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ERROR ANALYSIS OF PRESSURE-CORRECTION SCHEMES FOR THE TIME-DEPENDENT STOKES EQUATIONS WITH OPEN BOUNDARY CONDITIONS*

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Abstract. The incompressible Stokes equations with prescribed normal stress (open) boundary conditions on part of the boundary are considered. It is shown that the standard pressure-correction method is not suitable for approximating the Stokes equations with open boundary conditions, whereas the rotational pressure-correction method yields reasonably good error estimates. These results appear to be the first ever published for splitting schemes with open boundary conditions. Numerical results in agreement with the error estimates are presented.

Key words. Navier–Stokes and Stokes equations, incompressibility, pressure-correction methods, open boundary conditions, finite elements, spectral approximations

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1. Introduction.

In this paper we consider the time-dependent Navier–Stokes equations with normal stress boundary conditions prescribed on parts of the boundary. These conditions are usually imposed to model outflow boundaries or free surfaces. For Newtonian flows, the boundary conditions in question take the form

$$\left[\mathsf{p}n - \nu(\nabla\mathsf{u} + (\nabla\mathsf{u})^T)n\right]|_\Gamma = b,$$

where u is the velocity vector field, p is the pressure, Γ is the boundary of the domain Ω, n is the unit outward normal, and b is the prescribed data.

There are numerous ways to discretize the time-dependent incompressible Navier–Stokes equations in time. Undoubtedly, the most popular one consists of using projection methods. Most of these techniques are based on the original ideas of Chorin [2] and Temam [22]. They are usually fractional step methods composed of two substeps such that either the Laplacian of the velocity or the pressure gradient is made explicit in one substep and (implicitly) corrected in the other substep. In both cases, one substep always consists of the projection of some vector field onto a divergence-free space. Following the terminology introduced in [11], a scheme is classified as a pressure-correction (resp., velocity-correction) method if the pressure gradient (resp., Laplacian of the velocity) is treated explicitly in one substep and (implicitly) corrected in the other substep. In the present paper we restrict ourselves to pressure-correction methods. Each of the above two classes of methods has a standard form and a rotational form (see [9, 10]), and each of them can be implemented either in algebraic

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form (cf. [4, 5, 15]) or in differential form. However, to the best of our knowledge, no rigorous error analysis of any of these schemes with open boundary conditions is available in the literature. Moreover, there is some confusion in the literature over the performance of these methods with this type of boundary condition. The aim of this paper is to discuss some of these issues and to derive error estimates.

We show that the standard pressure-correction schemes, implemented either in algebraic form or in differential form (in fact, they can be shown to be equivalent), are not suitable for approximating the Navier–Stokes equations supplemented with open boundary conditions. However, we show that the rotational pressure-correction schemes yield reasonable error estimates. More precisely, assuming full regularity of the Stokes problem, the second-order rotational pressure-correction method yields $\mathcal{O}(\Delta t^{3/2})$ convergence rate for the velocity in the L^2-norm and $\mathcal{O}(\Delta t)$ convergence rate for both the velocity in the H^1-norm and the pressure in the L^2-norm. These estimates deteriorate if the Stokes problem does not possesses full regularity, as is probably the case in three dimensions.

2. Preliminaries. We shall consider the time-dependent Navier–Stokes equations on a finite time interval $[0, T]$ and in an open, connected, bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$, or 3) with a boundary Γ sufficiently smooth. We assume that the following nontrivial partition holds: $\Gamma = \Gamma_1 \cup \Gamma_2$, $\Gamma_1 \cap \Gamma_2 = \emptyset$, $\text{meas}(\Gamma_1) \neq \emptyset$, $\text{meas}(\Gamma_2) \neq \emptyset$.

2.1. Notation. We denote by $H^m(\Omega)$ and $\|\cdot\|_m$ ($m = 0, \pm 1, \ldots$) the standard Sobolev spaces and norms. In particular, the norm and inner product of $L^2(\Omega) = H^0(\Omega)$ are denoted by $\|\cdot\|_0$ and $(\cdot,\cdot)$, respectively. We shall also make use of fractional Sobolev spaces $H^s(\Omega)$ which are defined by interpolation. To account for homogeneous Dirichlet boundary conditions on Γ_1, we define

$$X = \{v \in H^1(\Omega) : v|_{\Gamma_1} = 0\}. \tag{2.1}$$

Owing to the Poincaré inequality, $\|\nabla v\|_0$ is a norm equivalent to $\|v\|_1$ for all $v \in X$. Henceforth, we redefine the norm $\|\cdot\|_1$ in X such that $\|v\|_1 := \|\nabla v\|_0$.

We introduce two spaces of incompressible vector fields,

$$H = \{v \in L^2(\Omega)^d;\ \nabla\cdot v = 0;\ v \cdot n|_{\Gamma_1} = 0\}, \tag{2.2}$$

$$V = \{v \in H^1(\Omega)^d;\ \nabla\cdot v = 0;\ v|_{\Gamma_1} = 0\}, \tag{2.3}$$

and we define P_H to be the L^2-orthogonal projection onto H, i.e.,

$$(u - P_H u, v) = 0 \quad \forall u \in L^2(\Omega)^d,\ \forall v \in H. \tag{2.4}$$

We also denote

$$N = \{q \in H^1(\Omega);\ q|_{\Gamma_2} = 0\}. \tag{2.5}$$

The following well-known lemma plays a key role in the analysis of projection methods.

LEMMA 2.1. *The following orthogonal decomposition of $L^2(\Omega)^d$ holds:*

$$L^2(\Omega)^d = H \oplus \nabla N. \tag{2.6}$$

Since the nonlinear term in the Navier–Stokes equations has a marginal influence on the splitting error, we shall hereafter consider only the time-dependent Stokes equations written in terms of velocity, u, and pressure, p:

$$\begin{cases} \partial_t \mathsf{u} + A\mathsf{u} + \nabla \mathsf{p} = f \quad \text{in } \Omega \times [0,T], \\ \nabla \cdot \mathsf{u} = 0 \quad \text{in } \Omega \times [0,T], \\ \mathsf{u}|_{\Gamma_1} = 0, \quad \text{and} \quad (\mathsf{p}n - \nu(D\mathsf{u})n)|_{\Gamma_2} = 0 \quad \text{in } [0,T], \\ \mathsf{u}|_{t=0} = u_0 \quad \text{in } \Omega. \end{cases} \tag{2.7}$$

Henceforth, the operators A and D may assume one of the two following forms:

$$Av = -2\nu \nabla \cdot Dv, \tag{2.8}$$

$$Dv = \begin{cases} \frac{1}{2}\nabla v, & \text{case 1,} \\ \frac{1}{2}(\nabla v + (\nabla v)^T), & \text{case 2.} \end{cases} \tag{2.9}$$

We recall that the symmetric positive definite bilinear form

$$a(u,v) = \nu(Du, Dv) \tag{2.10}$$

induces a norm on X that is equivalent to the H^1-norm. We denote by α the coercivity constant of a:

$$a(v,v) \geq \alpha \|\nabla v\|_0^2 \quad \forall v \in X. \tag{2.11}$$

In case 1, $\alpha = \nu$, whereas in case 2, $\alpha = c\nu$, where c is a constant that can be derived by using a Korn inequality; see, e.g., [1].

To simplify our presentation, we assume that the unique solution (u, p) to the above system is as smooth as needed.

To perform the temporal discretization of the problem, we define $\Delta t > 0$ to be a time step and we set $t^k = k\Delta t$ for $0 \leq k \leq K = [T/\Delta t]$. Let $\phi^0, \phi^1, \dots, \phi^K$ be a sequence of functions in some Hilbert space E. We denote by $\phi_{\Delta t}$ this sequence, and we use the following discrete norms:

$$\|\phi_{\Delta t}\|_{\ell^2(E)} := \left(\Delta t \sum_{k=0}^{K} \|\phi^k\|_E^2 \right)^{1/2}, \qquad \|\phi_{\Delta t}\|_{\ell^\infty(E)} := \max_{0 \leq k \leq K} \left(\|\phi^k\|_E \right). \tag{2.12}$$

We denote by c a generic constant that is independent of small parameters like ϵ, Δt, and h but possibly depends on the data and the solution. We shall use the expression $A \lesssim B$ to say that there exists a generic constant c such that $A \leq cB$.

Let μ be a positive real number. We shall repeatedly make use of the following interpolation result, whose proof is fairly standard and so we omit it due to the space limitation.

LEMMA 2.2. *For all $0 \leq s \leq 1$, there exists an operator $\mathcal{I}_{\mu,s} : H^s(\Omega) \longrightarrow H_0^1(\Omega)$ such that for all r in $H^s(\Omega)$ we have*

$$\|r - \mathcal{I}_{\mu,s} r\|_0 \lesssim \mu^{\frac{s}{2}} \|r\|_{H^s(\Omega)}, \tag{2.13}$$

$$\|\mathcal{I}_{\mu,s} r\|_1 \lesssim \mu^{-1+\frac{s}{2}} \|r\|_{H^s(\Omega)}. \tag{2.14}$$

2.1.1. The inverse of the Stokes operator and its regularity index. In this section we recall properties of the inverse of the Stokes operator. Let X' be the dual space of X. We denote by $\langle \cdot, \cdot \rangle$ the duality pairing between X' and X. The inverse of the Stokes operator, which we shall denote by $S : X' \longrightarrow X$, is defined as follows. For all v in X', $S(v) \in X$ is the solution to the dual problem

$$\begin{cases} a(w, S(v)) - (r, \nabla\cdot w) = \langle v, w\rangle & \forall w \in X, \\ (q, \nabla\cdot S(v)) = 0 & \forall q \in L^2(\Omega). \end{cases} \tag{2.15}$$

Obviously, we have

$$\forall v \in X', \qquad \|S(v)\|_1 + \|r\|_0 \le c\|v\|_{X'}. \tag{2.16}$$

It is well known that when Dirichlet boundary conditions on the velocity are enforced on the entire boundary and Ω is smooth or convex, we have $\|r\|_1 \lesssim \|v\|_0$ (see, for instance, [23]). In the present case, where boundary conditions are mixed, it is a nontrivial task to determine the regularity of r. It is generally expected that the H^1-regularity does not hold in the three-dimensional case. However, it is possible that regularity in some fractional Sobolev space holds. To account for this, we make the following definition.

