Research Article

Alexandre Ern and Jean-Luc Guermond*

Abstract Nonconforming Error Estimates and Application to Boundary Penalty Methods for Diffusion Equations and Time-Harmonic Maxwell's Equations

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Abstract: We devise a novel framework for the error analysis of finite element approximations to low-regularity solutions in nonconforming settings where the discrete trial and test spaces are not subspaces of their exact counterparts. The key is to use face-to-cell extension operators so as to give a weak meaning to the normal or tangential trace on each mesh face individually for vector fields with minimal regularity and then to prove the consistency of this new formulation by means of some recently-derived mollification operators that commute with the usual derivative operators. We illustrate the technique on Nitsche's boundary penalty method applied to a scalar diffusion equation and to the time-harmonic Maxwell's equations. In both cases, the error estimates are robust in the case of heterogeneous material properties. We also revisit the error analysis framework proposed by Gudi where a trimming operator is introduced to map discrete test functions into conforming test functions. This technique also gives error estimates for minimal regularity solutions, but the constants depend on the material properties through contrast factors.

Keywords: Finite Elements, Elliptic Equations, Error Estimates, Nonconforming Methods, Nitsche Method, Crouzeix–Raviart Method, Discontinuous Galerkin, Minimum Regularity

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1 Introduction

The error analysis of the finite element approximation of Partial Differential Equations (PDEs) is well understood; see, e.g., the textbooks [7, 8, 13]. The most basic result is Céa's Lemma [11] which is valid when the approximation setting is conforming (the discrete trial and test spaces are subspaces of their exact counterparts) and exactly consistent (the discrete forms are restrictions of the exact ones to the discrete spaces). Departures from this setting are usually handled in the literature by invoking Strang's Lemmas [27]. Strang's First Lemma assumes that the approximation setting is conforming but handles the case where the discrete forms differ from their exact counterpart. Strang's Second Lemma deals with nonconforming approximation settings and is frequently invoked in the literature for the error analysis of nonconforming techniques. For instance, many authors have adopted this approach to analyze discontinuous Galerkin (dG) methods (see, e.g., [12, 14] and the references therein).

Alexandre Ern, Université Paris-Est, CERMICS (ENPC), 77455 Marne-la-Vallée Cedex 2; and INRIA Paris, 2 rue Simone Iff, 75589 Paris, France, e-mail: alexandre.ern@enpc.fr
***Corresponding author: Jean-Luc Guermond,** Department of Mathematics, Texas A&M University 3368 TAMU, College Station, TX 77843, USA, e-mail: guermond@math.tamu.edu

One important shortcoming of Strang's Second Lemma is that one needs to insert the exact solution in the first argument of the discrete sesquilinear (or bilinear) form. Unfortunately, this is only possible if one assumes some additional regularity on the exact solution which often goes beyond the regularity provided by the weak formulation of the model problem at hand. For instance, when approximating a diffusion equation of the form $-\nabla\cdot(\kappa\nabla u) = f$ in some Lipschitz domain D in $\mathbb{R}^d$, one is essentially led to assume that $\kappa\nabla u \in \boldsymbol{H}^r(D)$ with $r > \frac{1}{2}$ so as to make sense of the normal component $\boldsymbol{n}\cdot(\kappa\nabla u)$ at the mesh interfaces. Although this assumption is not really restrictive for the Laplace equation in a polyhedron ($\kappa \equiv 1$), since elliptic regularity guarantees the existence of an index $r > \frac{1}{2}$ so that $u \in H^{1+r}(D)$, it becomes unrealistic in problems with discontinuous coefficients. Similarly, for the time-harmonic Maxwell's equations of the form $\tilde{\mu}\boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}$ in some Lipschitz domain D in $\mathbb{R}^3$, one is led to assume $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$ with $r > \frac{1}{2}$ so as to make sense of the tangential component $\boldsymbol{n}\times(\kappa\nabla\times\boldsymbol{A})$ at the mesh interfaces, but this assumption becomes unrealistic in problems with discontinuous coefficients. Let us mention in passing that we use boldface notation for $\mathbb{R}^d$-valued fields in D.

One possible way forward to overcome the limitations of Strang's Second Lemma has been proposed by Gudi [18]. The main idea is to introduce an operator that transforms the discrete test functions into elements of the exact test space. We call this operator a trimming operator, and we call the resulting error estimate a trimmed error estimate. The reason for our terminology is that one can view the elements in the kernel of the trimming operator as discrete (test) functions that are only needed to "stabilize" the bilinear form a_h, but do not contribute to the interpolatory properties of the approximation setting. We also observe that a trimming operator is one of the fundamental ingredients in the abstract setting recently devised by Veeser and Zanotti [28] to obtain quasi-optimal energy-norm error estimates for nonconforming finite element methods applied to symmetric elliptic PDEs. The trimmed error estimate in [18] (which is sometimes referred to as "medius analysis") has been applied to the Interior Penalty dG (IPDG) approximation of the Laplace equation with a source term $f \in L^2(D)$ (and to a fourth-order problem also in [18], to the Stokes equations in [3], and to the linear elasticity equations in [10]). In the present work, we show how to apply the trimmed error estimate to the diffusion equation with heterogeneous material property κ and source term $f \in L^q(D)$ with $q \in (2_*, 2]$, $2_* = \frac{2d}{2+d}$, and also to the time-harmonic Maxwell's equations with heterogeneous material properties $\tilde{\mu}, \kappa$ and source term $\boldsymbol{f} \in \boldsymbol{L}^2(D)$. For simplicity, we focus for both model problems on the use of H^1-conforming finite elements combined with the boundary penalty method of Nitsche [23] to enforce weakly Dirichlet boundary conditions. The main benefit of the trimmed error analysis is that it allows one to derive error estimates as soon as the exact solution is in $\{v \in H^1(D) \mid \nabla\cdot(\kappa\nabla v) \in L^q(D)\}$, $q \in (2_*, 2]$, for the diffusion equation, and as soon as the exact solution is in $\{\boldsymbol{A} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \nabla\times(\kappa\nabla\times\boldsymbol{A}) \in \boldsymbol{L}^2(D)\}$ for the time-harmonic Maxwell's equations.

One difficulty still remains with the trimmed error estimate in the case of strong contrasts in the material property κ since the error estimates feature a constant that is typically proportional to the square-root of a contrast factor associated with κ (and, in the case of Maxwell's equations, there is also a dependency on the square-root of a local magnetic Reynolds number). These dependencies originate from the usage of the trimming operator to perform some averaging to achieve the desired conformity property, but this averaging, in turn, precludes the derivation of stability and approximation properties for the trimming operator that are local to a mesh cell. To remedy this difficulty, we devise in this work a novel approach which avoids the use of any trimming operator and instead hinges on a decomposition of the discrete sesquilinear (or bilinear) form as $a_h(\cdot,\cdot) = \tilde{a}_h(\cdot,\cdot) + s_h(\cdot,\cdot)$, where $\tilde{a}_h(\cdot,\cdot)$ is meant to ensure a consistency property and $s_h(\cdot,\cdot)$ is added for stabilization purposes. The crucial ingredient is then to devise a form $a_\sharp(\cdot,\cdot)$ with the following key properties:

- $a_\sharp(\cdot, w_h)$ coincides with $\tilde{a}_h(\cdot, w_h)$ for any discrete function w_h when the first argument is discrete,
- $a_\sharp(\cdot, w_h)$ makes unambiguous sense when the first argument is a function with some minimal regularity,
- $a_\sharp(\cdot, w_h)$ enjoys a consistency property with the right-hand side of the discrete problem.

