Part X, Chapter 47

Symmetric operators, conforming approximation

The objective of this chapter is to study the approximation of eigenvalue problems associated with symmetric coercive differential operators using H^1-conforming finite elements. The goal is to derive error estimates on the eigenvalues and the eigenfunctions. The analysis is adapted from Raviart and Thomas [331] and uses relatively simple geometric arguments. The approximation of nonsymmetric eigenvalue problems using nonconforming techniques is studied in Chapter 48 using slightly more involved arguments.

47.1 Symmetric and coercive eigenvalue problems

In this section, we reformulate the eigenvalue problems introduced in §46.2 in a unified setting. This abstract setting will be used in §47.2 to analyze the approximation of these problems using H^1-conforming finite elements. We restrict ourselves to the real-valued setting since we are going to focus on symmetric operators.

47.1.1 Setting

Let D be a Lipschitz domain in $\mathbb{R}^d$. Let $L^2(D)$ be the real Hilbert space equipped with the inner product $(v,w)_{L^2(D)} := \int_D vw\,\mathrm{d}x$. Let V be a closed subspace of $H^1(D)$ which, depending on the boundary conditions that are enforced, satisfies $H_0^1(D) \subseteq V \subseteq H^1(D)$. We assume that V is equipped with a norm that is equivalent to that of $H^1(D)$. We also assume that the V-norm is rescaled so that the operator norm of the embedding $V \hookrightarrow L^2(D)$ is at most one, e.g., one could set $\|v\|_V := C_{\mathrm{PS}}^{-1}\ell_D\|\nabla v\|_{\boldsymbol{L}^2(D)}$ if $V := H_0^1(D)$, where C_{PS} is the constant from the Poincaré–Steklov inequality (31.12) in $H_0^1(D)$ and ℓ_D is a characteristic length associated with D, e.g., $\ell_D := \mathrm{diam}(D)$.

Let $a : V\times V \to \mathbb{R}$ be a symmetric bilinear form, i.e., $a(v,w) = a(w,v)$, satisfying the following coercivity and boundedness properties:

$$\alpha \|v\|_V^2 \leq a(v,v), \qquad |a(v,w)| \leq \|a\| \, \|v\|_V \|w\|_V, \tag{47.1}$$

for all $v, w \in V$, with $0 < \alpha \leq \|a\| < \infty$. For instance, we have $a(v,w) := \int_D (\mathbb{t}\nabla v)\cdot\nabla w \,\mathrm{d}x$ and $V := H_0^1(D)$ in (46.18), so that we can take $\alpha := \tau_\flat \ell_D^{-2}$ and $\|a\| := \tau_\sharp \ell_D^{-2}$, where $\tau_\flat$ and $\tau_\sharp$ are the smallest and the largest eigenvalues of $\mathbb{t}$ in D.

Our goal is to investigate the H^1-conforming approximation of the following eigenvalue problem:

$$\begin{cases} \text{Find } \psi \in V\backslash\{0\} \text{ and } \lambda \in \mathbb{R} \text{ such that} \\ a(\psi, w) = \lambda(\psi, w)_{L^2(D)}, \quad \forall w \in V. \end{cases} \tag{47.2}$$

Let $T : L^2(D) \to L^2(D)$ be the solution operator such that for all $u \in L^2(D)$,

$$a(T(u), w) := (u, w)_{L^2(D)}, \qquad \forall w \in V. \tag{47.3}$$

By proceeding as in §46.2.1, we conclude that T is symmetric and compact. We are then in the setting of Theorem 46.14 and Theorem 46.21.

Theorem 47.1 (Hilbert basis). *Under the above assumptions on the bilinear form a, the following properties hold true:*

- (i) $(\lambda, \psi) \in (0,\infty)\times V$ *is an eigenpair for the eigenvalue problem* (47.2) *iff* $(\lambda^{-1}, \psi) \in (0,\infty)\times V$ *is an eigenpair for T.*
- (ii) $\sigma_{\mathrm{p}}(T) \subset (0, \frac{1}{\alpha}]$.
- (iii) *The eigenvalue problem* (47.2) *has a countable sequence of isolated real positive eigenvalues that grows to infinity.*
- (iv) *It is possible to construct a Hilbert basis $(\psi_n)_{n\geq 1}$ of $L^2(D)$, where $(\lambda_n, \psi_n)_{n\geq 1}$ are the eigenpairs solving* (47.2) *(see Definition* 46.19*). (It is customary to enumerate the eigenpairs starting with $n \geq 1$.)*
- (v) $(\lambda_n^{-\frac{1}{2}}\psi_n)_{n\geq 1}$ *is a Hilbert basis of V equipped with the inner product $a(\cdot,\cdot)$.*

Proof. (i) Let (μ, ψ) be an eigenpair of T. Then $\|\psi\|_{L^2(D)}^2 = a(T(\psi), \psi) = \mu a(\psi, \psi)$, which implies that $\mu > 0$. This proves that $\sigma_{\mathrm{p}}(T) = \sigma(T)\backslash\{0\}$ and $\sigma_{\mathrm{p}}(T) \subset (0,\infty)$ (see Theorem 46.14(ii) and recall that $\dim(L^2(D)) = \infty$). Let (μ, ψ) be an eigenpair for T. Then $a(T(\psi), w) = \mu a(\psi, w) = (\psi, w)_{L^2(D)}$ for all $w \in V$. Since $\mu \neq 0$, we conclude that $a(\psi, w) = \mu^{-1}(\psi, w)_{L^2}$ for all $w \in V$, that is, (μ^{-1}, ψ) solves (47.2). The converse is also true: if (λ, ψ) is an eigenpair for (47.2), then the coercivity of a implies that $\lambda \neq 0$, and reasoning as above shows that (λ^{-1}, ψ) is an eigenpair of T.
(ii) Let (μ, ψ) be an eigenpair of T. The coercivity of a implies that $\|\psi\|_{L^2(D)}^2 = a(T(\psi), \psi) = \mu a(\psi, \psi) \geq \mu\alpha\|\psi\|_V^2 \geq \mu\alpha\|\psi\|_{L^2(D)}^2$, where the last bound follows from our assuming that the norm of the embedding $V \hookrightarrow L^2(D)$ is at most one. Hence, $\mu \in (0, \frac{1}{\alpha}]$.
(iii) The number of eigenvalues of T cannot be finite since the eigenspaces are finite-dimensional (see Theorem 46.13(ii)) and there exists a Hilbert basis of

$L^2(D)$ composed of eigenvectors of T (see Theorem 46.21). We are then in the third case described in Theorem 46.14(iii): the eigenvalues of T form a (countable) sequence that converges to zero. Hence, the eigenvalues of (47.2) grow to infinity.
(iv) This is a consequence of Theorem 46.21 and Item (iii) proved above.
(v) Let ψ_m, ψ_n be two members of the Hilbert basis $(\psi_k)_{k\geq 1}$ of $L^2(D)$. Recalling that (λ_m, ψ_m) and (λ_n, ψ_n) are eigenpairs of (47.2), we infer that

$$a(\lambda_m^{-\frac{1}{2}}\psi_m, \lambda_n^{-\frac{1}{2}}\psi_n) = \lambda_m^{\frac{1}{2}}\lambda_n^{-\frac{1}{2}}(\psi_m, \psi_n)_{L^2(D)} = \delta_{mn}.$$