Definition 2.1 (regularity index of the Stokes operator). *The regularity index of the Stokes operator is the largest number, s, such that for all $v \in L^2(\Omega)^d$, the solution $r \in L^2(\Omega)$ to the dual Stokes problem* (2.15) *satisfies* $\|r\|_{H^s(\Omega)} \lesssim \|v\|_0$.

We observe from (2.16) that $s \ge 0$, and it is clear that $s \le 1$. Hence, the case $s = 0$ is referred to as *no regularity* while the case $s = 1$ is referred to as *full regularity*. We refer to [14] for techniques to evaluate this index in two dimensions.

The operator S has interesting properties, as listed below.

Lemma 2.3. *For all v in X, all $0 < \gamma < 1$, and all $0 < \mu < 1$, we have*

$$a(v, S(v)) \ge (1-\gamma)\|v\|_0^2 - c(\gamma)\left(\mu^{2\alpha_1}\|\nabla\cdot v\|_0^2 + \mu^{-2\alpha_2}\|v - P_H v\|_0^2\right),$$

with $\alpha_1 = \frac{s}{2}$ and $\alpha_2 = 1 - \frac{s}{2}$ and s being the regularity index of the Stokes operator. In particular, for all $v \in V$, $(\nabla S(v), \nabla v) = \|v\|_0^2$.

Proof. Owing to the definition of $S(v)$ and to the fact $\mathcal{I}_{\varepsilon,s} r$ is zero on Γ_2, we have

$$\begin{aligned} a(v, S(v)) &= \|v\|_0^2 + (r, \nabla\cdot v) \\ &= \|v\|_0^2 + (r - \mathcal{I}_{\mu,s} r, \nabla\cdot v) + (\nabla \mathcal{I}_{\mu,s} r, v) \\ &= \|v\|_0^2 + (r - \mathcal{I}_{\mu,s} r, \nabla\cdot v) + (\nabla \mathcal{I}_{\mu,s} r, v - P_H v) \\ &\ge \|v\|_0^2 - \left(\mu^{\alpha_1}\|\nabla\cdot v\|_0 + \mu^{-\alpha_2}\|v - P_H v\|_0\right)\|r\|_{H^s(\Omega)}. \end{aligned}$$

Then using the fact that s is the regularity index of the Stokes operator (see Definition 2.1), we derive the desired bound. ◻

Lemma 2.4. *The bilinear form $X' \times X' \ni (v, w) \longmapsto \langle S(v), w\rangle := a(S(v), S(w)) \in \mathbb{R}$ induces a seminorm on X' that we denote by $|\cdot|_\star$, and*

$$\forall v \in X', \qquad |v|_\star = a(S(v), S(v))^{1/2} \lesssim \|v\|_{X'}.$$

Proof. It is clear that the bilinear form is symmetric, $\langle S(v), w\rangle = a(S(v), S(w)) = \langle S(w), v\rangle$, and positive, $\langle S(v), v\rangle = a(S(v), S(v))$; hence, $\langle S(v), w\rangle$ induces a seminorm on X'. Furthermore, $|v|_\star^2 = \langle S(v), v\rangle = a(S(v), S(v)) \lesssim \|v\|_{X'}^2$. The proof is complete. ◻

3. Standard pressure-correction methods. For purely Dirichlet boundary conditions, the second-order pressure-correction scheme is known to be one-order more accurate than the original projection scheme of Chorin–Temam (cf. [25, 3, 21, 7]). Using the second-order backward difference formula (BDF2) to discretize the time derivative, the second-order pressure-correction scheme takes the following form:

Set $u^0 = u_0$, $p^0 = \mathsf{p}|_{t=0}$, which can be computed from the data, and compute $(\tilde{u}^1, u^1, p^1)$ by using the scheme below with BDF2 replaced by the backward Euler formula. Then, for $k \geq 1$, compute $(\tilde{u}^{k+1}, u^{k+1}, p^{k+1})$ such that

$$\begin{cases} \dfrac{3\tilde{u}^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + A\tilde{u}^{k+1} + \nabla p^k = f(t^{k+1}), \\ \tilde{u}^{k+1}|_{\Gamma_1} = 0 \quad \text{and} \quad (p^k n - \nu(D\tilde{u}^{k+1})n)|_{\Gamma_2} = 0 \end{cases} \tag{3.1}$$

and

$$\begin{cases} \dfrac{3u^{k+1} - 3\tilde{u}^{k+1}}{2\Delta t} + \nabla(p^{k+1} - p^k) = 0, \\ \nabla\cdot u^{k+1} = 0, \\ u^{k+1}\cdot n|_{\Gamma_1} = 0 \quad \text{and} \quad (p^{k+1} - p^k)|_{\Gamma_2} = 0. \end{cases} \tag{3.2}$$

The first substep accounts for viscous effects, whereas the second one accounts for incompressibility. The second substep is usually referred to as the projection step, for it is a realization of the identity $u^{k+1} = P_H\tilde{u}^{k+1}$. We emphasize that it is essential, for stability considerations, that $(p^{k+1} - p^k)|_{\Gamma_2} = 0$ is enforced. Otherwise, (3.2) can not be interpreted as a projection step. Note that the boundary conditions in (3.2) lead to the series of equalities

$$\begin{aligned} &\frac{\partial}{\partial n}p^{k+1}|_{\Gamma_1} = \frac{\partial}{\partial n}p^k|_{\Gamma_1} = \cdots = \frac{\partial}{\partial n}p^1|_{\Gamma_1}, \\ &p^{k+1}|_{\Gamma_2} = p^k|_{\Gamma_2} = \cdots = p^1|_{\Gamma_2}, \end{aligned} \tag{3.3}$$

which are certainly inaccurate since they are almost never satisfied by the exact solution. In the purely Dirichlet case, i.e., $\Gamma_2 = \emptyset$, it is possible to deduce a reasonably good approximation result for the pressure in the L^2-norm. But when $\Gamma_2 \neq \emptyset$ the pressure approximation is severely degraded.

Not being aware of any published convergence result for the scheme (3.1)–(3.2), we shall prove the following result.

THEOREM 3.1. *If* (u, p)*, the solution to* (2.7)*, is smooth enough in space and time, the solution to* (3.1)–(3.2) *satisfies the following error estimates:*

$$\begin{aligned} \|\mathsf{p}_{\Delta t} - p_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^\infty(H^1(\Omega)^d)} &\lesssim \Delta t^{\frac{1}{2}}, \\ \|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} + \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} &\lesssim \Delta t^{\frac{s+1}{2}}, \end{aligned}$$

where s is the regularity index of the Stokes operator.

Proof. As will become clear in the course of the proof, using BDF2 instead of the backward Euler formula does not improve the accuracy in the presence of open boundary conditions. So to simplify the presentation, we consider the backward Euler formula for the time derivative:

$$\begin{cases} \dfrac{\tilde{u}^{k+1} - u^k}{\Delta t} + A\tilde{u}^{k+1} + \nabla p^k = f(t^{k+1}), \\ \tilde{u}^{k+1}|_{\Gamma_1} = 0 \quad \text{and} \quad (p^k n - \nu(D\tilde{u}^{k+1})n)|_{\Gamma_2} = 0 \end{cases} \tag{3.4}$$

and

$$
\begin{cases}
\dfrac{u^{k+1}-\tilde{u}^{k+1}}{\Delta t} + \nabla(p^{k+1}-p^k) = 0, \\
\nabla\cdot u^{k+1} = 0, \\
u^{k+1}\cdot n|_{\Gamma_1} = 0 \quad \text{and} \quad (p^{k+1}-p^k)|_{\Gamma_2} = 0.
\end{cases} \tag{3.5}
$$

Technically, the proof is very similar to those in Shen [21] and Guermond [6]; hence we show only those steps where the consistency error is degraded.

Let us introduce the interpolation operator $\mathcal{I}_{\Delta t,1} : H^1(\Omega) \longmapsto H^1_0(\Omega)$ defined in Lemma 2.2. This operator is such that for all r in $H^1(\Omega)$,

$$
\|\mathcal{I}_{\Delta t,1} r - r\|_0 \lesssim \Delta t^{\frac{1}{2}} \|r\|_1, \tag{3.6}
$$

$$
\|\nabla \mathcal{I}_{\Delta t,1} r\|_0 \lesssim \Delta t^{-\frac{1}{2}} \|r\|_1. \tag{3.7}
$$

Without introducing any essential extra error, we can take $p^0 = \mathcal{I}_{\Delta t,1}\mathsf{p}|_{t=0}$, which implies $p^k|_{\Gamma_2} = 0$ for all k.

Now we introduce the following notation:

$$
\begin{cases}
e^k = \mathsf{u}(t^k) - u^k, & \tilde{e}^k = \mathsf{u}(t^k) - \tilde{u}^k, \\
\psi^k = \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}) - p^k, & q^k = \mathcal{I}_{\Delta t,1}\mathsf{p}(t^k) - p^k.
\end{cases}
$$

The weak form of the error equation that corresponds to the viscous step (3.4) is given by

$$
\begin{aligned}
\frac{1}{\Delta t}(\tilde{e}^{k+1} - e^k, v) + a(\tilde{e}^{k+1}, v) - (\psi^k, \nabla\cdot v) &= (R(t^{k+1}), v) \\
&\quad + (\mathsf{p}(t^{k+1}) - \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}), \nabla\cdot v) \quad \forall v \in X,
\end{aligned}
$$

where $R(t^{k+1}) = \frac{1}{\Delta t}(u(t^{k+1}) - u(t^k)) - u_t(t^{k+1}) = \mathcal{O}(\Delta t)$. Note that the surface integrals resulting from the integration by parts cancel on both Γ_1 and Γ_2 due to the boundary conditions in (3.4).