The construction of $a_\sharp$ is achieved by giving a meaning by duality to the normal or tangential component of vector fields at the mesh faces using face-to-cell lifting operators which we construct herein following ideas similar to those in [1, 5]. Since the proof of the above key consistency property hinges on some recently-devised mollification operators, we call the resulting error estimate a mollified error estimate.

In the present work, we present an abstract setting for the mollified error analysis and then we show how to apply it to Nitsche's boundary penalty method to approximate the diffusion equation and the time-harmonic Maxwell's equations. In both cases, the error estimates are robust with respect to the contrast in material properties. The mollified error analysis is applicable as soon as the exact solution is in $\{v \in H^1(D) \mid \kappa\nabla v \in \boldsymbol{L}^p(D),\ \nabla\cdot(\kappa\nabla v) \in L^q(D)\}$, $p > 2$ and $q > \frac{2d}{2+d}$, for the diffusion equation, and as soon as the exact solution is in $\{\boldsymbol{A} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \kappa\nabla\times\boldsymbol{A} \in \boldsymbol{L}^p(D),\ \nabla\times(\kappa\nabla\times\boldsymbol{A}) \in \boldsymbol{L}^2(D)\}$, $p > 2$, for the time-harmonic Maxwell's equations. Owing to the Sobolev Embedding Theorem, the requirements that $\kappa\nabla v \in \boldsymbol{L}^p(D)$ or $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{L}^p(D)$, $p > 2$, hold true whenever $\kappa\nabla v \in \boldsymbol{H}^r(D)$ or $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$, $r > 0$, and these are minimal requirements to achieve some decay rate with respect to the mesh-size in the error estimate. Notice also that these requirements are in general compatible with the regularity pickup estimates available in the literature for the model problems at hand (see, e.g., [6, 20] for the Maxwell's equations).

This paper is organized as follows. In Section 2, we present the two model problems on which we will illustrate the present developments: the diffusion equation and the time-harmonic Maxwell's equations. In Section 3, we introduce the finite element setting and illustrate our abstract discrete setting on Nitsche's boundary penalty method for our two model problems. Section 4 is concerned with abstract error estimates. We first recall Strang's Lemmas, then we present Gudi's trimmed error estimate, and we finish with our novel mollified error estimate. Section 5 contains some useful analysis tools. We first recall some recent results from [15] on shrinking-based mollification operators that commute with the usual derivative operators (∇, $\nabla\times$, and $\nabla\cdot$). Then we present some inverse inequalities useful for the trimmed error analysis and some extension operators that are crucial for the mollified error analysis since they allow us to give a weak meaning to the normal or tangential component of vector fields. Finally, in Section 6 and in Section 7, we show how to apply the trimmed error estimate and the mollified error estimate to our two model problems from Section 3. Although we have focused for brevity on the application to Nitsche's boundary penalty method, we do not anticipate any significant difficulty in extending the present analysis to other nonconforming approximation methods, such as Crouzeix–Raviart-type finite elements and discontinuous Galerkin (dG) methods, since in all the cases the key issue is to give a suitable weak meaning to the normal or tangential trace of vector fields with minimal regularity.

2 Model Problem

We introduce in this section an abstract model problem and illustrate the setting on the diffusion equation and the time-harmonic Maxwell's equations.

2.1 Abstract Setting

Let V and W be two Banach spaces; to stay general, we consider linear spaces over the field of complex numbers. Let $a(\cdot,\cdot)$ be a bounded sesquilinear form on $V\times W$, and let $\ell(\cdot)$ be a bounded antilinear form on W, i.e., $\ell \in W'$. We consider the following abstract model problem: Find $u \in V$ such that

$$a(u, w) = \ell(w) \quad \text{for all } w \in W, \tag{2.1}$$

which we assume to be well-posed in the sense of Hadamard; that is to say, there is a unique solution and this solution depends continuously on the data. The well-posedness of the model problem (2.1) can be characterized by invoking Banach's Closed Range and Open Mapping Theorems; see [22] and [2, p. 112].

Theorem 2.1 (Banach–Nečas–Babuška (BNB)). *Assume that W is a reflexive Banach space. Problem (2.1) is well-posed if and only if*

$$\inf_{v\in V}\sup_{w\in W}\frac{|a(v,w)|}{\|v\|_V\|w\|_W} =: \alpha > 0 \qquad \textit{and} \qquad \forall w \in W,\quad [\forall v \in V,\ a(v,w) = 0] \implies [w = 0].$$

In particular, the a priori estimate $\|u\|_V \le \frac{1}{\alpha}\|\ell\|_{W'}$ holds true.

It is implicitly understood here and in what follows that the above infimum and supremum are taken over nonzero arguments.

2.2 Diffusion Equation

To illustrate the abstract setting introduced above, let us consider a bounded Lipschitz polyhedron D in $\mathbb{R}^d$ with $d \geq 2$. Let $f \in L^q(D)$ be a source term with $q \in (2_*, 2]$, $2_* := \frac{2d}{2+d}$ (so that $q \in (1, 2]$ if $d = 2$, and $q \in (\frac{6}{5}, 2]$ if $d = 3$). We consider the following model problem: Find $u : D \to \mathbb{R}$ such that

$$-\nabla\cdot(\kappa\nabla u) = f \quad \text{in } D, \qquad u = 0 \quad \text{on } \partial D, \tag{2.2}$$

where $\kappa \in L^\infty(D)$ takes values æin D in the interval $[\kappa_\flat, \kappa_\sharp]$ with $0 < \kappa_\flat \leq \kappa_\sharp < \infty$.

Let us introduce the Hilbert space

$$H^1(D) = \{v \in L^2(D) \mid \nabla v \in \boldsymbol{L}^2(D)\}$$

and its zero-trace subspace

$$H^1_0(D) = \{v \in H^1(D) \mid \gamma^{\mathrm{g}}(v) = 0\},$$

where $\gamma^{\mathrm{g}} : H^1(D) \to H^{\frac{1}{2}}(\partial D)$ is the well-known trace operator. To be dimensionally coherent, we equip the space $H^1(D)$ with the norm $\|v\|_{H^1(D)} = (\|v\|^2_{L^2(D)} + \ell_D^2\|\nabla v\|^2_{\boldsymbol{L}^2(D)})^{\frac{1}{2}}$, where ℓ_D is some length scale characteristic of D, e.g., the diameter of D. The model problem (2.2) fits the abstract setting of (2.1) with $V = W = H^1_0(D)$ and

$$a(v, w) := \int_D \kappa\nabla v\cdot\nabla w \,\mathrm{d}x, \quad \ell(w) := \int_D fw \,\mathrm{d}x,$$

and its well-posedness follows from the Lax–Milgram Lemma. In particular, we have

$$|a(v, w)| \leq \kappa_\sharp\|\nabla v\|_{\boldsymbol{L}^2(D)}\|\nabla w\|_{\boldsymbol{L}^2(D)},$$
$$a(v, v) \geq \kappa_\flat\|\nabla v\|^2_{\boldsymbol{L}^2(D)}$$

for all $v, w \in H^1_0(D)$. Note that $\|v\|_{H^1(D)} \leq (1 + C^{-2}_{\mathrm{PS},D})^{\frac{1}{2}}\ell_D\|\nabla v\|_{\boldsymbol{L}^2(D)}$ owing to the Poincaré–Steklov inequality $C_{\mathrm{PS},D}\|v\|_{L^2(D)} \leq \ell_D\|\nabla v\|_{\boldsymbol{L}^2(D)}$ for all $v \in H^1_0(D)$. Note also that a Sobolev embedding implies that $w \in L^{q'}(D)$ for all $w \in H^1(D)$, where q' is the conjugate number of q, i.e., $\frac{1}{q} + \frac{1}{q'} = 1$, so that the linear form $\ell(\cdot)$ is well-defined owing to Hölder's inequality.