Let W be the vector space composed of all the finite linear combinations of vectors in $\{\psi_n\}_{n\geq 1}$. We have to prove that W is dense in V. Let $f \in V'$ and assume that f annihilates W. Denoting by $(J_V^{\text{RF}})^{-1}(f)$ the Riesz–Fréchet representative of f in V equipped with the inner product $a(\cdot,\cdot)$, we have

$$\begin{aligned} 0 &= \langle f, \lambda_n^{-\frac{1}{2}}\psi_n\rangle_{V',V} = a((J_V^{\text{RF}})^{-1}(f), \lambda_n^{-\frac{1}{2}}\psi_n) = a(\lambda_n^{-\frac{1}{2}}\psi_n, (J_V^{\text{RF}})^{-1}(f)) \\ &= \lambda_n^{\frac{1}{2}}(\psi_n, (J_V^{\text{RF}})^{-1}(f))_{L^2(D)}, \end{aligned}$$

for all $n \geq 1$, where we used the symmetry of a. The above identity implies that $(J_V^{\text{RF}})^{-1}(f) = 0$ since W is dense in $L^2(D)$. Hence, $f = 0$. Corollary C.15 then implies that W is dense in V as claimed. □

The eigenvalues are henceforth counted with their multiplicity and ordered as follows: $\lambda_1 \leq \lambda_2 \leq \ldots$. Moreover, the associated eigenfunctions $\psi_1, \psi_2, \ldots$ are chosen and normalized as in Theorem 47.1(iv) so that $\|\psi_n\|_{L^2(D)} = 1$. The coercivity property of a implies that the eigenvalues are all positive and larger than or equal to α. Notice that since T is symmetric, the notions of algebraic and geometric multiplicity coincide, and for every eigenvalue $\lambda^{-1} \in \sigma_{\text{p}}(T)$, the multiplicity of λ is equal to $\dim(\lambda^{-1}I_{L^2(D)} - T)$.

47.1.2 Rayleigh quotient

We introduce in this section the notion of Rayleigh quotient which will be instrumental in the analysis of the H^1-conforming approximation technique presented in §47.2.

Definition 47.2 (Rayleigh quotient). *The Rayleigh quotient of a function $v \in V\backslash\{0\}$, relative to the bilinear form a, is defined as*

$$R(v) := \frac{a(v,v)}{\|v\|^2_{L^2(D)}}. \tag{47.4}$$

In this chapter, all the expressions involving $R(v)$ are understood with $v \neq 0$. For any functional $\mathcal{J} : V \to \mathbb{R}$, we write $\min_{v\in V} \mathcal{J}(v)$ instead of $\inf_{v\in V} \mathcal{J}(v)$ to indicate that the infimum is attained, i.e., if there exists a minimizer $v_* \in V$ such that $\mathcal{J}(v_*) = \inf_{v\in V} \mathcal{J}(v)$.

Proposition 47.3 (First eigenvalue). *Let λ_1 be the smallest eigenvalue of the problem* (47.2) *and let ψ_1 be a corresponding eigenfunction. Then we have*

$$\alpha \le \lambda_1 = R(\psi_1) = \min_{v\in V} R(v). \tag{47.5}$$

Proof. We have $\lambda_1 = R(\psi_1) \ge \inf_{v\in V} R(v) \ge \alpha$, where the first equality results from $a(\psi_1,\psi_1) = \lambda_1\|\psi_1\|^2_{L^2(D)}$ and the second from Theorem 47.1(ii). It remains to prove that $\inf_{v\in V} R(v) \ge \lambda_1$ (this also proves that the infimum of R over V is attained at ψ_1 since $\lambda_1 = R(\psi_1)$). Let $v \in V\backslash\{0\}$. Since $(\psi_n)_{n\ge1}$ is a Hilbert basis of $L^2(D)$ (see Theorem 47.1(iv)), the series $(\sum_{k\in\{1:n\}} \mathsf{W}_k\psi_k)_{n\ge1}$, with $\mathsf{W}_k := (v,\psi_k)_{L^2(D)}$, converges to v in $L^2(D)$ and we have $\|v\|^2_{L^2(D)} = \sum_{n\ge1}\mathsf{W}_n^2$. Furthermore, since $(\lambda_n^{-\frac12}\psi_n)_{n\ge1}$ is a Hilbert basis of V equipped with the inner product $a(\cdot,\cdot)$ (see Theorem 47.1(v)), the series $(\sum_{k\in\{1:n\}} \mathsf{V}_k\lambda_k^{-\frac12}\psi_k)_{n\ge1}$, with $\mathsf{V}_k := a(v,\lambda_k^{-\frac12}\psi_k)$, converges to v in V, and we have $a(v,v) = \sum_{n\ge1}\mathsf{V}_n^2$. But we also have $\mathsf{V}_n = a(v,\lambda_n^{-\frac12}\psi_n) = \lambda_n^{\frac12}(v,\psi_n)_{L^2(D)} = \lambda_n^{\frac12}\mathsf{W}_n$. Since $\lambda_1 \le \lambda_n$ for all $n\ge1$, we conclude that

$$R(v) = \frac{\sum_{n\ge1}\mathsf{V}_n^2}{\sum_{n\ge1}\mathsf{W}_n^2} = \frac{\sum_{n\ge1}\lambda_n\mathsf{W}_n^2}{\sum_{n\ge1}\mathsf{W}_n^2} \ge \lambda_1. \qquad \square$$

Proposition 47.4 (Min-max principle). *Let V_m denote the set of the subspaces of V having dimension m. For all $m\ge1$, we have*

$$\lambda_m = \min_{E_m\in V_m}\max_{v\in E_m} R(v) = \max_{E_{m-1}\in V_{m-1}}\min_{v\in E_{m-1}^\perp} R(v), \tag{47.6}$$

where for all $m>1$, $E_{m-1}^\perp$ denotes the orthogonal of E_{m-1} in $L^2(D)$ w.r.t. the L^2-inner product and $E_0 := \{0\}$ by convention.

