Part XI, Chapter 55

Stokes equations: Stable pairs (II)

In this chapter, we continue the study of stable finite element pairs that are suitable to approximate the Stokes equations. In doing so, we introduce another technique to prove the inf-sup condition that is based on a notion of macroelement. Recall that we assume that Dirichlet conditions are enforced on the velocity over the whole boundary, that D is a polyhedron in $\mathbb{R}^d$, and that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular sequence of affine meshes so that each mesh covers D exactly. In this chapter, we focus more specifically on the case where the discrete pressure space is a broken finite element space.

55.1 Macroelement techniques

In addition to the Fortin operator technique described in Lemma 54.1 and the method consisting of weakly controlling the pressure gradient described in Lemma 54.3, we now present a third method to establish the inf-sup condition between the discrete velocity space and the discrete pressure space. This method is based on a notion of macroelement.

We return to the abstract setting and consider two complex Banach spaces $\boldsymbol{V}$ and Q and a bounded sesquilinear form b on $\boldsymbol{V}\times Q$. Let $\boldsymbol{V}_{h0} \subset \boldsymbol{V}$ and $Q_h \subset Q$. Recall that $\|b\|$ denotes the boundedness constant of b on $\boldsymbol{V}\times Q$ and that the inf-sup condition (54.1) takes the form

$$\inf_{q_h\in Q_h}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q} =: \beta_h > 0. \tag{55.1}$$

Lemma 55.1 (Partition lemma). *Let $\boldsymbol{V}_{h0}^1, \boldsymbol{V}_{h0}^2$ be two subspaces of $\boldsymbol{V}_{h0}$ and Q_h^1, Q_h^2 be two subspaces of Q such that $Q_h = Q_h^1 + Q_h^2$. Let*

$$\beta_1 := \inf_{q_h\in Q_h^1}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}^1}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q}, \qquad \beta_2 := \inf_{q_h\in Q_h^2}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}^2}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q},$$

$$b_{12} := \sup_{q_h \in Q_h^1} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}^2} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|q_h\|_Q}, \qquad b_{21} := \sup_{q_h \in Q_h^2} \sup_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}^1} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}} \|q_h\|_Q}.$$

Assume that $0 < \beta_1\beta_2$ *and* $\lambda_1\lambda_2 < 1$ *with* $\lambda_1 := \frac{b_{12}}{\beta_2}$, $\lambda_2 := \frac{b_{21}}{\beta_1}$. *Then the inf-sup condition* (55.1) *holds true with* $\beta_h \geq \frac{1}{4}\min(\beta_1, \beta_2)$ *if* $\lambda_1 + \lambda_2 \leq 1$ *and with* $\beta_h \geq \frac{1}{64}(1 - \lambda_1\lambda_2)\|b\|^{-2}\min(\beta_1, \beta_2)^3$ *otherwise.*

Proof. Let $q_h := q_h^1 + q_h^2 \in Q_h \backslash \{0\}$. The definition of β_1, β_2 together with the assumption $0 < \beta_1\beta_2$ implies that there exists $\boldsymbol{v}_h^l \in \boldsymbol{V}_h^l$ so that $b(\boldsymbol{v}_h^l, q_h^l) = \|q_h^l\|_Q^2$ and $\beta_l\|\boldsymbol{v}_h^l\|_{\boldsymbol{V}} \leq \|q_h^l\|_Q$ for all $l \in \{1, 2\}$. We now investigate two cases: either $\lambda_1 + \lambda_2 \leq 1$ or $\lambda_1 + \lambda_2 > 1$.
(1) Let us assume that $\lambda_1 + \lambda_2 \leq 1$. Then, setting $\boldsymbol{v}_h := \boldsymbol{v}_h^1 + \boldsymbol{v}_h^2$ we have

$$\begin{aligned} b(\boldsymbol{v}_h, q_h) &= b(\boldsymbol{v}_h^1, q_h^1) + b(\boldsymbol{v}_h^2, q_h^1) + b(\boldsymbol{v}_h^1, q_h^2) + b(\boldsymbol{v}_h^2, q_h^2) \\ &\geq \|q_h^1\|_Q^2 + \|q_h^2\|_Q^2 - b_{12}\|\boldsymbol{v}_h^2\|_{\boldsymbol{V}}\|q_h^1\|_Q - b_{21}\|\boldsymbol{v}_h^1\|_{\boldsymbol{V}}\|q_h^2\|_Q \\ &\geq \|q_h^1\|_Q^2 + \|q_h^2\|_Q^2 - (\beta_2^{-1}b_{12} + \beta_1^{-1}b_{21})\|q_h^1\|_Q\|q_h^2\|_Q. \end{aligned}$$

Using that $\beta_2^{-1}b_{12} + \beta_1^{-1}b_{21} = \lambda_1 + \lambda_2 \leq 1$, we infer that

$$\begin{aligned} b(\boldsymbol{v}_h, q_h) &\geq \frac{1}{2}\|q_h^1\|_Q^2 + \frac{1}{2}\|q_h^2\|_Q^2 \geq \frac{1}{4}\big(\|q_h^1\|_Q + \|q_h^2\|_Q\big)^2 \\ &\geq \frac{1}{4}\|q_h\|_Q\big(\beta_1\|\boldsymbol{v}_h^1\|_{\boldsymbol{V}} + \beta_2\|\boldsymbol{v}_h^2\|_{\boldsymbol{V}}\big) \geq \frac{1}{4}\min(\beta_1, \beta_2)\|q_h\|_Q\|\boldsymbol{v}_h\|_{\boldsymbol{V}}, \end{aligned}$$

where we used the triangle inequality and the above bounds on $\|\boldsymbol{v}_h^l\|_{\boldsymbol{V}}$ for all $l \in \{1, 2\}$. The assertion then follows with $\beta_h \geq \frac{1}{4}\min(\beta_1, \beta_2)$.
(2) Let us now assume that $\lambda_1 + \lambda_2 > 1$. Without loss of generality, we assume that $\lambda_2 \geq \lambda_1$. Let $\sigma \in \mathbb{R}$, let $\boldsymbol{v}_h := \boldsymbol{v}_h^1 + \sigma\boldsymbol{v}_h^2$, let $\epsilon > 0$, and let us minorize $b(\boldsymbol{v}_h, q_h)$ as follows:

$$\begin{aligned} b(\boldsymbol{v}_h, q_h) &= b(\boldsymbol{v}_h^1, q_h^1) + \sigma b(\boldsymbol{v}_h^2, q_h^1) + b(\boldsymbol{v}_h^1, q_h^2) + \sigma b(\boldsymbol{v}_h^2, q_h^2) \\ &\geq \|q_h^1\|_Q^2 + \sigma\|q_h^2\|_Q^2 - b_{12}\|\boldsymbol{v}_h^2\|_{\boldsymbol{V}}\|q_h^1\|_Q - b_{21}\sigma\|\boldsymbol{v}_h^1\|_{\boldsymbol{V}}\|q_h^2\|_Q \\ &\geq \|q_h^1\|_Q^2 + \sigma\|q_h^2\|_Q^2 - (\beta_2^{-1}b_{12} + \sigma\beta_1^{-1}b_{21})\|q_h^1\|_Q\|q_h^2\|_Q \\ &\geq \Big(1 - \frac{\epsilon}{2}(\lambda_1 + \sigma\lambda_2)\Big)\|q_h^1\|_Q^2 + \Big(\sigma - \frac{1}{2\epsilon}(\lambda_1 + \sigma\lambda_2)\Big)\|q_h^2\|_Q^2. \end{aligned}$$

Let us show that we can choose σ and ϵ so that $\frac{\epsilon}{2}(\lambda_1 + \sigma\lambda_2) < 1$ and $\frac{1}{2\epsilon}(\lambda_1 + \sigma\lambda_2) < \sigma$. We consider the quadratic equation $\Psi(t) := (\lambda_1 + t\lambda_2)^2 - 4t = 0$. Since the discriminant, $16(1 - \lambda_1\lambda_2)$, is positive and $\lambda_2 \neq 0$, $\Psi(t)$ has two distinct roots, t_-, t_+, and Ψ is minimal at $\frac{1}{2}(t_- + t_+) = \frac{2-\lambda_1\lambda_2}{\lambda_2^2}$. Therefore, if we choose $\sigma := \frac{2-\lambda_1\lambda_2}{\lambda_2^2}$, we have $\Psi(\sigma) < 0$, i.e., $\frac{1}{2}(\lambda_1 + \sigma\lambda_2) < \frac{2\sigma}{\lambda_1+\sigma\lambda_2}$. We then define ϵ by setting $\epsilon\sigma := \frac{1}{2}(\frac{1}{2}(\lambda_1 + \sigma\lambda_2) + \frac{2\sigma}{\lambda_1+\sigma\lambda_2})$. This choice in turn implies that $\epsilon\sigma < \frac{2\sigma}{\lambda_1+\sigma\lambda_2}$, i.e., $\frac{\epsilon}{2}(\lambda_1 + \sigma\lambda_2) < 1$ and that $\frac{1}{2}(\lambda_1 + \sigma\lambda_2) < \epsilon\sigma$, i.e., $\frac{1}{2\epsilon}(\lambda_1 + \sigma\lambda_2) < \sigma$. We have thus proved that $c_1 := 1 - \frac{\epsilon}{2}(\lambda_1 + \sigma\lambda_2) > 0$

and $c_2 := \sigma - \frac{1}{2\epsilon}(\lambda_1 + \sigma\lambda_2) > 0$. Then we conclude as above