Taking $v = 2\Delta t\tilde{e}^{k+1}$ in the above equation and using (3.6), we can derive

$$
2\Delta t(\mathsf{p}(t^{k+1}) - \mathcal{I}_{\Delta t,1}\mathsf{p}(t^{k+1}), \nabla\cdot\tilde{e}^{k+1}) \lesssim \Delta t^2 + \alpha\Delta t\|\tilde{e}^{k+1}\|_1^2,
$$

$$
\|\tilde{e}^{k+1}\|_0^2 + \|\tilde{e}^{k+1} - e^k\|_0^2 + \alpha\Delta t\|\tilde{e}^{k+1}\|_1^2 - 2\Delta t(\psi^k, \nabla\cdot\tilde{e}^{k+1}) \le \|e^k\|_0^2 + c\Delta t^2. \tag{3.8}
$$

Note that the consistency error is degraded at this step; more precisely, a Δt factor is already missing in the above estimate.

The error equation corresponding to the projection step (3.5) can be written as

$$
\begin{cases}
\dfrac{1}{\Delta t}e^{k+1} + \nabla q^{k+1} = \dfrac{1}{\Delta t}\tilde{e}^{k+1} + \nabla\psi^k, \\
\nabla\cdot e^{k+1} = 0, \\
e^{k+1}\cdot n|_{\Gamma_1} = 0 \quad \text{and} \quad q^{k+1}|_{\Gamma_2} = 0.
\end{cases}
$$

Taking the square of the first relation above and multiplying the result by Δt^2, we infer

$$
\|e^{k+1}\|_0^2 + \Delta t^2\|\nabla q^{k+1}\|_0^2 = \|\tilde{e}^{k+1}\|_0^2 + \Delta t^2\|\nabla\psi^k\|_0^2 - 2\Delta t(\psi^k, \nabla\cdot\tilde{e}^{k+1}). \tag{3.9}
$$

Note that integration by parts can be performed on both sides owing to the fact that $q^{k+1}|_{\Gamma_2} = 0 = \psi^k|_{\Gamma_2}$. Now we have

$$\begin{aligned}\Delta t^2\|\nabla\psi^k\|_0^2 &= \Delta t^2\|\nabla q^k + \nabla(\mathcal{I}_{\Delta t}(\mathsf{p}(t^{k+1}) - \mathsf{p}(t^k)))\|_0^2,\\ &\le \Delta t^2(\|\nabla q^k\|_0^2 + c\Delta t^{1-\frac{1}{2}}\|\nabla q^k\|_0 + c'\Delta t^{2(1-\frac{1}{2})})\\ &\le \Delta t^2(1+\Delta t)\|\nabla q^k\|_0^2 + c\Delta t^2,\end{aligned}$$

where the consistency error is also degraded by a factor of $\mathcal{O}(\Delta t)$. Combining this result and the previous one, we have

$$\begin{aligned}\|e^{k+1}\|_0^2 + \Delta t^2\|\nabla q^{k+1}\|_0^2 \le \|\tilde{e}^{k+1}\|_0^2 + \Delta t^2(1+\Delta t)\|\nabla q^k\|_0^2\\ - 2\Delta t(\psi^{k+1}, \nabla\cdot\tilde{e}^{k+1}) + c\Delta t^2.\end{aligned} \tag{3.10}$$

The first error estimate of the theorem is obtained by combining (3.8) and (3.10), using the discrete Gronwall lemma, and repeating the whole argument for time increments. The second estimate can be derived by a duality argument similar to that used in the proof of Lemma 4.4. ☐

REMARK 3.1. *Note that the error on the pressure in the L^2-norm is $\mathcal{O}(\Delta t^{\frac{1}{2}})$, whereas it is $\mathcal{O}(\Delta t)$ when Dirichlet boundary conditions are enforced on the whole boundary. It is clear that the artificial Dirichlet boundary condition* (3.3) *is responsible for this poor convergence property. Since using an inexact factorization (cf.* [4, 5, 15, 16, 17, 13]*) of the discrete Stokes operator does not enforce the Dirichlet boundary condition on Γ_2* explicitly*, some authors have argued that the inexact factorization scheme does not suffer from the error due to the artificial Dirichlet boundary condition. However, it can be shown (see* [12] *for details) that the inexact factorization scheme actually enforces the artificial Dirichlet boundary condition* weakly *and hence suffers from the same accuracy loss as its PDE counterpart. In other words, mere algebraic manipulations cannot overcome essential difficulties encountered in functional analysis.*

REMARK 3.2. *Note that the need to integrate by parts the term $2\Delta t(\nabla\psi^{k+1}, \tilde{e}^{k+1})$ in* (3.9) *is critical, and it is made possible by enforcing the homogeneous Dirichlet boundary condition on the pressure at Γ_2 in the projection step* (3.2)*.*

We finish this section by recalling that to simulate outflow boundary conditions, an alternative set of conditions is $p|_{\Gamma_2} = 0$, $u \times n|_{\Gamma_2} = 0$. This set of conditions is not equivalent to the zero normal stress conditions studied above. Nevertheless, an interesting property of these boundary conditions is that they are compatible with the pressure-correction algorithm (3.1)–(3.2); i.e., they yield near optimal convergence rates. We refer to [8] for other technical details on this matter.

4. Rotational pressure-correction methods. In this section, we show that the rotational pressure-correction scheme introduced in [24] improves, by a factor of $\Delta t^{1/2}$, the error estimates of the standard pressure-correction scheme. It is proved in [11, 10] that when Dirichlet boundary conditions are enforced on the entire boundary, the same improvement holds. The main result is stated in Theorem 4.1.

4.1. Rotational form. When applied to problems with open boundary conditions on Γ_2, the rotational pressure-correction scheme takes the following form:

Set $u^0 = u_0$, $p^0 = \mathsf{p}|_{t=0}$, which can be computed from the data, and compute $(\tilde{u}^1, u^1, p^1)$ by using the scheme shown below with BDF2 replaced by the backward

Euler formula. Then, for $k \geq 1$, compute $(\tilde{u}^{k+1}, u^{k+1}, p^{k+1})$ such that

$$\begin{cases} \dfrac{3\tilde{u}^{k+1} - 4u^k + u^{k-1}}{2\Delta t} + A\tilde{u}^{k+1} + \nabla p^k = f(t^{k+1}), \\ \tilde{u}^{k+1}|_{\Gamma_1} = 0, \quad (p^k n - \nu(D\tilde{u}^{k+1})n)|_{\Gamma_2} = 0, \end{cases} \tag{4.1}$$

$$\begin{cases} \dfrac{3u^{k+1} - 3\tilde{u}^{k+1}}{2\Delta t} + \nabla\phi^{k+1} = 0, \\ \nabla\cdot u^{k+1} = 0, \\ u^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \phi^{k+1}|_{\Gamma_2} = 0. \end{cases} \tag{4.2}$$

$$\phi^{k+1} = p^{k+1} - p^k + \chi\nabla\cdot\tilde{u}^{k+1}, \tag{4.3}$$

where χ is a tunable positive coefficient.

REMARK 4.1. *As originally introduced in* [24]*, the coefficient χ was taken to be equal to α, defined in* (2.11)*, which is simply ν in the Newtonian case. The analysis performed in* [11, 10] *shows that this choice is sufficient to guarantee stability and convergence when Dirichlet boundary conditions are enforced. However, when natural boundary conditions are enforced on parts of the boundary, the analysis (see below) shows that χ should be chosen such that*

$$0 < \chi < 2\alpha \inf_{v\in X} \frac{\|\nabla v\|^2}{\|\nabla\cdot v\|^2}. \tag{4.4}$$

Owing to the inequality $\|\nabla\cdot v\|^2 \leq d\|\nabla v\|^2$, where d is the space dimension, it is sufficient to choose

$$0 < \chi < \frac{2}{d}\alpha. \tag{4.5}$$

4.2. A corresponding singularly perturbed system. To better understand the behavior of the scheme (4.1)–(4.3), we examine first a singularly perturbed system corresponding to the limiting case as $\Delta t \to 0$ (with $\varepsilon \sim \Delta t$). This system of PDEs is obtained by eliminating u^k from (4.1)–(4.2) and dropping some higher-order terms in ε:

$$\begin{cases} \partial_t u^\varepsilon + Au^\varepsilon + \nabla p^\varepsilon = f, \quad u^\varepsilon|_{\Gamma_1} = 0, \quad (p^\varepsilon n - \nu(Du^\varepsilon)n)|_{\Gamma_2} = 0, \\ \nabla\cdot u^\varepsilon - \varepsilon\nabla^2\phi^\varepsilon = 0, \quad \dfrac{\partial\phi^\varepsilon}{\partial n}|_{\Gamma_1} = 0, \quad \phi^\varepsilon|_{\Gamma_2} = 0, \\ \varepsilon\partial_t p^\varepsilon = \phi^\varepsilon - \chi\nabla\cdot u^\varepsilon, \end{cases} \tag{4.6}$$

with $u^\varepsilon|_{t=0} = u(0)$ and $p^\varepsilon(0) = p(0)$.