Proof. Let $W_m := \text{span}\{\psi_1,\ldots,\psi_m\}$. Using the notation $\mathsf{W}_k := (v,\psi_k)_{L^2(D)}$, a direct computation shows that

$$\min_{E_m\in V_m}\max_{v\in E_m} R(v) \le \max_{v\in W_m} R(v) = \max_{v\in W_m}\frac{\sum_{n\in\{1:m\}}\lambda_n\mathsf{W}_n^2}{\sum_{n\in\{1:m\}}\mathsf{W}_n^2} = \lambda_m.$$

Consider now any $E_m\in V_m$. A dimensional argument shows that there exists $w\ne0$ in $E_m\cap W_{m-1}^\perp$ (apply the rank nullity theorem to the L^2-orthogonal projection from E_m onto W_{m-1}). Since w can be written in the form $w = \sum_{n\ge m}\mathsf{W}_n\psi_n = \sum_{n\ge m}\lambda_n^{\frac12}\mathsf{W}_n\lambda_n^{-\frac12}\psi_n$, one shows by proceeding as in the proof of Proposition 47.3 that $R(w)\ge\lambda_m$. As a result, $\max_{v\in E_m} R(v)\ge\lambda_m$. Hence, $\min_{E_m\in V_m}\max_{v\in E_m} R(v)\ge\lambda_m$. This concludes the proof of the first equality in (47.6). See Exercise 47.4 for the proof of the second equality. $\square$

Remark 47.5 (Poincaré–Steklov constant). The best Poincaré–Steklov constant in $H_0^1(D)$ is $C_{\rm PS} := \inf_{v\in H_0^1(D)\setminus\{0\}} \frac{\ell_D\|\nabla v\|_{\boldsymbol{L}^2(D)}}{\|v\|_{L^2(D)}}$. Letting λ_1 be the smallest eigenvalue of the Laplacian with Dirichlet boundary conditions, Proposition 47.3 shows that $C_{\rm PS} = \ell_D\lambda_1^{\frac12}$, and the Poincaré–Steklov inequality becomes an equality when applied to the first eigenfunction ψ_1. $\square$

47.2 H^1-conforming approximation

In this section, we investigate the H^1-conforming finite element approximation of the spectral problem (47.2).

47.2.1 Discrete setting and algebraic viewpoint

We assume that D is a Lipschitz polyhedron in $\mathbb{R}^d$, and we consider a shape-regular sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ of affine meshes so that each mesh covers D exactly. Depending on the boundary conditions that are imposed in V, we denote by V_h the H^1-conforming finite element space based on $\mathcal{T}_h$ such that $V_h \subset V$ and $P_{k,0}^{\rm g}(\mathcal{T}_h) \subseteq V_h \subseteq P_k^{\rm g}(\mathcal{T}_h)$ with $k \geq 1$ (see §19.2.1 or §19.4). The approximate eigenvalue problem we consider is the following:

$$\begin{cases} \text{Find } \psi_h \in V_h\setminus\{0\} \text{ and } \lambda_h \in \mathbb{R} \text{ such that} \\ a(\psi_h, w_h) = \lambda_h(\psi_h, w_h)_{L^2(D)}, \quad \forall w_h \in V_h. \end{cases} \tag{47.7}$$

Let $I := \dim V_h$, let $\{\varphi_i\}_{i\in\{1:I\}}$ be the global shape functions in V_h, and let $\mathsf{U}_h \in \mathbb{R}^I$ be the coordinate vector of ψ_h relative to this basis. The discrete eigenvalue problem (47.7) can be recast as follows:

$$\begin{cases} \text{Find } \mathsf{U}_h \in \mathbb{R}^I\setminus\{0\} \text{ and } \lambda_h \in \mathbb{R} \text{ such that} \\ \mathcal{A}\mathsf{U}_h = \lambda_h\mathcal{M}\mathsf{U}_h, \end{cases} \tag{47.8}$$

where the *stiffness matrix* $\mathcal{A}$ and the *mass matrix* $\mathcal{M}$ have entries

$$\mathcal{A}_{ij} := a(\varphi_j, \varphi_i) \qquad \text{and} \qquad \mathcal{M}_{ij} := (\varphi_j, \varphi_i)_{L^2(D)}. \tag{47.9}$$

Both matrices are symmetric positive definite since they are Gram matrices (see also §28.1). Because $\mathcal{M}$ is not the identity matrix, the problem (47.8) is called *generalized eigenvalue problem*.

Proposition 47.6 (Spectral problems). (i) (47.7) *and* (47.8) *admit* I *(positive) eigenvalues (counted with their multiplicity)* $\{\lambda_{hi}\}_{i\in\{1:I\}}$. (ii) *The eigenfunctions* $\{\psi_{hi}\}_{i\in\{1:I\}} \subset V_h$ *in* (47.7) *can be chosen so that* $a(\psi_{hj},\psi_{hi}) = \lambda_{hi}\delta_{ij}$ *and* $(\psi_{hj},\psi_{hi})_{L^2(D)} = \delta_{ij}$. *Equivalently, the eigenvectors* $\{\mathsf{U}_{hi}\}_{i\in\{1:I\}} \subset \mathbb{R}^I$ *in* (47.8) *can be chosen so that* $\mathsf{U}_{hj}^{\mathsf{T}}\mathcal{A}\mathsf{U}_{hi} = \lambda_{hi}\delta_{ij}$ *and* $\mathsf{U}_{hj}^{\mathsf{T}}\mathcal{M}\mathsf{U}_{hi} = \delta_{ij}$.

Proof. (i) Since $\mathcal{A}$ is symmetric and $\mathcal{M}$ is symmetric positive definite, these two matrices can be simultaneously diagonalized. Let us recall the process for completeness. Let $\mathcal{Q}\mathcal{Q}^{\mathsf{T}}$ be the Cholesky factorization of $\mathcal{M}^{-1}$, i.e., $\mathcal{M} = \mathcal{Q}^{-\mathsf{T}}\mathcal{Q}^{-1}$. Since $\mathcal{Q}^{\mathsf{T}}\mathcal{A}\mathcal{Q}$ is real and symmetric, there exists an orthogonal matrix $\mathcal{P}$ (with $\mathcal{P}\mathcal{P}^{\mathsf{T}} = \mathbb{I}_I$), and a diagonal matrix Λ with diagonal entries $(\lambda_{hi})_{i\in\{1:I\}}$, such that $\mathcal{Q}^{\mathsf{T}}\mathcal{A}\mathcal{Q} = \mathcal{P}\Lambda\mathcal{P}^{-1}$. Then $\mathcal{A}\mathcal{Q}\mathcal{P} = \mathcal{Q}^{-\mathsf{T}}\mathcal{P}\Lambda = \mathcal{M}\mathcal{Q}\mathcal{P}\Lambda$. Let us set $\mathcal{U} := \mathcal{Q}\mathcal{P}$ and let $(\mathsf{U}_{hi})_{i\in\{1:I\}}$ be the columns of the matrix $\mathcal{U}$. The identity $\mathcal{A}\mathcal{U} = \mathcal{M}\mathcal{U}\Lambda$ is equivalent to

$$\mathcal{A}\mathsf{U}_{hi} = \lambda_{hi}\mathcal{M}\mathsf{U}_{hi}, \qquad \forall i \in \{1:I\},$$

showing that the λ_{hi}'s are the eigenvalues of the generalized eigenvalue problem (47.8) and the U_{hi}'s are the corresponding eigenvectors.
(ii) One readily sees that $\mathcal{U}^{\mathsf{T}}\mathcal{A}\mathcal{U} = \mathcal{P}^{\mathsf{T}}\mathcal{Q}^{\mathsf{T}}\mathcal{Q}^{-\mathsf{T}}\mathcal{P}\Lambda = \Lambda$ and $\mathcal{U}^{\mathsf{T}}\mathcal{M}\mathcal{U} = \mathcal{P}^{\mathsf{T}}\mathcal{Q}^{\mathsf{T}}\mathcal{Q}^{-\mathsf{T}}\mathcal{Q}^{-1}\mathcal{Q}\mathcal{P} = \mathbb{I}_I$. This proves the identities on the eigenvectors, and those on the eigenfunctions follow from the definitions of $\mathcal{A}$ and $\mathcal{M}$. $\square$

It is henceforth assumed that the eigenvalues are enumerated in increasing order $\lambda_{h1} \le \ldots \le \lambda_{hI}$, where each eigenvalue appears in this list as many times as its multiplicity. Moreover, the eigenfunctions are chosen and normalized as in Proposition 47.6(ii) so that $\|\psi_{hi}\|_{L^2(D)} = 1$.