$$\begin{aligned} b(\boldsymbol{v}_h, q_h) &\ge \frac{1}{2}\min(c_1, c_2)\|q_h\|_Q(\beta_1\|\boldsymbol{v}_h^1\|_{\boldsymbol{V}} + \beta_2\|\boldsymbol{v}_h^2\|_{\boldsymbol{V}}) \\ &\ge \frac{1}{2}\min(c_1, c_2)\min(\beta_1, \sigma^{-1}\beta_2)\|q_h\|_Q\|\boldsymbol{v}_h\|_{\boldsymbol{V}}, \end{aligned}$$

and the assertion follows with $\beta_h \ge \frac{1}{2}\min(c_1, c_2)\min(\beta_1, \sigma^{-1}\beta_2)$. Notice that $\lambda_2 \in [\frac{1}{2}, \frac{\|b\|}{\beta_1}]$ because $2\lambda_2 \ge \lambda_1 + \lambda_2 \ge 1$ and $b_{21} \le \|b\|$. Moreover, since $\sigma = \frac{2-\lambda_1\lambda_2}{\lambda_2^2}$ and $\epsilon = \frac{\lambda_2(3-\lambda_1\lambda_2)}{2(2-\lambda_1\lambda_2)}$, we obtain

$$c_1 = \frac{1-\lambda_1\lambda_2}{2(2-\lambda_1\lambda_2)}, \qquad c_2 = \frac{(1-\lambda_1\lambda_2)(2-\lambda_1\lambda_2)}{\lambda_2^2(3-\lambda_1\lambda_2)},$$

so that $c_1 \ge \frac{1}{4}(1-\lambda_1\lambda_2)$, $c_2 \ge \frac{\beta_1^2}{2\|b\|^2}(1-\lambda_1\lambda_2)$, $\sigma^{-1} \ge \frac{1}{8}$. Hence, we have $\beta_h \ge \frac{1}{32}\min(\frac{1}{2}, \frac{\beta_1^2}{\|b\|^2})(1-\lambda_1\lambda_2)\min(\beta_1, \beta_2) \ge \frac{1}{64}(1-\lambda_1\lambda_2)\frac{\min(\beta_1,\beta_2)^3}{\|b\|^2}$. □

Remark 55.2 (Inequality $\lambda_1\lambda_2 < 1$). This inequality, which amounts to $b_{12}b_{21} < \beta_1\beta_2$, is trivially satisfied if $b_{12}b_{21} = 0$, which is the case in many applications; see, e.g., Corollary 55.3 below. □

Let us illustrate the above result with the Stokes problem. We set $\boldsymbol{V} := \boldsymbol{H}_0^1(D)$, $Q := L_*^2(D)$, $\|\boldsymbol{v}\|_{\boldsymbol{V}} := |\boldsymbol{v}|_{\boldsymbol{H}^1(D)}$, $\|q\|_Q := \|q\|_{L^2(D)}$, and $b(\boldsymbol{v}, q) := -(\nabla\cdot\boldsymbol{v}, q)_{L^2(D)}$. Let $\mathcal{T}_h$ be a mesh in the sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$. Let $\mathcal{U}_h$ be a partition of the set $\mathcal{T}_h$. We call $\mathcal{U}_h$ *macroelement partition* and the members of $\mathcal{U}_h$ *macroelements*. For every macroelement $U \in \mathcal{U}_h$, we abuse the notation by writing U also for the set of the points composing the cells in the macroelement U. For all $U \in \mathcal{U}_h$, we define the following spaces:

$$\boldsymbol{V}_{h0}(U) := \{\boldsymbol{v}_h \in \boldsymbol{V}_{h0} \mid \boldsymbol{v}_{h|U} \in \boldsymbol{H}_0^1(U),\ \boldsymbol{v}_{h|D\setminus U} = 0\} \subset \boldsymbol{V}_{h0}, \tag{55.2a}$$

$$Q_h(U) := \{\mathbb{1}_U q_h \mid q_h \in Q_h\}, \tag{55.2b}$$

$$\overline{Q}_h(U) := \operatorname{span}(\mathbb{1}_U), \qquad \widetilde{Q}_h(U) := \{q_h \in Q_h(U) \mid \textstyle\int_U q_h \,\mathrm{d}x = 0\}, \tag{55.2c}$$

where $\mathbb{1}_U$ is the indicator function of U. We additionally define

$$\widetilde{Q}_h := \sum_{U\in\mathcal{U}_h} \widetilde{Q}_h(U), \qquad \overline{Q}_h := \sum_{U\in\mathcal{U}_h} \overline{Q}_h(U). \tag{55.3}$$

Corollary 55.3 (Macroelement partition). *Assume that for all $h \in \mathcal{H}$, there exists a partition of $\mathcal{T}_h$, say $\mathcal{U}_h$, such that*

$$\forall U \in \mathcal{U}_h, \qquad \inf_{q_h\in\widetilde{Q}_h(U)} \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}(U)} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q} =: \beta_{1h}(U) > 0, \tag{55.4a}$$

$$\inf_{q_h\in\overline{Q}_h} \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q} =: \beta_{2h} > 0. \tag{55.4b}$$

(i) *The inf-sup condition* (55.1) *is satisfied.* (ii) *If* $\inf_{h\in\mathcal{H}}\beta_{2h} > 0$ *and* $\inf_{h\in\mathcal{H}}\min_{U\in\mathcal{U}_h}\beta_{1h}(U) > 0$, *the inf-sup condition* (55.1) *holds uniformly w.r.t.* $h \in \mathcal{H}$.

Proof. The idea is to show that the assumptions of Lemma 55.1 are met.
(1) For all $q_h \in Q_h$ and all $U \in \mathcal{U}_h$, let us denote $\overline{q}_{hU} := \frac{1}{|U|}\int_U q_h\,\mathrm{d}x$. The identities $q_h = \sum_{U\in\mathcal{U}_h}\mathbb{1}_U q_h$ and $\mathbb{1}_U q_h = \mathbb{1}_U(q_h - \overline{q}_{hU}) + \overline{q}_{hU}\mathbb{1}_U$ show that $Q_h = Q_h^1 + Q_h^2$, with $Q_h^1 := \widetilde{Q}_h$ and $Q_h^2 := \overline{Q}_h$. Notice that this decomposition holds true whether Q_h is composed of discontinuous functions or not.
(2) Let us prove the first inf-sup condition from Lemma 55.1. Let $q_h \in Q_h^1 = \widetilde{Q}_h$. Then (55.4a) implies that for all $U \in \mathcal{U}_h$ there is $\boldsymbol{v}_h(U) \in \boldsymbol{V}_{h0}(U)$ s.t. $\nabla\cdot(\boldsymbol{v}_h(U)) = \mathbb{1}_U q_h$ and $\beta_{1h}(U)\|\boldsymbol{v}_h(U)\|_{\boldsymbol{V}} \le \|\mathbb{1}_U q_h\|_Q = \|q_h\|_{L^2(U)}$. Set $\boldsymbol{v}_h := \sum_{U\in\mathcal{U}_h}\boldsymbol{v}_h(U) \in \boldsymbol{V}_{h0}^1 := \sum_{U\in\mathcal{U}_h}\boldsymbol{V}_{h0}(U)$. Notice that $\boldsymbol{V}_{h0}^1 \subset \boldsymbol{V}_{h0}$ by construction. Using that $\big(\sum_{U\in\mathcal{U}_h}\|\boldsymbol{v}_h(U)\|_{\boldsymbol{V}}^2\big)^{\frac12} = \|\boldsymbol{v}_h\|_{\boldsymbol{V}}$, we infer that

$$\begin{aligned}
\int_D q_h\nabla\cdot\boldsymbol{v}_h\,\mathrm{d}x &= \sum_{U\in\mathcal{U}_h}\int_U q_h\nabla\cdot\boldsymbol{v}_h(U)\,\mathrm{d}x = \sum_{U\in\mathcal{U}_h}\|q_h\|_{L^2(U)}^2 \\
&= \|q_h\|_{L^2(D)}\Bigg(\sum_{U\in\mathcal{U}_h}\|q_h\|_{L^2(U)}^2\Bigg)^{\frac12} \ge \|q_h\|_Q\Bigg(\sum_{U\in\mathcal{U}_h}(\beta_{1h}(U))^2\|\boldsymbol{v}_h(U)\|_{\boldsymbol{V}}^2\Bigg)^{\frac12} \\
&\ge \beta_{1h}\|q_h\|_Q\Bigg(\sum_{U\in\mathcal{U}_h}\|\boldsymbol{v}_h(U)\|_{\boldsymbol{V}}^2\Bigg)^{\frac12} = \beta_{1h}\|q_h\|_Q\|\boldsymbol{v}_h\|_{\boldsymbol{V}},
\end{aligned}$$

$\beta_{1h} := \min_{U\in\mathcal{U}_h}\beta_{1h}(U) > 0$. Hence, $\inf_{q_h\in Q_h^1}\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}^1}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q} \ge \beta_{1h}$.
(3) The second inf-sup condition from Lemma 55.1 holds by assumption with $\boldsymbol{V}_{h0}^2 := \boldsymbol{V}_{h0}$, $Q_h^2 := \overline{Q}_h$, and the constant $\beta_{2h} > 0$.
(4) Finally, let us verify the last assumption by showing that $\lambda_1\lambda_2 := \frac{b_{12}b_{21}}{\beta_{1h}\beta_{2h}} = 0 < 1$. Let $\boldsymbol{v}_h := \sum_{U\in\mathcal{U}_h}\boldsymbol{v}_h(U) \in \boldsymbol{V}_{h0}^1$ and $q_h := \sum_{U\in\mathcal{U}_h}q_U\mathbb{1}_U \in Q_h^2$. We obtain

$$b(\boldsymbol{v}_h,q_h) = \sum_{U\in\mathcal{U}_h}q_U\int_U\nabla\cdot\boldsymbol{v}_h(U)\,\mathrm{d}x = 0,$$

since $\boldsymbol{v}_h(U) \in H_0^1(U)$ implies that $\int_U\nabla\cdot\boldsymbol{v}_h(U)\,\mathrm{d}x = 0$ for all $U \in \mathcal{U}_h$. Hence, $b_{21} = 0$. This completes the proof. $\square$