4.2.1. An estimate on $\nabla\cdot u^\varepsilon$. The following lemma is the key to obtaining improved error estimates.

LEMMA 4.1. *Provided* u *and* p *are smooth enough in time and space, we have*

$$\|\nabla\cdot u^\varepsilon\|_{L^\infty(L^2(\Omega)^d)} + \sqrt{\varepsilon}\|\nabla\phi^\varepsilon\|_{L^\infty(L^2(\Omega))} \lesssim \varepsilon^{\frac{5}{4}}.$$

Proof. We set $e = u^\varepsilon - \mathsf{u}$ and $q = p^\varepsilon - \mathsf{p}$. Subtracting (4.6) from (2.7), we find

$$e_t + Ae + \nabla q = 0; \quad e|_{\Gamma_1} = 0, \quad (qn - \nu(De)n)|_{\Gamma_2} = 0, \tag{4.7}$$

$$\nabla\cdot e - \varepsilon\nabla^2\phi^\varepsilon = 0, \quad \frac{\partial\phi^\varepsilon}{\partial n}|_{\Gamma_1} = 0, \quad \phi^\varepsilon|_{\Gamma_2} = 0, \tag{4.8}$$

$$\varepsilon q_t = \phi^\varepsilon - \chi\nabla\cdot e - \varepsilon\mathsf{p}_t, \tag{4.9}$$

with $e(0) = 0$ and $q(0) = 0$.

Taking the inner product of the time derivative of (4.7) with e_t, we find

$$\frac{1}{2}\partial_t\|e_t\|_0^2 + \alpha\|\nabla e_t\|_0^2 - (q_t, \nabla\cdot e_t) \le 0. \tag{4.10}$$

The inner product of (4.9) with $\nabla\cdot e_t$ yields

$$(q_t, \nabla\cdot e_t) = \frac{1}{\varepsilon}(\phi^\varepsilon, \nabla\cdot e_t) - (\mathsf{p}_t, \nabla\cdot e_t) - \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|^2, \tag{4.11}$$

and the inner product of the time derivative of (4.8) with ϕ^ε yields

$$\frac{1}{\varepsilon}(\phi^\varepsilon, \nabla\cdot e_t) = -(\nabla\phi_t^\varepsilon, \nabla\phi^\varepsilon). \tag{4.12}$$

The above two relations lead to

$$(q_t, \nabla\cdot e_t) = -\frac{1}{2}\partial_t\|\nabla\phi^\varepsilon\|_0^2 - (\mathsf{p}_t, \nabla\cdot e_t) - \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|^2. \tag{4.13}$$

Substituting this expression into (4.10) we obtain

$$\frac{1}{2}\partial_t\|e_t\|_0^2 + \alpha\|\nabla e_t\|_0^2 + \frac{1}{2}\partial_t\|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{2\varepsilon}\partial_t\|\nabla\cdot e\|_0^2 \le -(\mathsf{p}_t, \nabla\cdot e_t). \tag{4.14}$$

At this point, one would like to replace $\nabla\cdot e_t$ by $\varepsilon\nabla^2\phi_t^\varepsilon$ in $(\mathsf{p}_t, \nabla\cdot e_t)$ and integrate by parts. The integration by parts is not possible since neither p_t nor $\partial_n\phi_t^\varepsilon$ is zero at the boundary Γ_2. To account for this fact, we introduce the interpolation operator $\mathcal{J}_\varepsilon : H^1(\Omega) \longmapsto H_0^1(\Omega) \subset N$ such that $\mathcal{J}_\varepsilon = \mathcal{I}_{\sqrt{\varepsilon},1}$, where $\mathcal{I}_{\mu,s}$ has been defined in Lemma 2.2. Recall that for all r in $H^1(\Omega)$, Lemma 2.2 (with $\mu = \sqrt{\varepsilon}$, $s = 1$) yields

$$\|\mathcal{J}_\varepsilon r - r\|_0 \lesssim \varepsilon^{\frac{1}{4}}\|r\|_1, \qquad \|\nabla\mathcal{J}_\varepsilon r\|_0 \lesssim \varepsilon^{-\frac{1}{4}}\|r\|_1. \tag{4.15}$$

We rewrite (4.14) as

$$\frac{1}{2}\partial_t\left(\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2\right) + \alpha\|\nabla e_t\|_0^2 = -(\mathsf{p}_t - \mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\cdot e_t) + \varepsilon(\nabla\mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\phi_t^\varepsilon).$$

Note that we used the fact that $\mathcal{J}_\varepsilon\mathsf{p}_t$ is zero at Γ_2 to integrate by parts. This is the key argument in this proof. Since $e(0) = 0$ and $q(0) = 0$, we infer $e_t(0) = 0$. Since $\nabla\cdot u^\varepsilon(0) = \nabla\cdot u(0) = 0$, we derive from (4.8) that $\phi^\varepsilon(0) = 0$. By integrating in time between 0 and t, we obtain

$$\begin{aligned}
\frac{1}{2}&\left(\|e_t\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2 + \frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2\right) + \alpha\int_0^t \|\nabla e_t\|_0^2 d\tau \\
&\le -(\mathsf{p}_t - \mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\cdot e) + \int_0^t (\mathsf{p}_{\tau\tau} - \mathcal{J}_\varepsilon\mathsf{p}_{\tau\tau}, \nabla\cdot e)d\tau \\
&\quad + \varepsilon(\nabla\mathcal{J}_\varepsilon\mathsf{p}_t, \nabla\phi^\varepsilon) - \int_0^t \varepsilon(\nabla\mathcal{J}_\varepsilon\mathsf{p}_{\tau\tau}, \nabla\phi^\varepsilon)d\tau \\
&\le \frac{1}{4}\left(\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2\right) + \int_0^t \left(\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2 + \|\nabla\phi^\varepsilon\|_0^2\right)d\tau \\
&\quad + c\,\varepsilon\|\mathsf{p}_t - \mathcal{J}_\varepsilon\mathsf{p}_t\|_{L^\infty(0,t;L^2(\Omega))}^2 + c'\,\varepsilon^2\|\mathcal{J}_\varepsilon\mathsf{p}_t\|_{L^\infty(0,t;H^1(\Omega))}^2 \\
&\quad + c\,\varepsilon\|\mathsf{p}_{tt} - \mathcal{J}_\varepsilon\mathsf{p}_{tt}\|_{L^2(0,t;L^2(\Omega))}^2 + c'\,\varepsilon^2\|\mathcal{J}_\varepsilon\mathsf{p}_{tt}\|_{L^2(0,t;H^1(\Omega))}^2.
\end{aligned}$$

Using the estimates (4.15), we infer

$$\frac{1}{4}\left(\|e_t\|_0^2+\|\nabla\phi^\varepsilon\|_0^2+\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2\right)+\alpha\int_0^t\|\nabla e_t\|_0^2d\tau\le\int_0^t\left(\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2+\|\nabla\phi^\varepsilon\|_0^2\right)d\tau+c\,\varepsilon^{\frac{3}{2}}.$$

An application of the Gronwall lemma leads to

$$\|e_t\|_0^2+\|\nabla\phi^\varepsilon\|_0^2+\frac{\chi}{\varepsilon}\|\nabla\cdot e\|_0^2+\int_0^t\|\nabla e_\tau\|_0^2d\tau\lesssim\varepsilon^{\frac{3}{2}}. \tag{4.16}$$

The proof is complete. □

4.2.2. L^2-estimate on the velocity. An estimation of the error on the velocity in the L^2-norm is given by the following lemma.

LEMMA 4.2. *Provided* u *and* p *are smooth enough in time and space, then*

$$\|\mathsf{u}-u^\varepsilon\|_{L^2(L^2(\Omega)^d)}\lesssim\varepsilon^{\frac{5+s}{4}}, \tag{4.17}$$

where s is the regularity index of the Stokes operator.

Proof. We multiply (4.7) by $S(e)$. Owing to Lemma 2.4 we infer

$$\frac{1}{2}\partial_t|e|_\star^2+a(e,S(e))=0.$$

Using Lemma 2.3 with $\mu=\sqrt{\varepsilon}$, we obtain

$$\frac{1}{2}\partial_t|e|_\star^2+\frac{1}{2}\|e\|_0^2\lesssim\varepsilon^{\alpha_1}\|\nabla\cdot e\|_0^2+\varepsilon^{-\alpha_2}\|e-P_He\|_0^2.$$

From the definition of ϕ^ε, it is clear that $\varepsilon\nabla\phi^\varepsilon=e-P_He$; we then derive from the estimates in Lemma 4.1 that

$$\frac{1}{2}\partial_t|e|_\star^2+\frac{1}{2}\|e\|_0^2\lesssim\varepsilon^{\alpha_1}\|\nabla\cdot e\|_0^2+\varepsilon^{1-\alpha_2}\varepsilon\|\nabla\phi^\varepsilon\|_0^2\lesssim\varepsilon^{\frac{5}{2}}(\varepsilon^{\alpha_1}+\varepsilon^{1-\alpha_2}).$$

Since $\alpha_1=1-\alpha_2$, we find

$$\frac{1}{2}\partial_t|e|_\star^2+\frac{1}{2}\|e\|_0^2\lesssim\varepsilon^{\frac{5}{2}+\alpha_1}=\varepsilon^{\frac{5+s}{2}}.$$

The proof is completed using an integration in time. □

4.3. Error estimates for the time discrete case. The main result in this paper is the following.

THEOREM 4.1. *Let $0<\chi<\frac{2\alpha}{d}$. Assuming that the solution to* (2.7) *is smooth enough in time and space, the solution $(u^k,\tilde{u}^k,p^k)$ to* (4.1)–(4.3) *satisfies the estimates*

$$\|\mathsf{u}_{\Delta t}-u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)}+\|\mathsf{u}_{\Delta t}-\tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)}\lesssim\Delta t^{\frac{5+s}{4}},$$
$$\|\mathsf{u}_{\Delta t}-\tilde{u}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)}+\|\mathsf{p}_{\Delta t}-p_{\Delta t}\|_{\ell^2(L^2(\Omega))}\lesssim\Delta t^{\frac{3+s}{4}},$$

where s is the regularity index of the Stokes operator.

