important example of eigenvalue problem in physics is the Schrödinger equation. For instance, the normalized Schrödinger equation takes the following form for the one-dimensional quantum harmonic oscillator over $\mathbb{R}$:

$$-\frac{1}{2}\psi'' + \frac{1}{2}x^2\psi = E\psi \quad \text{in } \mathbb{R}. \tag{46.22}$$

The function $\psi$ is the wave function of the oscillator, and the quantity $\psi^2$ is its probability distribution function. The eigenvalue $E$ is called energy. This problem has a countable (quantified) set of eigenpairs

$$\psi_n(x) := \frac{1}{(2^n n!)^{\frac{1}{2}}\pi^{\frac{1}{4}}} e^{-\frac{x^2}{2}} H_n(x), \qquad E_n := n + \frac{1}{2}, \tag{46.23}$$

where $H_n(x) := (-1)^n e^{x^2} \frac{\mathrm{d}^n}{\mathrm{d}x^n} e^{-x^2}$ is the Hermite polynomial of order $n$. A natural functional space for this problem is

$$B^1(\mathbb{R}) := \{v \in H^1(\mathbb{R}) \mid xv \in L^2(\mathbb{R})\}. \tag{46.24}$$

In addition to being in $H^1(\mathbb{R})$, functions in $B^1(\mathbb{R})$ satisfy $\int_{\mathbb{R}} x^2 v^2(x)\,\mathrm{d}x < \infty$. It is shown in Exercise 46.8 that the embedding $B^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact, whereas it is shown in Exercise 46.7 that the embedding $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact. Hence, the sesquilinear form $a(v,w) = \int_{\mathbb{R}}(v'\overline{w}' + x^2 v\overline{w})\,\mathrm{d}x$ is bounded and coercive on $B^1(\mathbb{R})$, and the operator $T : B^1(\mathbb{R}) \to B^1(\mathbb{R})$ s.t. $a(T(u),w) = \int_{\mathbb{R}} u\overline{w}\,\mathrm{d}x$ for all $w \in B^1(\mathbb{R})$, is symmetric and compact.

The hydrogen atom is a model for which the Schrödinger equation has the following simple form:

$$-\frac{\hbar^2}{2m_{\mathrm{e}}}\Delta\psi - \frac{q^2}{4\pi\epsilon_0 r}\psi = E\psi \quad \text{in } \mathbb{R}^3. \tag{46.25}$$

Here, $\hbar$ is the Planck constant, $m_{\mathrm{e}}$ the mass of the electron, $\epsilon_0$ the permittivity of free space, $q$ the electron charge, and $r := \|\boldsymbol{x}\|_{\ell^2}$ the Euclidean distance of the electron to the nucleus. This problem is far more difficult than the one-dimensional quantum harmonic oscillator because the Coulomb potential $-\frac{q^2}{4\pi\epsilon_0 r}$ is negative and vanishes at infinity. The sign problem can be handled as for the Helmholtz problem (see Chapter 35) by invoking Gårding's inequality after making use of Hardy's inequality $|u|^2_{H^1(\mathbb{R}^d)} \geq \frac{(d-2)^2}{4}\int_{\mathbb{R}^d}\frac{u^2}{r^2}\,\mathrm{d}x$ for all $u \in H^1(\mathbb{R}^d)$. The spectrum of the solution operator is composed of the point spectrum and the continuous spectrum. The residual spectrum is empty because the solution operator is symmetric. There is a countable (quantified) set of eigenpairs. Using spherical coordinates, they are given for all $n \geq 1$ by

$$\psi_{n,l,m}(r,\theta,\phi) := C_{n,l} a_0^{-\frac{3}{2}} e^{-\frac{\rho}{2}} \rho^l L^{2l+1}_{n-l-1}(\rho) Y^m_l(\theta,\phi),$$
$$E_n := -\frac{\hbar^2}{2m_{\mathrm{e}}a_0^2}\frac{1}{n^2},$$

where $l \in \{0{:}n{-}1\}$, $m \in \{-l{:}l\}$, $C_{n,l} := \left(\frac{2}{n}\right)^{\frac{3}{2}}\left(\frac{(n-l-1)!}{2n((n+l)!)^3}\right)^{\frac{1}{2}}$, $a_0 := \frac{4\pi\epsilon_0\hbar^2}{m_{\mathrm{e}}q^2}$ is the Bohr radius, $\rho := \frac{2r}{na_0}$, $L^\gamma_\beta(r) := \frac{r^{-\gamma}e^r}{\beta!}\frac{\mathrm{d}^\beta}{\mathrm{d}r^\beta}(e^{-r}r^{\gamma+\beta})$ is the generalized Laguerre polynomial of degree $\beta$, and $Y^m_l$ is the spherical harmonic function of degree $l$ and order $m$.