47.2.2 Eigenvalue error analysis

Let $m \ge 1$ be a fixed natural number. We assume that h is small enough so that $m \le I$ (recall that $I := \dim(V_h)$ grows roughly like $(\ell_D/h)^d$ as $h \to 0$). Our objective is to estimate $|\lambda_{hm} - \lambda_m|$. Let us introduce the discrete solution map $G_h : V \to V_h$ defined s.t. $a(G_h(v) - v, v_h) = 0$ for all $v \in V$ and all v_h in V_h (see §26.3.4 and §32.1). Let $W_m := \operatorname{span}\{\psi_i\}_{i\in\{1:m\}}$ and let S_m be the unit sphere of W_m in $L^2(D)$. We define

$$\sigma_{hm} := \min_{v\in W_m\setminus\{0\}} \frac{\|G_h(v)\|_{L^2(D)}}{\|v\|_{L^2(D)}} = \min_{v\in S_m} \|G_h(v)\|_{L^2(D)}. \tag{47.10}$$

(Note that $\|G_h(v)\|_{L^2(D)}$ attains its infimum over S_m since S_m is compact.)

Lemma 47.7 (Comparing λ_m and λ_{hm}). *Let $m \in \{1:I\}$. Assume that $\sigma_{hm} \neq 0$. The following holds true:*

$$\lambda_m \le \lambda_{hm} \le \sigma_{hm}^{-2}\lambda_m. \tag{47.11}$$

Proof. Let $w_h = \sum_{i\in\{1:m\}} \mathsf{W}_i\psi_{hi} \in W_{hm} := \operatorname{span}\{\psi_{hi}\}_{i\in\{1:m\}}$, where the eigenfunctions are chosen and normalized as in Proposition 47.6(ii), so that $\|\psi_{hi}\|_{L^2(D)} = 1$. Then $R(w_h) = \sum_{i\in\{1:m\}} \lambda_{hi}\mathsf{W}_i^2 / \sum_{i\in\{1:m\}} \mathsf{W}_i^2$. We infer that $\lambda_{hm} = \max_{w_h\in W_{hm}} R(w_h)$, and the first inequality in (47.11) is a consequence of Proposition 47.4. Let us now prove the second inequality. We observe that $\ker(G_h) \cap W_m = \{0\}$ since $\sigma_{hm} \neq 0$ by assumption. Hence, the rank nullity theorem implies that $\dim(G_h(W_m)) = m$. Let $W_{h,m-1} = \text{span}\{\psi_{hi}\}_{i\in\{1:m-1\}}$ and consider the L^2-projection from $G_h(W_m)$ onto $W_{h,m-1}$. The rank nullity theorem implies that there is a nonzero vector $v_h \in G_h(W_m)$ such that v_h is L^2-orthogonal to $W_{h,m-1}$, so that $v_h = \sum_{i\in\{m:I\}} V_i\psi_{hi}$. It follows that $R(v_h) \geq \lambda_{hm}$. As a result, we have

$$\lambda_{hm} \leq R(v_h) \leq \max_{w_h\in G_h(W_m)} \frac{a(w_h, w_h)}{\|w_h\|^2_{L^2(D)}} = \max_{v\in W_m} \frac{a(G_h(v), G_h(v))}{\|G_h(v)\|^2_{L^2(D)}}.$$

Using that $a(G_h(v), G_h(v)) = a(v, G_h(v)) \leq a(v, v)^{\frac{1}{2}} a(G_h(v), G_h(v))^{\frac{1}{2}}$ since a is symmetric and coercive, we infer that $a(G_h(v), G_h(v)) \leq a(v, v)$. Recalling that $\max_{v\in W_m} R(v) = \lambda_m$, we conclude that

$$\begin{aligned}\lambda_{hm} &\leq \max_{v\in W_m} \frac{a(v, v)}{\|G_h(v)\|^2_{L^2(D)}} \leq \max_{v\in W_m} \frac{\|v\|^2_{L^2(D)}}{\|G_h(v)\|^2_{L^2(D)}} \max_{v\in W_m} R(v) \\ &= \sigma_{hm}^{-2} \max_{v\in W_m} R(v) = \sigma_{hm}^{-2}\lambda_m. \qquad \square\end{aligned}$$

Remark 47.8 (Guaranteed upper bound). It is remarkable that independently of the approximation space, but provided conformity holds true, i.e., $V_h \subset V$, each eigenvalue of the discrete problem (47.8) is larger than the corresponding eigenvalue of the exact problem (46.17). In other words, the discrete eigenvalue λ_{hm} is a guaranteed upper bound on the exact eigenvalue λ_m for all $m \in \{1:I\}$. Estimating computable lower bounds on the eigenvalues using conforming elements is more challenging. We refer the reader to Cancès et al. [104] for a literature overview and to Remark 48.13 when the approximation setting is nonconforming. □

Lemma 47.9 (Lower bound on σ_{hm}). *Let $m \in \{1:I\}$. Recall that S_m is the unit sphere of $W_m := \text{span}\{\psi_i\}_{i\in\{1:m\}}$ in $L^2(D)$ and recall that $G_h : V_h \to V$ is the discrete solution operator. The following holds true:*

$$\sigma^2_{hm} \geq 1 - 2\sqrt{m}\frac{\|a\|}{\lambda_1} \max_{v\in S_m} \|v - G_h(v)\|^2_V. \tag{47.12}$$

Proof. Let $v \in S_m$. Let $(\mathsf{V}_i)_{i\in\{1:m\}}$ be the coordinate vector of v relative to the basis $\{\psi_i\}_{i\in\{1:m\}}$. Since $(\psi_i, \psi_j)_{L^2(D)} = \delta_{ij}$, we have $\sum_{i\in\{1:m\}} \mathsf{V}_i^2 = \|v\|^2_{L^2(D)} = 1$. In addition, $\|G_h(v)\|^2_{L^2(D)}$ can be bounded from below as

$$\begin{aligned}\|G_h(v)\|^2_{L^2(D)} &= \|v\|^2_{L^2(D)} - 2(v, v - G_h(v))_{L^2(D)} + \|v - G_h(v)\|^2_{L^2(D)} \\ &\geq \|v\|^2_{L^2(D)} - 2(v, v - G_h(v))_{L^2(D)} \\ &= 1 - 2(v, v - G_h(v))_{L^2(D)}.\end{aligned} \tag{47.13}$$

Using that (λ_i, ψ_i) is an eigenpair, the symmetry of a, and the Galerkin orthogonality property satisfied by the discrete solution map, we have

$$(v, v - G_h(v))_{L^2(D)} = \sum_{i\in\{1:m\}} \mathsf{V}_i(\psi_i, v - G_h(v))_{L^2(D)}$$
$$= \sum_{i\in\{1:m\}} \frac{\mathsf{V}_i}{\lambda_i} a(\psi_i, v - G_h(v)) = \sum_{i\in\{1:m\}} \frac{\mathsf{V}_i}{\lambda_i} a(\psi_i - G_h(\psi_i), v - G_h(v)).$$