REMARK 4.2. *With full Stokes regularity, i.e., $s=1$, the L^2-norm of the error on the velocity is $\mathcal{O}(\Delta t^{\frac{3}{2}})$, and the H^1-norm of the error on the velocity and the L^2-norm of the error on the pressure are $\mathcal{O}(\Delta t)$. In view of Lemma 4.1 and of the first estimate in Lemma 4.3, we believe that the H^1-estimates can be improved up to $\mathcal{O}(\Delta t^{\frac{5}{4}})$ by a* *sophisticated argument using weighted seminorms in time as in* [18, 20]. *However, the details of this proof are beyond the scope of this paper. Numerical results reported in section* 5 *seem to confirm this conjecture, at least in two dimensions.*

The proof of Theorem 4.1 is carried out in a way similar to that of Theorem 4.1 in [10], but since there are several important differences in the proofs of the underlying lemmas, we give all the details. In particular the error analysis reveals why a homogeneous Dirichlet boundary condition must be enforced on ϕ^{k+1} on Γ_2; it explains also the origin of the factor χ in (4.3).

Let us first introduce some notation. For any sequence $\varphi^0, \varphi^1, \dots$, we set

$$\delta_t\varphi^k = \varphi^k - \varphi^{k-1}, \quad \delta_{tt}\varphi^k = \delta_t(\delta_t\varphi^k), \quad \delta_{ttt}\varphi^k = \delta_t(\delta_{tt}\varphi^k),$$

and

$$\begin{cases} e^k = \mathsf{u}(t^k) - u^k, & \tilde{e}^k = \mathsf{u}(t^k) - \tilde{u}^k, \\ \psi^k = \mathsf{p}(t^{k+1}) - p^k, & q^k = \mathsf{p}(t^k) - p^k. \end{cases} \tag{4.18}$$

It is straightforward to show that $(\tilde{u}^1, u^1, p^1)$ obtained by using the scheme (4.1)–(4.3), with BDF2 replaced by backward Euler, satisfies the following estimates:

$$\begin{aligned} \|e^1\|_0 + \|\tilde{e}^1\|_0 + \Delta t^{\frac{1}{2}}(\|\nabla e^1\|_0 + \|\nabla \tilde{e}^1\|_0) &\lesssim \Delta t^2, \\ \|q^1\|_0 &\lesssim \Delta t. \end{aligned} \tag{4.19}$$

Note that for any bilinear form $(\cdot,\cdot)$ and any sequences $a^0, a^1, \dots$, and $b^0, b^1, \dots$, the following holds:

$$\delta_t(a^{k+1}, b^{k+1}) = (\delta_t a^{k+1}, b^{k+1}) + (a^k, \delta_t b^{k+1}). \tag{4.20}$$

The error estimates of Theorem 4.1 are proved through a succession of lemmas. The following result is the discrete counterpart of Lemma 4.1.

LEMMA 4.3. *Under the hypotheses of Theorem* 4.1, *we have*

$$\begin{aligned} \|\nabla\cdot \tilde{u}_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} + \sqrt{\Delta t}\|\nabla \phi_{\Delta t}\|_{\ell^\infty(L^2(\Omega))} &\lesssim \Delta t^{\frac{5}{4}}, \\ \|\delta_t \tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} &\lesssim \Delta t^{\frac{7}{4}}, \\ \|\delta_t \tilde{e}_{\Delta t} - \delta_t e_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} &\lesssim \Delta t^{\frac{9}{4}}. \end{aligned}$$

Proof. Upon defining

$$R^k = \partial_t \mathsf{u}(t^k) - \frac{3\mathsf{u}(t^k) - 4\mathsf{u}(t^{k-1}) + \mathsf{u}(t^{k-2})}{2\Delta t}, \tag{4.21}$$

then, for $k \geq 2$, the equations that control the time increments of the errors are

$$\begin{cases} \dfrac{3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}}{2\Delta t} + A\delta_t\tilde{e}^{k+1} + \nabla\delta_t\psi^k = \delta_t R^{k+1}, \\ \delta_t\tilde{e}^{k+1}|_{\Gamma_1} = 0, \quad (\delta_t\psi^k n - \nu(D\delta_t\tilde{e}^{k+1})n)|_{\Gamma_2} = 0 \end{cases} \tag{4.22}$$

and

$$\begin{cases} \dfrac{3}{2\Delta t}\delta_t e^{k+1} - \nabla\phi^{k+1} = \dfrac{3}{2\Delta t}\delta_t\tilde{e}^{k+1} - \nabla\phi^k, \\ \nabla\cdot\delta_t e^{k+1} = 0, \\ \delta_t e^{k+1}\cdot n|_{\Gamma_1} = 0, \quad \phi^{k+1}|_{\Gamma_2} = \phi^k|_{\Gamma_2} = 0. \end{cases} \tag{4.23}$$

We take the inner product of (4.22) with $4\Delta t\, \delta_t \tilde{e}^{k+1}$ and obtain

$$\begin{aligned} 2(\delta_t\tilde{e}^{k+1}, 3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) + 4\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 \\ - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) = 4\Delta t(\delta_t\tilde{e}^{k+1}, \delta_t R^{k+1}) \\ \le \gamma\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + c\Delta t^7, \end{aligned} \tag{4.24}$$

where γ will be chosen later, and we have used the coercivity of the bilinear form a together with the fact that $\|\delta_t R^{k+1}\|_0 \lesssim \Delta t^3$. Note also that we have used the inequality $2ab \le \gamma a^2 + b^2/\gamma$, which holds for all $\gamma > 0$. We shall repeatedly use this standard trick hereafter without mentioning it anymore.

Let us denote $I = 2(\delta_t\tilde{e}^{k+1}, 3\delta_t\tilde{e}^{k+1} - 4\delta_t e^k + \delta_t e^{k-1})$; then we have

$$\begin{aligned} I = 6(\delta_t\tilde{e}^{k+1}, \delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}) + 2(\delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) \\ + 2(\delta_t e^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}). \end{aligned}$$

Let I_1, I_2, and I_3 be the three terms in the right-hand side. Using the algebraic identities

$$2(a^{k+1}, a^{k+1} - a^k) = |a^{k+1}|^2 + |a^{k+1} - a^k|^2 - |a^k|^2, \tag{4.25}$$

$$\begin{aligned} 2(a^{k+1}, 3a^{k+1} - 4a^k + a^{k-1}) = |a^{k+1}|^2 + |2a^{k+1} - a^k|^2 + |\delta_{tt}a^{k+1}|^2 \\ - |a^k|^2 - |2a^k - a^{k-1}|^2, \end{aligned} \tag{4.26}$$

we derive

$$\begin{aligned} I_1 &= 3\|\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t e^{k+1} - \delta_t\tilde{e}^{k+1}\|_0^2 - 3\|\delta_t e^{k+1}\|_0^2, \\ I_3 &= \|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2 - \|\delta_t e^k\|_0^2 - \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2. \end{aligned}$$

Owing to (4.23) and using the fact that $e^k \in H$, we derive the following equality:

$$\frac{3}{2\Delta t} I_2 = -2(\nabla\delta_t\phi^{k+1}, 3\delta_t e^{k+1} - 4\delta_t e^k + \delta_t e^{k-1}) = 0.$$

Collecting all the above results, we obtain

$$\begin{aligned} 3\|\delta_t\tilde{e}^{k+1}\|_0^2 - 3\|\delta_t e^{k+1}\|_0^2 + \|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 \\ + 3\|\delta_t e^{k+1} - \delta_t\tilde{e}^{k+1}\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2 \\ + (4-\gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) \\ \le c\,\Delta t^7 + \|\delta_t e^k\|_0^2 + \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2. \end{aligned} \tag{4.27}$$

Taking the square of (4.23) and integrating over the domain, we obtain

$$\begin{aligned} 3\|\delta_t e^{k+1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 = 3\|\delta_t\tilde{e}^{k+1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^k\|_0^2 \\ + 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \phi^k). \end{aligned} \tag{4.28}$$

Note that integration by parts on $(\delta_t e^{k+1}, \nabla\phi^{k+1})$ and $(\delta_t\tilde{e}^{k+1}, \nabla\phi^k)$ is legitimate because both $\phi^{k+1}|_{\Gamma_2}$ and $\phi^k|_{\Gamma_2}$ are zero. Since $\phi^k = p^k - p^{k-1} - \chi\nabla\cdot\tilde{e}^k$, we can bound the inner product in the right-hand side of (4.28) as follows:

$$\begin{aligned} 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \phi^k) &= 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, p^k - p^{k-1} - \chi\nabla\cdot\tilde{e}^k) \\ &= 2\chi\Delta t(-\|\nabla\cdot\tilde{e}^{k+1}\|_0^2 + \|\nabla\cdot\tilde{e}^k\|_0^2 + \|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2) \\ &\quad - 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\psi^k) + 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\mathsf{p}(t^{k+1})). \end{aligned} \tag{4.29}$$

To control the troublesome term $\Delta t\|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2$ we use

$$\chi\|\nabla\cdot v\|_0^2 \le 2\gamma'\alpha\|\nabla v\|_0^2 \quad \forall v\in X. \tag{4.30}$$

Due to the condition χ, (4.4), we know that the constant γ' is such that $0<\gamma'<1$. Summing (4.27), (4.28), and (4.29), and using (4.30), we finally obtain

$$\begin{aligned}
&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1}-\delta_t e^k\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2\\
&\qquad + (4-4\gamma'-\gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t(e^{k+1}-\tilde{e}^{k+1})\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2\\
&\quad \le \|\delta_t e^k\|_0^2 + \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^k\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^k\|_0^2\\
&\qquad + 4\Delta t(\nabla\cdot\delta_t\tilde{e}^{k+1}, \delta_t\mathsf{p}(t^{k+1})) + c\Delta t^7.
\end{aligned} \tag{4.31}$$