# Exercises

**Exercise 46.1 (Spectrum).** Let $L$ be a complex Banach space. Let $T \in \mathcal{L}(L)$. (i) Show that $(\lambda T)^* = \overline{\lambda} T^*$ for all $\lambda \in \mathbb{C}$. (ii) Show that $\sigma_{\rm r}(T) \subset \mathrm{conj}(\sigma_{\rm p}(T^*)) \subset \sigma_{\rm r}(T) \cup \sigma_{\rm p}(T)$. (*Hint*: use Corollary C.15.) (iii) Show that the spectral radius of $T$ verifies $r(T) \le \limsup_{n\to\infty} \|T^n\|_{\mathcal{L}(L)}^{\frac{1}{n}}$. (*Hint*: consider $\sum_{n\in\mathbb{N}} (\mu^{-1}T)^n$ and use the root test: the complex-valued series $\sum_{n\in\mathbb{N}} a_n$ converges absolutely if $\limsup_{n\to\infty} |a_n|^{\frac{1}{n}} < 1$.)

**Exercise 46.2 (Ascent, algebraic and geometric multiplicities).** (i) Let $T \in \mathcal{L}(L)$. Let $\mu$ be an eigenvalue of $T$ and let $K_i := \ker(\mu I_L - T)^i$ for all $i \in \mathbb{N}\backslash\{0\}$. Prove that $K_1 \subset K_2 \ldots$, and assuming that there is $j \ge 1$ s.t. $K_j = K_{j+1}$, show that $K_j = K_{j'}$ for all $j' > j$. (ii) Assume that $\mu$ has a finite ascent $\alpha$, and finite algebraic multiplicity $m$ and geometric multiplicity $g$. Show that $\alpha + g - 1 \le m \le \alpha g$. (*Hint*: letting $g_i := \dim(K_i)$ for all $i \in \{1{:}\alpha\}$, prove that $g_1 + i - 1 \le g_i$ and $g_i \le g_{i-1} + g_1$.) (iii) Compute the ascent, algebraic multiplicity, and geometric multiplicity of the eigenvalues of following matrices and verify the two inequalities from Step (i):

$$\begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

**Exercise 46.3 (Eigenspaces).** The following three questions are independent. (i) Suppose $V = V_1 \oplus V_2$ and consider $T \in \mathcal{L}(V)$ defined by $T(v_1 + v_2) := v_1$ for all $v_1 \in V_1$ and all $v_2 \in V_2$. Find all the eigenvalues and eigenspaces of $T$. (ii) Let $T \in \mathcal{L}(V)$. Assume that $S$ is invertible. Prove that $S^{-1}TS$ and $T$ have the same eigenvalues. What is the relationship between the eigenvectors of $T$ and those of $S^{-1}TS$? (iii) Let $V$ be a finite-dimensional vector space. Let $\{v_n\}_{n\in\{1:m\}} \subset V$, $m \ge 1$. Show that the vectors $\{v_n\}_{n\in\{1:m\}}$ are linearly independent iff there exists $T \in \mathcal{L}(V)$ such that $\{v_n\}_{n\in\{1:m\}}$ are eigenvectors of $T$ corresponding to distinct eigenvalues.

**Exercise 46.4 (Volterra operator).** Let $L := L^2((0,1);\mathbb{C})$ and let $T : L \to L$ be s.t. $T(f)(x) := \int_0^x f(t)\,\mathrm{d}t$ for a.e. $x \in (0,1)$. Notice that $T$ is a Hilbert–Schmidt operator, but this exercise is meant to be done without using this fact. (i) Show that $T^{\mathsf{H}}(g) = \int_x^1 g(t)\,\mathrm{d}t$ for all $g \in L^2((0,1);\mathbb{C})$. (ii) Show that $T$ is injective. (*Hint*: use Theorem 1.32.) (iii) Show that $0 \in \sigma_{\rm c}(T)$. (iv) Show that $\sigma_{\rm p}(T) = \emptyset$. (v) Prove that $\mu I_L - T$ is bijective if $\mu \neq 0$. (vi) Determine $\rho(T)$, $\sigma_{\rm p}(T)$, $\sigma_{\rm c}(T)$, $\sigma_{\rm r}(T)$. Do the same for $T^{\mathsf{H}}$.

**Exercise 46.5 (Riesz–Fréchet).** Let $H$ be a finite-dimensional complex Hilbert space with orthonormal basis $\{e_i\}_{i\in\{1:n\}}$ and inner product $(\cdot,\cdot)_H$. (i) Let $g$ be an antilinear form on $H$, i.e., $g \in H'$. Show that $(J_H^{\rm RF})^{-1}(g) = \sum_{i\in\{1:n\}} g(e_i) e_i$ with $g(e_i) := \langle g, e_i\rangle_{H',H}$, $\forall i \in \{1{:}n\}$. Is $(J_H^{\rm RF})^{-1} : H' \to H$

linear or antilinear? (ii) Let $g$ be a linear form on $H$. Show that $x_g := \sum_{i\in\{1:n\}} \overline{g(e_i)}e_i$ is s.t. $\langle g, y\rangle_{H',H} = \overline{(x_g, y)_H}$. Is the map $H' \ni g \mapsto x_g \in H$ linear or antilinear?