This implies that

$$\begin{aligned}(v, v - G_h(v))_{L^2(D)} &\le \frac{\|a\|}{\lambda_1}\|v - G_h(v)\|_V \sum_{i\in\{1:m\}} |\mathsf{V}_i|\|\psi_i - G_h(\psi_i)\|_V \\ &\le \frac{\|a\|}{\lambda_1} \max_{w\in S_m} \|w - G_h(w)\|_V^2 \sum_{i\in\{1:m\}} |\mathsf{V}_i| \\ &\le \sqrt{m}\frac{\|a\|}{\lambda_1} \max_{w\in S_m} \|w - G_h(w)\|_V^2,\end{aligned}$$

where we used the boundedness of a and $\lambda_1 \le \lambda_i$ for all $i \in \{1:m\}$ in the first bound, that $v \cup \{\psi_i\}_{i\in\{1:m\}} \subset S_m$ in the second bound, and the Cauchy–Schwarz inequality and $\sum_{i\in\{1:m\}} \mathsf{V}_i^2 = 1$ in the third bound. The expected estimate is obtained by inserting this bound into (47.13) and taking the infimum over $v \in S_m$ (recall that $\sigma_{hm} := \min_{v\in S_m} \|G_h(v)\|_{L^2(D)}$). □

Theorem 47.10 (Error on eigenvalues). *Let $m \in \mathbb{N}\backslash\{0\}$ and $c_1(m) := 4\sqrt{m}\frac{\|a\|}{\lambda_1}\frac{\|a\|}{\alpha}$. There is $h_0(m) > 0$ s.t. for all $h \in \mathcal{H} \cap (0, h_0(m)]$, we have $\sigma_{hm} \ge \frac{1}{2}$ and*

$$0 \le \lambda_{hm} - \lambda_m \le \lambda_m c_1(m) \max_{v\in S_m} \min_{v_h\in V_h} \|v - v_h\|_V^2. \tag{47.14}$$

Proof. (1) Since I grows unboundedly as $h \downarrow 0$, there is $h_0'(m) > 0$ s.t. $m \in \{1:I\}$ for all $h \in \mathcal{H} \cap (0, h_0'(m)]$, i.e., the pair $(\lambda_{hm}, \psi_{hm})$ exists for all $h \in \mathcal{H} \cap (0, h_0'(m)]$. Moreover, since the unit sphere S_m is compact, there is $v_*(m) \in S_m$ such that $\max_{v\in S_m} \|v - G_h(v)\|_V^2 = \|v_*(m) - G_h(v_*(m))\|_V^2$. The approximation property of the sequence $(V_h)_{h\in\mathcal{H}}$ implies that there is $h_0''(m) > 0$ such that $c_0(m)\|v_*(m) - G_h(v_*(m))\|_V^2 \le \frac{1}{2}$ for all $h \in \mathcal{H} \cap (0, h_0''(m)]$, with $c_0(m) := 2\sqrt{m}\frac{\|a\|}{\lambda_1}$. We now set $h_0(m) := \min(h_0'(m), h_0''(m))$. Observing that $\frac{1}{1-x} \le 1 + 2x$ for all $x \in [0, \frac{1}{2}]$, and applying this inequality to (47.12) with $x := c_0(m) \max_{v\in S_m} \|v - G_h(v)\|_V^2 \le \frac{1}{2}$, we infer that $\sigma_{hm}^{-2} \le 1 + 2c_0(m) \max_{v\in S_m} \|v - G_h(v)\|_V^2$. This implies in particular that $\sigma_{hm} \ge \frac{1}{\sqrt{2}} \ge \frac{1}{2}$ for all $h \in \mathcal{H} \cap (0, h_0(m)]$.
(2) Inserting the above bound into (47.11) yields

$$\lambda_{hm} - \lambda_m \le (\sigma_{hm}^{-2} - 1)\lambda_m \le 2\lambda_m c_0(m) \max_{v\in S_m} \|v - G_h(v)\|_V^2.$$

Since a is symmetric and coercive, Céa's lemma (Lemma 26.13) implies that

$$\|v - G_h(v)\|_V \le \left(\frac{\|a\|}{\alpha}\right)^{\frac{1}{2}} \min_{v_h\in V_h} \|v - v_h\|_V. \tag{47.15}$$

The assertion follows readily. □

Remark 47.11 (Units). One readily sees that $\frac{\|a\|}{\lambda_1}$ scales as $\|\cdot\|_{L^2(D)}^{-2}$, i.e., as ℓ_D^{-2d}. Since $\|\cdot\|_V^2$ also scales like ℓ_D^{2d} owing to our assumption on the boundedness of the embedding $V \hookrightarrow L^2(D)$, we infer that the factor $c_1(m)\max_{v\in S_m}\min_{v_h\in V_h}\|v - v_h\|_V^2$ is nondimensional. □

Remark 47.12 (Double rate). The elliptic regularity theory implies that for all $m \ge 1$, there are $s(m) > 0$ and c_m s.t. $\|\psi_m\|_{H^{1+s(m)}(D)} \le c_m$. Here, the value of $s(m)$ is not restricted to the interval $(0,1]$ since there is a bootstrapping phenomenon that allows $s(m)$ to be large. To illustrate this property, assume that D is of class $C^{r+1,1}$, $r \in \mathbb{N}$, and the bilinear form a is associated with an operator A satisfying the assumptions of Theorem 31.29. Let $s := r \mod 2 \in \{0,1\}$ and let $l^\sharp \in \mathbb{N}\backslash\{0\}$ be s.t. $2(l^\sharp - 1) + s = r$. Theorem 31.29 implies that there is $c_0(r)$ such that $\|A^{-1}(v)\|_{H^s(D)} \le c_0(r)\ell_D^2\|v\|_{L^2(D)}$ for all $v \in L^2(D)$, and there are $c_l(r)$, such that $\|A^{-1}(v)\|_{H^{2l+s}(D)} \le c_l(r)\ell_D^2\|v\|_{L^{2(l-1)+s}(D)}$ for all $v \in H^{2(l-1)+s}(D)$ and all $l \in \{1{:}l^\sharp\}$. Since $A(\psi_m) = \lambda_m\psi_m$, we obtain $\|\psi_m\|_{H^{r+2}(D)} = \|\psi_m\|_{H^{2l^\sharp+s}(D)} \le c_{l^\sharp}(r)\ldots c_1(r)c_0(r)(\lambda_m\ell_D^2)^{l^\sharp+1}\|\psi_m\|_{L^2(D)}$. Recalling the normalization $\|\psi_m\|_{L^2(D)} = 1$, this argument shows that if D is of class $C^{r+1,1}$, we have $\|\psi_m\|_{H^{1+s(m)}(D)} \le c_m$ with $s(m) := r+1$ and $c_m := c_{l^\sharp}(r)\ldots c_1(r)c_0(r)(\lambda_m\ell_D^2)^{l^\sharp+1}$. Recalling that k is the approximation degree of V_h, let $s_\flat(m) := \min(s(1),\ldots,s(m),k)$ for all $m \ge 1$, and $\chi(m) := \max_{v\in S_m}\ell_D^{1+s_\flat(m)}|v|_{H^{1+s_\flat(m)}(D)}$ (recall that S_m is the unit sphere of W_m in $L^2(D)$). The best-approximation estimates established in §22.3 and §22.4 imply that there exists c_{app} such that the following holds true for all $h \in \mathcal{H}\cap(0,h_0(m)]$:

$$\max_{v\in S_m}\min_{v_h\in V_h}\|v - v_h\|_V \le c_{\text{app}}\,\chi(m)(h/\ell_D)^{s_\flat(m)}.$$