At this point, we are formally at the same stage as (4.14). To integrate by parts in time the term $(\nabla\cdot\delta_t\tilde{e}^{k+1},\delta_t\mathsf{p}(t^{k+1}))$, we use (4.20) as follows:

$$(\nabla\cdot\delta_t\tilde{e}^{k+1},\delta_t\mathsf{p}(t^{k+1})) = \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathsf{p}(t^{k+1})) - (\nabla\cdot\tilde{e}^k,\delta_{tt}\mathsf{p}(t^{k+1})).$$

Next, we use the interpolation operator defined in (4.15). Let us denote $\mathcal{R}^{k+1} = \mathsf{p}(t^{k+1}) - \mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1}))$ (where $\mathcal{J}_{\Delta t} = \mathcal{I}_{\sqrt{\Delta t},1}$). Then we have

$$\frac{1}{\Delta t}\|\delta_{tt}\mathcal{R}^{k+1}\|_0^2 + \|\nabla\delta_{tt}\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1}))\|_0^2 \lesssim \Delta t^{\frac{7}{2}}.$$

Since $\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1}))$ is zero on Γ_2, we have

$$\begin{aligned}
(\nabla\cdot\delta_t\tilde{e}^{k+1},\delta_t\mathsf{p}(t^{k+1})) &= \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1})))\\
&\quad - (\nabla\cdot\tilde{e}^k,\delta_{tt}\mathcal{R}^{k+1}) - (\nabla\cdot\tilde{e}^k,\delta_{tt}\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1})))\\
&= \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{2\Delta t}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1})))\\
&\quad - (\nabla\cdot\tilde{e}^k,\delta_{tt}\mathcal{R}^{k+1}) - \frac{2\Delta t}{3}(\nabla\phi^k,\nabla\delta_{tt}\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1})))\\
&\le \delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{2\Delta t}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1})))\\
&\quad + \frac{\chi\Delta t}{2}\|\nabla\cdot\tilde{e}^k\|_0^2 + \frac{\Delta t^2}{3}\|\nabla\phi^k\|_0^2 + c\Delta t^{\frac{7}{2}}.
\end{aligned}$$

By inserting this bound into (4.31), we obtain

$$\begin{aligned}
&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1}-\delta_t e^k\|_0^2 + \frac{4}{3}\Delta t^2\|\nabla\phi^{k+1}\|_0^2 + 2\chi\Delta t\|\nabla\cdot\tilde{e}^{k+1}\|_0^2\\
&\qquad + (4-4\gamma'-\gamma)\alpha\Delta t\|\nabla\delta_t\tilde{e}^{k+1}\|_0^2 + 3\|\delta_t(e^{k+1}-\tilde{e}^{k+1})\|_0^2 + \|\delta_{ttt}e^{k+1}\|_0^2\\
&\quad \le \|\delta_t e^k\|_0^2 + \|2\delta_t e^k - \delta_t e^{k-1}\|_0^2\\
&\qquad + \frac{4}{3}\Delta t^2(1+\Delta t)\|\nabla\phi^k\|_0^2 + 2\chi\Delta t(1+\Delta t)\|\nabla\cdot\tilde{e}^k\|_0^2\\
&\qquad + 4\Delta t\delta_t(\nabla\cdot\tilde{e}^{k+1},\delta_t\mathcal{R}^{k+1}) + \frac{8\Delta t^2}{3}\delta_t(\nabla\phi^{k+1},\nabla\delta_t\mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1}))) + c\,\Delta t^{\frac{9}{2}}.
\end{aligned}$$

Summing up the relation above for $l = 2, \dots, k$ and taking into account (4.19), we obtain

$$\begin{aligned}
&\|\delta_t e^{k+1}\|_0^2 + \|2\delta_t e^{k+1} - \delta_t e^k\|_0^2 + \frac{4}{3}\Delta t^2 \|\nabla \phi^{k+1}\|_0^2 + 2\chi\Delta t \|\nabla\cdot \tilde{e}^{k+1}\|_0^2 \\
&\qquad + (4 - 4\gamma' - \gamma)\alpha \Delta t \sum_{l=2}^{k} \|\nabla \delta_t \tilde{e}^{l+1}\|_0^2 + 3\sum_{l=2}^{k} \|\delta_t e^{l+1} - \delta_t \tilde{e}^{l+1}\|_0^2 \\
&\quad \leq c\left(\|\delta_t e^2\|_0^2 + \|2\delta_t e^2 - \delta_t e^1\|_0^2 + \Delta t^2 \|\nabla \phi^2\|_0^2 + \Delta t \|\nabla\cdot \tilde{e}^2\|_0^2 + \Delta t^{\frac{7}{2}} \right) \\
&\qquad + \Delta t \sum_{l=2}^{k} \left(\frac{4}{3}\Delta t^2 \|\nabla \phi^l\|_0^2 + 2\chi \Delta t \|\nabla\cdot \tilde{e}^l\|_0^2 \right) \\
&\qquad - 4\Delta t (\nabla\cdot \tilde{e}^{k+1}, \delta_t \mathcal{R}^{k+1}) - \frac{8\Delta t^2}{3} (\nabla \phi^{k+1}, \nabla \delta_t \mathcal{J}_{\Delta t}(\mathsf{p}(t^{k+1}))) \\
&\qquad + 4\Delta t (\nabla\cdot \tilde{e}^2, \delta_t \mathcal{R}^2) + \frac{8\Delta t^2}{3} (\nabla \phi^2, \nabla \delta_t \mathcal{J}_{\Delta t}(\mathsf{p}(t^2))) \\
&\quad \leq c\Delta t^{\frac{7}{2}} + \frac{2}{3}\Delta t^2 \|\nabla \phi^{k+1}\|_0^2 + \chi \Delta t \|\nabla\cdot \tilde{e}^{k+1}\|_0^2 \\
&\qquad + \Delta t \sum_{l=2}^{k} \left(\frac{4}{3}\Delta t^2 \|\nabla \phi^l\|_0^2 + 2\chi \Delta t \|\nabla\cdot \tilde{e}^l\|_0^2 \right).
\end{aligned}$$

Since $0 < \gamma' < 1$, we can choose γ such that $4 - 4\gamma' - \gamma \geq 0$. Then an application of the discrete Gronwall lemma yields the desired result. ∎

REMARK 4.3. *Note that to balance the term* $-(\nabla\cdot\delta_t\tilde{e}^{k+1}, \psi^k)$ *in* (4.27) *it is necessary to integrate by parts the term* $(\delta_t \tilde{e}^{k+1}, \nabla\phi^k)$ *in* (4.28). *This is possible only because the Dirichlet boundary condition* $\phi^k|_{\Gamma_2} = 0$ *is enforced. This fact is the main reason why we enforce a homogeneous Dirichlet boundary condition on* ϕ^{k+1} *in* (4.2). *This argument shows the importance of the error analysis (or stability analysis) performed in the proof of Lemma* 4.3. *The necessity of the Dirichlet boundary condition also becomes clear when one understands that* (4.2) *is a realization of* $u^{k+1} = P_H \tilde{u}^{k+1}$, *since the orthogonal complement of* H *is* ∇N *according to Lemma* 2.1.

REMARK 4.4. *The introduction of the parameter* χ *together with the bound* (4.4) *is justified by step* (4.30). *Whether the bound* (4.4) *is sharp is not yet clear.*

LEMMA 4.4. *Under the hypotheses of Theorem* 4.1, *we have*

$$\|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} + \|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{5+s}{4}}.$$

Proof. By using the relation $e^l = \tilde{e}^l + \frac{2\Delta t}{3}\nabla\phi^l$, for all $l \geq 2$, one obtains

$$\left\{ \begin{aligned} &\frac{3\tilde{e}^{k+1} - 4\tilde{e}^k + \tilde{e}^{k-1}}{2\Delta t} + A\tilde{e}^{k+1} + \nabla\gamma^k = R^{k+1}, \\ &\tilde{e}^{k+1}|_{\Gamma_1} = 0, \quad (\gamma^k n - \nu(D\tilde{e}^{k+1})n)|_{\Gamma_2} = 0, \end{aligned} \right. \tag{4.32}$$

where $\nabla\gamma^k$ stands for the collection of all the gradient terms.

