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Analysis of the edge finite element approximation of the Maxwell equations with low regularity solutions

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ABSTRACT

We derive $\boldsymbol{H}(\mathrm{curl})$-error estimates and improved $\boldsymbol{L}^2$-error estimates for the Maxwell equations approximated using edge finite elements. These estimates only invoke the expected regularity pickup of the exact solution in the scale of the Sobolev spaces, which is typically lower than $\frac{1}{2}$ and can be arbitrarily close to 0 when the material properties are heterogeneous. The key tools for the analysis are commuting quasi-interpolation operators in $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces and, most crucially, newly-devised quasi-interpolation operators delivering optimal estimates on the decay rate of the best-approximation error for functions with Sobolev smoothness index arbitrarily close to 0. The proposed analysis entirely bypasses the technique known in the literature as the discrete compactness argument.

1. Introduction

The objective of this paper is to review some recent results concerning the approximation of the Maxwell equations using edge finite elements. One important difficulty is the modest regularity pickup of the exact solution in the scale of the Sobolev spaces which is typically lower than $\frac{1}{2}$ and can be arbitrarily close to 0 when the material properties are heterogeneous. We show that the difficulties induced by the lack of stability of the canonical interpolation operators in $\boldsymbol{H}(\mathrm{curl})$- and $\boldsymbol{H}(\mathrm{div})$-conforming finite element spaces can be overcome by invoking recent results on commuting quasi-interpolation operators and newly devised quasi-interpolation operators that deliver optimal estimates on the decay rate of the best-approximation error in those spaces. In addition to a curl-preserving lifting operator introduced by Monk [1, p. 249–250], the commuting quasi-interpolation operators are central to establish a discrete counterpart of the Poincaré–Steklov inequality (bounding the $\boldsymbol{L}^2$-norm of a divergence-free field by the $\boldsymbol{L}^2$-norm of its curl), as already shown in the pioneering work of Arnold et al. [2, §9.1] on Finite Element Exterior Calculus. It is therefore possible to bypass entirely the technique known in the literature as the discrete compactness argument (Kikuchi [3], Monk and Demkowicz [4], Caorsi et al. [5]). The novelty here is the use of quasi-interpolation operators devised by the authors in [6] that give optimal decay rates of the approximation error in fractional Sobolev spaces with a smoothness index that can be arbitrarily small. This allows us to establish optimal $\boldsymbol{H}(\mathrm{curl})$-norm and $\boldsymbol{L}^2$-norm error estimates that do not invoke additional regularity assumptions on the exact solution other than those resulting from the model problem at hand. Optimality is understood here in the sense of the decay rates with respect to the mesh-size; the constants in the error estimates can depend on the heterogeneity ratio of the material properties. Note that all the above quasi-interpolation operators are available with or without prescription of essential boundary conditions.

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The paper is organized as follows. Notation and technical results are given in Section 2. The main results from this section are Theorem 2.2, which states the existence of optimal commuting quasi-interpolation operators, and Theorems 2.3 and 2.4, which give decay estimates of the best approximation in fractional Sobolev norms. Section 3 is concerned with standard facts about the Maxwell equations. In particular, we state our main assumptions on the model problem and briefly recall standard approximation results for the Maxwell equations that solely rely on a coercivity argument. The new results announced above are collected in Section 4 and in Section 5. After establishing the discrete Poincaré–Steklov inequality in Theorem 4.5, our main results are Theorem 4.8 for the $\boldsymbol{H}(\text{curl})$-error estimate and Theorem 5.3 for the improved $\boldsymbol{L}^2$-error estimate. Both results do not invoke regularity assumptions on the exact solution other than those resulting from the model problem at hand.

2. Preliminaries

We recall in this section some notions of functional analysis and approximation using finite elements that will be invoked in the paper. The space dimension is 3 in the entire paper ($d = 3$) and D is an open, bounded, and connected Lipschitz subset in $\mathbb{R}^3$.

2.1. Functional spaces

We are going to make use of the standard L^2-based Sobolev spaces $H^m(D)$, $m \in \mathbb{N}$. The vector-valued counterpart of $H^m(D)$ is denoted $\boldsymbol{H}^m(D)$. We additionally introduce the vector-valued spaces

$$\boldsymbol{H}(\text{curl}; D) := \{\boldsymbol{b} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{b} \in \boldsymbol{L}^2(D)\}, \tag{2.1}$$

$$\boldsymbol{H}(\text{div}; D) := \{\boldsymbol{b} \in \boldsymbol{L}^2(D) \mid \nabla\cdot\boldsymbol{b} \in L^2(D)\}. \tag{2.2}$$

To be dimensionally coherent, we equip these Hilbert spaces with the norms

$$\|b\|_{H^1(D)} := (\|b\|^2_{L^2(D)} + \ell_D^2\|\nabla b\|^2_{L^2(D)})^{\frac12}, \tag{2.3}$$

$$\|\boldsymbol{b}\|_{\boldsymbol{H}(\text{curl};D)} := (\|\boldsymbol{b}\|^2_{L^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{b}\|^2_{L^2(D)})^{\frac12}, \tag{2.4}$$

$$\|\boldsymbol{b}\|_{\boldsymbol{H}(\text{div};D)} := (\|\boldsymbol{b}\|^2_{L^2(D)} + \ell_D^2\|\nabla\cdot\boldsymbol{b}\|^2_{L^2(D)})^{\frac12}, \tag{2.5}$$

where ℓ_D is some characteristic dimension of D, say the diameter of D for instance. In this paper we are also going to use fractional Sobolev norms with smoothness index $s \in (0, 1)$, defined as follows:

$$\|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} := (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^{2s}|\boldsymbol{b}|^2_{\boldsymbol{H}^s(D)})^{\frac12}, \tag{2.6}$$

where $|\cdot|_{\boldsymbol{H}^s(D)}$ is the Sobolev–Slobodeckij semi-norm applied componentwise. Similarly, for any $s > 0$, $s \in \mathbb{R}\setminus\mathbb{N}$, and $p \in [1,\infty)$, the norm of the Sobolev space $W^{s,p}(D)$ is defined by $\|v\|_{W^{s,p}(D)} := (\|v\|^p_{W^{m,p}(D)} + \ell_D^{sp}\sum_{|\alpha|=m}|\partial^\alpha v|^p_{W^{\sigma,p}(D)})^{\frac1p}$ with $\|v\|_{W^{m,p}(D)} := (\sum_{|\alpha|\le m}\ell_D^{|\alpha|p}\|\partial^\alpha v\|^p_{L^p(D)})^{\frac1p}$ where $m := \lfloor s\rfloor \in \mathbb{N}$, $\sigma := m - s \in (0, 1)$.

2.2. Traces

In order to make sense of the boundary conditions, we introduce trace operators. Let $\gamma^{\text{g}} : H^1(D) \to H^{\frac12}(\partial D)$ be the (full) trace operator. It is known that γ^{g} is surjective. Let $\langle\cdot,\cdot\rangle_{\partial D}$ denote the duality pairing between $\boldsymbol{H}^{-\frac12}(\partial D) := (\boldsymbol{H}^{\frac12}(\partial D))'$ and $\boldsymbol{H}^{\frac12}(\partial D)$. We define the tangential trace operator $\gamma^{\text{c}} : \boldsymbol{H}(\text{curl}; D) \to \boldsymbol{H}^{-\frac12}(\partial D)$ as follows:

$$\langle\gamma^{\text{c}}(\boldsymbol{v}), \boldsymbol{l}\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla\times\boldsymbol{w}(\boldsymbol{l})\,\text{d}x - \int_D (\nabla\times\boldsymbol{v})\cdot\boldsymbol{w}(\boldsymbol{l})\,\text{d}x, \tag{2.7}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\text{curl}; D)$, all $\boldsymbol{l} \in \boldsymbol{H}^{\frac12}(\partial D)$ and all $\boldsymbol{w}(\boldsymbol{l}) \in \boldsymbol{H}^1(D)$ such that $\gamma^{\text{g}}(\boldsymbol{w}(\boldsymbol{l})) = \boldsymbol{l}$. One readily verifies that the definition (2.7) is independent of the choice of $\boldsymbol{w}(\boldsymbol{l})$, that $\gamma^{\text{c}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\times\boldsymbol{n}$ when $\boldsymbol{v}$ is smooth, and that the map γ^{c} is bounded.

We define similarly the normal trace map $\gamma^{\text{d}} : \boldsymbol{H}(\text{div}; D) \to H^{-\frac12}(\partial D)$ by

$$\langle\gamma^{\text{d}}(\boldsymbol{v}), l\rangle_{\partial D} := \int_D \boldsymbol{v}\cdot\nabla q(l)\,\text{d}x + \int_D (\nabla\cdot\boldsymbol{v})q(l)\,\text{d}x, \tag{2.8}$$

for all $\boldsymbol{v} \in \boldsymbol{H}(\text{div}; D)$, all $l \in H^{\frac12}(\partial D)$, and all $q(l) \in H^1(D)$ such that $\gamma^{\text{g}}(q(l)) = l$. Here $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the duality pairing between $H^{-\frac12}(\partial D)$ and $H^{\frac12}(\partial D)$. One can verify that the definition (2.8) is independent of the choice of $q(l)$, that $\gamma^{\text{d}}(\boldsymbol{v}) = \boldsymbol{v}_{|\partial D}\cdot\boldsymbol{n}$ when $\boldsymbol{v}$ is smooth, and that the map γ^{d} is bounded.

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Theorem 2.1 (*Kernel of Trace Operators*). *Let* $H_0^1(D) := \overline{C_0^\infty(D)}^{H^1(D)}$, $\boldsymbol{H}_0(curl; D) := \overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{H}(curl;D)}$, *and* $\boldsymbol{H}_0(div; D) := \overline{\boldsymbol{C}_0^\infty(D)}^{\boldsymbol{H}(div;D)}$. *Then, we have*

$$H_0^1(D) = \ker(\gamma^g), \tag{2.9a}$$
$$\boldsymbol{H}_0(curl; D) = \ker(\gamma^c), \tag{2.9b}$$
$$\boldsymbol{H}_0(div; D) = \ker(\gamma^d). \tag{2.9c}$$

Proof. The first identity is well-known, see e.g., Brezis [7, p. 315]. The second one and the third one have been established in [8, Thm. 4.9] in Lipschitz domains. □

2.3. Generic finite element setting

Let $(\mathcal{T}_h)_{h>0}$ be a shape-regular sequence of affine meshes. To avoid technical questions regarding hanging nodes, we also assume that the meshes cover D exactly and that they are matching, i.e., for all cells $K, K' \in \mathcal{T}_h$ such that $K \neq K'$ and $K \cap K' \neq \emptyset$, the set $K \cap K'$ is a common vertex, edge, or face of both K and K' (with obvious extensions in higher space dimensions). Given a mesh $\mathcal{T}_h$, the elements in $K \in \mathcal{T}_h$ are closed sets in $\mathbb{R}^3$ by convention, and they are all assumed to be constructed from a single reference cell $\widehat{K}$ through affine, bijective, geometric transformations $\boldsymbol{T}_K : \widehat{K} \to K$.

The set of the mesh faces is denoted $\mathcal{F}_h$ and is partitioned into the subset of the interfaces denoted $\mathcal{F}_h^\circ$ and the subset of the boundary faces denoted $\mathcal{F}_h^\partial$. Each interface F is oriented by choosing one unit vector $\boldsymbol{n}_F$. The boundary faces are oriented by using the outward normal vector. Given an interface $F \in \mathcal{F}_h^\circ$, we denote by K_l and K_r the two cells such that $F = K_l \cap K_r$ and $\boldsymbol{n}_F$ points from K_l to K_r. This convention allows us to define the notion of jump across F as follows

$$[\![v]\!]_F(\boldsymbol{x}) = v_{|K_l}(\boldsymbol{x}) - v_{|K_r}(\boldsymbol{x}) \qquad \text{a.e. } \boldsymbol{x} \text{ in } F. \tag{2.10}$$

We consider three types of reference elements in the sense of Ciarlet as follows: $(\widehat{K}, \widehat{P}^{\rm g}, \widehat{\Sigma}^{\rm g})$, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm c}, \widehat{\Sigma}^{\rm c})$ and $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm d}, \widehat{\Sigma}^{\rm d})$. We think of $(\widehat{K}, \widehat{P}^{\rm g}, \widehat{\Sigma}^{\rm g})$ as a scalar-valued finite element with some degrees of freedom that require point evaluations, for instance $(\widehat{K}, \widehat{P}^{\rm g}, \widehat{\Sigma}^{\rm g})$ could be a Lagrange element. The finite element $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm c}, \widehat{\Sigma}^{\rm c})$ is vector-valued with some degrees of freedom that require to evaluate integrals over edges. Typically, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm c}, \widehat{\Sigma}^{\rm c})$ is a Nédélec-type or edge element. Likewise, the finite element $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm d}, \widehat{\Sigma}^{\rm d})$ is vector-valued with some of degrees of freedom that require evaluation of integrals over faces. Typically, $(\widehat{K}, \widehat{\boldsymbol{P}}^{\rm d}, \widehat{\Sigma}^{\rm d})$ is a Raviart–Thomas-type element. The reader is referred to Hiptmair [9] for an overview of a canonical construction of the above finite elements.