Remark 55.4 (Assumption (55.4a)**).** For all $q_h \in Q_h$, let $\overline{q}_h \in \overline{Q}_h$ be defined s.t. $\overline{q}_{h|U} := \overline{q}_{hU} := \frac{1}{|U|}\int_U q_h\,\mathrm{d}x$ for all $U \in \mathcal{U}_h$. Since $\int_U q_h\nabla\cdot\boldsymbol{v}_h\,\mathrm{d}x = \int_U(q_h - \overline{q}_{hU})\nabla\cdot\boldsymbol{v}_h\,\mathrm{d}x$ for all $\boldsymbol{v}_h \in \boldsymbol{V}_{h0}(U)$ and all $U \in \mathcal{U}_h$, the assumption (55.4a) means that for all $q_h \in Q_h$, we have $\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}(U)}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \ge \beta_{1h}(U)\|q_{h|U} - \overline{q}_{hU}\|_Q$. Then the argument in Step (2) of the proof of Corollary 55.3 shows that $\sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}^1}\frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \ge \beta_{1h}\|q_h - \overline{q}_h\|_Q$ for all $q_h \in Q_h$, where we have set $\beta_{1h} := \min_{U\in\mathcal{U}_h}\beta_{1h}(U)$. $\square$

Notice that $\widetilde{Q}_h \subset Q_h$ and $\overline{Q}_h \subset Q_h$ when Q_h is composed of discontinuous functions, but the above theory does not require that Q_h be composed of discontinuous finite elements. It turns out that the assumption (55.4b) can be relaxed if Q_h is H^1-conforming.

Proposition 55.5 (Macroelement, continuous pressures). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular mesh sequence. Assume that there exists a macroelement partition $\mathcal{U}_h$ for every mesh $\mathcal{T}_h$. Assume that every $U \in \mathcal{U}_h$ can be mapped by an affine mapping to a reference set $\widehat{U}$ and that the sequence $\{\mathcal{U}_h\}_{h\in\mathcal{H}}$ is shape-regular. Assume that $\inf_{h\in\mathcal{H}} \max_{U\in\mathcal{U}_h} \operatorname{card}\{K \subset U\} < \infty$. Assume that $Q_h \subset H^1(D)\cap L^2_*(D)$ and that the following holds true that for all $h \in \mathcal{H}$:*

$$\forall U \in \mathcal{U}_h, \qquad \inf_{q_h\in\widetilde{Q}_h(U)} \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}(U)} \frac{|b(\boldsymbol{v}_h, q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}\|q_h\|_Q} =: \beta_{1h}(U) > 0. \tag{55.5}$$

(i) *The inf-sup condition* (55.1) *is satisfied.* (ii) *If* $\inf_{h\in\mathcal{H}} \min_{U\in\mathcal{U}_h} \beta_{1h}(U) > 0$*, the inf-sup condition* (55.1) *holds uniformly w.r.t.* $h \in \mathcal{H}$.

Proof. See Brezzi and Bathe [91, Prop. 4.1] and Exercise 55.7. □

Remark 55.6 (Literature). Macroelement techniques have been introduced in a series of works by Boland and Nicolaides [67], Girault and Raviart [217, §II.1.4], Stenberg [352, 354, 353]. This theory is further refined in Qin [328, Chap. 3]. In particular, Lemma 55.1 is established in [328, Thm. 3.4.1]. It is possible to generalize the macroelement technique to situations where the macroelements are not disjoint provided one assumes that each cell K belongs to a finite set of macroelements with cardinality bounded from above uniformly w.r.t. $h \in \mathcal{H}$. This type of technique can be used in particular to prove the stability of the generalized Taylor–Hood elements $(\boldsymbol{\mathbb{P}}_k, \mathbb{P}_{k-1})$, $(\boldsymbol{\mathbb{Q}}_k, \mathbb{Q}_{k-1})$, $k \geq 2$. We refer the reader to Boffi et al. [65, §8.8] for a thorough discussion on this topic. □

55.2 Discontinuous pressures and bubbles

We investigate in this section finite element pairs based on simplicial meshes. The pressure approximation is discontinuous and stability is achieved by enriching the velocity space.

55.2.1 Discontinuous pressures

Since the functional space for the pressure is $Q := L^2_*(D)$, the approximation setting remains conforming for the pressure. The discrete pressure space is typically the broken polynomial space (see §18.1.2)

$$P^{\mathrm{b}}_{l,*}(\mathcal{T}_h) := \{q_h \in L^2_*(D) \mid \forall K \in \mathcal{T}_h,\, q_h \circ \boldsymbol{T}_K \in \mathbb{P}_{l,d}\}, \tag{55.6}$$

for some $l \in \mathbb{N}$ and where $\boldsymbol{T}_K : \widehat{K} \to K$ is the geometric mapping. The $(\mathbb{P}_k, \mathbb{P}_l^{\mathrm{b}})$ pair refers to the choice of finite element space $\boldsymbol{V}_{h0} := \boldsymbol{P}_{k,0}^{\mathrm{g}}(\mathcal{T}_h)$ for the velocity and $Q_h := P_{l,*}^{\mathrm{b}}(\mathcal{T}_h)$ for the pressure. The stable finite element pairs investigated herein are the $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ and the $(\mathbb{P}_2\text{-bubble}, \mathbb{P}_1^{\mathrm{b}})$ pairs.

Remark 55.7 (Local mass balance). Working with discontinuous pressures is interesting since it becomes possible to test the discrete mass conservation equation against a function supported in a single mesh cell $K \in \mathcal{T}_h$. This leads to the local mass balance $\int_K (\psi_K^{\mathrm{g}})^{-1}(q) \nabla\cdot \boldsymbol{u}_h \,\mathrm{d}x = \int_K (\psi_K^{\mathrm{g}})^{-1}(q) g \,\mathrm{d}x$ for all $q \in \mathbb{P}_{k,d}$ with $\psi_K^{\mathrm{g}}(q) := q \circ \boldsymbol{T}_K$, see Exercise 55.1. $\square$

55.2.2 The $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ pair

Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular family of affine simplicial meshes. Recalling that we are enforcing homogeneous Dirichlet conditions on the velocity, the $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ pair gives to the following approximation spaces:

$$\boldsymbol{V}_{h0} := \boldsymbol{P}_{2,0}^{\mathrm{g}}(\mathcal{T}_h), \qquad Q_h := P_{0,*}^{\mathrm{b}}(\mathcal{T}_h). \tag{55.7}$$

This simple finite element pair satisfies the inf-sup condition (55.1) uniformly w.r.t. $h \in \mathcal{H}$ in dimension two, but it has little practical interest since it is does not provide optimal convergence results. Nevertheless it is an important building block for other more useful finite element pairs. Let $\boldsymbol{V} := \boldsymbol{W}_0^{1,p}(D)$ be equipped with the norm $\|\boldsymbol{v}\|_{\boldsymbol{V}} := |\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$ and let $Q := L_*^{p'}(D)$ be equipped with the norm $\|q\|_Q := \|q\|_{L^{p'}(D)}$, where $p, p' \in (1, \infty)$ are s.t. $\frac{1}{p} + \frac{1}{p'} = 1$.

Lemma 55.8 (Stability). *Assume that $d = 2$. The $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ pair satisfies the inf-sup condition (55.1) uniformly w.r.t. $h \in \mathcal{H}$.*

Proof. We construct a Fortin operator by using the decomposition defined in Lemma 54.2 and by invoking Lemma 54.1 to conclude. The operator $\boldsymbol{\Pi}_{2h} : \boldsymbol{V} \to \boldsymbol{V}_{h0}$ is defined as follows. Let $\boldsymbol{v} \in \boldsymbol{V}_{h0}$. We set $\boldsymbol{\Pi}_{2h}(\boldsymbol{v})(\boldsymbol{z}) := \boldsymbol{0}$ for all $\boldsymbol{z} \in \mathcal{V}_h^\circ$ ($\mathcal{V}_h^\circ$ is the collection of the internal vertices of the mesh), and $\boldsymbol{\Pi}_{2h}(\boldsymbol{v})(\boldsymbol{m}_F) := \frac{3}{2|F|}\int_F \boldsymbol{v}\,\mathrm{d}s$ for all $F \in \mathcal{F}_h^\circ$ ($\mathcal{F}_h^\circ$ is the collection of the mesh interfaces), where $\boldsymbol{m}_F$ is the barycenter of F. This entirely defines $\boldsymbol{\Pi}_{2h}(\boldsymbol{v})$ in $\boldsymbol{V}_{h0}$ since $d = 2$. Notice that $\boldsymbol{v}_{|F} \in \boldsymbol{L}^1(F)$ for all $\boldsymbol{v} \in \boldsymbol{W}_0^{1,p}(D)$ and all $F \in \mathcal{F}_h^\circ$ so that the above construction is meaningful. Then we set $\boldsymbol{\Pi}_{1h} := \boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$, where $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}$ is the $\mathbb{R}^d$-valued version of the $W_0^{1,p}$-conforming quasi-interpolation operator introduced in §22.4.2. This means that $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{av}}(\boldsymbol{v}) := \sum_{i\in\{1:d\}} \mathcal{I}_{h0}^{\mathrm{av}}(v_i)\boldsymbol{e}_i$, where $\boldsymbol{v} := \sum_{i\in\{1:d\}} v_i \boldsymbol{e}_i$ and $\{\boldsymbol{e}_i\}_{i\in\{1:d\}}$ is the canonical Cartesian basis of $\mathbb{R}^d$. The rest of the proof consists of verifying that the assumptions (i)–(iii) from Lemma 54.2 are met; see Exercise 55.2. $\square$