Remark 2.2 (Extensions). Most of what is said in the paper generalizes when lower-order terms are added to the PDE in (2.2), κ is tensor-valued, and non-homogeneous Dirichlet conditions are imposed.

2.3 Time-Harmonic Maxwell's Equations

As a second example to illustrate the abstract setting introduced above, we consider the time-harmonic Maxwell's equations in a bounded Lipschitz polyhedron D in $\mathbb{R}^3$. Let $\boldsymbol{f} \in \boldsymbol{L}^2(D)$ be a source term. We consider the following model problem: Find $\boldsymbol{A} : D \to \mathbb{R}^3$ such that

$$\widetilde{\mu}\boldsymbol{A} + \nabla\times(\kappa\nabla\times\boldsymbol{A}) = \boldsymbol{f}, \quad \boldsymbol{A}_{|\partial D}\times\boldsymbol{n} = \boldsymbol{0}. \tag{2.3}$$

We assume that $\widetilde{\mu} \in L^\infty(D; \mathbb{C})$, $\kappa \in BV(D; \mathbb{C}) \cap L^\infty(D; \mathbb{C})$, and we set

$$\mu_\sharp = \operatorname*{ess\,sup}_{\boldsymbol{x}\in D} |\widetilde{\mu}(\boldsymbol{x})| \quad \text{and} \quad \kappa_\sharp = \operatorname*{ess\,sup}_{\boldsymbol{x}\in D} |\kappa(\boldsymbol{x})|.$$

We also assume the following positivity condition: There are real numbers $\theta, \mu_\flat > 0$, and $\kappa_\flat > 0$ so that, letting $\mu_r := \mathbb{R}(e^{i\theta}\widetilde{\mu})$ and $\kappa_r := \mathbb{R}(e^{i\theta}\kappa)$, we have

$$\operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \mu_r(\boldsymbol{x}) \geq \mu_\flat \quad \text{and} \quad \operatorname*{ess\,inf}_{\boldsymbol{x}\in D} \kappa_r(\boldsymbol{x}) \geq \kappa_\flat. \tag{2.4}$$

The positivity condition (2.4) fails when the two complex numbers $\tilde{\mu}$ and κ are collinear and point in opposite directions. If it is the case, the model problem (2.3) is an eigenvalue problem, otherwise it is a boundary-value problem. The model problem (2.3) can be derived from the Maxwell's equations in the time-harmonic regime, i.e., under the assumption that the time variation is of the form $e^{i\omega t}$, where ω is the angular frequency and $i^2 = -1$. One example is the Helmholtz problem where $\boldsymbol{A}$ stands for the electric field, $\tilde{\mu} = -\omega^2\epsilon + i\omega\sigma$ with ϵ the electric permittivity and σ the electric conductivity, $\kappa = \mu^{-1}$ with μ the magnetic permeability, and $\boldsymbol{f} = -i\omega\boldsymbol{j}_s$ with $\boldsymbol{j}_s$ an imposed current. Another example is the eddy-current problem where $\boldsymbol{A}$ stands for the magnetic field, $\tilde{\mu} = i\omega\mu$, $\kappa = \sigma^{-1}$, and $\boldsymbol{f} = \nabla\times(\sigma^{-1}\boldsymbol{j}_s)$.

Let us introduce the Hilbert space

$$\boldsymbol{H}(\mathrm{curl}; D) = \{\boldsymbol{b} \in \boldsymbol{L}^2(D) \mid \nabla\times\boldsymbol{b} \in \boldsymbol{L}^2(D)\}$$

and its zero-trace subspace

$$\boldsymbol{H}_0(\mathrm{curl}; D) = \{\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D) \mid \gamma^{\mathrm{c}}(\boldsymbol{b}) = \boldsymbol{0}\},$$

where $\gamma^{\mathrm{c}} : \boldsymbol{H}(\mathrm{curl}; D) \to \boldsymbol{H}^{-\frac{1}{2}}(\partial D) := (\boldsymbol{H}^{\frac{1}{2}}(\partial D))'$ is the tangential trace operator such that

$$\langle \gamma^{\mathrm{c}}(\boldsymbol{b}), \boldsymbol{l}\rangle_{\partial D} := \int_D \boldsymbol{b}\cdot\nabla\times\boldsymbol{w}(\boldsymbol{l})\,\mathrm{d}x - \int_D (\nabla\times\boldsymbol{b})\cdot\boldsymbol{w}(\boldsymbol{l})\,\mathrm{d}x$$

for all $\boldsymbol{b} \in \boldsymbol{H}(\mathrm{curl}; D)$ and all $\boldsymbol{l} \in \boldsymbol{H}^{\frac{1}{2}}(\partial D)$, where $\boldsymbol{w}(\boldsymbol{l}) \in \boldsymbol{H}^1(D)$ is a lifting of $\boldsymbol{l}$ such that $\gamma^{\mathrm{g}}(\boldsymbol{w}(\boldsymbol{l})) = \boldsymbol{l}$ (componentwise) and $\langle\cdot,\cdot\rangle_{\partial D}$ denotes the duality pairing between $\boldsymbol{H}^{-\frac{1}{2}}(\partial D)$ and $\boldsymbol{H}^{\frac{1}{2}}(\partial D)$. Note that $\gamma^{\mathrm{c}}(\boldsymbol{b}) = \boldsymbol{b}_{|\partial D}\times\boldsymbol{n}$ whenever the field $\boldsymbol{b}$ is smooth. To be dimensionally coherent, we equip the space $\boldsymbol{H}(\mathrm{curl}; D)$ with the norm $\|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)} = (\|\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)} + \ell_D^2\|\nabla\times\boldsymbol{b}\|^2_{\boldsymbol{L}^2(D)})^{\frac{1}{2}}$. The model problem (2.3) fits the abstract setting of (2.1) with $V = W = \boldsymbol{H}_0(\mathrm{curl}; D)$ and

$$a(\boldsymbol{v}, \boldsymbol{b}) := \int_D (\tilde{\mu}\boldsymbol{v}\cdot\overline{\boldsymbol{b}} + \kappa\nabla\times\boldsymbol{v}\cdot\nabla\times\overline{\boldsymbol{b}})\,\mathrm{d}x, \qquad \ell(\boldsymbol{b}) := \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}\,\mathrm{d}x,$$

and its well-posedness follows from the Lax–Milgram Lemma. In particular, we have

$$|a(\boldsymbol{v}, \boldsymbol{b})| \le \max(\mu_\sharp, \ell_D^{-2}\kappa_\sharp)\|\boldsymbol{v}\|_{\boldsymbol{H}(\mathrm{curl};D)}\|\boldsymbol{b}\|_{\boldsymbol{H}(\mathrm{curl};D)},$$
$$\Re(e^{i\theta}a(\boldsymbol{b}, \boldsymbol{b})) \ge \min(\mu_\flat, \ell_D^{-2}\kappa_\flat)\|\boldsymbol{b}\|^2_{\boldsymbol{H}(\mathrm{curl};D)}$$

for all $\boldsymbol{v}, \boldsymbol{b} \in \boldsymbol{H}_0(\mathrm{curl}; D)$.

Remark 2.3 (Extensions). Most of what is said in the paper generalizes when the non-homogeneous Dirichlet condition $\gamma^{\mathrm{c}}(\boldsymbol{A}) = \boldsymbol{g}$ is enforced in (2.3) with $\boldsymbol{g}$ in the range of the trace map γ^{c}.