At this point we do not need to know the exact structure of the above elements, but we are going to assume that they satisfy some commuting properties. More precisely, let $s > \frac{3}{2}$ and let us consider the following functional spaces:

$$\check{V}^{\rm g}(\widehat{K}) = \{f \in H^s(\widehat{K}) \mid \nabla f \in \boldsymbol{H}^{s-\frac{1}{2}}(\widehat{K})\}, \tag{2.11a}$$
$$\check{\boldsymbol{V}}^{\rm c}(\widehat{K}) = \{\boldsymbol{g} \in \boldsymbol{H}^{s-\frac{1}{2}}(\widehat{K}) \mid \nabla \times \boldsymbol{g} \in \boldsymbol{H}^{s-1}(\widehat{K})\}, \tag{2.11b}$$
$$\check{\boldsymbol{V}}^{\rm d}(\widehat{K}) = \{\boldsymbol{g} \in \boldsymbol{H}^{s-1}(\widehat{K}) \mid \nabla \cdot \boldsymbol{g} \in L^1(\widehat{K})\}. \tag{2.11c}$$

Let $\mathcal{I}_{\widehat{K}}^{\rm g}, \mathcal{I}_{\widehat{K}}^{\rm c}, \mathcal{I}_{\widehat{K}}^{\rm d}$ be the canonical interpolation operators associated with the above reference elements. Let $k \in \mathbb{N}$ and let $\mathbb{P}_k(\mathbb{R}^3; \mathbb{R})$ be the vector space composed of the trivariate polynomials of degree at most k. We set $\widehat{P}^{\rm b} := \mathbb{P}_k(\mathbb{R}^3; \mathbb{R})$ and let $\mathcal{I}_{\widehat{K}}^{\rm b}$ be the L^2-projection onto $\widehat{P}^{\rm b}$. We now state a key structural property that must be satisfied by the above Ciarlet triples by assuming that the following diagram commutes:

$$\begin{array}{ccccccc}
\check{V}^{\rm g}(\widehat{K}) & \xrightarrow{\ \nabla\ } & \check{\boldsymbol{V}}^{\rm c}(\widehat{K}) & \xrightarrow{\ \nabla\times\ } & \check{\boldsymbol{V}}^{\rm d}(\widehat{K}) & \xrightarrow{\ \nabla\cdot\ } & L^1(\widehat{K}) \\
\downarrow \mathcal{I}_{\widehat{K}}^{\rm g} & & \downarrow \mathcal{I}_{\widehat{K}}^{\rm c} & & \downarrow \mathcal{I}_{\widehat{K}}^{\rm d} & & \downarrow \mathcal{I}_{\widehat{K}}^{\rm b} \\
\widehat{P}^{\rm g} & \xrightarrow{\ \nabla\ } & \widehat{\boldsymbol{P}}^{\rm c} & \xrightarrow{\ \nabla\times\ } & \widehat{\boldsymbol{P}}^{\rm d} & \xrightarrow{\ \nabla\cdot\ } & \widehat{P}^{\rm b}
\end{array} \tag{2.12}$$

In order to construct conforming approximation spaces based on $(\mathcal{T}_h)_{h>0}$ using the above reference elements, we introduce the following linear maps:

$$\psi_K^{\rm g}(v) = v \circ \boldsymbol{T}_K, \tag{2.13a}$$
$$\psi_K^{\rm c}(\boldsymbol{v}) = \mathbb{J}_K^{\mathsf{T}}(\boldsymbol{v} \circ \boldsymbol{T}_K), \tag{2.13b}$$
$$\psi_K^{\rm d}(\boldsymbol{v}) = \det(\mathbb{J}_K)\,\mathbb{J}_K^{-1}(\boldsymbol{v} \circ \boldsymbol{T}_K), \tag{2.13c}$$
$$\psi_K^{\rm b}(v) = \det(\mathbb{J}_K)(v \circ \boldsymbol{T}_K), \tag{2.13d}$$

where $\psi_K^{\rm g}$ is the pullback by $\boldsymbol{T}_K$, and $\psi_K^{\rm c}$ and $\psi_K^{\rm d}$ are the contravariant and covariant Piola transformations, respectively. With these definitions in hand, we set

$$P^{\rm g}(\mathcal{T}_h) := \{v_h \in L^1(D) \mid \psi_K^{\rm g}(v_{h|K}) \in \widehat{P}^{\rm g},\ \forall K \in \mathcal{T}_h,\ [\![v_h]\!]_F^{\rm g} = 0,\ \forall F \in \mathcal{F}_h^\circ\}, \tag{2.14a}$$

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$$\boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \psi_K^{\mathrm{c}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{c}},\ \forall K \in \mathcal{T}_h,\ [\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} = \boldsymbol{0},\ \forall F \in \mathcal{F}_h^\circ\}, \tag{2.14b}$$

$$\boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) := \{\boldsymbol{v}_h \in \boldsymbol{L}^1(D) \mid \psi_K^{\mathrm{d}}(\boldsymbol{v}_{h|K}) \in \widehat{\boldsymbol{P}}^{\mathrm{d}},\ \forall K \in \mathcal{T}_h,\ [\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}} = 0,\ \forall F \in \mathcal{F}_h^\circ\}, \tag{2.14c}$$

$$P^{\mathrm{b}}(\mathcal{T}_h) := \{v_h \in L^1(D) \mid \psi_K^{\mathrm{b}}(v_{h|K}) \in \widehat{P}^{\mathrm{b}},\ \forall K \in \mathcal{T}_h\}, \tag{2.14d}$$

where $[\![v_h]\!]_F^{\mathrm{g}} := [\![v_h]\!]_F$, $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{c}} := [\![\boldsymbol{v}_h]\!]_F \times \boldsymbol{n}_F$, and $[\![\boldsymbol{v}_h]\!]_F^{\mathrm{d}} := [\![\boldsymbol{v}_h]\!]_F \cdot \boldsymbol{n}_F$. Finally, to be able to account for boundary conditions, we define

$$P_0^{\mathrm{g}}(\mathcal{T}_h) := P^{\mathrm{g}}(\mathcal{T}_h) \cap H_0^1(D), \tag{2.15a}$$

$$\boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) := \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathrm{curl}; D), \tag{2.15b}$$

$$\boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h) := \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathrm{div}; D). \tag{2.15c}$$

2.4. Best approximation and commuting quasi-interpolation

Until recently, the stability and the convergence analysis of finite element techniques for the approximation of the Maxwell equations was made difficult by the absence of optimal approximation results. The root of the difficulty was that the equivalent of the Clément/Scott–Zhang quasi-interpolation operator was not available for $\boldsymbol{H}(\mathrm{curl}; D)$-conforming and $\boldsymbol{H}(\mathrm{div}; D)$-conforming elements. Moreover, no clear best-approximation estimate in fractional Sobolev norms was known. We summarize in this section some of the most recent results in this direction.

The bases for the construction of stable, commuting, and quasi-interpolation projectors have been laid out in Schöberl [10,11] and Christiansen [12], where stability and commutation are achieved by composing the canonical finite element interpolation operators with some mollification technique. Then, following Schöberl [13], the projection property over finite element spaces is obtained by composing these operators with the inverse of their restriction to the said spaces. An important extension of this construction allowing the possibility of using shape-regular mesh sequences and boundary conditions has been achieved by Christiansen and Winther [14] (see also Arnold et al. [2, §5.4] where this work was prefigured). Further variants of this construction have lately been proposed. For instance in Christiansen [15], the quasi-interpolation projector has the additional property of preserving polynomials locally, up to a certain degree, and in Falk and Winther [16], it is defined locally. The results of [14] have been revisited in [8] by invoking shrinking-based mollification operators which do not require any extension outside the domain.

In order to stay general, we introduce an integer q with the convention that $q = 1$ when we work with scalar-valued functions and $q = 3$ when we work with vector-valued functions. For instance, we denote by $\mathbb{P}_k(\mathbb{R}^3; \mathbb{R}^q)$ the vector space composed of the trivariate polynomials with values in $\mathbb{R}^q$. The quasi-interpolation results mentioned above can be summarized as follows:

Theorem 2.2 (*Stable, Commuting Projection*). *Let $P(\mathcal{T}_h)$ be one of the finite element spaces introduced in (2.14)–(2.15). Then there exists a quasi-interpolation operator $\mathcal{J}_h : L^1(D; \mathbb{R}^q) \to P(\mathcal{T}_h)$ with the following properties:*

(i) *$P(\mathcal{T}_h)$ is pointwise invariant under $\mathcal{J}_h$.*
(ii) *Let $p \in [1, \infty]$. There is c, uniform w.r.t. h, such that $\|\mathcal{J}_h\|_{\mathcal{L}(L^p;L^p)} \le c$ and*

$$\|f - \mathcal{J}_h(f)\|_{L^p(D;\mathbb{R}^q)} \le c \inf_{f_h \in P(\mathcal{T}_h)} \|f - f_h\|_{L^p(D;\mathbb{R}^q)}, \tag{2.16}$$

for all $f \in L^p(D; \mathbb{R}^q)$;
(iii) *$\mathcal{J}_h$ commutes with the standard differential operators, i.e., the following diagrams commute:*

$$\begin{array}{ccccccc}
H^1(D) & \xrightarrow{\nabla} & \boldsymbol{H}(\mathrm{curl}; D) & \xrightarrow{\nabla\times} & \boldsymbol{H}(\mathrm{div}; D) & \xrightarrow{\nabla\cdot} & L^2(D) \\
\downarrow \mathcal{J}_h^{\mathrm{g}} & & \downarrow \mathcal{J}_h^{\mathrm{c}} & & \downarrow \mathcal{J}_h^{\mathrm{d}} & & \downarrow \mathcal{J}_h^{\mathrm{b}} \\
P^{\mathrm{g}}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}^{\mathrm{d}}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}(\mathcal{T}_h)
\end{array} \tag{2.17}$$

$$\begin{array}{ccccccc}
H_0^1(D) & \xrightarrow{\nabla} & \boldsymbol{H}_0(\mathrm{curl}; D) & \xrightarrow{\nabla\times} & \boldsymbol{H}_0(\mathrm{div}; D) & \xrightarrow{\nabla\cdot} & L^2(D) \\
\downarrow \mathcal{J}_{h0}{}^{\mathrm{g}} & & \downarrow \mathcal{J}_{h0}{}^{\mathrm{c}} & & \downarrow \mathcal{J}_{h0}{}^{\mathrm{d}} & & \downarrow \mathcal{J}_h{}^{\mathrm{b}} \\
P_0^{\mathrm{g}}(\mathcal{T}_h) & \xrightarrow{\nabla} & \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) & \xrightarrow{\nabla\times} & \boldsymbol{P}_0{}^{\mathrm{d}}(\mathcal{T}_h) & \xrightarrow{\nabla\cdot} & P^{\mathrm{b}}(\mathcal{T}_h)
\end{array} \tag{2.18}$$

Proof. See e.g., [8, Thm. 6.5]. □

For the estimate (2.16) to be useful we need an estimate on the decay rate of best-approximation error. This question is answered by the following two results:

Theorem 2.3 (*Best Approximation*)**.** *Let $P(\mathcal{T}_h)$ be one of the finite element spaces introduced in (2.14). Let k be the largest integer such that $\mathbb{P}_k(\mathbb{R}^3;\mathbb{R}^q)\subset\widehat{P}$. There exists a uniform constant c such that*

$$\inf_{w_h\in P(\mathcal{T}_h)}\|v-w_h\|_{L^p(D;\mathbb{R}^q)}\le c\,h^r|v|_{W^{r,p}(D;\mathbb{R}^q)}, \tag{2.19}$$

for all $r\in[0,k+1]$, all $p\in[1,\infty)$ if $r\notin\mathbb{N}$ or all $p\in[1,\infty]$ if $r\in\mathbb{N}$, and all $v\in W^{r,p}(D;\mathbb{R}^q)$.