Remark 55.9 (Literature). The reader is referred to Boffi et al. [65, §8.4.3] for other details on the $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ pair. In general, this pair is not stable in dimension 3, but it is shown in Zhang and Zhang [404] that one can construct special families of tetrahedral meshes for which stability holds. □

55.2.3 The $(\mathbb{P}_2\text{-bubble}, \mathbb{P}_1^{\mathrm{b}})$ pair

Let $\widehat{b}$ be the bubble function defined in (54.5) and $\widehat{\boldsymbol{P}} := \mathbb{P}_{2,d} \oplus (\mathrm{span}\{\widehat{b}\})^d$. Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular family of affine simplicial meshes. Recalling that we are enforcing homogeneous Dirichlet conditions on the velocity, the $(\mathbb{P}_2\text{-bubble}, \mathbb{P}_1^{\mathrm{b}})$ pair gives the following approximation spaces:

$$\boldsymbol{V}_{h0} := \boldsymbol{P}_{2,0}^{\mathrm{g}}(\mathcal{T}_h) \oplus \boldsymbol{B}_h, \qquad Q_h := P_{1,*}^{\mathrm{b}}(\mathcal{T}_h), \tag{55.8}$$

with $\boldsymbol{B}_h := \bigoplus_{K\in\mathcal{T}_h}(\mathrm{span}\{b_K\})^d$ and $b_K := \widehat{b}\circ\boldsymbol{T}_K$ is the bubble function associated with the mesh cell $K\in\mathcal{T}_h$. Notice that

$$\boldsymbol{V}_{h0} := \{\boldsymbol{v}_h \in \boldsymbol{C}^0(\overline{D}) \mid \forall K\in\mathcal{T}_h,\ \boldsymbol{v}_h\circ\boldsymbol{T}_K \in \widehat{\boldsymbol{P}},\ \boldsymbol{v}_{h|\partial D} = \boldsymbol{0}\}. \tag{55.9}$$

Since the pressure is locally $\mathbb{P}_1$ on each simplex and globally discontinuous, its local degrees of freedom can be taken to be its mean value and its gradient in each mesh cell. A conventional representation is shown in Figure 55.1. We have the following result (see Boffi et al. [65, p. 488]).

Proposition 55.10 ($\mathbb{P}_2$-bubble, $\mathbb{P}_1^{\mathrm{b}}$). *The $(\mathbb{P}_2\text{-bubble}, \mathbb{P}_1^{\mathrm{b}})$ pair satisfies the inf-sup condition (55.1) uniformly w.r.t. $h\in\mathcal{H}$. Moreover, this pair leads to the same error estimates as the Taylor–Hood element, that is, $\mu|\boldsymbol{u}-\boldsymbol{u}_h|_{\boldsymbol{H}^1(D)} + \|p-p_h\|_{L^2(D)} \le ch^2(\mu|\boldsymbol{u}|_{\boldsymbol{H}^3(D)} + |p|_{H^2(D)})$, and if the assumptions of Theorem 53.19 are met for some $s\in(0,1]$, then $\mu\|\boldsymbol{u}-\boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \le ch^{2+s}\ell_D^{1-s}(\mu|\boldsymbol{u}|_{\boldsymbol{H}^3(D)} + |p|_{H^2(D)})$.*

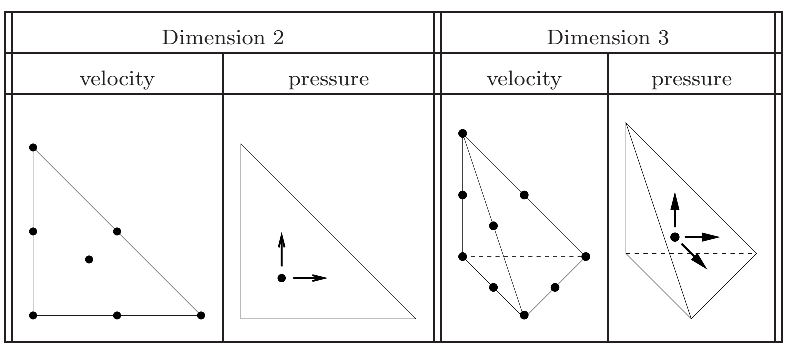

Fig. 55.1 Conventional representation of the $(\mathbb{P}_2\text{-bubble}, \mathbb{P}_1^{\mathrm{b}})$ pair in dimensions two (left) and three (right, only visible degrees of freedom of the velocity are shown). Among various possibilities, the degrees of freedom for the pressure here are the mean value (indicated by a dot) and the d components of the gradient (indicated by arrows).

Remark 55.11 (Literature). The ($\mathbb{P}_2$-bubble, $\mathbb{P}_1^{\mathrm{b}}$) pair is also called conforming *Crouzeix–Raviart mixed finite element* [151]. □

55.3 Scott–Vogelius elements and generalizations

Let $k \geq 1$. The $(\mathbb{P}_k, \mathbb{P}_{k-1}^{\mathrm{b}})$ pair is interesting since $\nabla{\cdot}\boldsymbol{P}_{k,0}^{\mathrm{g}}(\mathcal{T}_h) \subset P_{k-1,*}^{\mathrm{b}}(\mathcal{T}_h)$, which implies that any vector field in $\boldsymbol{P}_{k,0}^{\mathrm{g}}(\mathcal{T}_h)$ whose divergence is L^2-orthogonal to $P_{k-1,*}^{\mathrm{b}}(\mathcal{T}_h)$ is exactly divergence-free.

55.3.1 Special meshes

In general, the $(\mathbb{P}_k, \mathbb{P}_{k-1}^{\mathrm{b}})$ pair does not satisfy the inf-sup condition (55.1) (e.g., we have seen in §53.4.3 that for $k = 1$, this pair suffers from locking). However, it is possible to construct special meshes so that this element satisfies the inf-sup condition (55.1) uniformly w.r.t. $h \in \mathcal{H}$ for some k. Let us now introduce some special meshes to substantiate this claim. Various two-dimensional examples of such meshes are shown in Figure 55.2.

Irregular crisscross: A two-dimensional triangulation $\mathcal{T}_h$ is said to be an *irregular crisscross* mesh if it is obtained from a matching mesh of $D \subset \mathbb{R}^2$ composed of quadrangles, where each quadrilateral cell is divided along its two diagonals; see the leftmost panel in Figure 55.2.

Simplicial barycentric $(d+1)$-sected: We say that $\mathcal{T}_h$ is a *simplicial barycentric $(d+1)$-sected* mesh in $\mathbb{R}^d$ if $\mathcal{T}_h$ is obtained after refinement of a simplicial matching mesh by subdividing each initial simplex into $(d+1)$ subsimplices by connecting the barycenter with the $(d+1)$ vertices. Simplicial barycentric $(d+1)$-sected meshes are also called *Hsieh–Clough–Tocher (HCT)* meshes in the literature; see the second panel from the left in Figure 55.2.

Twice quadrisected crisscrossed: We say that a two-dimensional triangulation $\mathcal{T}_h$ is *twice quadrisected crisscrossed* if it is formed as follows. First, the polygon D is partitioned into a matching mesh of quadrangles, say $\mathcal{Q}_{4h}$. Then, each quadrangle in $\mathcal{Q}_{4h}$ is divided into four new quadrangles by connecting the point at the intersection of its two diagonals with the midpoint on each of its edges. The mesh Q_{2h} thus formed is subdivided once more by repeating this process. Finally, $\mathcal{T}_h$ is obtained by dividing each quadrangle in $\mathcal{Q}_h$ along its two diagonals, thereby giving 4 triangles per quadrangular cell in $\mathcal{Q}_h$, or 64 triangles for each quadrangle in Q_{4h}; see the third panel from the left in Figure 55.2.

Powell–Sabin: A simplicial mesh of a polygon or polyhedron D is said to be a *Powell–Sabin* mesh if it is constructed as follows. For instance, assuming that the space dimension is two, let $\mathcal{T}_h$ be an affine simplicial matching mesh of D. For each triangle $K \in \mathcal{T}_h$, let $\boldsymbol{c}_K$ be the center of the inscribed circle of K and assume that $\boldsymbol{c}_K \in K$ for all $K \in \mathcal{T}_h$. We then divide K into three triangles by connecting $\boldsymbol{c}_K$ to the three vertices of K (this is similar to an HCT triangulation). Each of the newly created triangles is divided again by connecting $\boldsymbol{c}_K$ to $\boldsymbol{c}_{K_1}$, $\boldsymbol{c}_{K_2}$, and $\boldsymbol{c}_{K_3}$, where K_1, K_2, and K_3 are the three neighbors of K (or $\boldsymbol{c}_K$ is connected to the midpoint of the edge if the corresponding neighbor does not exist). The same construction can be done in $\mathbb{R}^d$ as shown in Zhang [403, Fig. 1]. This construction is illustrated in the rightmost panel in Figure 55.2.

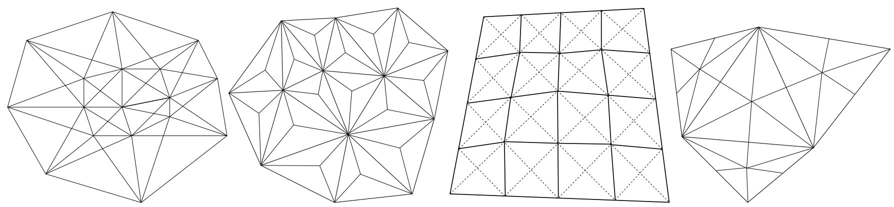

Fig. 55.2 Irregular crisscross mesh (left). Simplicial barycentric trisected mesh also called Hsieh–Clough–Tocher (HCT) mesh (center left). One quadrangular cell that is twice quadrisected and crisscrossed (center right). Powell–Sabin mesh (right).