Owing to Theorem 47.10, this implies that

$$0 \le \lambda_{hm} - \lambda_m \le \lambda_m c_1(m)c_{\text{app}}^2\chi(m)^2(h/\ell_D)^{2s_\flat(m)}. \tag{47.16}$$

In the best-case scenario where $s(n) \ge k$ for all $n \in \{1{:}m\}$, we have $s_\flat(m) = k$ so that the convergence rate for the error on λ_m is $\mathcal{O}(h^{2k})$, i.e., this error converges at a rate that is double that of the best-approximation error on the eigenvectors in the H^1-norm; see Remark 47.16 below. Note that the convergence rate on λ_m in (47.16) depends on the smallest smoothness index of all the eigenfunctions $\{\psi_n\}_{n\in\{1:m\}}$. This shortcoming is circumvented with the more general theory presented in Chapter 48, where the convergence rate on λ_m only depends on the smoothness index of the eigenfunctions associated with λ_m. Note also that since $c_1(m)$ grows unboundedly with m, (47.16) shows that when h is fixed the accuracy of the approximation decreases as m increases. □

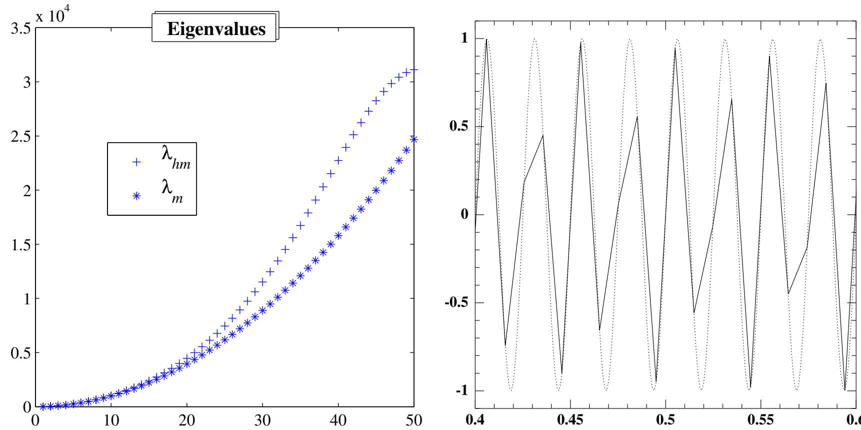

Fig. 47.1 $\mathbb{P}_1$ approximation of the eigenvalues of the Laplacian in one dimension. Left: discrete and exact eigenvalues, $I := 50$. Right: Graph of the 80th exact (dashed line) and discrete (solid line) eigenfunctions in the interval $(0.4, 0.6)$, $I := 100$.

Example 47.13 (1D Laplacian). Let us consider the eigenvalue problem for the one-dimensional Laplacian discretized using $\mathbb{P}_1$ Lagrange elements on a uniform mesh on $D := (0, 1)$. It is shown in Exercise 47.5 that $\lambda_m = m^2\pi^2$ and $\lambda_{hm} = \frac{6}{h^2}\frac{1-\cos(m\pi h)}{2+\cos(m\pi h)}$ for all $m \geq 1$. The left panel of Figure 47.1 shows the first 50 exact eigenvalues and the 50 discrete eigenvalues on a mesh having $I := 50$ internal vertices. The exact eigenvalues are approximated from above as predicted in Lemma 47.7. Observe that only the first eigenvalues are approximated accurately. The reason for this is that the eigenfunctions corresponding to large eigenvalues oscillate too much to be represented accurately on the mesh as illustrated in the right panel of Figure 47.1. A rule of thumb is that a meshsize smaller than $\frac{\sqrt{\epsilon}}{m}$ must be used to approximate the m-th eigenvalue with relative accuracy ϵ, i.e., $|\lambda_{hm} - \lambda_m| < \epsilon\lambda_m$. For instance, only the first 10 eigenvalues are approximated within 1% accuracy when $I := 100$. We refer the reader to Exercise 47.5 for further details. □

47.2.3 Eigenfunction error analysis

The goal of this section is to estimate the approximation error on the eigenfunctions. We first estimate this error in the L^2-norm and then in the H^1-norm. Let $m \geq 1$ be a fixed natural number, and let us assume as in the previous section that the meshsize $h \in \mathcal{H}$ is small enough so that $m \leq I$ and

$\sigma_{hm} > 0$ (see Theorem 47.10). For the sake of simplicity, we also assume that the eigenvalue λ_m is simple, and we set $\gamma_m := 2\max_{i\in\mathbb{N}\backslash\{0,m\}} \frac{\lambda_m}{|\lambda_m-\lambda_i|}$. Observe that $\gamma_m = 2\max(\frac{\lambda_m}{\lambda_m-\lambda_{m-1}}, \frac{\lambda_m}{\lambda_{m+1}-\lambda_m})$. Since $\lambda_{hi} \to \lambda_i$ as $h \to 0$ for all $i \in \{1{:}m{+}1\}$ (see Theorem 47.10), there exists $h_0(m) > 0$ so that $\frac{\lambda_m}{|\lambda_m-\lambda_{hi}|} \leq \gamma_m$ for all $i \in \{1{:}m+1\}\backslash\{m\}$ and all $h \in \mathcal{H}\cap(0,h_0(m)]$. Moreover, using that $|\lambda_m - \lambda_{hi}| \leq |\lambda_m - \lambda_{m+1}|$ for all $i \geq m+1$, we infer that the following holds true for all $h \in \mathcal{H}\cap(0,h_0(m)]$:

$$\max_{\substack{i\in\{1:I\}\\ i\neq m}} \frac{\lambda_m}{|\lambda_m-\lambda_{hi}|} \leq \gamma_m. \tag{47.17}$$

Theorem 47.14 (L^2-error on eigenfunctions). *Let $m \in \mathbb{N}\backslash\{0\}$. Assume that λ_m is simple and let $h_0(m) > 0$ be s.t.* (47.17) *holds true. Let $c_2(m) := 2(1+\gamma_m)$. There is an eigenfunction ψ_m such that the following holds true for all $h \in \mathcal{H}\cap(0,h_0(m)]$:*

$$\|\psi_m - \psi_{hm}\|_{L^2(D)} \leq c_2(m)\|\psi_m - G_h(\psi_m)\|_{L^2(D)}. \tag{47.18}$$