Proof. See [6, Cor. 5.4]. □

Theorem 2.4 (*Best Approximation with Boundary Conditions*)**.** *Let $P_0(\mathcal{T}_h)$ be one of the finite element spaces introduced in (2.15) and let γ be the trace operator from Section 2.2 associated with $P_0(\mathcal{T}_h)$. Let k be the largest integer such that $\mathbb{P}_k(\mathbb{R}^3;\mathbb{R}^q)\subset\widehat{P}$. There exists a uniform constant c, that depends on $|rp-1|$, such that*

$$\inf_{w_h\in P_0(\mathcal{T}_h)}\|v-w_h\|_{L^p(D;\mathbb{R}^q)}\le\begin{cases}ch^r|v|_{W^{r,p}(D;\mathbb{R}^q)}, & \forall v\in W^{r,p}_{0,\gamma}(D;\mathbb{R}^q)\text{ if } rp>1,\\ ch^r\ell_D^{-r}\|v\|_{W^{r,p}(D;\mathbb{R}^q)}, & \forall v\in W^{r,p}(D;\mathbb{R}^q)\text{ if } rp<1,\end{cases} \tag{2.20}$$

where $W^{r,p}_{0,\gamma}(D;\mathbb{R}^q):=\{v\in W^{r,p}(D;\mathbb{R}^q)\mid\gamma(v)=0\}$, for all $r\in[0,k+1]$, all $p\in[1,\infty)$ if $r\notin\mathbb{N}$ or all $p\in[1,\infty]$ if $r\in\mathbb{N}$.

Proof. See [6, Cor. 6.5]. □

Localized versions of the above results and best-approximation error estimates for higher-order Sobolev semi-norms can be found in [6]. These results are proved by constructing quasi-interpolation operators in a unified way. This construction is done in two steps and consists of composing an elementwise projection onto the broken finite element space with a smoothing operator based on the averaging of the degrees of freedom on the broken space. We also refer the reader to Ciarlet [17] for similar estimates for the Scott–Zhang quasi-interpolation operator in the context of scalar-valued finite elements and $rp>1$.

Remark 2.5 (*Edge Elements*)**.** To put the above results in perspective with the literature, let us observe that the canonical interpolation operator for edge elements is only stable in $\boldsymbol{H}^s(D)$ for $s>1$. Using the techniques in Amrouche et al. [18], one can show that this operator is also stable in the space $\{\boldsymbol{v}\in\boldsymbol{H}^s(D)\mid\nabla\times\boldsymbol{v}\in\boldsymbol{L}^p(D)\}$ with $s>\frac{1}{2}$ and $p>2$, see, e.g., Boffi and Gastaldi [19]. In contrast with these results, Theorem 2.3 states that any function $\boldsymbol{v}$ in $\boldsymbol{H}^s(D)$, with s arbitrarily close to zero, can be optimally approximated in $\boldsymbol{P}^c(\mathcal{T}_h)$. Let us also observe that the best-approximation result from Theorem 2.4 can be used with functions that are not smooth enough to have a well-defined trace at the boundary and that these functions are approximated by finite element functions that do satisfy a boundary condition. □

3. Maxwell's equations

In this section we recall standard facts about the Maxwell equations that will be used later in the paper. For an introduction to the subject, the reader is referred to Bossavit [20, Chap. 1] or Monk [21, Chap. 1].

3.1. The model problem

Maxwell's equations consist of a set of partial differential equations giving a macroscopic description of electromagnetic phenomena. More precisely, these equations describe how the electric field, $\boldsymbol{E}$, the magnetic field $\boldsymbol{H}$, the electric displacement field, $\boldsymbol{D}$, and the magnetic induction (sometimes called magnetic flux density), $\boldsymbol{B}$, interact through the action of currents, $\boldsymbol{j}$, and charges, ρ:

$$\partial_t\boldsymbol{D}-\nabla\times\boldsymbol{H}=-\boldsymbol{j}\quad\text{(Ampère's law)}, \tag{3.1a}$$
$$\partial_t\boldsymbol{B}+\nabla\times\boldsymbol{E}=\boldsymbol{0}\quad\text{(Faraday's law of induction)}, \tag{3.1b}$$
$$\nabla\cdot\boldsymbol{D}=\rho\quad\text{(Gauss' law for electricity)}, \tag{3.1c}$$
$$\nabla\cdot\boldsymbol{B}=0\quad\text{(Gauss' law for magnetism)}. \tag{3.1d}$$

Note that (3.1c)–(3.1d) can be viewed as constraints on the time-evolution problem (3.1a)–(3.1b). If $(\nabla\cdot\boldsymbol{B})_{|t=0}=0$, then taking the divergence of (3.1b) implies that (3.1d) holds at all times. Similarly, assuming that $(\nabla\cdot\boldsymbol{D})_{|t=0}=\rho_{|t=0}$ and that the charge conservation equation $\partial_t\rho+\nabla\cdot\boldsymbol{j}=0$ holds at all times implies that (3.1c) holds at all times.

The system (3.1) is closed by relating the fields through constitutive laws describing microscopic mechanisms of polarization and magnetization:

$$\boldsymbol{D}-\varepsilon_0\boldsymbol{E}=\boldsymbol{P},\qquad\boldsymbol{B}=\mu_0(\boldsymbol{H}+\boldsymbol{M}), \tag{3.2}$$

where ε_0 and μ_0 are the electric permittivity and the magnetic permeability of vacuum; the fields $\boldsymbol{P}$ and $\boldsymbol{M}$ are the polarization and the magnetization, respectively. These quantities are the average representatives at macroscopic scale of

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complicated microscopic interactions, i.e., they need to be modeled and measured. For instance, $\boldsymbol{P} = \boldsymbol{0}$ and $\boldsymbol{M} = \boldsymbol{0}$ in vacuum, and it is common to use $\boldsymbol{P} = \varepsilon_0\varepsilon_r\boldsymbol{E}$ and $\boldsymbol{M} = \mu_r\boldsymbol{H}$ to model isotropic homogeneous dielectric and magnetic materials, where ε_r is the electric susceptibility and μ_r is the magnetic susceptibility. The current and charge density, $\boldsymbol{j}$ and ρ, are a priori given, but it is also possible to make these quantities depend on the other fields through phenomenological mechanisms. For instance, it is possible to further decompose the current into one component that depends on the material and another one that is a source; the simplest model doing that is Ohm's law: $\boldsymbol{j} = \boldsymbol{j}_s + \sigma\boldsymbol{E}$; σ is the electrical conductivity and $\boldsymbol{j}_s$ is an imposed current.

We now formulate Maxwell's equations in two different regimes: the harmonic regime, leading to the Helmholtz problem, and the eddy current limit. We henceforth assume that

$$\boldsymbol{D} = \epsilon\boldsymbol{E} \quad \text{and} \quad \boldsymbol{B} = \mu\boldsymbol{H}, \tag{3.3}$$

where ϵ and μ may be space-dependent.

3.2. The Helmholtz problem

We first consider Maxwell's equations in the harmonic regime. Using the convention $i^2 = -1$, the time-dependence is assumed to be of the type $e^{i\omega t}$ where ω is the angular frequency. Letting $(\partial D_\mathrm{d}, \partial D_\mathrm{n})$ be a partition of the boundary ∂D of D, the time-harmonic version of (3.1a)–(3.1b) is

$$i\omega\epsilon\boldsymbol{E} + \sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{j}_s, \qquad \text{in } D, \tag{3.4a}$$
$$i\omega\mu\boldsymbol{H} + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \qquad \text{in } D, \tag{3.4b}$$
$$\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{d}, \quad \boldsymbol{E}_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{n}, \qquad \text{on } \partial D. \tag{3.4c}$$

The dependent variables are the electric field, $\boldsymbol{E}$, and the magnetic field, $\boldsymbol{H}$. The data are the conductivity, σ, the permittivity, ϵ, the permeability, μ, the current, $\boldsymbol{j}_s$, and the boundary data $\boldsymbol{a}_\mathrm{d}$ and $\boldsymbol{a}_\mathrm{n}$. The material coefficients ϵ and μ are allowed to be complex-valued. The system (3.4) models, for instance, a microwave oven; see e.g., [20, Chap. 9]. The conditions $\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{0}$ and $\boldsymbol{E}_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{0}$ are usually called perfect conductor and perfect magnetic conductor boundary conditions, respectively.

Let us assume that the modulus of the magnetic permeability μ is bounded away from zero uniformly in D. It is then possible to eliminate $\boldsymbol{H}$ by using $\boldsymbol{H} = i(\omega\mu)^{-1}\nabla\times\boldsymbol{E}$, and the resulting system takes the following form:

$$(-\omega^2\epsilon + i\omega\sigma)\boldsymbol{E} + \nabla\times(\mu^{-1}\nabla\times\boldsymbol{E}) = -i\omega\boldsymbol{j}_s, \qquad \text{in } D, \tag{3.5a}$$
$$(\nabla\times\boldsymbol{E})_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = -i\omega\mu\boldsymbol{a}_\mathrm{d}, \quad \boldsymbol{E}_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{n}, \qquad \text{on } \partial D. \tag{3.5b}$$

If $\sigma = 0$, this problem leads to two different situations depending on whether ω is a resonance frequency of the domain D or not. If it is the case, the above problem is an eigenvalue problem, otherwise it is a boundary-value problem.

3.3. Eddy current problem

When the time scale of interest, say τ, is such that the ratio $\epsilon/(\tau\sigma)$ is very small, i.e., $\epsilon/(\tau\sigma) \ll 1$, it is legitimate to neglect the so-called displacement current density (i.e., Maxwell's correction, $\partial_t\boldsymbol{D}$). This situation occurs in particular in systems with moving parts (either solid or fluids) with a characteristic speed that is significantly slower than the speed of light. Assuming again that $(\partial D_\mathrm{d}, \partial D_\mathrm{n})$ forms a partition of the boundary ∂D of D, the resulting system takes the following form:

$$\sigma\boldsymbol{E} - \nabla\times\boldsymbol{H} = -\boldsymbol{j}_s, \qquad \text{in } D, \tag{3.6a}$$
$$\partial_t(\mu\boldsymbol{H}) + \nabla\times\boldsymbol{E} = \boldsymbol{0}, \qquad \text{in } D, \tag{3.6b}$$
$$\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{d}, \quad \boldsymbol{E}_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{n}, \qquad \text{on } \partial D. \tag{3.6c}$$

The system (3.6) arises in magneto-hydrodynamics (MHD); in this case, $\boldsymbol{j}_s$ is further decomposed into $\boldsymbol{j}_s = \boldsymbol{j}_s' + \sigma\boldsymbol{u}\times\boldsymbol{B}$ where $\boldsymbol{u}$ is the velocity of the fluid occupying the domain D, i.e., the actual current is decomposed into $\boldsymbol{j} = \boldsymbol{j}_s' + \sigma(\boldsymbol{E} + \boldsymbol{u}\times\boldsymbol{B})$.

Let us assume that the conductivity σ is bounded uniformly from below in the domain D by a positive constant $\sigma_{\min}$. It is then possible to eliminate the electric field from (3.6) by using $\boldsymbol{E} = \sigma^{-1}(\nabla\times\boldsymbol{H} - \boldsymbol{j}_s)$. The new system to be solved is re-written as follows:

$$\partial_t(\mu\boldsymbol{H}) + \nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H}) = \nabla\times(\sigma^{-1}\boldsymbol{j}_s), \qquad \text{in } D, \tag{3.7a}$$
$$\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{d}, \quad (\sigma^{-1}\nabla\times\boldsymbol{H})_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{c}_\mathrm{n}, \qquad \text{on } \partial D, \tag{3.7b}$$

where $\boldsymbol{c}_\mathrm{n} = \boldsymbol{a}_\mathrm{n} + \sigma^{-1}\boldsymbol{j}_{s|\partial D_\mathrm{n}}\times\boldsymbol{n}$. At this point, it is possible to further simplify the problem by assuming that either the time evolution is harmonic, $\boldsymbol{H}(\boldsymbol{x}, t) = \boldsymbol{H}_\mathrm{sp}(\boldsymbol{x})e^{i\omega t}$, or the time derivative is approximated as follows $\partial_t\boldsymbol{H}(\boldsymbol{x}, t) \approx (\Delta t)^{-1}(\boldsymbol{H}(\boldsymbol{x}, t) - \boldsymbol{H}(\boldsymbol{x}, t - \Delta t))$, where Δt is the time step of the time discretization. The above system then reduces to solving the following problem:

$$\widetilde{\mu}\boldsymbol{H} + \nabla\times(\sigma^{-1}\nabla\times\boldsymbol{H}) = \boldsymbol{f}, \qquad \text{in } D, \tag{3.8a}$$
$$\boldsymbol{H}_{|\partial D_\mathrm{d}}\times\boldsymbol{n} = \boldsymbol{a}_\mathrm{d}, \quad (\sigma^{-1}\nabla\times\boldsymbol{H})_{|\partial D_\mathrm{n}}\times\boldsymbol{n} = \boldsymbol{c}_\mathrm{n}, \qquad \text{on } \partial D, \tag{3.8b}$$

after appropriately renaming the dependent variable and the data, say either $\widetilde{\mu} := i\omega\mu$ and $\boldsymbol{f} = \nabla\times(\sigma^{-1}\boldsymbol{j}_s)$, or $\widetilde{\mu} := \mu(\Delta t)^{-1}$ and $\boldsymbol{f} := \nabla\times(\sigma^{-1}\boldsymbol{j}_s) + \widetilde{\mu}\boldsymbol{H}(x, t - \Delta t)$, etc.; note that $\nabla\cdot\boldsymbol{f} = 0$ in both cases.