55.3.2 Stable $(\mathbb{P}_k, \mathbb{P}^{\mathrm{b}}_{k-1})$ pairs on special meshes

The stability of the $(\mathbb{P}_k, \mathbb{P}^{\mathrm{b}}_{k-1})$ pair has been thoroughly investigated in dimension two by Scott and Vogelius [345].

Lemma 55.12 ($(\mathbb{P}_k, \mathbb{P}^{\mathrm{b}}_{k-1})$, $k \geq 4$, $d = 2$). *Let $d = 2$ and $k \geq 4$. Assume that the mesh sequence $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is quasi-uniform. Assume also that any pair of edges meeting at an internal vertex does not form a straight line. (An internal vertex violating this property is called singular vertex; see Exercise 54.3.) The $(\mathbb{P}_k, \mathbb{P}^{\mathrm{b}}_{k-1})$ pair satisfies the inf-sup condition* (55.1) *uniformly w.r.t. $h \in \mathcal{H}$.*

Proof. See [345, Thm. 5.1]. □

There are extensions of the above result to the $(\mathbb{P}_3, \mathbb{P}^{\mathrm{b}}_2)$ pair, the $(\mathbb{P}_2, \mathbb{P}^{\mathrm{b}}_1)$ pair, and the $(\mathbb{P}_1, \mathbb{P}^{\mathrm{b}}_0)$ pair in dimension two on some of the special meshes described above; see Qin [328].

Lemma 55.13 (Crisscross meshes, $k \in \{2, 3\}$, $d = 2$). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of irregular crisscross meshes. Then the $(\mathbb{P}_2, \mathbb{P}^{\mathrm{b}}_1)$ pair and the $(\mathbb{P}_3, \mathbb{P}^{\mathrm{b}}_2)$ pair have as many spurious pressure modes as singular vertices, but the velocity approximation is optimal, and the pressure approximation in the L^2-orthogonal complement to the spurious modes is optimal.*

Proof. See [328, Thm. 4.3.1 & 6.2.1]. □

Lemma 55.14 (HCT meshes, $k \in \{2, 3\}$, $d = 2$). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of barycentric trisected triangulations. Then the $(\mathbb{P}_2, \mathbb{P}^{\mathrm{b}}_1)$ pair and the $(\mathbb{P}_3, \mathbb{P}^{\mathrm{b}}_2)$ pair satisfy the inf-sup condition* (55.1) *uniformly w.r.t. $h \in \mathcal{H}$, and therefore lead to optimal error estimates.*

Proof. These statements are proved in Qin [328, Thm. 4.6.1 & 6.4.1]. We detail the proof for the $(\mathbb{P}_2, \mathbb{P}_1^{\rm b})$ pair since it illustrates the use of the macroelement technique from Corollary 55.3. Here, $\boldsymbol{V}_{h0} := \mathbb{P}^{\rm g}_{2,0}(\mathcal{T}_h)$ and $Q_h := \mathbb{P}^{\rm b}_{1,*}(\mathcal{T}_h)$.
(1) Let $(\mathcal{U}_h)_{h\in\mathcal{H}}$ be the sequence of triangulations that is used to create $(\mathcal{T}_h)_{h\in\mathcal{H}}$ by barycentric trisection. For every triangle $U \in \mathcal{U}_h$, we consider the spaces $\boldsymbol{V}_{h0}(U)$, $Q_h(U)$, $\overline{Q}_h(U)$, and $\widetilde{Q}_h(U)$ defined in (55.2). We also consider the spaces $\widetilde{Q}_h$, $\overline{Q}_h$ defined in (55.3). We are going to prove the inf-sup conditions (55.4a) and (55.4b) in Corollary 55.3.
(2) Proof of (55.4b). We have $\overline{Q}_h := \sum_{U\in\mathcal{U}_h} \overline{Q}_h(U) = P^{\rm b}_{0,*}(\mathcal{U}_h)$. Since, as established in Lemma 55.8, the $(\boldsymbol{P}^{\rm g}_{2,0}(\mathcal{U}_h), P^{\rm b}_{0,*}(\mathcal{U}_h))$ pair satisfies an inf-sup condition uniformly w.r.t. $h \in \mathcal{H}$, and $\boldsymbol{P}^{\rm g}_{2,0}(\mathcal{U}_h) \subset \boldsymbol{P}^{\rm g}_{2,0}(\mathcal{T}_h) =: \boldsymbol{V}_{h0}$, we infer that the inf-sup condition (55.4b) is satisfied uniformly w.r.t. $h \in \mathcal{H}$.
(3) Proof of (55.4a). Let $\widehat{U}$ be the reference simplex in $\mathbb{R}^2$. For every $U \in \mathcal{U}_h$, let $\boldsymbol{T}_U : \widehat{U} \to U$ be the corresponding affine geometric mapping. Let us set

$$\begin{aligned}
&\boldsymbol{V}(\widehat{U}) := \{\boldsymbol{\psi}^{\rm d}_U(\boldsymbol{v}_h) \mid \boldsymbol{v}_h \in \boldsymbol{V}_{h0}(U)\},\\
&Q(\widehat{U}) := \{\psi^{\rm g}_U(q_h) \mid q_h \in Q_h(U)\}, \qquad \widetilde{Q}(\widehat{U}) := \{\psi^{\rm g}_U(q_h) \mid q_h \in \widetilde{Q}_h(U)\},
\end{aligned}$$

where $\psi^{\rm g}_U$ is the pullback by $\boldsymbol{T}_U$ and $\boldsymbol{\psi}^{\rm d}_U$ is the contravariant Piola transformation, i.e., $\psi^{\rm g}_U(q) := q \circ \boldsymbol{T}_U$ and $\boldsymbol{\psi}^{\rm d}_U(\boldsymbol{v}) := \det(\mathbb{J}_U)\mathbb{J}_U^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_U)$ (see Definition 9.8). One can verify that both spaces $\boldsymbol{V}(\widehat{U})$ and $\widetilde{Q}(\widehat{U})$ are 8-dimensional, whereas the space $Q(\widehat{U})$ is 9-dimensional. Let $\widehat{B} : \boldsymbol{V}(\widehat{U}) \to Q(\widehat{U})$ be defined by $(\widehat{B}(\widehat{\boldsymbol{v}}), \widehat{q})_{L^2(\widehat{U})} = \int_{\widehat{U}} \widehat{q}(\widehat{\boldsymbol{x}})\nabla{\cdot}\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})\,{\rm d}\widehat{x}$ for all $(\widehat{\boldsymbol{v}}, \widehat{q}) \in \boldsymbol{V}(\widehat{U}){\times}Q(\widehat{U})$. A lengthy but straightforward computation (see Exercise 55.5) shows that $\mathrm{im}(\widehat{B})^{\perp} = \mathrm{span}(\mathbb{1}_{\widehat{U}})$, where $^{\perp}$ means the L^2-orthogonal complement in $Q(\widehat{U})$. Since $\widetilde{Q}(\widehat{U}) = (\mathrm{span}(\mathbb{1}_{\widehat{U}}))^{\perp}$, this result implies that $\widehat{B} : \boldsymbol{V}(\widehat{U}) \to \widetilde{Q}(\widehat{U})$ is surjective. (Actually, $\widehat{B}$ is bijective since $\dim(\boldsymbol{V}(\widehat{U})) = \dim(\widetilde{Q}(\widehat{U}))$.) Hence, we have

$$\inf_{\widehat{q}\in\widetilde{Q}(\widehat{U})} \sup_{\widehat{\boldsymbol{v}}\in\boldsymbol{V}(\widehat{U})} \frac{|\int_{\widehat{U}} \widehat{q}(\widehat{\boldsymbol{x}})\nabla{\cdot}\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})\,{\rm d}\widehat{x}|}{\|\widehat{q}\|_{Q(\widehat{U})}\|\widehat{\boldsymbol{v}}\|_{\boldsymbol{V}(\widehat{U})}} =: \widehat{\beta}_1 > 0,$$

with $\|\boldsymbol{v}\|_{\boldsymbol{V}(\widehat{U})} := |\widehat{\boldsymbol{v}}|_{\boldsymbol{H}^1(\widehat{U})}$ and $\|\widehat{q}\|_{Q(\widehat{U})} := \|\widehat{q}\|_{L^2(\widehat{U})}$. Using the scaling inequality (11.7b) and the regularity of the mesh sequence $(\mathcal{U}_h)_{h\in\mathcal{H}}$, we infer that there is $c_1 > 0$ s.t. $c_1\|\boldsymbol{v}\|_{\boldsymbol{V}}\|q\|_Q \le \|\widehat{\boldsymbol{v}}\|_{\boldsymbol{V}(\widehat{U})}\|\widehat{q}\|_{Q(\widehat{U})}$ for all $\boldsymbol{v} \in \boldsymbol{V}_{h0}(U)$, all $q \in Q_h(U)$, all $U \in \mathcal{U}_h$, and all $h \in \mathcal{H}$. Since $\int_{\widehat{U}} \widehat{q}(\widehat{\boldsymbol{x}})\nabla{\cdot}\widehat{\boldsymbol{v}}(\widehat{\boldsymbol{x}})\,{\rm d}\widehat{x} = \int_U q(\boldsymbol{x})\nabla{\cdot}\boldsymbol{v}(\boldsymbol{x})\,{\rm d}x$ (see Exercise 14.3(i)), we infer that

$$\inf_{q\in\widetilde{Q}(U)} \sup_{\boldsymbol{v}\in\boldsymbol{V}_{h0}(U)} \frac{|\int_U q(\boldsymbol{x})\nabla{\cdot}\boldsymbol{v}(\boldsymbol{x})\,{\rm d}x|}{\|q\|_{Q_h}\|\boldsymbol{v}\|_{\boldsymbol{V}}} =: \beta_1 \ge c_1\widehat{\beta}_2 > 0, \tag{55.10}$$

i.e., the inf-sup condition (55.4a) is satisfied uniformly w.r.t. $h \in \mathcal{H}$. □

The analysis of the $(\mathbb{P}_1, \mathbb{P}_0^{\mathrm{b}})$ pair is a little bit more subtle since filtering the spurious pressure modes is not enough to approximate the velocity and the pressure properly on general meshes, but filtering is sufficient on twice quadrisected crisscrossed meshes or Powell–Sabin meshes.