Proof. Recall that $G_h(\psi_m) = \sum_{i\in\{1:I\}} \mathsf{V}_i\psi_{hi}$ with $\mathsf{V}_i := (G_h(\psi_m),\psi_{hi})_{L^2(D)}$. Let us set $v_{hm} := \mathsf{V}_m\psi_{hm}$ so that $G_h(\psi_m)-v_{hm} = \sum_{i\in\{1:I\}\backslash\{m\}} \mathsf{V}_i\psi_{hi}$. Since the bilinear form a is symmetric and (λ_{hi},ψ_{hi}) is a discrete eigenpair, we have

$$\begin{aligned}\mathsf{V}_i = \frac{1}{\lambda_{hi}}a(\psi_{hi},G_h(\psi_m)) &= \frac{1}{\lambda_{hi}a(G_h(\psi_m),\psi_{hi})} \\ &= \frac{1}{\lambda_{hi}}a(\psi_m,\psi_{hi}) = \frac{\lambda_m}{\lambda_{hi}}(\psi_m,\psi_{hi})_{L^2(D)},\end{aligned}$$

where we used the definition of G_h and that (λ_m,ψ_m) is an eigenpair. This implies that

$$\begin{aligned}(\lambda_{hi}-\lambda_m)\mathsf{V}_i = \lambda_{hi}\mathsf{V}_i - \lambda_m\mathsf{V}_i &= \lambda_m(\psi_m,\psi_{hi})_{L^2(D)} - \lambda_m\mathsf{V}_i \\ &= \lambda_m(\psi_m,\psi_{hi})_{L^2(D)} - \lambda_m(G_h(\psi_m),\psi_{hi})_{L^2(D)} \\ &= \lambda_m(\psi_m - G_h(\psi_m),\psi_{hi})_{L^2(D)}.\end{aligned}$$

Hence, we have $\mathsf{V}_i = \frac{\lambda_m}{\lambda_{hi}-\lambda_m}(\psi_m - G_h(\psi_m),\psi_{hi})_{L^2(D)}$ for all $i \in \{1{:}I\}\backslash\{m\}$. Since the discrete eigenfunctions $\{\psi_{hi}\}_{i\in\{1:I\}}$ are L^2-orthonormal, we obtain

$$\begin{aligned}\|G_h(\psi_m) - v_{hm}\|^2_{L^2(D)} = \sum_{\substack{i\in\{1:I\}\\ i\neq m}} \mathsf{V}_i^2 &\leq \gamma_m^2 \sum_{\substack{i\in\{1:I\}\\ i\neq m}} (\psi_m - G_h(\psi_m),\psi_{hi})^2_{L^2(D)} \\ &\leq \gamma_m^2\|\psi_m - G_h(\psi_m)\|^2_{L^2(D)},\end{aligned} \tag{47.19}$$

where the first bound follows from (47.17) and the last one from Bessel's inequality $\sum_{i\in\{1:I\}}(\psi_m - G_h(\psi_m),\psi_{hi})^2_{L^2(D)} \leq \|\psi_m - G_h(\psi_m)\|^2_{L^2(D)}$. Let us now estimate $\|\psi_{hm} - v_{hm}\|_{L^2(D)}$. Since $\|\psi_{hm}\|_{L^2(D)} = 1$, we have

$$\begin{aligned}\|\psi_{hm} - v_{hm}\|_{L^2(D)} = \|(1 - \mathsf{V}_m)\psi_{hm}\|_{L^2(D)} &= |\mathsf{V}_m - 1| \\ &= |(G_h(\psi_m), \psi_{hm})_{L^2(D)} - 1|.\end{aligned}$$

Assume that ψ_{hm} is chosen so that $\mathsf{V}_m = (G_h(\psi_m), \psi_{hm})_{L^2(D)} \geq 0$. Then we have $\|v_{hm}\|_{L^2(D)} = |\mathsf{V}_m| = (G_h(\psi_m), \psi_{hm})_{L^2(D)}$, and $\|\psi_{hm} - v_{hm}\|_{L^2(D)} = |\|v_{hm}\|_{L^2(D)} - 1|$. Since the triangle inequality implies that

$$\|\psi_m\|_{L^2(D)} - \|\psi_m - v_{hm}\|_{L^2(D)} \leq \|v_{hm}\|_{L^2(D)} \leq \|\psi_m\|_{L^2(D)} + \|\psi_m - v_{hm}\|_{L^2(D)},$$

and since $\|\psi_m\|_{L^2(D)} = 1$, we infer that $|\|v_{hm}\|_{L^2(D)} - 1| \leq \|\psi_m - v_{hm}\|_{L^2(D)}$. This implies that

$$\|\psi_{hm} - v_{hm}\|_{L^2(D)} = |\|v_{hm}\|_{L^2(D)} - 1| \leq \|\psi_m - v_{hm}\|_{L^2(D)}.$$

Invoking the triangle inequality, the above bound, and the triangle inequality one more time gives

$$\begin{aligned}&\|\psi_m - \psi_{hm}\|_{L^2(D)} \\ &\leq \|\psi_m - G_h(\psi_m)\|_{L^2(D)} + \|G_h(\psi_m) - v_{hm}\|_{L^2(D)} + \|\psi_{hm} - v_{hm}\|_{L^2(D)} \\ &\leq \|\psi_m - G_h(\psi_m)\|_{L^2(D)} + \|G_h(\psi_m) - v_{hm}\|_{L^2(D)} + \|\psi_m - v_{hm}\|_{L^2(D)} \\ &\leq 2(\|\psi_m - G_h(\psi_m)\|_{L^2(D)} + \|G_h(\psi_m) - v_{hm}\|_{L^2(D)}) \\ &\leq 2(1 + \gamma_m)\|\psi_m - G_h(\psi_m)\|_{L^2(D)},\end{aligned}$$

where the last bound follows from (47.19). Using the definition of $c_2(m)$ leads to the expected estimate. □

Theorem 47.15 (H^1-error on eigenfunctions). *Let $m \in \mathbb{N}\backslash\{0\}$. Assume that λ_m is simple and let $h_0(m) > 0$ be s.t. (47.14) and (47.17) hold for all $h \in \mathcal{H} \cap (0, h_0(m)]$. There is an eigenfunction ψ_m such that the following holds true for all $h \in \mathcal{H} \cap (0, h_0(m)]$:*

$$\|\psi_m - \psi_{hm}\|_V \leq c_3(m) \max_{v \in S_m} \min_{v_h \in V_h} \|v - v_h\|_V, \tag{47.20}$$

where $c_3(m) := (\frac{\lambda_m}{\alpha})^{\frac{1}{2}}(c_1(m) + c_2(m)^2 \frac{\|a\|}{\alpha})^{\frac{1}{2}}$ is independent of $h \in \mathcal{H}$.