3 Discrete Problem

We now formulate a discrete version of problem (2.1) by using the Galerkin method. The central idea in the Galerkin method consists of replacing the infinite-dimensional spaces V and W by finite-dimensional spaces V_h and W_h that are members of sequences of spaces $(V_h)_{h\to 0}$, $(W_h)_{h\to 0}$ endowed with some approximation properties as $h \to 0$. The norms in V_h and W_h are denoted by $\|\cdot\|_{V_h}$ and $\|\cdot\|_{W_h}$, respectively. The discrete problem is formulated as follows: Find $u_h \in V_h$ such that

$$a_h(u_h, w_h) = \ell_h(w_h) \quad \text{for all } w_h \in W_h, \tag{3.1}$$

where $a_h(\cdot,\cdot)$ is a bounded sesquilinear form on $V_h\times W_h$ and $\ell_h(\cdot)$ is a bounded antilinear form on W_h; note that $a_h(\cdot,\cdot)$ and $\ell_h(\cdot)$ possibly differ from $a(\cdot,\cdot)$ and $\ell(\cdot)$, respectively. We henceforth assume that $\dim(V_h) = \dim(W_h)$ and that

$$\inf_{v_h\in V_h}\sup_{w_h\in W_h}\frac{|a_h(v_h, w_h)|}{\|v_h\|_{V_h}\|w_h\|_{W_h}} =: \alpha_h > 0 \quad \text{for all } h > 0, \tag{3.2}$$

so that the discrete problem (3.1) is well-posed.

3.1 Finite Element Setting

Let $(\mathcal{T}_h)_{h>0}$ be a shape-regular sequence of meshes; we assume that each mesh covers D exactly. To avoid technical questions regarding hanging nodes, we also suppose that each mesh is matching, i.e., for all cells $K, K' \in \mathcal{T}_h$ such that $K \neq K'$ and $K \cap K' \neq \emptyset$, the set $K \cap K'$ is a common vertex, edge, or face of both K and K' (with obvious extensions in higher space dimensions). Given a mesh $\mathcal{T}_h$, the elements $K \in \mathcal{T}_h$ are closed sets in $\mathbb{R}^d$ by convention, and they are all assumed to be constructed from a single reference cell $\widehat{K}$ through affine, bijective, geometric transformations $\boldsymbol{T}_K : \widehat{K} \to K$. For a mesh cell $K \in \mathcal{T}_h$, we define $\check{\mathcal{T}}_K$ to be the collection of the mesh cells in $\mathcal{T}_h$ that touch K, i.e., the mesh cells that share a vertex, an edge or a face (in dimension 3) with K, plus K itself. We define $D_K := \operatorname{int}(\bigcup_{K' \in \check{\mathcal{T}}_K} K')$; note that the number of cells composing $\check{\mathcal{T}}_K$ is uniformly bounded owing to the shape-regularity of the mesh sequence.

The set of the mesh faces is denoted $\mathcal{F}_h$. This set is partitioned into the subset of the interfaces denoted $\mathcal{F}_h^\circ$ and the subset of the boundary faces denoted $\mathcal{F}_h^\partial$. Each interface F is oriented by choosing one unit normal vector $\boldsymbol{n}_F$. The boundary faces are oriented by using the outward normal vector that we denote $\boldsymbol{n}$. Given an interface $F \in \mathcal{F}_h^\circ$, we denote by K_l (left cell) and K_r (right cell) the two cells such that $F = K_l \cap K_r$ and $\boldsymbol{n}_F$ points from K_l to K_r. This convention allows us to define the notion of jump across F for any smooth enough function v as follows:

$$[\![v]\!]_F(\boldsymbol{x}) := v_{|K_l}(\boldsymbol{x}) - v_{|K_r}(\boldsymbol{x}) \quad \text{æ}\boldsymbol{x} \text{ in } F.$$

We consider a reference finite element in the sense of Ciarlet $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$. (The superscript $^{\mathrm{g}}$ is intended to remind us that this finite element will be used to build a finite-dimensional subspace composed of functions whose *gradient* in D is integrable.) We think of $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ as a scalar-valued finite element with some degrees of freedom that require point evaluations, for instance $(\widehat{K}, \widehat{P}^{\mathrm{g}}, \widehat{\Sigma}^{\mathrm{g}})$ could be a Lagrange finite element. The local shape functions are denoted $(\widehat{\theta}_i)_{i \in \mathcal{N}}$; recall that $\sigma_i(\widehat{\theta}_j) = \delta_{ij}$ for all $\sigma_i \in \widehat{\Sigma}^{\mathrm{g}}$, and all $i, j \in \mathcal{N}$. At this point, we do not need to know the exact structure of the reference element. One typically assumes that there exists $k \in \mathbb{N}$ such that $\mathbb{P}_{k,d} \subset \widehat{P}^{\mathrm{g}}$, where $\mathbb{P}_{k,d}$ is the vector space composed of the d-variate polynomials of degree at most k.

In order to construct H^1-conforming approximation spaces based on $(\mathcal{T}_h)_{h>0}$ using the above reference finite element, we introduce the pullback by the geometric map $\boldsymbol{T}_K$ which we denote by ψ_K^{g}, i.e., $\psi_K^{\mathrm{g}}(v) = v \circ \boldsymbol{T}_K$. Then we set

$$P^{\mathrm{g}}(\mathcal{T}_h) := \{v_h \in L^1(D) \mid v_{h|K} \in P_K \text{ for all } K \in \mathcal{T}_h,\ [\![v_h]\!]_F = 0 \text{ for all } F \in \mathcal{F}_h^\circ\}, \tag{3.3a}$$

$$P_0^{\mathrm{g}}(\mathcal{T}_h) := P^{\mathrm{g}}(\mathcal{T}_h) \cap H_0^1(D), \tag{3.3b}$$

where $P_K := (\psi_K^{\mathrm{g}})^{-1}(\widehat{P}^{\mathrm{g}})$. Let $\mathcal{F}_K$ be the collection of the faces of K, and for all $F \in \mathcal{F}_K$, let $\gamma_{K,F}$ be the corresponding trace map. For the above construction of $P^{\mathrm{g}}(\mathcal{T}_h)$ to be meaningful, we assume that for any mesh interface $F \in \mathcal{F}_h^\circ$ such that $F = K_l \cap K_r$, we have $\gamma_{K_l,F}(P_{K_l}) = \gamma_{K_r,F}(P_{K_r}) =: P_F$. We call P_F the finite element trace space. For instance, if $\widehat{P} = \mathbb{P}_{k,d}$ and $\widehat{K}$ is a simplex, then P_F is composed of the restriction of d-variate polynomials of degree at most k to F.

Remark 3.1 (Reference Cell). The construction of the H^1-conforming space $P^{\mathrm{g}}(\mathcal{T}_h)$ by means of a reference cell $\widehat{K}$ is classical in the context of finite elements. On polyhedral meshes, one can also consider H^1-conforming spaces defined locally in each cell of the mesh, as in the Virtual Element Method [4].

3.2 Boundary Penalty for the Diffusion Equation

We are going to illustrate our results on the so-called boundary penalty method of Nitsche [23]. Let us first consider the diffusion equation from Section 2.2. To avoid technicalities, we assume that there is a partition of D into M disjoint Lipschitz polyhedra $D_1, \dots, D_M$ so that $\kappa_{|D_i}$ is constant for all $1 \leq i \leq M$, and we assume that the meshes in $(\mathcal{T}_h)_{h>0}$ are fitted to this partition, so that, for all $h > 0$ and all $K \in \mathcal{T}_h$, $\kappa_{|K}$ is constant; we use the notation $\kappa_K := \kappa_{|K}$.