As in the time continuous case, we make use of the inverse Stokes operator. By taking the inner product of (4.32) with $4\Delta t S(\tilde{e}^{k+1})$ and using the identity (4.26), we obtain

$$\begin{aligned} &|\tilde{e}^{k+1}|_\star^2 + |2\tilde{e}^{k+1} - \tilde{e}^k|_\star^2 + |\delta_{tt}\tilde{e}^{k+1}|_\star^2 + 4\Delta t\, a(\tilde{e}^{k+1}, S(\tilde{e}^{k+1})) \\ &\qquad = 4\Delta t\, (R^{k+1}, S(\tilde{e}^{k+1})) + |\tilde{e}^k|_\star^2 + |2\tilde{e}^k - \tilde{e}^{k-1}|_\star^2. \end{aligned}$$

Using Lemma 2.3 with $\mu = \sqrt{\Delta t}$ and Lemma 4.3, we infer

$$\begin{aligned}4a(\tilde{e}^{k+1}, S(\tilde{e}^{k+1})) &\geq 2\|\tilde{e}^{k+1}\|_0^2 - c(\Delta t^{\alpha_1}\,\|\nabla\cdot\tilde{e}^{k+1}\|_0^2 + \Delta t^{-\alpha_2}\,\|\tilde{e}^{k+1} - e^{k+1}\|^2)\\ &\geq 2\|\tilde{e}^{k+1}\|_0^2 - c(\Delta t^{\alpha_1}\,\|\nabla\cdot\tilde{e}^{k+1}\|_0^2 + \Delta t^{1-\alpha_2}\,\Delta t\|\nabla\phi^{k+1}\|^2\\ &\geq 2\|\tilde{e}^{k+1}\|_0^2 - c\Delta t^{\alpha_1+\frac{5}{2}} \geq 2\|\tilde{e}^{k+1}\|_0^2 - c\Delta t^{\frac{5+s}{2}}.\end{aligned}$$

We also derive from the Cauchy–Schwarz inequality and (2.16) that

$$4\Delta t(R^{k+1}, S(\tilde{e}^{k+1})) \leq c\Delta t\|R^{k+1}\|_{X'}^2 + \Delta t\|\tilde{e}^{k+1}\|_0^2 \leq c'\Delta t^5 + \Delta t\|\tilde{e}^{k+1}\|_0^2.$$

Combining these two estimates, we obtain

$$|\tilde{e}^{k+1}|_\star^2 + |2\tilde{e}^{k+1} - \tilde{e}^k|_\star^2 + \Delta t\|\tilde{e}^{k+1}\|_0^2 \leq |\tilde{e}^k|_\star^2 + |2\tilde{e}^k - \tilde{e}^{k-1}|_\star^2 + c\Delta t^{1+\frac{5+s}{2}}.$$

The desired result is now an easy consequence of the discrete Gronwall lemma. The estimate on $\|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_0$ is obtained by using the triangular inequality $\|\mathsf{u}_{\Delta t} - u_{\Delta t}\|_0 \leq \|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_0 + \frac{2\Delta t}{3}\|\nabla\phi_{\Delta t}\|_0$ (derived from (4.2)) and Lemma 4.3. ◻

The key for obtaining improved estimates on $\|\tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)}$ and $\|q_{\Delta t}\|_{\ell^2(L^2(\Omega))}$ is to derive an improved estimate on $\frac{1}{2\Delta t}(3\delta_t\tilde{e}^{k+1} - 4\delta_t\tilde{e}^k + \delta_t\tilde{e}^{k-1})$. To this end, for any sequence of functions $\phi^0, \phi^1, \ldots$, we define

$$D_t\phi^{k+1} := \frac{1}{2}(3\phi^{k+1} - 4\phi^k + \phi^{k-1}).$$

LEMMA 4.5. *Under the hypotheses of Theorem* 4.1*, we have*

$$\Delta t^{-1}\|(D_t\tilde{e})_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Proof. We use the same argument as in the proof of the L^2-estimate, but we use it on the time increment $\delta_t\tilde{e}^{k+1}$. For $k \geq 2$ we have

$$\frac{3\delta_t\tilde{e}^{k+1} - 4\delta_t\tilde{e}^k + \delta_t\tilde{e}^{k-1}}{2\Delta t} + A\delta_t\tilde{e}^{k+1} + \nabla\delta_t\gamma^{k+1} = \delta_t R^{k+1}.$$

Taking the inner product of the above relation with $4\Delta t S(\delta_t\tilde{e}^{k+1})$, using Lemma 2.3 with $\mu = \sqrt{\Delta t}$, and repeating the same arguments as in the previous lemma, we obtain

$$\begin{aligned}&|\delta_t\tilde{e}^{k+1}|_\star^2 + |2\delta_t\tilde{e}^{k+1} - \delta_t\tilde{e}^k|_\star^2 + |\delta_{ttt}\tilde{e}^{k+1}|_\star^2 + \Delta t\|\delta_t\tilde{e}^{k+1}\|_0^2\\ &\quad\leq c\Delta t\|\delta_t R^{k+1}\|_0^2 + c\Delta t(\Delta t^{\alpha_1}\|\nabla\cdot\delta_t\tilde{e}^{k+1}\|_0^2 + \Delta t^{-\alpha_2}\|\delta_t\tilde{e}^{k+1} - \delta_t e^{k+1}\|_0^2)\\ &\qquad+ |\delta_t\tilde{e}^k|_\star^2 + |2\delta_t\tilde{e}^k - \delta_t\tilde{e}^{k-1}|_\star^2.\end{aligned}$$

Applying the discrete Gronwall lemma, and using the initial estimates and Lemma 4.3, we obtain

$$\|\delta_t\tilde{e}_{\Delta t}\|_{l^2(L^2(\Omega)^d)}^2 \lesssim \Delta t^{\frac{7+s}{2}}.$$

We conclude by using the fact that $2D_t\tilde{e}^{k+1} = 3\delta_t\tilde{e}^{k+1} - \delta_t\tilde{e}^k$. ◻

We are now in position to prove the remaining claims in Theorem 4.1.

LEMMA 4.6. *Under the hypotheses of Theorem* 4.1*, we have*

$$\|\mathsf{u}_{\Delta t} - \tilde{u}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} + \|\mathsf{p}_{\Delta t} - p_{\Delta t}\|_{\ell^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Proof. By adding the viscous step and the projection step, it is clear that we have

$$\begin{cases} A\tilde{e}^{k+1} + \nabla(q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1}) = h^{k+1}, \\ \nabla\cdot\tilde{e}^{k+1} = g^{k+1}, \quad \tilde{e}^{k+1}|_{\Gamma_1} = 0, \quad ((q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1})n - (D\tilde{e}^{k+1})n)|_{\Gamma_2} = 0, \end{cases} \tag{4.33}$$

where

$$h^{k+1} = R^{k+1} - \frac{D_t e^{k+1}}{\Delta t}, \qquad g^{k+1} = -\frac{2\Delta t}{3}\nabla^2\phi^{k+1}. \tag{4.34}$$

Owing to Lemma 4.3, we have

$$\|g^{k+1}\|_0 = \|\nabla\cdot\tilde{e}^{k+1}\|_0 \lesssim \Delta t^{\frac{5}{4}} \qquad \forall k. \tag{4.35}$$

Since $e^k = P_H\tilde{e}^k$, owing to Lemma 4.5, we infer

$$\Delta t^{-1}\|\delta_t e_{\Delta t}\|_{l^2(L^2(\Omega)^d)} \le \Delta t^{-1}\|\delta_t \tilde{e}_{\Delta t}\|_{l^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Hence, we have

$$\|h_{\Delta t}\|_{\ell^2(X')} \lesssim \|R_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} + \Delta t^{-1}\|D_t\tilde{e}_{\Delta t}\|_{\ell^2(L^2(\Omega)^d)} \lesssim \Delta t^{\frac{3+s}{4}}. \tag{4.36}$$

Now, we apply the following standard stability result for nonhomogeneous Stokes systems to (4.33) (cf. [23]):

$$\|\tilde{e}^{k+1}\|_1 + \|(q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1})\|_0 \lesssim \|h^{k+1}\|_{X'} + \|g^{k+1}\|_0. \tag{4.37}$$

Owing to (4.35) and (4.36), we derive

$$\|\tilde{e}_{\Delta t}\|_{\ell^2(H^1(\Omega)^d)} + \|(q + \chi\nabla\cdot\tilde{e})\|_{\ell^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}.$$

Then, from

$$\|q^{k+1}\|_0 \le \|q^{k+1} + \chi\nabla\cdot\tilde{e}^{k+1}\|_0 + \chi\|\nabla\cdot\tilde{e}^{k+1}\|_0,$$

we derive $\|q_{\Delta t}\|_{l^2(L^2(\Omega))} \lesssim \Delta t^{\frac{3+s}{4}}$. ∎

Thus, all the results in Theorem 4.1 have been proved.

5. Numerical results and discussions.

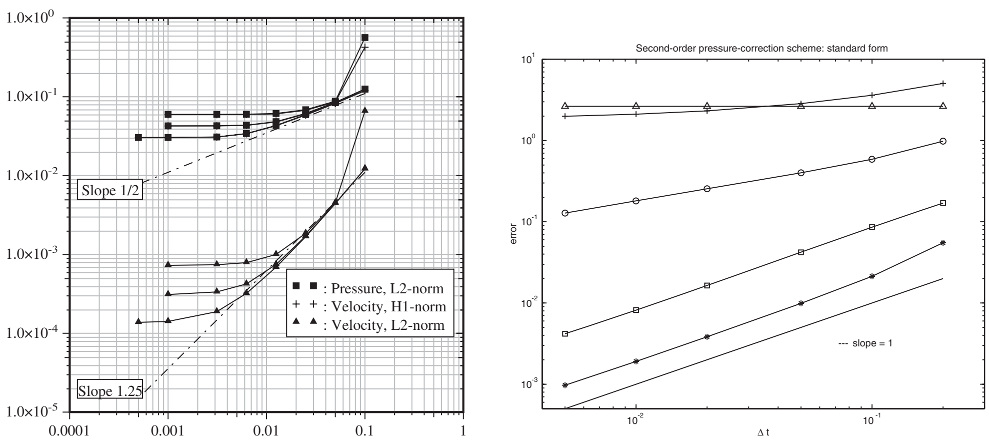
5.1. Standard pressure-correction scheme. We take the exact solution (u_1, u_2, p) of the linearized Navier–Stokes equations to be

$$\mathsf{u}_1(x,y,t) = \sin x \sin(y+t), \;\; \mathsf{u}_2(x,y,t) = \cos x\cos(y+t), \;\; \mathsf{p}(x,y,t) = \cos x \sin(y+t).$$

We set $\Omega =]0,1[^2$, $\Gamma_2 = \{(x,y) \in \Gamma,\, x = 0\}$. This solution satisfies the following open boundary conditions:

$$-\partial_x \mathsf{u}_2|_{\Gamma_2} = 0, \qquad \mathsf{p} - \partial_x\mathsf{u}_1|_{\Gamma_2} = 0.$$

To confirm the results in Theorem 3.1, we have carried out convergence tests in time using $\mathbb{P}_2/\mathbb{P}_1$ finite elements as well as the $\mathbb{P}_N^2 \times \mathbb{P}_{N-2}$ Legendre–Galerkin method [19] (where $\mathbb{P}_k$ denotes the space of polynomials of degree less than or equal



Finite elements with $h = 1/20$, $1/40$, and $1/80$. Legendre–Galerkin with $N = 40$.