Proof. Owing to the coercivity of a, we infer that

$$\begin{aligned}\alpha\|\psi_m - \psi_{hm}\|_V^2 &\leq a(\psi_m - \psi_{hm}, \psi_m - \psi_{hm}) \\ &= \lambda_{hm} + \lambda_m - 2\lambda_m(\psi_m, \psi_{hm})_{L^2(D)} \\ &= \lambda_{hm} - \lambda_m + \lambda_m\|\psi_m - \psi_{hm}\|_{L^2(D)}^2,\end{aligned}$$

since $\|\psi_m\|_{L^2(D)} = \|\psi_{hm}\|_{L^2(D)} = 1$ implies that $\|\psi_m - \psi_{hm}\|_{L^2(D)}^2 = 2 - 2(\psi_m, \psi_{hm})_{L^2(D)}$. The inequality (47.20) is obtained by estimating $(\lambda_{hm} - \lambda_m)$

and $\|\psi_m - \psi_{hm}\|^2_{L^2(D)}$. The estimate on $(\lambda_{hm} - \lambda_m)$ is given by (47.14) in Theorem 47.10, and Theorem 47.14 gives $\|\psi_m - \psi_{hm}\|_{L^2(D)} \leq c_2(m)\|\psi_m - G_h(\psi_m)\|_{L^2(D)}$. We observe that

$$\begin{aligned}\|\psi_m - G_h(\psi_m)\|_{L^2(D)} \leq \|\psi_m - G_h(\psi_m)\|_V &\leq \max_{v\in S_m} \|v - G_h(v)\|_v \\ &\leq \left(\frac{\|a\|}{\alpha}\right)^{\frac{1}{2}} \min_{v_h\in V_h} \|v - v_h\|_V,\end{aligned}$$

where the last bound follows from (47.15) (Céa's lemma). Putting everything together leads to the expected estimate. □

Remark 47.16 (Convergence rates). Let us use the notation of Remark 47.12. Assume that the eigenvalue λ_m is simple. We can then invoke the estimates from Theorem 47.14 and Theorem 47.15. The best-approximation estimates in the H^1-norm established in §22.3 and §22.4 and the Aubin–Nitsche lemma (Lemma 32.11) imply that the following holds true for all $h \in \mathcal{H} \cap (0, h_0(m)]$:

$$\|\psi_m - \psi_{hm}\|_{L^2(D)} \leq \check{c}_2(m)\chi(m)(h/\ell_D)^{s_\flat(m)+s}, \tag{47.21a}$$

$$\|\psi_m - \psi_{hm}\|_{H^1(D)} \leq \check{c}_3(m)\chi(m)(h/\ell_D)^{s_\flat(m)}, \tag{47.21b}$$

where the constants $\check{c}_2(m), \check{c}_3(m)$ have the same dependencies w.r.t. m as the constants $c_2(m), c_3(m)$, and $\chi(m)$ is defined in Remark 47.12. The best possible convergence rates are obtained when $s_n(m) \geq k$ for all $n \in \{1{:}m\}$ so that $s_\flat(m) = k$, yielding the rates $\mathcal{O}(h^{k+1})$ in the L^2-norm and $\mathcal{O}(h^k)$ in the H^1-norm. Moreover, it can be shown that if λ_m has multiplicity p, i.e., $\lambda_m = \lambda_{m+1} = \ldots = \lambda_{m+p-1}$, then there exists an eigenfunction $\psi_m^\dagger \in \operatorname{span}\{\psi_m, \ldots, \psi_{m+p-1}\}$ with $\|\psi_m^\dagger\|_{L^2(D)} = 1$ such that (47.21) holds true with ψ_m replaced by $\psi_m^\dagger$. Note that (47.21) shows that when h is fixed, the accuracy of the approximation decreases as m increases, since $c_2(m), c_3(m)$ grow unboundedly with m. □

Exercises

Exercise 47.1 (Real eigenvalues). Consider the eigenvalue problem: Find $\psi \in H^1_0(D;\mathbb{C})\backslash\{0\}$ and $\lambda \in \mathbb{C}$ s.t. $\int_D (\nabla\psi\cdot\nabla\overline{w} + \psi\overline{w})\,\mathrm{d}x = \lambda \int_D \psi\overline{w}\,\mathrm{d}x$ for all $w \in H^1_0(D;\mathbb{C})$. Prove directly that λ is real. (*Hint*: test with $w := \psi$.)

Exercise 47.2 (Smallest eigenvalue). Let $D_1 \subset D_2$ be two Lipschitz domains in $\mathbb{R}^d$. Let $a_i : H^1_0(D_i)\times H^1_0(D_i) \to \mathbb{R}$, $i \in \{1, 2\}$, be two symmetric, coercive, bounded bilinear forms. Assume that $a_1(v, w) = a_2(\widetilde{v}, \widetilde{w})$ for all $v, w \in H^1_0(D_1)$, where $\widetilde{v}, \widetilde{w}$ denote the extension by zero of v, w, respectively. Let $\lambda_1(D_i)$ be the smallest eigenvalue of the eigenvalue problem: Find

$\psi \in H_0^1(D_i)\backslash\{0\}$ and $\lambda \in \mathbb{R}$ s.t. $a_i(\psi, w) = \lambda(\psi, w)_{L^2(D_i)}$ for all $w \in H_0^1(D_i)$. Prove that $\lambda_1(D_2) \leq \lambda_1(D_1)$. (*Hint*: use Proposition 47.3.)

Exercise 47.3 (Continuity of eigenvalues). Consider the setting defined in §47.1. Let $a_1, a_2 : V \times V \to \mathbb{R}$ be two symmetric, coercive, bounded bilinear forms. Let $A_1, A_2 : V \to V'$ be the linear operators defined by $\langle A_i(v), w\rangle_{V',V} := a_i(v, w)$, $i \in \{1, 2\}$, for all $v, w \in A$. Let $\lambda_k(a_1)$ and $\lambda_k(a_2)$ be the k-th eigenvalues, respectively. Prove that $|\lambda_k(a_1) - \lambda_k(a_2)| \leq \sup_{v \in S} |\langle (A_1 - A_2)(v), v\rangle_{V',V}|$, where S is the unit sphere in $L^2(D)$. (*Hint*: use the min-max principle.)

Exercise 47.4 (Max-min principle). Prove the second equality in (47.6). (*Hint*: let $E_{m-1} \in V_{m-1}$ and observe that $E_{m-1}^{\perp} \cap W_m \neq \{0\}$.)

Exercise 47.5 (Laplacian, 1D). Consider the spectral problem for the 1D Laplacian on $D := (0, 1)$. (i) Show that the eigenpairs (λ_m, ψ_m) are $\lambda_m = m^2\pi^2$, $\psi_m(x) = \sin(m\pi x)$, for all $x \in D$ and all $m \geq 1$. (ii) Consider a uniform mesh of D of size $h := \frac{1}{I+1}$ and H^1-conforming $\mathbb{P}_1$ finite elements. Compute the stiffness matrix $\mathcal{A}$ and the mass matrix $\mathcal{M}$. (iii) Show that the eigenvalues of the discrete problem (47.8) are $\lambda_{hm} = \frac{6}{h^2}(\frac{1-\cos(m\pi h)}{2+\cos(m\pi h)})$ for all $m \in \{1{:}I\}$. (*Hint*: consider the vectors $(\sin(\pi hml))_{l \in \{1:I\}}$ for all $m \in \{1{:}I\}$.)

Exercise 47.6 (Stiffness matrix). Assume that the mesh sequence $(\mathcal{T}_h)_{h \in \mathcal{H}}$ is quasi-uniform. Estimate from below the smallest eigenvalue of the stiffness matrix $\mathcal{A}$ defined in (47.9) and estimate from above its largest eigenvalue. (*Hint*: see §28.2.3.)