Let $V_h := P^{\mathrm{g}}(\mathcal{T}_h)$ be the H^1-conforming finite element space based on $\mathcal{T}_h$ introduced in (3.3). For the diffusion equation, the discrete forms $a_h(\cdot,\cdot)$ and $\ell_h(\cdot)$ are defined by

$$a_h(v_h, w_h) := \int_D \kappa\nabla v_h{\cdot}\nabla w_h \,\mathrm{d}x - \int_{\partial D} (\boldsymbol{n}{\cdot}\kappa\nabla v_h) w_h \,\mathrm{d}s + \int_{\partial D} \eta_h v_h w_h \,\mathrm{d}s, \tag{3.4a}$$

$$\ell_h(w_h) := \int_D f w_h \,\mathrm{d}x \tag{3.4b}$$

for all $v_h, w_h \in V_h$. It is useful to decompose the discrete bilinear form as

$$a_h(\cdot,\cdot) = \tilde{a}_h(\cdot,\cdot) + s_h(\cdot,\cdot),$$

where

$$\tilde{a}_h(v_h, w_h) := \int_D \kappa\nabla v_h{\cdot}\nabla w_h \,\mathrm{d}x - \int_{\partial D} (\boldsymbol{n}{\cdot}\kappa\nabla v_h) w_h \,\mathrm{d}s, \tag{3.5a}$$

$$s_h(v_h, w_h) := \int_{\partial D} \eta_h v_h w_h \,\mathrm{d}s. \tag{3.5b}$$

The discrete bilinear form $\tilde{a}_h(\cdot,\cdot)$ is meant to ensure a consistency property, and the discrete bilinear form $s_h(\cdot,\cdot)$ is added for stabilization purposes. The penalty parameter is defined by setting $\eta_h := \eta_0 \rho_h$, where the user-dependent factor $\eta_0 > 0$ has yet to be chosen large enough (see Lemma 3.2 below) and where

$$\rho_{h|F} := \frac{\kappa_{K_F}}{h_F} \quad \text{for all } F \in \mathcal{F}_h^\partial, \tag{3.6}$$

where K_F is the unique mesh cell having F as a face.

We equip the space V_h with the following norm:

$$\|v_h\|_{V_h} := \Big(\|\kappa^{\frac{1}{2}}\nabla v_h\|^2_{\boldsymbol{L}^2(D)} + \|\rho_h^{\frac{1}{2}} v_h\|^2_{L^2(\partial D)} \Big)^{\frac{1}{2}} \quad \text{for all } v_h \in V_h. \tag{3.7}$$

Since $\|v_h\|_{V_h} = 0$ implies that v_h is constant on D and vanishes on ∂D, and hence vanishes everywhere in D, we infer that $\|\cdot\|_{V_h}$ is indeed a norm on V_h. Furthermore, owing to the assumed shape-regularity of the mesh sequence, there is c_I, uniform with respect to h (but depending on the shape-regularity of the mesh sequence and on the reference finite element), such that

$$\|v_h\|_{L^2(F)} \le c_I h_F^{-\frac{1}{2}} \|v_h\|_{L^2(K_F)}$$

for all $v_h \in V_h$ and all $F \in \mathcal{F}_h^\partial$. The following stability result is classical; we simply state it without proof (see, e.g., [12, Lem. 4.12] for a proof in the context of dG methods).

Lemma 3.2 (Coercivity and Well-Posedness). *Suppose that η_h is defined by* (3.6) *with $\eta_0 > \frac{1}{4} n_\partial c_I^2$, where n_∂ is the maximum number of boundary faces that a mesh cell can have ($n_\partial \le d$ for simplicial meshes). Then the following coercivity property holds true:*

$$a_h(v_h, v_h) \ge \alpha \|v_h\|^2_{V_h} \quad \textit{for all } v_h \in V_h$$

with $\alpha := \frac{\eta_0 - \frac{1}{4} n_\partial c_I^2}{1+\eta_0}$. Consequently, the discrete problem (3.1) *is well-posed for the diffusion equation.*

3.3 Boundary penalty for Maxwell's equations

Nitsche's boundary penalty method can also be applied to the time-harmonic Maxwell's equations from Section 2.3. We assume that there is a partition of D into M disjoint Lipschitz polyhedra $D_1, \dots, D_M$ so that $\widetilde{\mu}_{|D_i}$ and $\kappa_{|D_i}$ are constant for all $1 \le i \le M$, and we assume that the meshes in $(\mathcal{T}_h)_{h>0}$ are fitted to this partition, so that, for all $h > 0$ and all $K \in \mathcal{T}_h$, $\widetilde{\mu}_{|K}$ and $\kappa_{|K}$ are constant; we use the notation $\mu_{\sharp,K} := |\widetilde{\mu}_{|K}|$, $\mu_{r,K} := \mu_{r|K}$, $\kappa_{\sharp,K} := |\kappa_{|K}|$, and $\kappa_{r,K} := \kappa_{r|K}$, where we recall that $\mu_r := \mathbb{R}(e^{i\theta}\widetilde{\mu})$ and $\kappa_r := \mathbb{R}(e^{i\theta}\kappa)$.

Let $\boldsymbol{V}_h = \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h)$ be the $\boldsymbol{H}^1$-conforming finite element space based on $\mathcal{T}_h$, where $\boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h)$ is the vector-valued version of the finite element space $P^{\mathrm{g}}(\mathcal{T}_h)$ considered above for the diffusion equation. For the time-harmonic Maxwell's equations, the discrete forms $a_h(\cdot,\cdot) = \tilde{a}_h(\cdot,\cdot) + s_h(\cdot,\cdot)$ and $\ell_h(\cdot)$ are defined by

$$\tilde{a}_h(\boldsymbol{v}_h, \boldsymbol{b}_h) := \int_D (\tilde{\mu}\boldsymbol{v}_h\cdot\overline{\boldsymbol{b}}_h + \kappa\nabla\times\boldsymbol{v}_h\cdot\nabla\times\overline{\boldsymbol{b}}_h)\,\mathrm{d}x + \int_{\partial D} (\boldsymbol{n}\times(\kappa\nabla\times\boldsymbol{v}_h))\cdot\overline{\boldsymbol{b}}_h\,\mathrm{d}s, \tag{3.8a}$$

$$s_h(\boldsymbol{v}_h, \boldsymbol{b}_h) := \int_{\partial D} \eta_h(\boldsymbol{v}_h\times\boldsymbol{n})\cdot(\overline{\boldsymbol{b}}_h\times\boldsymbol{n})\,\mathrm{d}s, \tag{3.8b}$$

$$\ell_h(\boldsymbol{b}_h) := \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{b}}_h\,\mathrm{d}x \tag{3.8c}$$

for all $\boldsymbol{v}_h, \boldsymbol{b}_h \in \boldsymbol{V}_h$. The discrete sesquilinear form $\tilde{a}_h(\cdot,\cdot)$ is meant to ensure a consistency property, and the discrete sesquilinear form $s_h(\cdot,\cdot)$ is added for stabilization purposes. The penalty parameter is defined by setting

$$\eta_h = \eta_0 e^{-i\theta}\rho_h,$$

where the user-dependent factor $\eta_0 > 0$ has yet to be chosen large enough (see Lemma 3.3 below), and where

$$\rho_{h|F} := \frac{|\kappa_{K_F}|^2}{\kappa_{r,K_F} h_F} \quad \text{for all } F \in \mathcal{F}_h^\partial, \tag{3.9}$$

where K_F is the unique mesh cell having F as a face.

We equip the space $\boldsymbol{V}_h$ with the following norm:

$$\|\boldsymbol{b}_h\|_{\boldsymbol{V}_h} := \Big(\|\mu_r^{\frac{1}{2}}\boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + \|\kappa_r^{\frac{1}{2}}\nabla\times\boldsymbol{b}_h\|^2_{\boldsymbol{L}^2(D)} + \|\rho_h^{\frac{1}{2}}(\boldsymbol{b}_h\times\boldsymbol{n})\|^2_{\boldsymbol{L}^2(\partial D)}\Big)^{\frac{1}{2}} \quad \text{for all } \boldsymbol{b}_h \in \boldsymbol{V}_h. \tag{3.10}$$

The following stability result is proved using the same arguments as in the proof of Lemma 3.2.