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3.4. Abstract problem

The Helmholtz problem and the eddy current problem have a very similar structure. For simplicity, we restrict the scope to Dirichlet boundary conditions; the techniques presented below can be adapted to handle Neumann boundary conditions as well. After lifting the Dirichlet boundary condition and making appropriate changes of notation, the above two problems (3.5) and (3.8) can be reformulated in the following common form: Find $\boldsymbol{A}$ such that

$$\widetilde{\mu}\boldsymbol{A} + \nabla \times (\kappa \nabla \times \boldsymbol{A}) = \boldsymbol{f}, \qquad \boldsymbol{A}_{|\partial D} \times \boldsymbol{n} = \boldsymbol{0}. \tag{3.9}$$

We henceforth assume that $\boldsymbol{f} \in \boldsymbol{L}^2(D)$ and that $\nabla \cdot \boldsymbol{f} = 0$. Note that, by taking the divergence of the PDE in (3.9), this property implies that

$$\nabla \cdot (\widetilde{\mu}\boldsymbol{A}) = 0. \tag{3.10}$$

This additional condition on $\boldsymbol{A}$ plays a key role in Section 4 and in Section 5. Concerning the material properties $\widetilde{\mu}$ and κ, we make three assumptions: (i) Boundedness: $\widetilde{\mu}, \kappa \in L^\infty(D; \mathbb{C})$ and we set $\mu_\sharp = \operatorname{ess\,sup}_{\boldsymbol{x}\in D}|\widetilde{\mu}(\boldsymbol{x})|$ and $\kappa_\sharp = \operatorname{ess\,sup}_{\boldsymbol{x}\in D}|\kappa(\boldsymbol{x})|$; (ii) Rotated positivity: there are real numbers θ, $\mu_\flat > 0$, and $\kappa_\flat > 0$ so that

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\widetilde{\mu}(\boldsymbol{x})) \geq \mu_\flat \quad \text{and} \quad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \Re(\mathrm{e}^{i\theta}\kappa(\boldsymbol{x})) \geq \kappa_\flat. \tag{3.11}$$

We define the heterogeneity ratios $\mu_{\sharp/\flat} := \frac{\mu_\sharp}{\mu_\flat}$ and $\kappa_{\sharp/\flat} := \frac{\kappa_\sharp}{\kappa_\flat}$. We also define the magnetic Reynolds number $\gamma_{\widetilde{\mu},\kappa} = \mu_\sharp \ell_D^2 \kappa_\sharp^{-1}$. (Obviously, if the material is highly contrasted, several magnetic Reynolds numbers can be defined.) (iii) Piecewise smoothness: there is a partition of D into M disjoint Lipschitz subdomains $D_1, \ldots, D_M$ so that $\widetilde{\mu}_{|D_i} \in W^{1,\infty}(D_i)$ and $\kappa_{|D_i} \in W^{1,\infty}(D_i)$ for all $1 \leq i \leq M$.

Remark 3.1 (*Assumption (3.11)*)**.** This assumption holds, e.g., if the coefficient κ is real and if $\widetilde{\mu} = \rho_\mu e^{i\theta_\mu}$ with $\operatorname{ess\,inf}_{\boldsymbol{x}\in D} \rho_\mu(\boldsymbol{x}) = \rho_\flat > 0$ and $\theta_\mu(\boldsymbol{x}) \in [\theta_{\min}, \theta_{\max}] \subset (-\pi, \pi)$ a.e. in D with $\delta := \theta_{\max} - \theta_{\min} < \pi$; indeed, one can take $\theta = -\frac{1}{2}(\theta_{\min} + \theta_{\max})\frac{\pi}{2\pi-\delta}$, $\mu_\flat = \min(\cos(\theta_{\min} + \theta), \cos(\theta_{\max} + \theta))\rho_\flat$ and $\kappa_\flat = \cos(\theta)\operatorname{ess\,inf}_{\boldsymbol{x}\in D} \kappa(\boldsymbol{x})$. For instance, this is the case for the Helmholtz problem (where $\widetilde{\mu} = -\omega^2\epsilon + i\omega\sigma$ and $\kappa = \mu^{-1}$) and for the eddy current problem (where $\widetilde{\mu} := i\omega\mu$ or $\widetilde{\mu} := \mu(\Delta t)^{-1}$ and $\kappa = \sigma^{-1}$). An important example when the condition (3.11) does not hold is when the two complex numbers $\widetilde{\mu}$ and κ are collinear and point in opposite directions; in this case, (3.9) is an eigenvalue problem. □

3.5. Basic weak formulation and approximation

We recall in this section standard approximation results for (3.9) that solely rely on a coercivity argument. More subtle arguments are invoked in Section 4 and in Section 5; in particular, we do not use here that $\nabla \cdot \boldsymbol{f} = 0$.

A weak formulation of (3.9) is obtained by multiplying the equation by the complex conjugate of a smooth test function $\boldsymbol{b}$ with zero tangential component, integrating the result over D, and integrating by parts:

$$\int_D (\widetilde{\mu}\boldsymbol{A} \cdot \overline{\boldsymbol{b}} + \kappa \nabla \times \boldsymbol{A} \cdot \nabla \times \overline{\boldsymbol{b}})\,\mathrm{d}x = \int_D \boldsymbol{f} \cdot \overline{\boldsymbol{b}}\,\mathrm{d}x.$$

The integral on the left-hand side makes sense if $\boldsymbol{A}, \boldsymbol{b} \in \boldsymbol{H}(\text{curl}; D)$. We introduce the following closed subspace of $\boldsymbol{H}(\text{curl}; D)$ to account for the boundary conditions:

$$\boldsymbol{V}_0 := \boldsymbol{H}_0(\text{curl}; D) = \{\boldsymbol{b} \in \boldsymbol{H}(\text{curl}; D) \mid \gamma^{c}(\boldsymbol{b}) = \boldsymbol{0}\}. \tag{3.12}$$

The weak formulation of (3.9) is the following:

$$\begin{cases} \text{Find } \boldsymbol{A} \in \boldsymbol{V}_0 \text{ such that} \\ a_{\widetilde{\mu},\kappa}(\boldsymbol{A}, \boldsymbol{b}) = \ell(\boldsymbol{b}), \quad \forall \boldsymbol{b} \in \boldsymbol{V}_0, \end{cases} \tag{3.13}$$

where the sesquilinear form $a_{\widetilde{\mu},\kappa}$ and the antilinear form ℓ are defined as follows:

$$a_{\widetilde{\mu},\kappa}(\boldsymbol{a}, \boldsymbol{b}) := \int_D (\widetilde{\mu}\boldsymbol{a} \cdot \overline{\boldsymbol{b}} + \kappa \nabla \times \boldsymbol{a} \cdot \nabla \times \overline{\boldsymbol{b}})\,\mathrm{d}x, \tag{3.14a}$$

$$\ell(\boldsymbol{b}) := \int_D \boldsymbol{f} \cdot \overline{\boldsymbol{b}}\,\mathrm{d}x. \tag{3.14b}$$

Theorem 3.2 (*Well-posedness*)**.** *Assume that $\boldsymbol{f} \in \boldsymbol{L}^2(D)$, $\widetilde{\mu}, \kappa \in L^\infty(D; \mathbb{C})$, and that (3.11) holds. Then the sesquilinear form $a_{\widetilde{\mu},\kappa}$ is coercive:*

$$\Re(e^{i\theta} a_{\widetilde{\mu},\kappa}(\boldsymbol{b}, \boldsymbol{b})) \geq \min(\mu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(curl;D)}, \tag{3.15}$$

for all $\boldsymbol{b} \in \boldsymbol{H}(\text{curl}; D)$, and the problem (3.13) is well-posed.

Proof. We apply the complex version of the Lax–Milgram Lemma. The coercivity follows from (3.11), and the boundedness of $a_{\widetilde{\mu},\kappa}$ and ℓ are consequences of $\widetilde{\mu}, \kappa \in L^\infty(D;\mathbb{C})$ and $\boldsymbol{f} \in \boldsymbol{L}^2(D)$. □

Let us now briefly recall elementary approximation results solely based on the coercivity of the sesquilinear form $a_{\widetilde{\mu},\kappa}$. We consider a shape-regular sequence of affine meshes $(\mathcal{T}_h)_{h>0}$ of D and the associated approximation setting introduced in Sections 2.3–2.4. In the rest of the paper, k is the largest integer such that $\mathbb{P}_k(\mathbb{R}^3;\mathbb{R}^3) \subset \widehat{\boldsymbol{P}}^{\mathrm{c}} \cap \widehat{\boldsymbol{P}}^{\mathrm{d}}$. The finite element space we are going to work with is defined by

$$\boldsymbol{V}_{h0} = \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h) = \{\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{c}}(\mathcal{T}_h) \mid \boldsymbol{b}_{h|\partial D} \times \boldsymbol{n} = \boldsymbol{0}\}. \tag{3.16}$$

Observe that the Dirichlet boundary condition is strongly enforced. The discrete counterpart of (3.13) is formulated as follows:

$$\begin{cases} \text{Find } \boldsymbol{A}_h \in \boldsymbol{V}_{h0} \text{ such that} \\ a_{\widetilde{\mu},\kappa}(\boldsymbol{A}_h, \boldsymbol{b}_h) = \ell(\boldsymbol{b}_h), \quad \forall \boldsymbol{b}_h \in \boldsymbol{V}_{h0}. \end{cases} \tag{3.17}$$

Theorem 3.2 together with the Lax–Milgram Lemma and the conformity property $\boldsymbol{V}_{h0} \subset \boldsymbol{V}_0$ implies that (3.17) has a unique solution.

Theorem 3.3 (*Error Estimate*)**.** *Under the assumptions of Theorem 3.2, the following holds true:*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(curl;D)} \le \frac{\max(\mu_\sharp, \ell_D^{-2}\kappa_\sharp)}{\min(\mu_\flat, \ell_D^{-2}\kappa_\flat)} \inf_{\boldsymbol{b}_h \in \boldsymbol{V}_{h0}} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{H}(curl;D)}. \tag{3.18}$$

Moreover, assuming that there is $r \in (0, k+1]$ so that $\boldsymbol{A} \in \boldsymbol{H}^r(D)$ and $\nabla \times \boldsymbol{A} \in \boldsymbol{H}^r(D)$, the following error estimate holds true:

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(curl;D)} \le c\, h^r (|\boldsymbol{A}|_{\boldsymbol{H}^r(D)} + \ell_D |\nabla \times \boldsymbol{A}|_{\boldsymbol{H}^r(D)}), \tag{3.19}$$

where the constant c is uniform w.r.t. h and proportional to $\frac{\max(\mu_\sharp, \ell_D^{-2}\kappa_\sharp)}{\min(\mu_\flat, \ell_D^{-2}\kappa_\flat)}$.