Lemma 55.15 ($(\mathbb{P}_1, \mathbb{P}_0^{\mathrm{b}})$). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular mesh sequence of either twice quadrisected crisscrossed meshes or Powell–Sabin meshes. Then the $(\mathbb{P}_1, \mathbb{P}_0^{\mathrm{b}})$ pair optimally approximates the velocity of the Stokes problem (i.e., first-order in the $\boldsymbol{H}^1$-seminorm) and the approximation of the pressure is optimal as well after post-processing the spurious pressure modes.*

Proof. See Qin [328, Thm. 7.4.2], Zhang [402]. □

Three-dimensional extensions of the above results are available.

Lemma 55.16 ($(\mathbb{P}_k, \mathbb{P}_{k-1}^{\mathrm{b}})$, $d = 3$). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of simplicial barycentric quadrisected meshes in $\mathbb{R}^3$. The $(\mathbb{P}_k, \mathbb{P}_{k-1}^{\mathrm{b}})$ pair is uniformly stable for all $k \geq 3$.*

Proof. See Zhang [401, Thm. 5]. □

Lemma 55.17 ($(\mathbb{P}_2, \mathbb{P}_1^{\mathrm{b}})$, $d = 3$). *Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of Powell–Sabin simplicial meshes in $\mathbb{R}^3$. The $(\mathbb{P}_2, \mathbb{P}_1^{\mathrm{b}})$ pair optimally approximates the velocity and after post-processing the spurious modes, the approximation of the pressure is optimal as well.*

Proof. See Zhang [403, Thm. 4.1]. □

55.4 Nonconforming and hybrid methods

In this section, we review some nonconforming and some hybrid discretization methods. Let us start with a nonconforming approximation technique based on the Crouzeix–Raviart finite element studied in Chapter 36. Let $(\mathcal{T}_h)_{h\in\mathcal{H}}$ be a shape-regular sequence of affine simplicial meshes. Let $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ be the Crouzeix–Raviart finite element space with homogeneous Dirichlet conditions (see (36.8)). Recall that $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ is composed of piecewise affine functions with continuous mean value across the mesh interfaces and zero mean value at the boundary faces. The $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair gives the following approximation spaces:

$$\boldsymbol{V}_{h0} := \boldsymbol{P}_{1,0}^{\mathrm{CR}}(\mathcal{T}_h), \qquad Q_h := P_{0,*}^{\mathrm{b}}(\mathcal{T}_h), \tag{55.11}$$

where $\boldsymbol{P}_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$ is composed of vector-valued functions with each Cartesian component in $P_{1,0}^{\mathrm{CR}}(\mathcal{T}_h)$. Observe that $\boldsymbol{V}_{h0}$ is nonconforming in $\boldsymbol{W}_0^{1,p}(D)$. The conventional representation of the $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair is shown in Figure 55.3.

To avoid technicalities related to the discrete version of Korn's inequality in $\boldsymbol{V}_{h0}$ (see §42.4.1), we assume in this section that the momentum equation in the Stokes equations is written in the Laplacian (or Cauchy–Navier)

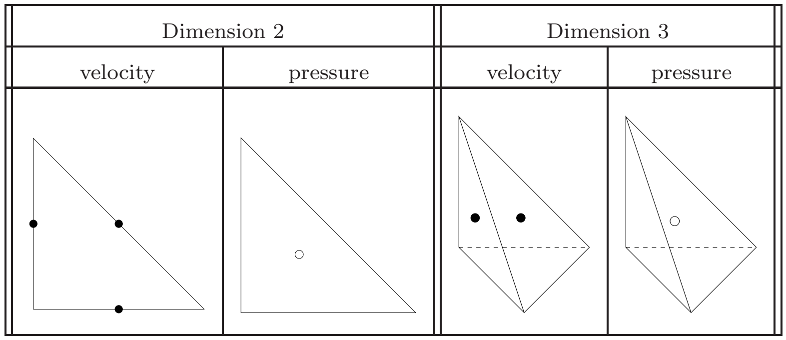

Fig. 55.3 Conventional representation of the $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair in dimensions two (left) and three (right, only visible velocity degrees of freedom are shown). The pressure degree of freedom is the average over each mesh cell.

form (see Remark 53.3), i.e., we replace the bilinear form a defined in (53.5) by $a(\boldsymbol{v}, \boldsymbol{w}) := \int_D \mu\nabla\boldsymbol{v}{:}\nabla\boldsymbol{w}\,\mathrm{d}x$. Since $\boldsymbol{V}_{h0}$ is nonconforming, we define the following discrete counterparts of the bilinear forms a and b:

$$a_h(\boldsymbol{v}_h, \boldsymbol{w}_h) := \sum_{K\in\mathcal{T}_h} \int_K \mu\nabla\boldsymbol{v}_h{:}\nabla\boldsymbol{w}_h\,\mathrm{d}x, \quad b_h(\boldsymbol{v}_h, q_h) := -\sum_{K\in\mathcal{T}_h} \int_K q_h\nabla{\cdot}\boldsymbol{v}_h\,\mathrm{d}x,$$

and consider the following discrete problem:

$$\begin{cases} \text{Find } \boldsymbol{u}_h \in \boldsymbol{V}_{h0} \text{ and } p_h \in Q_h \text{ such that} \\ a_h(\boldsymbol{u}_h, \boldsymbol{v}_h) + b_h(\boldsymbol{v}_h, p_h) = F(\boldsymbol{v}_h), \quad & \forall \boldsymbol{v}_h \in \boldsymbol{V}_{h0}, \\ b_h(\boldsymbol{u}_h, q_h) = G(q_h), & \forall q_h \in Q_h, \end{cases} \tag{55.12}$$

where the linear forms on the right-hand side are defined as before as $F(\boldsymbol{v}_h) := \int_D \boldsymbol{f}{\cdot}\boldsymbol{v}_h\,\mathrm{d}x$ and $G(q_h) := -\int_D g q_h\,\mathrm{d}x$. Let $p \in (1,\infty)$ and let us equip $\boldsymbol{V}_{h0}$ with the mesh-dependent norm $|\boldsymbol{v}_h|^p_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)} := \sum_{K\in\mathcal{T}_h} |\boldsymbol{v}_h|^p_{\boldsymbol{W}^{1,p}(K)}$ (the same reasoning as in the proof of Lemma 36.4 shows that $\boldsymbol{v}_h \mapsto |\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)}$ is indeed a norm on $\boldsymbol{V}_{h0}$).

Lemma 55.18 (Stability). *Let $p, p' \in (1,\infty)$ be s.t. $\frac{1}{p} + \frac{1}{p'} = 1$. There is β_0 such that for all $h \in \mathcal{H}$,*

$$\inf_{q_h\in Q_h} \sup_{\boldsymbol{v}_h\in\boldsymbol{V}_{h0}} \frac{|b_h(\boldsymbol{v}_h, q_h)|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)}\|q_h\|_{L^{p'}(D)}} \geq \beta_0 > 0. \tag{55.13}$$

Proof. For all $r \in L^p_*(D)$, there is $\boldsymbol{v}_r \in \boldsymbol{W}^{1,p}_0(D)$ s.t. $\nabla{\cdot}\boldsymbol{v}_r = r$ and $|\boldsymbol{v}_r|_{\boldsymbol{W}^{1,p}(D)} \leq c\|r\|_{L^p(D)}$ (see Remark 53.10). Let $\boldsymbol{\mathcal{I}}^{\mathrm{CR}}_{h0} : \boldsymbol{W}^{1,p}_0(D) \to \boldsymbol{V}_{h0}$ be the vector-valued Crouzeix–Raviart interpolation operator. Owing to the local commuting property established in Exercise 36.1, we have $b_h(\boldsymbol{\mathcal{I}}^{\mathrm{CR}}_{h0}(\boldsymbol{v}_r) -$

$\boldsymbol{v}_r, q_h) = 0$ for all $q_h \in Q_h$. Since

$$\int_D q_h r \,\mathrm{d}x = b(\boldsymbol{v}_r, q_h) = b_h(\boldsymbol{v}_r, q_h) = b_h(\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v}_r), q_h),$$

we infer that

$$\begin{aligned}\|q_h\|_{L^{p'}(D)} &\le \sup_{r\in L^p_*(D)} \frac{|\int_D q_h r \,\mathrm{d}x|}{\|r\|_{L^p(D)}} = \sup_{r\in L^p_*(D)} \frac{|b_h(\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v}_r), q_h)|}{\|r\|_{L^p(D)}} \\ &\le \sup_{\boldsymbol{v}_h\in \boldsymbol{V}_{h0}} \frac{|b_h(\boldsymbol{v}_h, q_h)|}{|\boldsymbol{v}_h|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)}} \times \sup_{r\in L^p_*(D)} \frac{|\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v}_r)|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)}}{\|r\|_{L^p(D)}}.\end{aligned}$$

Using the $\boldsymbol{W}^{1,p}_0$-stability of $\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}$ (see Lemma 36.1 with $r := 0$) together with the above bound on $\boldsymbol{v}_r$, we conclude that $\sup_{r\in L^p_*(D)} \frac{|\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v}_r)|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)}}{\|r\|_{L^p(D)}}$ is uniformly bounded w.r.t. $h \in \mathcal{H}$. This proves the expected inf-sup condition. □