Lemma 3.3 (Coercivity and Well-Posedness). *Suppose that η_h is defined by (3.9) with $\eta_0 > \frac{1}{4}n_\partial c_I^2$. Then the following coercivity property holds true:*

$$\Re(e^{i\theta}a_h(\boldsymbol{b}_h, \boldsymbol{b}_h)) \ge \alpha\|\boldsymbol{b}_h\|^2_{\boldsymbol{V}_h} \quad \textit{for all } \boldsymbol{b}_h \in \boldsymbol{V}_h,$$

with $\alpha := \frac{\eta_0 - \frac{1}{4}n_\partial c_I^2}{1+\eta_0}$. Consequently, the discrete problem (3.1) is well-posed for the Maxwell's equations.

4 Abstract Error Estimates

There are many ways to investigate the approximation properties of the above discrete problem (3.1). Since u_h may not be a member of V, it follows that u and u_h may be objects of different nature. This poses the question of defining a common ground for the discrete solution u_h and the exact solution u to measure the error. For this purpose, we assume that it is meaningful to define the linear space

$$V_\flat := V + V_h.$$

We equip the space $V_\flat$ with a norm denoted $\|\cdot\|_{V_\flat}$ which we assume extends the discrete norm $\|\cdot\|_{V_h}$ to $V_\flat$, i.e., there exists a real number $c_\flat$ so that

$$\|v_h\|_{V_\flat} \le c_\flat\|v_h\|_{V_h} \quad \text{for all } v_h \in V_h. \tag{4.1}$$

The goal of this section is to bound the error $u - u_h$ using the $\|\cdot\|_{V_\flat}$-norm. Note that even in the conforming case where $V_\flat$ and V coincide as linear spaces, choosing $\|\cdot\|_{V_\flat}$ to be different from $\|\cdot\|_V$ can be useful for the error analysis.

4.1 A Basic Error Identity

Our starting point is the following (relatively straightforward) error identity. Recall that the norm of any antilinear form $\phi_h \in W_h' := \mathcal{L}(W_h;\mathbb{C})$ is defined by $\|\phi_h\|_{W_h'} := \sup_{w_h\in W_h} \frac{|\phi_h(w_h)|}{\|w_h\|_{W_h}}$.

Lemma 4.1 (Error Identity). *Assume that the discrete inf-sup condition* (3.2) *is satisfied. Then the following identity holds true:*

$$\|u-u_h\|_{V_\flat} = \inf_{v_h\in V_h}\Big[\|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\|\delta_h(v_h)\|_{W_h'}\Big], \tag{4.2}$$

where $\delta_h : V_h \to W_h'$, which we call consistency error, is defined by

$$\langle \delta_h(v_h), w_h\rangle_{W_h',W_h} := \ell_h(w_h) - a_h(v_h,w_h). \tag{4.3}$$

Proof. Let $v_h \in V_h$. The triangle inequality, (4.1), stability, and the fact that $a_h(u_h,w_h) = \ell_h(w_h)$ for all $w_h \in W_h$ imply that

$$\begin{aligned}\|u-u_h\|_{V_\flat} &\le \|u-v_h\|_{V_\flat} + \|u_h-v_h\|_{V_\flat} \le \|u-v_h\|_{V_\flat} + c_\flat\|u_h-v_h\|_{V_h}\\ &\le \|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\sup_{w_h\in W_h}\frac{|a_h(u_h-v_h,w_h)|}{\|w_h\|_{W_h}}\\ &= \|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\sup_{w_h\in W_h}\frac{|\langle\delta_h(v_h),w_h\rangle_{W_h',W_h}|}{\|w_h\|_{W_h}}.\end{aligned}$$

Since v_h is arbitrary in V_h and recalling the definition of the norm of the discrete antilinear form $\delta_h(v_h)$, we conclude that $\|u-u_h\|_{V_\flat} \le r_h$, where r_h denotes the right-hand side of (4.2). Finally, taking $v_h = u_h$ in the infimum and observing that $\delta_h(u_h)$ vanishes identically on W_h, we infer that $\|u-u_h\|_{V_\flat} = r_h$. □

4.2 Strang's Lemmas

The traditional form of Strang's First Lemma consists of assuming that the approximation setting is conforming; that is to say, $V_h \subset V$ and $W_h \subset W$. This implies that the linear spaces V and $V_\flat$ coincide; however, these spaces may be equipped with different norms.

Lemma 4.2 (Strang 1). *Assume the following:*
(i) *$V_h \subset V$ and $W_h \subset W$.*
(ii) *The sesquilinear form $a(\cdot,\cdot)$ is bounded on $V_\flat\times W_h$ with norm*

$$\|a\|_{V_\flat,W_h} := \sup_{v\in V_\flat}\sup_{w_h\in W_h}\frac{|a(v,w_h)|}{\|v\|_{V_\flat}\|w_h\|_{W_h}}.$$

Then the following error estimate holds true:

$$\|u-u_h\|_{V_\flat} \le \inf_{v_h\in V_h}\Big[\Big(1 + c_\flat\frac{\|a\|_{V_\flat,W_h}}{\alpha_h}\Big)\|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\|\delta_h^{\mathrm{St1}}\|_{W_h'}\Big]$$

with $\delta_h^{\mathrm{St1}} : V_h \to W_h'$ defined by

$$\langle\delta_h^{\mathrm{St1}}(v_h), w_h\rangle_{W_h',W_h} := \ell_h(w_h) - \ell(w_h) + a(v_h,w_h) - a_h(v_h,w_h).$$

Proof. This is an easy consequence of the error identity (4.2) after one has observed that

$$\ell_h(w_h) - a_h(v_h,w_h) = \ell_h(w_h) - \ell(w_h) + a(u,w_h) + [a(v_h,w_h) - a(v_h,w_h)] - a_h(v_h,w_h),$$

since $a(u,w_h) = \ell(w_h)$ for all $w_h \in W_h \subset W$. One concludes by invoking the boundedness of a on $V_\flat\times W_h$. □

The main inconvenient of the above estimate is that is assumes that the discrete setting is conforming. This shortcoming is traditionally addressed in the literature by invoking Strang's Second Lemma where one supposes that the discrete sesquilinear form $a_h(\cdot,\cdot)$ can be extended as a bounded sesquilinear form $a_\flat(\cdot,\cdot)$ on $V_\flat \times W_h$.

Lemma 4.3 (Strang 2). *Assume that the discrete sesquilinear form $a_h(\cdot,\cdot)$ admits a bounded extension $a_\flat(\cdot,\cdot)$ on $V_\flat \times W_h$ with norm*

$$\|a_\flat\|_{V_\flat,W_h} := \sup_{v\in V_\flat}\sup_{w_h\in W_h}\frac{|a_\flat(v,w_h)|}{\|v\|_{V_\flat}\|w_h\|_{W_h}}.$$

Then the following error estimate holds true:

$$\|u-u_h\|_{V_\flat} \le \Big(1 + c_\flat\frac{\|a_\flat\|_{V_\flat,W_h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_{V_\flat} + \frac{c_\flat}{\alpha_h}\|\ell_h - a_\flat(u,\cdot)\|_{W_h'}.$$

Proof. This is also an easy consequence of the error identity (4.2) after one writes

$$\ell_h(w_h) - a_h(v_h,w_h) = \ell_h(w_h) + [a_\flat(u,w_h) - a_\flat(u,w_h)] - a_\flat(v_h,w_h),$$

and uses the boundedness of $a_\flat$ on $V_\flat \times W_h$. □

The key problem with the above estimate is that, in general, it is not possible to extend $a_h(\cdot,\cdot)$ to $V_\flat \times W_h$ unless one requires some regularity assumption on the exact solution. For instance, for the boundary penalty method, this requirement is $\kappa\nabla u \in \boldsymbol{H}^r(D)$ with $r > \frac{1}{2}$ in the case of the diffusion equation, and it is $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$ with $r > \frac{1}{2}$ in the case of the Maxwell equations. These requirements are unrealistic if the model coefficients are nonsmooth.