Proof. The sesquilinear form $a_{\widetilde{\mu},\kappa}$ is bounded on $\boldsymbol{V}_0 \times \boldsymbol{V}_0$ with

$$|a_{\widetilde{\mu},\kappa}(\boldsymbol{a}, \boldsymbol{b})| \le \max(\mu_\sharp, \ell_D^{-2}\kappa_\sharp) \|\boldsymbol{a}\|_{\boldsymbol{H}(\mathrm{curl};D)} \|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)}, \tag{3.20}$$

and it is coercive on $\boldsymbol{V}_{h0}$ with coercivity constant $\min(\mu_\flat, \ell_D^{-2}\kappa_\flat)$. Hence, using the abstract error estimate from Xu and Zikatanov [22, Thm. 2], we obtain (3.18). The second inequality (3.19) is a consequence of Theorem 2.2 together with the estimates of the best approximation error in Theorems 2.3 and 2.4. More precisely, we estimate from above the infimum in (3.18) by taking $\boldsymbol{b}_h = \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A}) \in \boldsymbol{V}_{h0}$. Using the notation from Sections 2.3–2.4, we have $\boldsymbol{V}_{h0} = \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)$ and

$$\begin{aligned} \|\boldsymbol{A} - \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A})\|_{\boldsymbol{H}(\mathrm{curl};D)} &\le \|\boldsymbol{A} - \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla \times (\boldsymbol{A} - \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A}))\|_{\boldsymbol{L}^2(D)} \\ &= \|\boldsymbol{A} - \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \ell_D \|\nabla \times \boldsymbol{A} - \mathcal{J}_{h0}^{\mathrm{d}}(\nabla \times \boldsymbol{A})\|_{\boldsymbol{L}^2(D)} \\ &\le c \inf_{\boldsymbol{b}_h \in \boldsymbol{P}_0^{\mathrm{c}}(\mathcal{T}_h)} \|\boldsymbol{A} - \boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)} + c' \ell_D \inf_{\boldsymbol{d}_h \in \boldsymbol{P}_0^{\mathrm{d}}(\mathcal{T}_h)} \|\nabla \times \boldsymbol{A} - \boldsymbol{d}_h\|_{\boldsymbol{L}^2(D)} \\ &\le c\, h^r (|\boldsymbol{A}|_{\boldsymbol{H}^r(D)} + \ell_D |\nabla \times \boldsymbol{A}|_{\boldsymbol{H}^r(D)}). \end{aligned}$$

This completes the proof. □

Remark 3.4 (*Convergence Rate*)**.** An alternative proof of (3.19) is given in Ciarlet [23, Prop. 4] using subtle decompositions of the subspaces $\boldsymbol{X}_{0\widetilde{\mu}}$ and $\boldsymbol{X}_{*\kappa^{-1}}$ defined by (3.22) and (3.24). The main idea is that fields in these spaces can be decomposed into a regular part which can be approximated using the Nédélec interpolation operator and a singular part that crucially takes the form of the gradient of some potential that can be approximated using the Scott–Zhang quasi-interpolation operator.

Finally, let us see whether an improved estimate on $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2(D)}$ can be obtained by the Aubin–Nitsche duality argument. It is at this point that we realize that the approach we have taken so far is too simplistic. To better understand the problem, let us recall a fundamental result that relates the Aubin–Nitsche Lemma to the compactness of the embedding $V \hookrightarrow L$ where L is the pivot space.

Theorem 3.5 (*Aubin–Nitsche = Compactness*)**.** *Let $V \hookrightarrow L$ be two Hilbert spaces with continuous embedding. Let $a : V \times V \to \mathbb{C}$ be a continuous and coercive sesquilinear from. Let $(V_h)_{h>0}$ be a sequence of conforming finite-dimensional approximation spaces. Let $G_h : V \to V_h \subset L$ be the discrete solution map defined by $a(G_h(v) - v, v_h) = 0$ for all $v_h \in V_h$. Then, $\lim_{h\to 0}\left(\sup_{v \in V\setminus V_h} \frac{\|G_h(v)-v\|_L}{\|G_h(v)-v\|_V}\right) = 0$ iff the embedding $V \hookrightarrow L$ is compact.*

Proof. See Sayas [24, Thm. 1.1]. □

Let us illustrate this result in the present setting. We have $\boldsymbol{V}_0 = \boldsymbol{H}_0(\text{curl}; D)$ and $\boldsymbol{L} = \boldsymbol{L}^2(D)$, and Theorem 3.5 tells us that the Aubin–Nitsche argument provides an extra rate of convergence in the $\boldsymbol{L}^2$-norm if and only if the embedding $\boldsymbol{H}_0(\text{curl}; D) \hookrightarrow \boldsymbol{L}^2(D)$ is compact, which is false. The conclusion of this argumentation is that we should try to find a space smaller than $\boldsymbol{H}_0(\text{curl}; D)$ where $\boldsymbol{A}$ lives and that embeds compactly into $\boldsymbol{L}^2(D)$. We are going to see in Lemma 3.6 that a good candidate is essentially $\boldsymbol{H}_0(\text{curl}; D) \cap \boldsymbol{H}(\text{div}; D)$, as pointed out in Weber [25, Thm. 2.1–2.3].

3.6. Regularity pickup

We now recall some key regularity results related to the curl operator and we use them to infer some regularity pickup in the scale of Sobolev spaces for $\boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ where $\boldsymbol{A}$ is the unique solution to (3.9).

Recall that $\boldsymbol{V}_0 = \boldsymbol{H}_0(\text{curl}; D)$ and consider the subspace $\boldsymbol{X}_{0\widetilde{\mu}} := \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid \nabla \cdot (\widetilde{\mu}\boldsymbol{b}) = 0\}$. Setting

$$M_0 := H^1_0(D), \tag{3.21}$$

a distribution argument shows that we can equivalently define $\boldsymbol{X}_{0\widetilde{\mu}}$ by

$$\boldsymbol{X}_{0\widetilde{\mu}} = \{\boldsymbol{b} \in \boldsymbol{V}_0 \mid (\widetilde{\mu}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0, \ \forall m \in M_0\}, \tag{3.22}$$

where $(\cdot,\cdot)_{\boldsymbol{L}^2(D)}$ denotes the inner product in $\boldsymbol{L}^2(D)$. Let us also set

$$M_* := \{q \in H^1(D) \mid (q, 1)_{L^2(D)} = 0\}, \tag{3.23}$$

and define the following subspace:

$$\boldsymbol{X}_{*\kappa^{-1}} = \{\boldsymbol{b} \in \boldsymbol{H}(\text{curl}; D) \mid (\kappa^{-1}\boldsymbol{b}, \nabla m)_{\boldsymbol{L}^2(D)} = 0, \ \forall m \in M_*\}. \tag{3.24}$$

Lemma 3.6 (*Regularity Pickup*). *Let D be an open, bounded, and connected Lipschitz subset in $\mathbb{R}^3$. (i) Assume that the boundary ∂D is connected and that $\widetilde{\mu}$ is piecewise smooth as specified in* Section 3.4. *Then there exist $s > 0$ and $\check{C}_D > 0$ (depending on D and the heterogeneity ratio $\mu_{\sharp/\flat}$ but not on $\mu_\flat$ alone) such that*

$$\check{C}_D \ell_D^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^s(D)} \le \|\nabla \times \boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\widetilde{\mu}}. \tag{3.25}$$

(ii) Assume that D is simply connected and that κ is piecewise smooth as specified in Section 3.4. *Then there exist $s' > 0$ and $\check{C}'_D > 0$ (depending on D and the heterogeneity ratio $\kappa_{\sharp/\flat}$ but not on $\kappa_\flat$ alone) such that*

$$\check{C}'_D \ell_D^{-1} \|\boldsymbol{b}\|_{\boldsymbol{H}^{s'}(D)} \le \|\nabla \times \boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{*\kappa^{-1}}. \tag{3.26}$$

The above inequalities, proved in Jochmann [26] and Bonito et al. [27], generalize the classical results due to Birman and Solomyak [28, Thm. 3.1] and Costabel [29, Thm. 2], where the material properties are assumed to be either constant or smooth and in this case the smoothness index is $s = \frac{1}{2}$. One even has $s \in (\frac{1}{2}, 1]$ if D is a Lipschitz polyhedron (see Amrouche et al. [18, Prop. 3.7]) and $s = 1$ if D is convex (see Amrouche et al. [18, Thm. 2.17]). We also refer the reader to Ciarlet [23, Thm. 16] for particular situations in heterogeneous materials for which local smoothness with index $s > \frac{1}{2}$ can be established.

Let us now examine the consequences of Lemma 3.6 on the Sobolev regularity of $\boldsymbol{A}$ and $\nabla \times \boldsymbol{A}$ where $\boldsymbol{A}$ is the unique solution to (3.9). Recalling (3.10), we infer that $\boldsymbol{A}$ is actually a member of $\boldsymbol{X}_{0\widetilde{\mu}}$. Owing to (3.25), we infer that there is $s > 0$ so that

$$\boldsymbol{A} \in \boldsymbol{H}^s(D). \tag{3.27}$$

Furthermore, the field $\boldsymbol{R} := \kappa \nabla \times \boldsymbol{A}$ is in $\boldsymbol{X}_{*\kappa^{-1}}$, so that we deduce from (3.26) that there is $s' > 0$ so that $\boldsymbol{R} \in \boldsymbol{H}^{s'}(D)$. In addition, the material property κ being piecewise smooth, we infer that the following multiplier property holds (see Jochmann [26, Lem. 2] and Bonito et al. [27, Prop. 2.1]): There exist $\tau > 0$ and $C_{\kappa^{-1}}$ such that

$$|\kappa^{-1}\boldsymbol{\xi}|_{\boldsymbol{H}^{\tau'}(D)} \le C_{\kappa^{-1}} |\boldsymbol{\xi}|_{\boldsymbol{H}^{\tau'}(D)}, \qquad \forall \boldsymbol{\xi} \in \boldsymbol{H}^{\tau}(D), \quad \forall \tau' \in [0, \tau]. \tag{3.28}$$

Letting $s'' := \min(s', \tau) > 0$, we conclude that

$$\nabla \times \boldsymbol{A} \in \boldsymbol{H}^{s''}(D). \tag{3.29}$$

4. Coercivity revisited

To cope with the loss of coercivity of the sesquilinear form $a_{\widetilde{\mu},\kappa} : \boldsymbol{V}_{h0} \times \boldsymbol{V}_{h0} \to \mathbb{C}$ when the lower bound on $\widetilde{\mu}$ becomes very small, we derive in this section a sharper coercivity property in a proper subspace of $\boldsymbol{V}_{h0}$. The loss of coercivity occurs in the following situations: (i) In the low frequency limit ($\omega \to 0$) when $\widetilde{\mu} = i\omega\mu$, as in the eddy current problem; (ii) If $\kappa \in \mathbb{R}$ and $\sigma \ll \omega\epsilon$ when $\widetilde{\mu} = -\omega^2\epsilon + i\omega\sigma$, as in the Helmholtz problem.

4.1. Continuous Poincaré–Steklov inequality

Recall the subspace $\boldsymbol{X}_{0\widetilde{\mu}}$ of $\boldsymbol{V}_0 = \boldsymbol{H}_0(\mathrm{curl}; D)$ defined in (3.22).

Lemma 4.1 (*Helmholtz Decomposition*)**.** *The following direct sum holds true:*

$$\boldsymbol{V}_0 = \boldsymbol{X}_{0\widetilde{\mu}} \oplus \nabla M_0. \tag{4.1}$$

Proof. Let $\boldsymbol{b} \in \boldsymbol{V}_0$ and let $p \in M_0$ be the unique solution to the following problem: $\int_D \widetilde{\mu}\nabla p \cdot \nabla \overline{q}\,\mathrm{d}x = \int_D \widetilde{\mu}\boldsymbol{b} \cdot \nabla \overline{q}\,\mathrm{d}x$ for all $q \in M_0$. The assumptions on $\widetilde{\mu}$ indeed imply that there is a unique solution to this problem. Then we set $\boldsymbol{v} = \boldsymbol{b} - \nabla p$ and observe that $\boldsymbol{v} \in \boldsymbol{X}_{0\widetilde{\mu}}$. The sum is direct because if $\boldsymbol{v} + \nabla p = \boldsymbol{0}$, then $\int_D \widetilde{\mu}\nabla p \cdot \nabla \overline{p}\,\mathrm{d}x = 0$ owing to (3.22), which in turn implies that $p = 0$ and $\boldsymbol{v} = \boldsymbol{0}$. □

Lemma 4.2 (*Poincaré–Steklov*)**.** *Assume that the boundary ∂D is connected and that $\widetilde{\mu}$ is piecewise smooth as specified in Section 3.4. Then the following Poincaré–Steklov inequality holds:*

$$\check{C}_{\mathrm{P},D}\ell_D^{-1}\|\boldsymbol{b}\|_{\boldsymbol{L}^2(D)} \le \|\nabla \times \boldsymbol{b}\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{b} \in \boldsymbol{X}_{0\widetilde{\mu}}, \tag{4.2}$$

where $\check{C}_{\mathrm{P},D}$ can depend on D and the heterogeneity ratio $\mu_{\sharp/\flat}$ but not on $\mu_\flat$ alone.