Remark 55.19 (Convergence rate). The $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair is first-order accurate. More precisely, let $(\boldsymbol{u}, p)$ solve (53.6) and assume that $\boldsymbol{u} \in \boldsymbol{H}^2(D) \cap \boldsymbol{H}^1_0(D)$, $p \in H^1(D) \cap L^2_*(D)$. Then the solution to (53.14) with $(\boldsymbol{V}_{h0}, Q_h)$ defined in (55.11) satisfies $\mu\|\nabla_h(\boldsymbol{u}-\boldsymbol{u}_h)\|_{\mathbb{L}^2(D)} + \|p-p_h\|_{L^2(D)} \le ch(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$. Moreover, if the assumptions of Theorem 53.19 are met for some $s \in (0, 1]$, we have $\mu\|\boldsymbol{u} - \boldsymbol{u}_h\|_{\boldsymbol{L}^2(D)} \le ch^{1+s}\ell_D^{1-s}(\mu|\boldsymbol{u}|_{\boldsymbol{H}^2(D)} + |p|_{H^1(D)})$; see Exercise 55.4. □

Remark 55.20 (Literature). The $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair has been introduced by Crouzeix and Raviart [151]. A quadrilateral nonconforming mixed finite element has been introduced by Rannacher and Turek [330, 366]. □

Remark 55.21 (Fortin operator). The proof of Lemma 55.18 shows that the Crouzeix–Raviart interpolation operator acts as a nonconforming Fortin operator. Indeed, we have $\nabla{\cdot}(\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v})) = \Pi_K^0(\nabla{\cdot}\boldsymbol{v})$ for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p}_0(D)$ and all $K \in \mathcal{T}_h$ (see Exercise 36.1), and since any $q_h \in Q_h$ is piecewise constant, this implies that $b_h(\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v}) - \boldsymbol{v}, q_h) = 0$. Moreover, there is $\gamma_0 > 0$ s.t. $\gamma_0|\boldsymbol{\mathcal{I}}_{h0}^{\mathrm{CR}}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(\mathcal{T}_h)} \le |\boldsymbol{v}|_{\boldsymbol{W}^{1,p}(D)}$ for all $\boldsymbol{v} \in \boldsymbol{W}^{1,p}(D)$ and all $h \in \mathcal{H}$. □

An arbitrary-order discretization of the Stokes equations can be done by using the hybrid high-order (HHO) method introduced in §39.1. The method uses face-based and cell-based velocities together with discontinuous cell-based pressures. Let $k \in \mathbb{N}$ denote the degree of the velocity and pressure unknowns. As in Di Pietro et al. [169], one can take any $k \ge 0$ if one uses the Cauchy–Navier form of the momentum equation (see Remark 53.3). If one uses instead the formulation based on the linearized strain tensor (i.e., (53.1a) with (53.2)), then one can adapt the HHO method for the linear elasticity equations from Di Pietro and Ern [166] (see §42.4.3). In this case, one takes $k \ge 1$ since the analysis invokes a Korn inequality in each mesh cell. In practice, the size of the linear system can be significantly reduced since one can eliminate locally all the cell-based velocities and all the (cell-based) pressures up to a constant in each cell. The size of the linear system is thus reduced to $\dim(\mathbb{P}_{k,d-1})\times d\times N_{\mathrm{f}} + N_{\mathrm{c}}$, where N_{f} and N_{c} are the number of mesh faces and cells, respectively. Other methods using similar discrete unknowns are the hybridizable discontinuous Galerkin (HDG) methods developed by Egger and Waluga [184], Cockburn and Shi [132], and the related weak Galerkin methods from Wang and Ye [387]. See also Lehrenfeld and Schöberl [281] for HDG methods with $\boldsymbol{H}(\mathrm{div})$-velocities and Jeon et al. [254] for hybridized finite elements.

Remark 55.22 (Well-balanced scheme). For the $(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$ pair, the discrete velocity fields are divergence-free locally in each mesh cell, but since $\boldsymbol{V}_{h0}$ is nonconforming in $\boldsymbol{H}(\mathrm{div}; D)$ (the normal component of fields in $\boldsymbol{V}_{h0}$ can jump across the mesh interfaces), these fields are generally not divergence-free in D. Recalling Remark 53.22, this means that the discretization is not well-balanced, and this can lead to a poor velocity approximation in problems with large curl-free body forces. This issue has been addressed in Linke [283], where a well-balanced scheme is designed by using a lifting operator mapping the velocity test functions to the lowest-order Raviart–Thomas space in order to test the body forces in the discrete momentum balance equation. A similar modification is possible for the HHO discretization by using a lifting operator mapping the velocity test functions to the Raviart–Thomas space of the same degree as the face-based velocities; see [169]. $\square$

55.5 Stable pairs with $\mathbb{Q}_k$-based velocities

It is possible to used mixed finite elements based on quadrangular and hexahedral meshes. Since the literature on the topic is vast and this chapter is just meant to be a brief overview of the field, we only mention a few results. We assume in the entire section that $(\mathcal{T}_h)_{h\in\mathcal{H}}$ is a shape-regular sequence of affine meshes composed of cuboids. We start with a negative result.

Lemma 55.23 ($(\mathbb{Q}_k, \mathbb{Q}_{k-1}^{\mathrm{b}})$). *The $(\mathbb{Q}_k, \mathbb{Q}_{k-1}^{\mathrm{b}})$ pair composed of continuous $\mathbb{Q}_k$ elements for the velocity and discontinuous $\mathbb{Q}_{k-1}$ elements for the pressure does not satisfy the inf-sup condition for all $k \geq 1$.*

Proof. This result is established in Brezzi and Falk [92, Thm. 3.2]. A proof is proposed in Exercise 55.3. $\square$

It is possible to save the situation by removing some degrees of freedom in the pressure space. This can be done by considering the polynomial space $\mathbb{P}_l^{\mathrm{b}}$ instead of $\mathbb{Q}_l^{\mathrm{b}}$ with $l \in \{0, 1\}$.

Lemma 55.24 ($(\mathbb{Q}_2, \mathbb{P}_0^{\mathrm{b}})$). *The $(\mathbb{Q}_2, \mathbb{P}_0^{\mathrm{b}})$ pair satisfies the inf-sup condition* (53.15) *uniformly w.r.t. $h \in \mathcal{H}$ in $\mathbb{R}^2$.*

Proof. The proof is the same as that for the $(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$ pair. For every face/edge $F \in \mathcal{F}_h$ and every $\boldsymbol{v}_h \in \mathbb{Q}^{\mathrm{g}}_{2,0}(\mathcal{T}_h)$, $\boldsymbol{v}_{|F}\cdot\boldsymbol{n}_F$ is quadratic and one can use Simpson's quadrature rule to compute $\int_F \boldsymbol{v}_h\cdot\boldsymbol{n}_F \,\mathrm{d}s$; see Exercise 55.2. □

Lemma 55.25 ($(\mathbb{Q}_2, \mathbb{P}_1^{\mathrm{b}})$). *The $(\mathbb{Q}_2, \mathbb{P}_1^{\mathrm{b}})$ pair satisfies the inf-sup condition (53.15) uniformly w.r.t. $h \in \mathcal{H}$ in $\mathbb{R}^2$ and yields the same error estimates as the Taylor–Hood mixed finite element.*

Proof. The proof is similar to that of the $(\mathbb{P}_2, \mathbb{P}_1^{\mathrm{b}})$ pair. The reader is referred to Boffi et al. [65, §8.6.3.1] for other details and a literature review. □

Remark 55.26 ($\mathbb{Q}_1$ geometric transformation). Let us assume that for all $K \in \mathcal{T}_h$, the geometric finite element that is used to construct the cells in $\mathcal{T}_h$ is the Lagrange $\mathbb{Q}_1$ element; see §8.1. Then the $(\mathbb{Q}_2, \mathbb{P}_1^{\mathrm{b}})$ pair satisfies the inf-sup condition (53.15) uniformly w.r.t. $h \in \mathcal{H}$ in $\mathbb{R}^2$ (the proof is the same as that of Lemma 55.25), but, as shown in Arnold et al. [22], the approximation properties are suboptimal since in this case the polynomial space $\mathbb{P}_1$ is not rich enough to ensure optimal approximability of the pressure. □

Exercises

Exercise 55.1 (Local mass balance). Let $\boldsymbol{u}_h \in \boldsymbol{V}_{h0}$ and $g \in L^2_*(D)$ satisfy $\int_D q_h \nabla\cdot\boldsymbol{u}_h \,\mathrm{d}x = \int_D q_h g \,\mathrm{d}x$ for all $q_h \in P^{\mathrm{b}}_{k,*}(\mathcal{T}_h)$. Show that $\int_K (\psi^{\mathrm{g}}_K)^{-1}(q)\nabla\cdot\boldsymbol{u}_h \,\mathrm{d}x = \int_K (\psi^{\mathrm{g}}_K)^{-1}(q) g \,\mathrm{d}x$ for all $q \in \mathbb{P}_{k,d}$ and all $K \in \mathcal{T}_h$ with $\psi^{\mathrm{g}}_K(q) := q\circ\boldsymbol{T}_K$. (*Hint*: use that $\int_D \nabla\cdot\boldsymbol{u}_h \,\mathrm{d}x = \int_D g \,\mathrm{d}x = 0$.)

Exercise 55.2 ($(\mathbb{P}_2, \mathbb{P}_0^{\mathrm{b}})$). Complete the proof of Lemma 55.8. (*Hint*: to show that the assumption (ii) from Lemma 54.2 is met, prove that $\int_F (\boldsymbol{v} - \boldsymbol{\Pi}_{2h}(\boldsymbol{v}))\,\mathrm{d}s = \boldsymbol{0}$ for all $F \in \mathcal{F}^{\circ}_h$ using Simpson's quadrature rule; to show that the assumption (iii) is met, show first that $|\boldsymbol{\Pi}_{2h}(\boldsymbol{v})|_{\boldsymbol{W}^{1,p}(K)} \le c h_K^{\frac{1}{p}-1} \sum_{F\in\mathcal{F}^{\circ}_K} \|\boldsymbol{v}\|_{\boldsymbol{L}^p(F)}$ and then invoke the multiplicative trace inequality (12.16).)