4.3 Alternative Error Estimates

In this subsection, we present two alternative error estimates that avoid extending the discrete sesquilinear form $a_h(\cdot,\cdot)$ to $V_\flat \times W_h$. We still need a regularity assumption on the exact solution, but this assumption is milder than that required to extend $a_h(\cdot,\cdot)$. To stay general, we formalize this regularity assumption by assuming that $u \in V_{\mathrm{s}}$ where V_{s} is a dense subspace of V. We set

$$V_\sharp := V_{\mathrm{s}} + V_h,$$

and we note that $V_\sharp$ is a subspace of $V_\flat$. We equip the space $V_\sharp$ with a norm $\|\cdot\|_{V_\sharp}$ that we suppose to be (slightly) stronger than the norm $\|\cdot\|_{V_\flat}$ restricted to $V_\sharp$; specifically, we assume that

$$\|v\|_{V_\flat} \le c_\flat\|v\|_{V_\sharp} \quad \text{for all } v \in V_\sharp. \tag{4.4}$$

We use the same constant $c_\flat$ in (4.4) and in (4.1) to simplify the notation; we could consider two constants and call $c_\flat$ the largest of the two. We refer the reader to Section 6.2 and Section 7.2 where examples for the spaces V_{s} and $V_\sharp$ and the corresponding norms are given. Our starting point is the following result where we do not separate the notions of consistency and boundedness by triangle inequalities.

Lemma 4.4 (Key Error Estimate). *Assume that the exact solution u is in V_{s}. Assume the following* consistency/boundedness *property: There is a real number $\omega_{\sharp h}$ so that*

$$\|\delta_h(v_h)\|_{W_h'} \le \omega_{\sharp h}\|u-v_h\|_{V_\sharp} \quad \textit{for all } v_h \in V_h, \tag{4.5}$$

with $\delta_h : V_h \to W_h'$ defined by (4.3). *Then the following holds true:*

$$\|u-u_h\|_{V_\flat} \le c_\flat\Big(1 + \frac{\omega_{\sharp h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_{V_\sharp}. \tag{4.6}$$

Moreover, if the following bound holds true for some real number $c_\sharp$ uniform with respect to h:

$$\|v_h\|_{V_\sharp} \le c_\sharp\|v_h\|_{V_h} \quad \textit{for all } v_h \in V_h, \tag{4.7}$$

then we have the quasi-optimal error estimate

$$\|u-u_h\|_{V_\sharp} \le \Big(1 + c_\sharp\frac{\omega_{\sharp h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_{V_\sharp}. \tag{4.8}$$

Proof. The error estimate (4.6) is a direct consequence of Lemma 4.1 and consistency/boundedness. For the proof of (4.8), we proceed as in the proof of Lemma 4.1, and we write

$$\begin{aligned}\|u-u_h\|_{V_\sharp} &\leq \|u-v_h\|_{V_\sharp} + \|v_h-u_h\|_{V_\sharp}\\ &\leq \|u-v_h\|_{V_\sharp} + c_\sharp\|v_h-u_h\|_{V_h}\\ &\leq \|u-v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h}\sup_{w_h\in W_h}\frac{|a_h(u_h-v_h,w_h)|}{\|w_h\|_{W_h}}\\ &= \|u-v_h\|_{V_\sharp} + \frac{c_\sharp}{\alpha_h}\|\delta_h(v_h)\|_{W_h'}\\ &\leq \|u-v_h\|_{V_\sharp} + \frac{c_\sharp\omega_{\sharp h}}{\alpha_h}\|u-v_h\|_{V_\sharp},\end{aligned}$$

and we conclude by taking the infimum over $v_h \in V_h$. □

Remark 4.5 (Quasi-Optimality). The error estimate (4.8) is said to be quasi-optimal since the same norm is used to measure the error and the best-approximation error of the solution in V_h.

4.3.1 Trimmed Error Estimate

One possible way forward to overcome the limitations of Strang's Second Lemma has been proposed by Gudi [18]. The key idea is to introduce a so-called trimming operator $T : W_h \to W \cap W_h$ that transforms the discrete test functions into (discrete) objects that are conforming in W.

Lemma 4.6 (Trimmed Error Estimate). *Assume that the exact solution u is in V_{S}. Consider any map*

$$T : W_h \to W \cap W_h$$

such that the following properties hold true:

(i) *There exists a real number $\omega^{\mathrm{tri}}_{V_\sharp,W_h}$ so that*

$$\|a(u,T(\cdot)) - a_h(v_h,T(\cdot))\|_{W_h'} \leq \omega^{\mathrm{tri}}_{V_\sharp,W_h}\|u-v_h\|_{V_\sharp} \quad \text{for all } v_h \in V_h. \tag{4.9}$$

(ii) *There exists a real number $\varpi^{\mathrm{tri}}_{V_\sharp,W_h}$ so that*

$$\|\ell_h - \ell\circ T - a_h(v_h,(I-T)(\cdot))\|_{W_h'} \leq \varpi^{\mathrm{tri}}_{V_\sharp,W_h}\|u-v_h\|_{V_\sharp} \quad \text{for all } v_h \in V_h, \tag{4.10}$$

where I is the identity operator in W_h.

Then the following error estimate holds true:

$$\|u-u_h\|_{V_\flat} \leq c_\flat\Big(1 + \frac{\omega^{\mathrm{tri}}_{V_\sharp,W_h} + \varpi^{\mathrm{tri}}_{V_\sharp,W_h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_{V_\sharp}. \tag{4.11}$$

Moreover, if the discrete norm equivalence (4.7) holds true, we have the quasi-optimal error estimate

$$\|u-u_h\|_{V_\sharp} \leq \Big(1 + c_\sharp\frac{\omega^{\mathrm{tri}}_{V_\sharp,W_h} + \varpi^{\mathrm{tri}}_{V_\sharp,W_h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u-v_h\|_{V_\sharp}.$$

Proof. We observe that, for all $v_h \in V_h$ and all $w_h \in W_h$, we have

$$\ell_h(w_h) - a_h(v_h,w_h) = \ell_h(w_h) - \ell(T(w_h)) + a(u,T(w_h)) - [a_h(v_h,T(w_h)) - a_h(v_h,T(w_h))] - a_h(v_h,w_h),$$

since $a(u,T(w_h)) = \ell(T(w_h))$ for all $w_h \in W_h$. Owing to properties (4.9) and (4.10), we infer that the consistency/boundedness property (4.5) holds true with $\omega_{\sharp h} = \omega^{\mathrm{tri}}_{V_\sharp,W_h} + \varpi^{\mathrm{tri}}_{V_\sharp,W_h}$. The assertions then follow from the key error estimates of Lemma 4.4. □

Remark 4.7 (Conforming Case). Whenever $W_h \subset W$, one can take T to be the canonical injection $W_h \hookrightarrow W$. In this case, the abstract error estimate (4.11) differs from that derived in Strang's First Lemma. The reason for this is that we have used different triangle inequalities to derive (4.11).