Proof. This is a direct consequence of (3.25). □

The bound (4.2) is the coercivity property that we need. Indeed, (4.2) implies the following series of inequalities for all $\boldsymbol{b} \in \boldsymbol{X}_{0\widetilde{\mu}}$:

$$\begin{aligned}\Re(e^{i\theta}a_{\widetilde{\mu},\kappa}(\boldsymbol{b},\boldsymbol{b})) &\ge \mu_\flat\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} \ge \kappa_\flat\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)}\\ &\ge \frac{1}{2}\kappa_\flat(\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \check{C}^2_{\mathrm{P},D}\ell_D^{-2}\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})\\ &\ge \frac{1}{2}\kappa_\flat\ell_D^{-2}\min(1,\check{C}^2_{\mathrm{P},D})\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}.\end{aligned} \tag{4.3}$$

This shows that the sesquilinear form $a_{\widetilde{\mu},\kappa}$ is coercive on $\boldsymbol{X}_{0\widetilde{\mu}}$ with a parameter depending on the heterogeneity ratio $\mu_{\sharp/\flat}$ but not on $\mu_\flat$ alone (whereas the coercivity parameter on $\boldsymbol{V}_0$ is $\min(\mu_\flat, \ell_D^{-2}\kappa_\flat)$, see (3.15)).

4.2. Discrete Poincaré–Steklov inequality

We show in this section that the ideas of Section 4.1 can be reproduced at the discrete level when working with $\boldsymbol{H}(\mathrm{curl})$-conforming finite elements. We consider again the discrete problem (3.17).

Our first step is to realize a discrete counterpart of (4.1) in order to weakly control the divergence of the discrete vector fields in $\boldsymbol{V}_{h0}$. Let us introduce the H_0^1-conforming space

$$M_{h0} := P_0^{\mathrm{g}}(\mathcal{T}_h) = \{q_h \in H_0^1(D) \mid \psi_K^{\mathrm{g}}(q_{|K}) \in \widehat{P}^{\mathrm{g}},\ \forall K \in \mathcal{T}_h\}. \tag{4.4}$$

Note that the commutative diagram (2.12) implies that $\nabla M_{h0} \subset \boldsymbol{V}_{h0}$, i.e., the polynomial degrees of the approximations in M_{h0} and $\boldsymbol{V}_{h0}$ are compatible. In practice, the polynomial degree of the reference finite element $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ is $(k+1)$, i.e., $\mathbb{P}_{k+1}(\mathbb{R}^3;\mathbb{R}) \subset \widehat{P}$.

Since it is not reasonable to consider the space $\{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid \nabla\cdot(\widetilde{\mu}\boldsymbol{b}_h) = 0\}$ because the normal component of $\widetilde{\mu}\boldsymbol{b}_h$ may jump across the mesh interfaces, we are going to consider instead the space

$$\boldsymbol{X}_{h0\widetilde{\mu}} := \{\boldsymbol{b}_h \in \boldsymbol{V}_{h0} \mid (\widetilde{\mu}\boldsymbol{b}_h, \nabla m_h)_{\boldsymbol{L}^2(D)} = 0,\ \forall m_h \in M_{h0}\}. \tag{4.5}$$

The subtlety here is that $\boldsymbol{X}_{h0\widetilde{\mu}}$ is not a subspace of $\boldsymbol{X}_{0\widetilde{\mu}}$, i.e., the approximation setting is nonconforming.

Lemma 4.3 (*Discrete Helmholtz Decomposition*)**.** *The following direct sum holds true:*

$$\boldsymbol{V}_{h0} = \boldsymbol{X}_{h0\widetilde{\mu}} \oplus \nabla M_{h0}. \tag{4.6}$$

Proof. The proof is similar to that of Lemma 4.1 since $\nabla M_{h0} \subset \boldsymbol{V}_{h0}$. □

Lemma 4.4 (*Discrete Solution*)**.** *Let $\boldsymbol{A}_h \in \boldsymbol{V}_{h0}$ be the unique solution to (3.17). Then, $\boldsymbol{A}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$.*

Proof. We must show that $\int_D \widetilde{\mu}\boldsymbol{A}_h \cdot \nabla m_h\,\mathrm{d}x = 0$ for all $m_h \in M_{h0}$. Since $\nabla m_h \in \nabla M_{h0} \subset \boldsymbol{V}_{h0}$, ∇m_h is an admissible test function in (3.17). Recalling that $\nabla\cdot\boldsymbol{f} = 0$, we infer that $0 = \ell(\nabla m_h) = a_{\widetilde{\mu},\kappa}(\boldsymbol{A}_h, \nabla m_h) = \int_D \widetilde{\mu}\boldsymbol{A}_h \cdot \nabla m_h\,\mathrm{d}x$ since $\nabla \times (\nabla m_h) = \boldsymbol{0}$. This completes the proof. □

We now establish a discrete counterpart to the Poincaré–Steklov inequality (4.2). This result is not straightforward since $\boldsymbol{X}_{h0\widetilde{\mu}}$ is not a subspace of $\boldsymbol{X}_{0\widetilde{\mu}}$. The key tools that we are going to invoke are the commuting quasi-interpolation projectors from Theorem 2.2.

Theorem 4.5 (*Discrete Poincaré–Steklov*). *Under the assumptions of Lemma 4.2, there is a uniform constant $\check{C}_{P,\mathcal{T}_h} > 0$ (depending on $\check{C}_{P,D}$, the polynomial degree k, the shape-regularity of $\mathcal{T}_h$, and the heterogeneity ratio $\mu_{\sharp/\flat}$, but not on $\mu_\flat$ alone) such that*

$$\check{C}_{P,\mathcal{T}_h}\ell_D^{-1}\|\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \le \|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}. \tag{4.7}$$

Proof. Let $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$ be a nonzero discrete field. Let $\phi(\boldsymbol{x}_h) \in M_0 = H^1_0(D)$ be the solution to the following well-posed Poisson problem:

$$(\widetilde{\mu}\nabla\phi(\boldsymbol{x}_h), \nabla m)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}\boldsymbol{x}_h, \nabla m)_{\boldsymbol{L}^2(D)}, \qquad \forall m \in M_0.$$

Let us define $\boldsymbol{\xi}(\boldsymbol{x}_h) := \boldsymbol{x}_h - \nabla\phi(\boldsymbol{x}_h)$. This definition implies that $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\widetilde{\mu}}$. Upon invoking the commuting quasi-interpolation operators $\mathcal{J}^{c}_{h0}$ and $\mathcal{J}^{d}_{h0}$ introduced in Theorem 2.2, we now observe that

$$\boldsymbol{x}_h - \mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)) = \mathcal{J}^{c}_{h0}(\boldsymbol{x}_h - \boldsymbol{\xi}(\boldsymbol{x}_h)) = \mathcal{J}^{c}_{h0}(\nabla(\phi(\boldsymbol{x}_h))) = \nabla(\mathcal{J}^{g}_{h0}(\phi(\boldsymbol{x}_h))), \tag{4.8}$$

where we have used that $\mathcal{J}^{c}_{h0}(\boldsymbol{x}_h) = \boldsymbol{x}_h$ and the commuting properties of the operators $\mathcal{J}^{g}_{h0}$ and $\mathcal{J}^{c}_{h0}$. Since $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$, we infer that $(\widetilde{\mu}\boldsymbol{x}_h, \nabla(\mathcal{J}^{g}_{h0}(\phi(\boldsymbol{x}_h))))_{\boldsymbol{L}^2(D)} = 0$, so that

$$\begin{aligned}(\widetilde{\mu}\boldsymbol{x}_h, \boldsymbol{x}_h)_{\boldsymbol{L}^2(D)} &= (\widetilde{\mu}\boldsymbol{x}_h, \boldsymbol{x}_h - \mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)} + (\widetilde{\mu}\boldsymbol{x}_h, \mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)}\\ &= (\widetilde{\mu}\boldsymbol{x}_h, \mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)}.\end{aligned}$$

Multiplying by $e^{i\theta}$, taking the real part, and using the Cauchy–Schwarz inequality, we infer that

$$\mu_\flat\|\boldsymbol{x}_h\|^2_{\boldsymbol{L}^2(D)} \le \mu_\sharp\|\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}\|\mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)}.$$

The uniform boundedness of $\mathcal{J}^{c}_{h0}$ on $\boldsymbol{L}^2(D)$ together with the Poincaré–Steklov inequality (4.2) implies that

$$\|\mathcal{J}^{c}_{h0}(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)} \le \|\mathcal{J}^{c}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)} \le \|\mathcal{J}^{c}_{h0}\|_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\check{C}^{-1}_{P,D}\ell_D\|\nabla\times\boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)},$$

so that (4.7) holds with $\check{C}_{P,\mathcal{T}_h} = \mu^{-1}_{\sharp/\flat}\|\mathcal{J}^{c}_{h0}\|^{-1}_{\mathcal{L}(\boldsymbol{L}^2;\boldsymbol{L}^2)}\check{C}_{P,D}$. □

Remark 4.6 (*Alternative Proofs*). There are many ways to prove the discrete Poincaré–Steklov inequality (4.7). One route described in Hiptmair [30, §4.2] hinges on (subtle) regularity estimates from Amrouche et al. [18, Lemma 4.7]. Another one, which avoids invoking regularity estimates, is based on the so-called discrete compactness argument of Kikuchi [3] and further developed by Monk and Demkowicz [4] and Caorsi et al. [5]. The proof based on the discrete compactness argument is not constructive but relies instead on an argument by contradiction. The technique used in the proof of Theorem 4.5, inspired from Arnold et al. [2, Thm. 5.11] and Arnold et al. [31, Thm 3.6], relies on the existence of the stable commuting quasi-interpolation projectors $\mathcal{J}^{c}_{h}$ and $\mathcal{J}^{c}_{h0}$. It was observed in Boffi [32] that the existence of commuting quasi-interpolation operators within the discrete de Rham complex would imply the discrete compactness property. □

4.3. Error analysis in the $\boldsymbol{H}$(curl)-norm

We are now in a position to revisit the error analysis of Section 3.5. Let us first show that $\boldsymbol{X}_{h0\widetilde{\mu}}$ has the same approximation properties as $\boldsymbol{V}_{h0}$ in $\boldsymbol{X}_{0\widetilde{\mu}}$.

Lemma 4.7 (*Approximation in $\boldsymbol{X}_{h0\widetilde{\mu}}$*). *The following holds true:*

$$\inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}}\|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(curl;D)} \le c\,\mu_{\sharp/\flat}\inf_{\boldsymbol{b}_h\in\boldsymbol{V}_{h0}}\|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{H}(curl;D)}, \quad \forall \boldsymbol{A}\in\boldsymbol{X}_{0\widetilde{\mu}}, \tag{4.9}$$

where the constant c is uniform with respect to h and the model parameters.