**Exercise 46.6 (Symmetric operator).** Let $L$ be a complex Hilbert space and $T \in \mathcal{L}(L)$ be a symmetric operator. (i) Show that $\sigma(T) \subset \mathbb{R}$. (*Hint*: compute $\Im((T(v) - \mu v, v)_L$ and show that $|\Im(\mu)|\|v\|_L^2 \leq |(T(v) - \mu v, v)_L|$ for all $v \in L$.) (ii) Prove that $\sigma_{\mathrm{r}}(T) = \emptyset$. (*Hint*: apply Corollary C.15.) (iii) Show that the ascent of each $\mu \in \sigma_{\mathrm{p}}(T)$ is equal to 1. (*Hint*: compute $\|(\mu I_L - T)(x)\|_L^2$ with $x \in \ker(\mu I_L - T)^2$.)

**Exercise 46.7 ($H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact).** (i) Let $\chi(x) := 1 + x$ if $-1 \leq x \leq 0$, $\chi(x) := 1 - x$ if $0 \leq x \leq 1$ and $\chi(x) := 0$ if $|x| \geq 1$. Show that $\chi \in H^1(\mathbb{R})$. (ii) Let $v_n(x) := \chi(x - n)$ for all $n \in \mathbb{N}$. Show that $(v_n)_{n\in\mathbb{N}}$ converges weakly to 0 in $L^2(\mathbb{R})$ (see Definition C.28). (iii) Show that the embedding $H^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is not compact. (*Hint*: argue by contradiction using Theorem C.23.)

**Exercise 46.8 ($B^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact).** (i) Show that the embedding $B^1(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact, where $B^1(\mathbb{R}) := \{v \in H^1(\mathbb{R}) \mid xv \in L^2(\mathbb{R})\}$. (*Hint*: let $(u_n)_{n\in\mathbb{N}}$ be a bounded sequence in $B^1(\mathbb{R})$, build nested subsets $J_k \subset \mathbb{N}$, $\forall k \in \mathbb{N}\backslash\{0\}$, s.t. the sequence $(u_{n|(-k,k)})_{n\in J_k}$ converges in $L^2(-k, k)$.) (ii) Give a sufficient condition on $\alpha \in \mathbb{R}$ so that $B^1_\alpha(\mathbb{R}) \hookrightarrow L^2(\mathbb{R})$ is compact, where $B^1_\alpha(\mathbb{R}) := \{v \in H^1(\mathbb{R}) \mid |x|^\alpha v \in L^2(\mathbb{R})\}$.

**Exercise 46.9 (Hausdorff–Toeplitz theorem).** The goal of this exercise is to prove that the numerical range of a bounded linear operator in a Hilbert space is convex; see also Gustafson [231]. Let $L$ be a complex Hilbert space and let $S_L(1) := \{x \in L \mid \|x\|_L = 1\}$ be the unit sphere in $L$. Let $T \in \mathcal{L}(L)$ and let $W(T) := \{\alpha \in \mathbb{C} \mid \exists x \in S_L(1),\ \alpha = (T(x), x)_L\}$ be the numerical range of $T$. Let $\gamma, \mu \in W(T)$, $\gamma \neq \mu$, and $x_1, x_2 \in S_L(1)$ be s.t. $(T(x_1), x_1)_L = \gamma$, $(T(x_2), x_2)_L = \mu$. Let $T' := \frac{1}{\mu-\gamma}(T - \gamma I_L)$. (i) Compute $(T'(x_1), x_1)_L$ and $(T'(x_2), x_2)_L$. (ii) Prove that there exists $\theta \in [0, 2\pi)$ s.t. $\Im(e^{\mathrm{i}\theta}(T'(x_1), x_2)_L + e^{-\mathrm{i}\theta}(T'(x_2), x_1)_L) = 0$. (iii) Let $x'_1 := e^{\mathrm{i}\theta}x_1$. Compute $(T'(x'_1), x'_1)_L$. (iv) Let $\lambda \in [0, 1]$. Show that the following problem has at least one solution: Find $\alpha, \beta \in \mathbb{R}$ s.t. $\|\alpha x'_1 + \beta x_2\|_L = 1$ and $(T'(\alpha x'_1 + \beta x_2), \alpha x'_1 + \beta x_2)_L = \lambda$. (*Hint*: view the two equations as those of an ellipse and an hyperbola, respectively, and determine how these curves cross the axes.) (v) Prove that $W(T)$ is convex. (*Hint*: compute $(T(\alpha x'_1 + \beta x_2), \alpha x'_1 + \beta x_2)_L$.)