FIG. 5.1. *Errors vs.* Δt*, standard pressure-correction scheme: Note that the curves corresponding to the error on the velocity in* H^1*-norm and the pressure in* L^2*-norm almost coincide.*

to k). We use the standard BDF2 pressure-correction scheme, which enforces a homogeneous Dirichlet boundary condition on the pressure increment at the open boundary in the projection step.

For the finite elements, the errors at $t = 1$ for three meshes ($h = 1/20$, $1/40$, $1/80$) and $5.10^{-4} \leq \Delta t \leq 10^{-1}$ are reported in the left panel of Figure 5.1. Note that the error for small time steps is dominated by the spatial discretization error. The reference slope represents the asymptotic convergence rate as $h \to 0$.

For the Legendre–Galerkin method, the results with $N = 40$ are reported in the right panel of Figure 5.1. For the range of time steps explored, the spatial discretization error is negligible compared to the time discretization error.

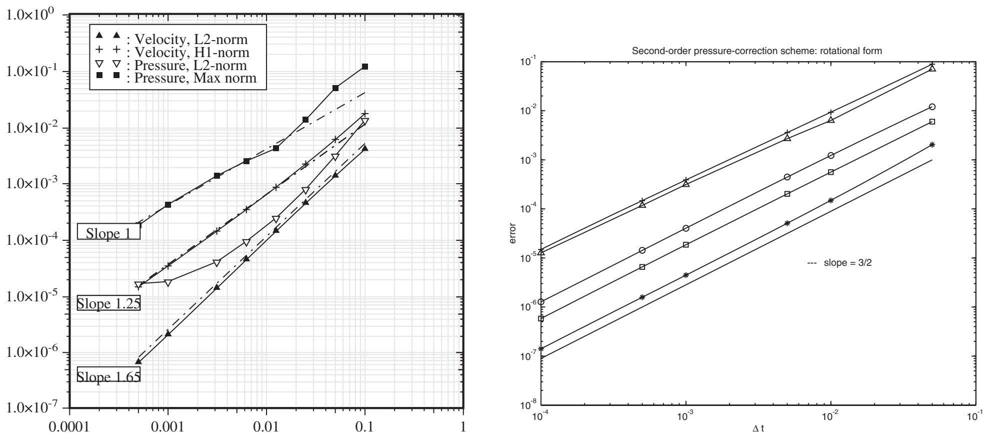
These tests clearly indicate that the L^2-error of the velocity (resp., the pressure) is of order Δt (resp., $\Delta t^{\frac{1}{2}}$), which are consistent with Theorem 3.1.

5.2. Rotational pressure-correction scheme. We again use the analytical solution described above to test the time accuracy of the rotational pressure-correction scheme (4.1)–(4.3).

We first report the results with $\mathbb{P}_2/\mathbb{P}_1$ finite elements. We use $h = 1/80$ to guarantee that the error in space is significantly smaller than the splitting error. The results are reported in the left panel of Figure 5.2. The convergence rate of the error on the velocity in the L^2-norm is close to $\mathcal{O}(\Delta t^{3/2})$, and that of the H^1-norm behaves like $\mathcal{O}(\Delta t^{5/4})$, which is higher than the $\mathcal{O}(\Delta t)$ rate predicted by Theorem 4.1 (see Remark 4.2 and Lemma 4.3). The convergence rate of the error on the pressure in the L^∞-norm is $\mathcal{O}(\Delta t)$, and that of the L^2-norm is between $\mathcal{O}(\Delta t)$ and $\mathcal{O}(\Delta t^{\frac{3}{2}})$. These rates are mostly consistent with the error estimates in Theorem 4.1. The accuracy saturation observed for small time steps comes from the spatial discretization error.

The results using the Legendre–Galerkin method are reported in the right panel of Figure 5.2. We note that the convergence rate for the error on the velocity in the L^2-norm is of order $O(\Delta t^{\frac{3}{2}})$, as predicted by Theorem 4.1. The convergence rates on all the other quantities are also close to $O(\Delta t^{\frac{3}{2}})$, which is higher than what Theorem 4.1 predicts (see Remark 4.2).

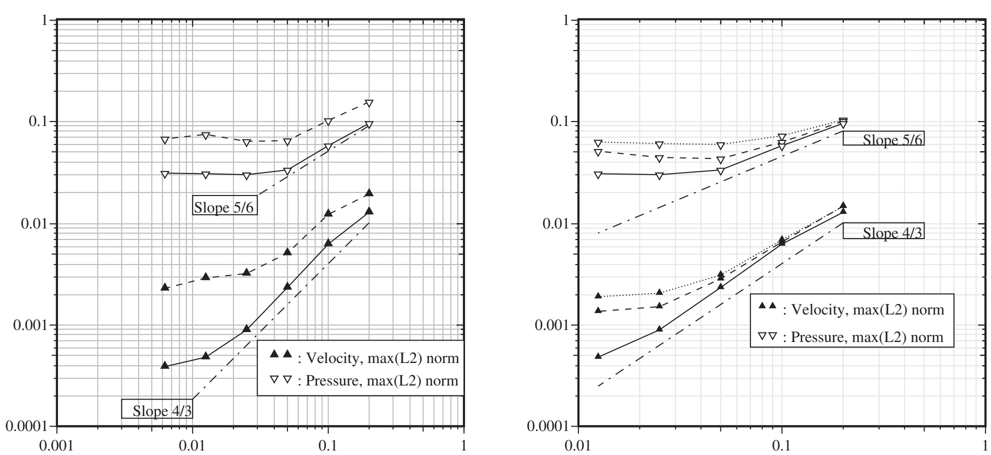
To complete this series of tests, we have performed convergence tests in three



Velocity: (▲) L^2-norm; (+) H^1-norm.
Pressure: (▽) L^2-norm; (■) L^∞-norm.

Velocity: (*) L^2-norm; (□) H^1-norm.
Pressure: (○) L^2; (+) H^1; (△) L^∞.

FIG. 5.2. *Rotational pressure-correction scheme: Left, finite elements; errors at $t = 1$ vs. Δt (using $h = 1/80$). Right, spectral method; error vs. Δt with $N = 40$ fixed.*



$h = 1/40$. Standard form (dashes) vs. rotational form (solid line)

$h = 1/10, 1/20, 1/40$. Rotational form, $\chi = 2/3$.

FIG. 5.3. *Pressure-correction scheme with $\mathbb{P}_2/\mathbb{P}_1$ finite elements in three dimensions. Errors vs. Δt. Velocity: (▲) L^2-norm. Pressure: (▽) L^2-norm.*

dimensions using $\mathbb{P}_2/\mathbb{P}_1$ finite elements. The boundary conditions and the source term in the Stokes equations are set so that the solution is given by

$$\mathsf{u}_1(x,y,z,t) = \sin x \sin(y+z+t),\ \mathsf{u}_2(x,y,z,t) = \cos x \cos(y+z+t),$$
$$\mathsf{u}_3(x,y,z,t) = \cos(x)\sin(y+t),\ \mathsf{p}(x,y,t) = \cos x \sin(y+z+t).$$

Both the standard and the rotational forms of the BDF2 pressure-correction scheme were tested. We show in Figure 5.3 the maximum in time of the L^2-norm of the errors on the velocity and the pressure for both schemes. On the left panel we compare the standard and rotational forms of the scheme using $h = 1/40$. Unfortunately, using a higher uniform resolution in space was not possible due to the high cost of the computations. The grid with a stepsize $h = 1/40$ already contains close to

500,000 $\mathbb{P}_2$ nodes. On the right panel we show the errors for the rotational form of the scheme using three different meshes: $h = 1/10$, $1/20$, $1/40$. The convergence rates of the standard version of the scheme are clearly lower than those of the rotational form. The slopes for both the velocity and the pressure errors obtained with the rotational form of the scheme are slightly lower than the best possible estimate following from the claim of Theorem 4.1. The rates $\mathcal{O}(\Delta t^{\frac{4}{3}})$ and $\mathcal{O}(\Delta t^{\frac{5}{6}})$ seem to correspond to a regularity index $s < 1$.

6. Concluding remarks. In this paper, we have analyzed pressure-correction schemes for approximating the incompressible Navier–Stokes equations with prescribed normal stress boundary conditions enforced on parts of the boundary. Our conclusions are twofold.

First, we have shown that the convergence rates of standard pressure-correction methods are too poor to be recommendable for approximating the Navier–Stokes equations in these circumstances. The main reason for the poor accuracy is that an *artificial* homogeneous Dirichlet boundary condition on the pressure has to be imposed to ensure stability.

Second, we have shown that the rotational pressure-correction method leads to reasonably good error estimates. More precisely, assuming full regularity of the Stokes problem, we have shown that the second-order rotational pressure-correction method yields $\mathcal{O}(\Delta t^{3/2})$ accuracy for the velocity in the L^2-norm and $\mathcal{O}(\Delta t)$ accuracy for the velocity in the H^1-norm and the pressure in the L^2-norm. To the best of our knowledge, the results presented in this paper are the first published convergence estimates for a splitting method solving the time-dependent Stokes equations with open boundary conditions.

Finally, it is clear that even though the second-order rotational pressure-correction method yields the best error estimates to date, these are still suboptimal and more research is needed to find a splitting scheme with better properties.

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