Proof. Let $\boldsymbol{A} \in \boldsymbol{X}_{0\widetilde{\mu}}$, and let $p_h \in M_{h0}$ be the unique solution to the following discrete Poisson problem: $(\widetilde{\mu}\nabla p_h, \nabla q_h)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}\mathcal{J}^{c}_{h0}(\boldsymbol{A}), \nabla q_h)_{\boldsymbol{L}^2(D)}$ for all $q_h \in M_{h0}$. Let us define $\boldsymbol{y}_h = \mathcal{J}^{c}_{h0}(\boldsymbol{A}) - \nabla p_h$. By construction, $\boldsymbol{y}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$ and $\nabla\times\boldsymbol{y}_h = \nabla\times\mathcal{J}^{c}_{h0}(\boldsymbol{A})$. Hence, $\|\nabla\times(\boldsymbol{A}-\boldsymbol{y}_h)\|_{\boldsymbol{L}^2(D)} = \|\nabla\times(\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A}))\|_{\boldsymbol{L}^2(D)}$. Moreover, since $\nabla\cdot(\widetilde{\mu}\boldsymbol{A}) = 0$, we infer that

$$(\widetilde{\mu}\nabla p_h, \nabla p_h)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}\mathcal{J}^{c}_{h0}(\boldsymbol{A}), \nabla p_h)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}(\mathcal{J}^{c}_{h0}(\boldsymbol{A})-\boldsymbol{A}), \nabla p_h)_{\boldsymbol{L}^2(D)},$$

which in turn implies that $\|\nabla p_h\|_{\boldsymbol{L}^2(D)} \le \mu_{\sharp/\flat}\|\mathcal{J}^{c}_{h0}(\boldsymbol{A})-\boldsymbol{A}\|_{\boldsymbol{L}^2(D)}$. The above argument shows that

$$\|\boldsymbol{A}-\boldsymbol{y}_h\|_{\boldsymbol{L}^2(D)} \le \|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)} + \|\mathcal{J}^{c}_{h0}(\boldsymbol{A})-\boldsymbol{y}_h\|_{\boldsymbol{L}^2(D)}$$

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$$\begin{aligned}&\le \|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}+\|\nabla p_h\|_{\boldsymbol{L}^2(D)}\\&\le (1+\mu_{\sharp/\flat})\|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}.\end{aligned}$$

In conclusion, we have proved that

$$\begin{aligned}\inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}}\|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}&\le \|\boldsymbol{A}-\boldsymbol{y}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}\\&\le (1+\mu_{\sharp/\flat})\|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{H}(\mathrm{curl};D)}.\end{aligned}$$

Upon invoking (2.16) and the commutative diagrams (2.18), we infer that

$$\begin{aligned}\|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{H}(\mathrm{curl};D)}&\le \|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}+\ell_D\|\nabla\times(\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A}))\|_{\boldsymbol{L}^2(D)}\\&= \|\boldsymbol{A}-\mathcal{J}^{c}_{h0}(\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}+\ell_D\|\nabla\times\boldsymbol{A}-\mathcal{J}^{d}_{h0}(\nabla\times\boldsymbol{A})\|_{\boldsymbol{L}^2(D)}\\&\le c\inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{c}_0(\mathcal{T}_h)}\|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}+c'\ell_D\inf_{\boldsymbol{d}_h\in\boldsymbol{P}^{d}_0(\mathcal{T}_h)}\|\nabla\times\boldsymbol{A}-\boldsymbol{d}_h\|_{\boldsymbol{L}^2(D)}\\&\le c\inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{c}_0(\mathcal{T}_h)}\|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{L}^2(D)}+c'\ell_D\inf_{\boldsymbol{b}_h\in\boldsymbol{P}^{c}_0(\mathcal{T}_h)}\|\nabla\times(\boldsymbol{A}-\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(D)},\end{aligned}$$

where the last bound follows by restricting the minimization set to $\boldsymbol{P}^{c}_0(\mathcal{T}_h)$ since $\nabla\times\boldsymbol{P}^{c}_0(\mathcal{T}_h)\subset\boldsymbol{P}^{d}_0(\mathcal{T}_h)$. The conclusion follows readily. □

Theorem 4.8 (*Error Estimate*). *Assume that the boundary ∂D is connected and that $\widetilde{\mu}$ is piecewise smooth as specified in* Section 3.4. *Then the following error estimate holds true:*

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(curl;D)}\le c\ \max(1,\gamma_{\widetilde{\mu},\kappa})\inf_{\boldsymbol{b}_h\in\boldsymbol{V}_{h0}}\|\boldsymbol{A}-\boldsymbol{b}_h\|_{\boldsymbol{H}(curl;D)},\tag{4.10}$$

where the constant c is uniform with respect to h and can depend on the discrete Poincaré–Steklov constant $\check{C}_{\mathrm{P},\mathcal{T}_h}$ and the heterogeneity ratios $\mu_{\sharp/\flat}$ and $\kappa_{\sharp/\flat}$, and where $\gamma_{\widetilde{\mu},\kappa}=\mu_\sharp\ell_D^2\kappa_\sharp^{-1}$ is the magnetic Reynolds number.

Proof. Owing to Lemma 4.4, $\boldsymbol{A}_h$ solves the following problem: Find $\boldsymbol{A}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}$ such that $a_{\widetilde{\mu},\kappa}(\boldsymbol{A}_h,\boldsymbol{x}_h)=\ell(\boldsymbol{x}_h)$, for all $\boldsymbol{x}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}$. Using the discrete Poincaré–Steklov inequality (4.7) and proceeding as in (4.3), we infer that

$$\Re(e^{i\theta}a_{\widetilde{\mu},\kappa}(\boldsymbol{x}_h,\boldsymbol{x}_h))\ge\frac{1}{2}\kappa_\flat\ell_D^{-2}\min(1,\check{C}^2_{\mathrm{P},\mathcal{T}_h})\|\boldsymbol{x}_h\|^2_{\boldsymbol{H}(\mathrm{curl};D)},$$

for all $\boldsymbol{x}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}$. Hence, the above problem is well-posed. Recalling the boundedness property (3.20) of the sesquilinear form $a_{\widetilde{\mu},\kappa}$ and invoking again the abstract error estimate from Xu and Zikatanov [22, Thm. 2] leads to

$$\|\boldsymbol{A}-\boldsymbol{A}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}\le\frac{2\max(\mu_\sharp,\ell_D^{-2}\kappa_\sharp)}{\kappa_\flat\ell_D^{-2}\min(1,\check{C}^2_{\mathrm{P},\mathcal{T}_h})}\inf_{\boldsymbol{x}_h\in\boldsymbol{X}_{h0\widetilde{\mu}}}\|\boldsymbol{A}-\boldsymbol{x}_h\|_{\boldsymbol{H}(\mathrm{curl};D)}.$$

We conclude the proof by invoking Lemma 4.7. □

Remark 4.9 (*Neumann Boundary Condition*). The above analysis can be adapted to account for the Neumann boundary condition $(\kappa\nabla\times\boldsymbol{A})_{|\partial D}\times\boldsymbol{n}=\boldsymbol{0}$. This condition implies that $(\nabla\times(\kappa\nabla\times\boldsymbol{A}))_{|\partial D}\cdot\boldsymbol{n}=0$. Moreover, assuming that $\boldsymbol{f}\cdot\boldsymbol{n}_{|\partial D}=0$, and taking the normal component of the equation $\widetilde{\mu}\boldsymbol{A}+\nabla\times(\kappa\nabla\times\boldsymbol{A})=\boldsymbol{f}$ at the boundary gives $\boldsymbol{A}\cdot\boldsymbol{n}_{|\partial D}=0$. Since $\nabla\cdot\boldsymbol{f}=0$, we also have that $\nabla\cdot(\widetilde{\mu}\boldsymbol{A})=0$. Using a distribution argument shows that $\boldsymbol{A}\in\boldsymbol{X}_{*\widetilde{\mu}}$ where $\boldsymbol{X}_{*\widetilde{\mu}}:=\{\boldsymbol{b}\in\boldsymbol{H}(\mathrm{curl};D)\mid(\widetilde{\mu}\boldsymbol{b},\nabla m)_{\boldsymbol{L}^2(D)}=0,\ \forall m\in M_*\}$ and M_* is defined in (3.23). The discrete spaces that must be used are now $\boldsymbol{V}_h=\boldsymbol{P}^{c}(\mathcal{T}_h)$ and $M_{h*}=P^{g}(\mathcal{T}_h)\cap M_*$. Using $\boldsymbol{V}_h$ for the discrete trial and test spaces in the weak formulation, one then deduces that

$$\boldsymbol{A}_h\in\boldsymbol{X}_{h*\widetilde{\mu}}:=\{\boldsymbol{b}_h\in\boldsymbol{V}_h\mid(\widetilde{\mu}\boldsymbol{b}_h,\nabla m_h)_{\boldsymbol{L}^2(D)}=0,\ \forall m_h\in M_{h*}\}.\tag{4.11}$$

The Poincaré–Steklov inequality (4.7) still holds if the assumption that ∂D is connected is replaced by the assumption that D is simply connected. The error analysis from Theorem 4.8 can be readily adapted. □

Remark 4.10 (*Helmholtz Problem*). The assumption (3.11) can be replaced by assuming that $\operatorname{ess\,inf}_{\boldsymbol{x}\in D}|\kappa|\ge\kappa_\flat>0$ and $\widetilde{\mu}$ is not an eigenvalue of the operator $\nabla\times(\kappa\nabla\times)$ equipped with the appropriate boundary condition. In this case, (3.13) is a Helmholtz-type boundary-value problem. This problem can be analyzed by using commuting quasi-interpolation operators as done in Arnold et al. [2, §9.1] for Neumann boundary conditions, or by using a constructive proof relying on Hilbertian bases as done in Ciarlet [33]. Note that the convergence rates derived in [33] require a smoothness index $s>\frac{1}{2}$ owing to the use of the Nédélec interpolation operator; therefore, the present quasi-interpolation operator can be combined with these results to treat more general situations regarding the heterogeneity of the material. □

5. The duality argument for edge elements

Our goal in this section is to estimate $(\boldsymbol{A} - \boldsymbol{A}_h)$ in the $\boldsymbol{L}^2$-norm using a duality argument that invokes a weak control on the divergence. The subtlety is that, as already mentioned, the setting is nonconforming since $\boldsymbol{X}_{h0\widetilde{\mu}}$ is not a subspace of $\boldsymbol{X}_{0\widetilde{\mu}}$. Recalling Theorem 3.5, the compactness that is required from the functional setting to obtain a better convergence rate in $\boldsymbol{L}^2(D)$ will result from Lemma 3.6 and the compact embedding $\boldsymbol{H}^s(D) \hookrightarrow \boldsymbol{L}^2(D)$, $s > 0$. In this section, we are going to use both inequalities (3.25) and (3.26) from Lemma 3.6; therefore, we assume that the boundary ∂D is connected and that D is simply connected, and that both $\widetilde{\mu}$ and κ are piecewise smooth. Recalling the results of Section 3.6 with smoothness indices $s, s' > 0$ and the index $\tau > 0$ from the multiplier property (3.28) and letting $s'' = \min(s', \tau)$, we have $\boldsymbol{A} \in \boldsymbol{H}^s(D)$ and $\nabla \times \boldsymbol{A} \in \boldsymbol{H}^{s''}(D)$ with $s, s'' > 0$. In what follows, we set

$$\sigma := \min(s, s''). \tag{5.1}$$

Recall the magnetic Reynolds number $\gamma_{\widetilde{\mu},\kappa} = \mu_\sharp \ell_D^2 \kappa_\sharp^{-1}$ and let us set $\hat{\gamma}_{\widetilde{\mu},\kappa} = \max(1, \gamma_{\widetilde{\mu},\kappa})$.

Let us first start with an approximation result on the curl-preserving lifting operator $\boldsymbol{\xi} : \boldsymbol{X}_{h0\widetilde{\mu}} \to \boldsymbol{X}_{0\widetilde{\mu}}$ defined in the proof of Theorem 4.5. Recall that, for all $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$, the field $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\widetilde{\mu}}$ is such $\boldsymbol{\xi}(\boldsymbol{x}_h) = \boldsymbol{x}_h - \nabla\phi(\boldsymbol{x}_h)$ with $\phi(\boldsymbol{x}_h) \in H_0^1(D)$; hence $\nabla \times \boldsymbol{\xi}(\boldsymbol{x}_h) = \nabla \times \boldsymbol{x}_h$.

Lemma 5.1 (*Curl-preserving Lifting*). *Let $s > 0$ be the smoothness index introduced in (3.25). Then, the following holds true:*

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \le c\, h^s \ell_D^{1-s} \|\nabla \times \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}, \qquad \forall \boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}, \tag{5.2}$$

where the constant c is uniform with respect to h and can depend on the constant $\check{C}_D$ from (3.25) and the heterogeneity ratio $\mu_{\sharp/\flat}$.