Exercise 55.3 ($(\mathbb{Q}_k, \mathbb{Q}^{\mathrm{b}}_{k-1})$). (i) Justify Lemma 55.23 for $k := 2$ by constructing a counterexample. (*Hint*: given an interior vertex of a uniform Cartesian mesh, consider the patch composed of the four square cells sharing this vertex, and find an oscillating pressure field using (ii) from Exercise 54.3.) (ii) Generalize the argument for all $k \ge 2$.

Exercise 55.4 ($(\mathbb{P}_1^{\mathrm{CR}}, \mathbb{P}_0^{\mathrm{b}})$). Justify the claim in Remark 55.19. (*Hint*: see the proof of Theorem 36.11.)

Exercise 55.5 ($(\mathbb{P}_2, \mathbb{P}_1^{\mathrm{b}})$, HCT mesh). Using the notation from the proof of Lemma 55.14, the goal is to prove that $\mathrm{im}(\widehat{B})^{\perp} = \mathrm{span}(\mathbb{1}_{\widehat{U}})$. Let $\widehat{\boldsymbol{z}}_1 :=$

$(0,0)$, $\widehat{\boldsymbol{z}}_2 := (1,0)$, $\widehat{\boldsymbol{z}}_3 := (0,1)$, $\widehat{\boldsymbol{z}}_4 := (\frac{1}{3},\frac{1}{3})$. Consider the triangles $\widehat{K}_1 := \mathrm{conv}(\widehat{\boldsymbol{z}}_1,\widehat{\boldsymbol{z}}_2,\widehat{\boldsymbol{z}}_4)$, $\widehat{K}_2 := \mathrm{conv}(\widehat{\boldsymbol{z}}_2,\widehat{\boldsymbol{z}}_3,\widehat{\boldsymbol{z}}_4)$, and $\widehat{K}_3 := \mathrm{conv}(\widehat{\boldsymbol{z}}_3,\widehat{\boldsymbol{z}}_1,\widehat{\boldsymbol{z}}_4)$. Let $p \in P_1^{\mathrm{b}}(\widehat{U})$ with the reference macroelement $\widehat{U} := \{\widehat{K}_1,\widehat{K}_2,\widehat{K}_3\}$, and set

$$
\begin{aligned}
p_1 &:= p_{|\widehat{K}_1}(\widehat{\boldsymbol{z}}_1),\ p_2 := p_{|\widehat{K}_1}(\widehat{\boldsymbol{z}}_2),\ p_3 := p_{|\widehat{K}_1}(\widehat{\boldsymbol{z}}_4),\\
q_1 &:= p_{|\widehat{K}_2}(\widehat{\boldsymbol{z}}_2),\ q_2 := p_{|\widehat{K}_2}(\widehat{\boldsymbol{z}}_3),\ q_3 := p_{|\widehat{K}_2}(\widehat{\boldsymbol{z}}_4),\\
s_1 &:= p_{|\widehat{K}_3}(\widehat{\boldsymbol{z}}_3),\ s_2 := p_{|\widehat{K}_3}(\widehat{\boldsymbol{z}}_1),\ s_3 := p_{|\widehat{K}_3}(\widehat{\boldsymbol{z}}_4).
\end{aligned}
$$

Let $\widehat{\boldsymbol{m}}_{14} := \frac{1}{2}(\widehat{\boldsymbol{z}}_1+\widehat{\boldsymbol{z}}_4)$, $\widehat{\boldsymbol{m}}_{24} := \frac{1}{2}(\widehat{\boldsymbol{z}}_2+\widehat{\boldsymbol{z}}_4)$, $\widehat{\boldsymbol{m}}_{34} := \frac{1}{2}(\widehat{\boldsymbol{z}}_3+\widehat{\boldsymbol{z}}_4)$. Let $\boldsymbol{u} \in \boldsymbol{P}_{2,0}^{\mathrm{g}}(\widehat{U})$ and set $(u_7,v_7)^{\mathsf{T}} := \boldsymbol{u}(\widehat{\boldsymbol{m}}_{14})$, $(u_8,v_8)^{\mathsf{T}} := \boldsymbol{u}(\widehat{\boldsymbol{m}}_{24})$, $(u_9,v_9)^{\mathsf{T}} := \boldsymbol{u}(\widehat{\boldsymbol{m}}_{34})$, $(u_{10},v_{10})^{\mathsf{T}} := \boldsymbol{u}(\widehat{\boldsymbol{z}}_4)$. (i) Show (or accept as a fact) that

$$
\begin{aligned}
&\int_{\widehat{K}_1} p\nabla\cdot\boldsymbol{u}\,\mathrm{d}\widehat{x} = (-u_7+u_8+4v_7+2v_8)p_1\\
&+(-u_7+u_8+v_7+5v_8)p_2 + (-2u_7+2u_8-v_7+v_8+3v_{10})p_3.
\end{aligned}
$$

(*Hint*: compute the $\mathbb{P}_2$ shape functions on $\widehat{K}_1$ associated with the nodes $\widehat{\boldsymbol{m}}_{14}$, $\widehat{\boldsymbol{m}}_{24}$, and $\widehat{\boldsymbol{z}}_4$.) (ii) Let $\boldsymbol{T}_{\widehat{K}_2} : \widehat{K}_1 \to \widehat{K}_2$, $\boldsymbol{T}_{\widehat{K}_3} : \widehat{K}_1 \to \widehat{K}_3$ be the geometric mappings s.t.

$$
\boldsymbol{T}_{\widehat{K}_2}(\widehat{\boldsymbol{x}}) := \widehat{\boldsymbol{z}}_2 + \begin{pmatrix} -1 & -1\\ 1 & 0\end{pmatrix}(\widehat{\boldsymbol{x}}-\widehat{\boldsymbol{z}}_1), \quad \boldsymbol{T}_{\widehat{K}_3}(\widehat{\boldsymbol{x}}) := \widehat{\boldsymbol{z}}_3 + \begin{pmatrix} 0 & 1\\ -1 & -1\end{pmatrix}(\widehat{\boldsymbol{x}}-\widehat{\boldsymbol{z}}_1).
$$

Verify that $\boldsymbol{T}_{\widehat{K}_i}$ maps the vertices of $\widehat{K}_1$ to the vertices of $\widehat{K}_i$ for $i \in \{2,3\}$. (iii) Compute the contravariant Piola tranformations $\boldsymbol{\psi}^{\mathrm{d}}_{\widehat{K}_2}(\boldsymbol{v})$ and $\boldsymbol{\psi}^{\mathrm{d}}_{\widehat{K}_3}(\boldsymbol{v})$. (iv) Compute $\int_{\widehat{K}_i} p\nabla\cdot\boldsymbol{u}\,\mathrm{d}\widehat{x}$ for $i \in \{2,3\}$. (*Hint*: use Steps (i) and (iii), and $\int_{\widehat{K}_i} q\nabla\cdot\boldsymbol{v}\,\mathrm{d}\widehat{x} = \int_{\widehat{K}_1} \psi^{\mathrm{g}}_{K_i}(q)\nabla\cdot(\boldsymbol{\psi}^{\mathrm{d}}_{K_i}(\boldsymbol{v}))\,\mathrm{d}\widehat{x}$ (see Exercise 14.3(i)).) (v) Write the linear system corresponding to the statement $(\widehat{B}(\boldsymbol{u}),p)_{L^2(\widehat{U})} := \int_{\widehat{U}} p\nabla\cdot\boldsymbol{u}\,\mathrm{d}\widehat{x} = 0$ for all $\boldsymbol{u} \in \boldsymbol{P}^{\mathrm{g}}_{2,0}(\widehat{U})$, and compute $\mathrm{im}(\widehat{B})^{\perp}$.

Exercise 55.6 (Macroelement partition). Reprove Corollary 55.3 without invoking the partition lemma (Lemma 55.1). (*Hint*: see Brezzi and Bathe [91, Prop.4.2].)

Exercise 55.7 (Macroelement, continuous pressure). Let the assumptions of Proposition 55.5 hold true. (i) Show that there are $c_1, c_2 > 0$ s.t. $\sup_{\boldsymbol{v}_h\in \boldsymbol{V}_{h0}} \frac{|b(\boldsymbol{v}_h,q_h)|}{\|\boldsymbol{v}_h\|_{\boldsymbol{V}}} \ge c_1\beta_D\|q_h\|_Q - c_2\big(\sum_{U\in\mathcal{U}_h} h_U^2|q_h|^2_{H^1(U)}\big)^{\frac{1}{2}}$, for all $q_h \in Q_h$ and all $h \in \mathcal{H}$. (*Hint:* use the quasi-interpolation operator $\boldsymbol{\mathcal{I}}^{\mathrm{av}}_{h0}$ and proceed as in the proof of Lemma 54.3.) (ii) Setting $\overline{q}_{hU} := \frac{1}{|U|}\int_U q_h\,\mathrm{d}x$, show that there is c s.t. $|q_{h|U}|_{H^1(U)} \le c\|q_h - \overline{q}_{hU}\|_{L^2(\widehat{U})}$ for all $U \in \mathcal{U}_h$ and all $h \in \mathcal{H}$. (*Hint*: use Lemma 11.7 and the affine geometric mapping $\boldsymbol{T}_U : \widehat{U} \to U$.) (iii) Prove Corollary 55.5. (*Hint*: use Remark 55.4. See also Brezzi and Bathe [91, Prop 4.1].)