Proof. Let $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$, and let us set $\boldsymbol{e}_h := \boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h$. We have seen in the proof of Theorem 4.5 that $\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h \in \nabla M_{h0}$, see (4.8). This, in turn, implies that $(\widetilde{\mu}\boldsymbol{e}_h, \mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h)_{\boldsymbol{L}^2(D)} = 0$ since $\boldsymbol{\xi}(\boldsymbol{x}_h) \in \boldsymbol{X}_{0\widetilde{\mu}}$, $M_{h0} \subset M_0$, and $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$. Since $\boldsymbol{e}_h = (I - \mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\xi}(\boldsymbol{x}_h)) + (\mathcal{J}_{h0}^{\mathrm{c}}(\boldsymbol{\xi}(\boldsymbol{x}_h)) - \boldsymbol{x}_h)$, we infer that

$$(\widetilde{\mu}\boldsymbol{e}_h, \boldsymbol{e}_h)_{\boldsymbol{L}^2(D)} = (\widetilde{\mu}\boldsymbol{e}_h, (I - \mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\xi}(\boldsymbol{x}_h)))_{\boldsymbol{L}^2(D)},$$

thereby implying that $\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \le \mu_{\sharp/\flat}\|(I - \mathcal{J}_{h0}^{\mathrm{c}})(\boldsymbol{\xi}(\boldsymbol{x}_h))\|_{\boldsymbol{L}^2(D)}$. Using the approximation properties of $\mathcal{J}_{h0}^{\mathrm{c}}$ yields

$$\|\boldsymbol{e}_h\|_{\boldsymbol{L}^2(D)} \le c\, \mu_{\sharp/\flat} h^s |\boldsymbol{\xi}(\boldsymbol{x}_h)|_{\boldsymbol{H}^s(D)},$$

and we conclude using the bound $|\boldsymbol{\xi}(\boldsymbol{x}_h)|_{\boldsymbol{H}^s(D)} \le \check{C}_D \ell_D^{1-s}\|\nabla \times \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)}$ which follows from (3.25). □

Lemma 5.2 (*Adjoint Solution*). *Let $\boldsymbol{y} \in \boldsymbol{X}_{0\widetilde{\mu}}$ and let $\boldsymbol{\zeta} \in \boldsymbol{X}_{0\widetilde{\mu}}$ be the (unique) solution to the (adjoint) problem $\widetilde{\mu}\boldsymbol{\zeta} + \nabla \times (\kappa \nabla \times \boldsymbol{\zeta}) = \mu_\flat^{-1}\widetilde{\mu}\boldsymbol{y}$. Then,*

$$|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le c\, \mu_\sharp^{-1} \gamma_{\widetilde{\mu},\kappa} \ell_D^{-\sigma} \|\boldsymbol{y}\|_{L^2(D)}, \tag{5.3a}$$

$$|\nabla \times \boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le c\, \mu_\sharp^{-1} \gamma_{\widetilde{\mu},\kappa} \hat{\gamma}_{\widetilde{\mu},\kappa} \ell_D^{-1-\sigma} \|\boldsymbol{y}\|_{L^2(D)}, \tag{5.3b}$$

where the constant c is uniform with respect to h and can depend on the constants $\check{C}_{\mathrm{P},D}$ from (4.2), $\check{C}_D$, $\check{C}'_D$ from (3.25)–(3.26), and the heterogeneity ratios $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, $\kappa_\sharp C_{\kappa^{-1}}$.

Proof. Testing the adjoint problem against $\boldsymbol{\zeta}$, we observe that $\kappa_\flat\|\nabla \times \boldsymbol{\zeta}\|^2_{\boldsymbol{L}^2(D)} \le \mu_{\sharp/\flat}\|\boldsymbol{y}\|_{L^2(D)}\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$, so that using the Poincaré–Steklov inequality (4.2) to bound $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ by $\|\nabla \times \boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$, we infer that

$$\|\nabla \times \boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\mu_{\sharp/\flat}\check{C}_{\mathrm{P},D}^{-1}\ell_D\|\boldsymbol{y}\|_{L^2(D)}. \tag{5.4}$$

Invoking (3.25) with $\sigma \le s$ yields

$$|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \le \check{C}_D^{-1}\ell_D^{1-\sigma}\|\nabla \times \boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\mu_{\sharp/\flat}\check{C}_D^{-1}\check{C}_{\mathrm{P},D}^{-1}\ell_D^{2-\sigma}\|\boldsymbol{y}\|_{L^2(D)},$$

which proves (5.3a) since $\kappa_\flat^{-1}\ell_D^2 = \kappa_{\sharp/\flat}\mu_\sharp^{-1}\gamma_{\widetilde{\mu},\kappa}$. Let us now prove (5.3b). Invoking (3.26) with $\sigma \le s'$ for $\boldsymbol{b} = \kappa\nabla \times \boldsymbol{\zeta}$, which is a member of $\boldsymbol{X}_{*\kappa^{-1}}$, we infer that

$$\check{C}'_D\ell_D^{-1+\sigma}|\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)} \le \|\nabla \times \boldsymbol{b}\|_{\boldsymbol{L}^2(D)} = \|\nabla \times (\kappa\nabla \times \boldsymbol{\zeta})\|_{\boldsymbol{L}^2(D)} \le \mu_{\sharp/\flat}\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)} + \mu_\sharp\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)},$$

by definition of the adjoint solution $\boldsymbol{\zeta}$ and the triangle inequality. Invoking again the Poincaré–Steklov inequality (4.2) to bound $\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ by $\|\nabla \times \boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)}$ and using (5.4) yields

$$\|\boldsymbol{\zeta}\|_{\boldsymbol{L}^2(D)} \le \kappa_\flat^{-1}\mu_{\sharp/\flat}\check{C}_{\mathrm{P},D}^{-2}\ell_D^2\|\boldsymbol{y}\|_{L^2(D)}.$$

As a result, we obtain

$$\check{C}'_D\ell_D^{-1+\sigma}|\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)} \le \mu_{\sharp/\flat}(1 + \mu_\sharp\kappa_\flat^{-1}\check{C}_{\mathrm{P},D}^{-2}\ell_D^2)\|\boldsymbol{y}\|_{\boldsymbol{L}^2(D)},$$

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and we can conclude the proof of (5.3b) since $|\nabla \times \boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} \leq C_{\kappa^{-1}} |\boldsymbol{b}|_{\boldsymbol{H}^\sigma(D)}$ owing to the multiplier property (3.28) and $\sigma \leq \tau$. □

We can now state the main result of this section.

Theorem 5.3 (*Improved L^2-error Estimate*). *The following holds true:*

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2} \leq c \inf_{\boldsymbol{v}_h \in \boldsymbol{V}_{h0}} (\|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2} + \hat{\gamma}^3_{\widetilde{\mu},\kappa} h^\sigma \ell_D^{-\sigma} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\text{curl})}), \tag{5.5}$$

where the constant c is uniform with respect to h and can depend on the constants $\check{C}_{P,D}$ from (4.2), $\check{C}_D$, $\check{C}'_D$ from (3.25)–(3.26), and the heterogeneity ratios $\mu_{\sharp/\flat}$, $\kappa_{\sharp/\flat}$, $\kappa_\sharp C_{\kappa^{-1}}$.

Proof. In this proof, we use the symbol c to denote a generic positive constant that can have the same parametric dependences as in the above statement. Let $\boldsymbol{v}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$ and let us set $\boldsymbol{x}_h := \boldsymbol{A}_h - \boldsymbol{v}_h$; observe that $\boldsymbol{x}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$. Let $\boldsymbol{\zeta} \in \boldsymbol{X}_{0\widetilde{\mu}}$ be the solution to the adjoint problem $\widetilde{\mu}\boldsymbol{\zeta} + \nabla \times (\kappa \nabla \times \boldsymbol{\zeta}) = \mu_\flat^{-1} \widetilde{\mu} \boldsymbol{\xi}(\boldsymbol{x}_h)$, where $\boldsymbol{\xi} : \boldsymbol{X}_{h0\widetilde{\mu}} \to \boldsymbol{X}_{0\widetilde{\mu}}$ is the curl-preserving lifting operator considered above.
(1) Let us first estimate $\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}$ from above. Recalling that $\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h = -\nabla\phi(\boldsymbol{x}_h)$ and that $(\widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h), \boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h)_{L^2(D)} = -(\widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h), \nabla\phi(\boldsymbol{x}_h))_{L^2(D)} = 0$, we infer that

$$\begin{aligned}
(\boldsymbol{\xi}(\boldsymbol{x}_h), \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} &= (\boldsymbol{x}_h, \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} \\
&= (\boldsymbol{A} - \boldsymbol{v}_h, \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + (\boldsymbol{A}_h - \boldsymbol{A}, \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} \\
&= (\boldsymbol{A} - \boldsymbol{v}_h, \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + \mu_\flat a_{\widetilde{\mu},\kappa}(\boldsymbol{A}_h - \boldsymbol{A}, \boldsymbol{\zeta}) \\
&= (\boldsymbol{A} - \boldsymbol{v}_h, \widetilde{\mu}\boldsymbol{\xi}(\boldsymbol{x}_h))_{\boldsymbol{L}^2(D)} + \mu_\flat a_{\widetilde{\mu},\kappa}(\boldsymbol{A}_h - \boldsymbol{A}, \boldsymbol{\zeta} - \mathcal{J}^{c}_{h0}(\boldsymbol{\zeta})),
\end{aligned}$$

where we have used Galerkin's orthogonality to pass from the third to the fourth line. Since we have $|a_{\widetilde{\mu},\kappa}(\boldsymbol{a}, \boldsymbol{b})| \leq \kappa_\sharp \ell_D^{-2} \hat{\gamma}_{\widetilde{\mu},\kappa} \|\boldsymbol{a}\|_{\boldsymbol{H}(\text{curl};D)} \|\boldsymbol{b}\|_{\boldsymbol{H}(\text{curl};D)}$ owing to (3.20), we infer from the commutation and approximation results from Theorems 2.2 and 2.4 that

$$\begin{aligned}
\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|^2_{\boldsymbol{L}^2(D)} \leq{}& \mu_{\sharp/\flat} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} \|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)} \\
&+ c\, \kappa_\sharp \ell_D^{-2} \hat{\gamma}_{\widetilde{\mu},\kappa} h^\sigma \|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} (|\boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)} + \ell_D |\nabla \times \boldsymbol{\zeta}|_{\boldsymbol{H}^\sigma(D)}).
\end{aligned}$$

Invoking the bounds from Lemma 5.2 on the adjoint solution with $\boldsymbol{y} = \boldsymbol{\xi}(\boldsymbol{x}_h)$, we conclude that

$$\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)} \leq \mu_{\sharp/\flat} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + c\, \hat{\gamma}^2_{\widetilde{\mu},\kappa} h^\sigma \ell_D^{-\sigma} \|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)}. \tag{5.6}$$

(2) The triangle inequality, together with the identity $\boldsymbol{A} - \boldsymbol{A}_h = \boldsymbol{A} - \boldsymbol{v}_h - \boldsymbol{x}_h$, implies that

$$\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{L}^2} \leq \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} + \|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}.$$

We use Lemma 5.1 to bound the second term on the right-hand side as follows:

$$\begin{aligned}
\|\boldsymbol{\xi}(\boldsymbol{x}_h) - \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} &\leq c\, h^\sigma \ell_D^{1-\sigma} \|\nabla \times \boldsymbol{x}_h\|_{\boldsymbol{L}^2(D)} \\
&\leq c\, h^\sigma \ell_D^{1-\sigma} (\|\nabla \times (\boldsymbol{A} - \boldsymbol{v}_h)\|_{\boldsymbol{L}^2(D)} + \|\nabla \times (\boldsymbol{A} - \boldsymbol{A}_h)\|_{\boldsymbol{L}^2(D)}),
\end{aligned}$$

and we use (4.10) to infer that $\|\boldsymbol{A} - \boldsymbol{A}_h\|_{\boldsymbol{H}(\text{curl};D)} \leq c\hat{\gamma}_{\widetilde{\mu},\kappa} \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{H}(\text{curl};D)}$. For the third term on the right-hand side, we use the bound on $\|\boldsymbol{\xi}(\boldsymbol{x}_h)\|_{\boldsymbol{L}^2(D)}$ estimated above in Step (1). We conclude by taking the infimum over $\boldsymbol{v}_h \in \boldsymbol{X}_{h0\widetilde{\mu}}$, and we use Lemma 4.7 to extend the infimum over $\boldsymbol{V}_{h0}$. □

Remark 5.4 (*Curl-preserving Lifting*). The idea of the construction of the curl-preserving lifting operator invoked in the proof of Theorems 4.5 and 5.3 is rooted in Monk [1, p. 249–250]. The statement in Lemma 5.1 is similar to that in Monk [21, Lem 7.6], but the proof we give is greatly simplified by using the commuting quasi-interpolation operators from Theorem 2.2. The curl-preserving lifting of $\boldsymbol{A} - \boldsymbol{A}_h$ is invoked in Arnold et al. [2, Eq. (9.9)] and denoted therein by $\boldsymbol{\psi}$. The estimate of $\|\boldsymbol{\psi}\|_{\boldsymbol{L}^2(D)}$ given one line above [2, Eq. (9.11)] is similar to (5.6) and is obtained by invoking the commuting quasi-interpolation operators constructed in [2, §5.4] for natural boundary conditions. Note that contrary to what is done in the above reference, we invoke the curl-preserving lifting of $\boldsymbol{A}_h - \boldsymbol{v}_h$ instead of $\boldsymbol{A} - \boldsymbol{A}_h$ and make use of Lemma 5.1, which simplifies the argument. Furthermore, the statement of Theorem 5.3 is similar to that of Zhong et al. [34, Thm. 4.1], but again the proof that we propose is simplified by using the commuting quasi-interpolation operators; moreover, the setting presented in this paper accounts for heterogeneous coefficients since it holds true for any smoothness index σ smaller than $\frac{1}{2}$. □

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