4.3.2 Mollified Error Estimate

Although the trimmed error estimate presented in the previous subsection can overcome some shortcomings encountered with the use of Strang's Lemmas, as illustrated by the examples in Section 6 and in Section 7, we will also see that some difficulties remain. In particular, it is not always easy to construct a trimming operator in the context of Maxwell's equations when one does not use edge elements and the faces of the domain D are not orthogonal to one of the coordinate axes. Moreover, it is not simple to construct a trimming operator that exhibits suitable stability properties that are robust in the case of highly-contrasted coefficients. The goal of this subsection is to present a new approach for the error analysis that attempts to remedy these difficulties.

Lemma 4.8 (Mollified Error Estimate). *Assume that the exact solution u is in V_{s}. Recall the decomposition $a_h(\cdot,\cdot) = \tilde{a}_h(\cdot,\cdot) + s_h(\cdot,\cdot)$. Assume that there is a sesquilinear form $a_\sharp(\cdot,\cdot)$ on $V_\sharp \times W_h$ that is bounded on $V_\sharp \times W_h$, i.e.,*

$$\|a_\sharp(v,\cdot)\|_{W_h'} \le \omega^{\mathrm{mol}}_{V_\sharp,W_h}\|v\|_{V_\sharp} \quad \text{for all } v \in V_\sharp, \tag{4.12}$$

and such that the following two identities hold true:

$$a_\sharp(v_h,w_h) = \tilde{a}_h(v_h,w_h) \quad \text{for all } (v_h,w_h) \in V_h \times W_h, \tag{4.13a}$$

$$a_\sharp(u,w_h) = \ell_h(w_h) \quad \text{for all } w_h \in W_h. \tag{4.13b}$$

Assume moreover that there exists a real number σ_{V_h,W_h} so that

$$\|s_h(v_h,\cdot)\|_{W_h'} \le \sigma_{V_h,W_h}\|v - v_h\|_{V_\sharp} \quad \text{for all } v_h \in V_h \text{ and all } v \in V. \tag{4.14}$$

Then the following error estimate holds true:

$$\|u - u_h\|_{V_\flat} \le c_\flat\Big(1 + \frac{\omega^{\mathrm{mol}}_{V_\sharp,W_h} + \sigma_{V_h,W_h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u - v_h\|_{V_\sharp}.$$

Moreover, if the discrete norm equivalence (4.7) *holds true, we have the quasi-optimal error estimate*

$$\|u - u_h\|_{V_\sharp} \le \Big(1 + c_\sharp\frac{\omega^{\mathrm{mol}}_{V_\sharp,W_h} + \sigma_{V_h,W_h}}{\alpha_h}\Big)\inf_{v_h\in V_h}\|u - v_h\|_{V_\sharp}.$$

Proof. We observe that, for all $v_h \in V_h$ and all $w_h \in W_h$, we have

$$\langle\delta_h(v_h), w_h\rangle_{W_h',W_h} = a_\sharp(u - v_h, w_h) - s_h(v_h, w_h),$$

where we used (4.13). Invoking now (4.12) and (4.14), we infer that the consistency/boundedness property (4.5) holds true with $\omega_{\sharp h} = \omega^{\mathrm{mol}}_{V_\sharp,W_h} + \sigma_{V_h,W_h}$. The assertions then follow from the key error estimates of Lemma 4.4. □

Remark 4.9 (Terminology). We call the estimates from Lemma 4.8 mollified error estimates since the proof of (4.13b) hinges on the use of suitable mollification operators; we refer the reader to the examples presented in Section 6.2 and in Section 7.2.

5 Analysis Tools

We introduce in this section some analysis tools that are useful to realize the above program. These tools include commuting mollification operators in Section 5.1, inverse inequalities on faces in Section 5.2, and the localization of weak traces to faces in Section 5.3. The results of Section 5.2 are useful in the context of the trimmed error estimates, and the results of Section 5.1 and of Section 5.3 are useful in the context of the mollified error estimates. The results from Section 5.2 and Section 5.3 invoke the shape-regularity of the mesh sequence. They can be extended to polyhedral mesh sequences admitting a simplicial submesh that belongs to a shape-regular sequence in the usual sense and such that all the polyhedral cells and polygonal faces are covered by a finite number of tetrahedra and triangles with uniformly the same size.

5.1 Mollification Operators

Smoothing by mollification (i.e., by convolution with a smooth kernel) is an important tool for the analysis and approximation of PDEs that has been introduced by Leray [21, p. 206], Sobolev [26, p. 487], and Friedrichs [17, pp. 136–139]. The goal of this subsection is to define mollification operators that commute with the usual differential operators, and that converge optimally when the function to be smoothed is defined over a Lipschitz domain D in $\mathbb{R}^d$. We use the shrinking technique of D in [15] to avoid the need to extend the function to be smoothed outside D.

The starting point is to observe that [19, Proposition 2.3] implies the existence of a vector field $\boldsymbol{j} \in \boldsymbol{C}^\infty(\mathbb{R}^d)$ that is globally transversal on ∂D (i.e., there is a real number $\gamma > 0$ such that $\boldsymbol{n}(\boldsymbol{x})\cdot\boldsymbol{j}(\boldsymbol{x}) \geq \gamma$ at æpoint $\boldsymbol{x}$ on ∂D, where $\boldsymbol{n}$ is the unit normal vector pointing outward D) and $\|\boldsymbol{j}(\boldsymbol{x})\|_{\ell^2} = 1$ for all $\boldsymbol{x} \in \partial D$. Then one can show the following:

(i) The map
$$\boldsymbol{\varphi}_\delta : \mathbb{R}^d \ni \boldsymbol{x} \mapsto \boldsymbol{x} - \delta \boldsymbol{j}(\boldsymbol{x}) \in \mathbb{R}^d$$
is in $C^\infty(\mathbb{R}^d)$ for all $\delta \in [0,1]$.

(ii) For all $k \in \mathbb{N}$, there is c such that $\max_{\boldsymbol{x}\in D} \|D^k \boldsymbol{\varphi}_\delta(\boldsymbol{x}) - D^k \boldsymbol{x}\|_{\ell^2} \leq c \ell_D^{-k}\delta$ for all $\delta \in [0,1]$.

(iii) There is $r > 0$ so that
$$\boldsymbol{\varphi}_\delta(D) + B(\boldsymbol{0}, \delta r) \subset D \quad \text{for all } \delta \in [0,1]. \tag{5.1}$$

Let us consider the following kernel:
$$\rho(\boldsymbol{y}) = \begin{cases} \eta \exp\big(-\frac{1}{1-\|\boldsymbol{y}\|_{\ell^2}^2}\big), & \text{if } \|\boldsymbol{y}\|_{\ell^2} < 1, \\ 0, & \text{if } \|\boldsymbol{y}\|_{\ell^2} \geq 1, \end{cases}$$
where η is chosen so that $\int_{\mathbb{R}^d} \rho(\boldsymbol{y})\,\mathrm{d}y = \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y})\,\mathrm{d}y = 1$. Let $\delta \in [0,1]$ and let $f \in L^1(D;\mathbb{R}^q)$ with $q = 1$ if we consider scalar-valued functions and $q = d$ if we consider vector-valued functions.

We define a mollification operator as follows:
$$\mathcal{K}_\delta(f)(\boldsymbol{x}) := \int_{B(\boldsymbol{0},1)} \rho(\boldsymbol{y})\mathbb{K}_\delta(\boldsymbol{x}) f(\boldsymbol{\varphi}_\delta(\boldsymbol{x}) + (\delta r)\boldsymbol{y})\,\mathrm{d}y \quad \text{for all } \boldsymbol{x} \in D, \tag{5.2}$$
where $\mathbb{K}_\delta : D \to \mathbb{R}^{q\times q}$ is a smooth field. Note that the definition (5.2) makes sense owing to (5.1). The examples we have in mind for the field $\mathbb{K}_\delta$ (inspired by Schöberl [24, 25]) are $\mathbb{K}_\delta^{\mathrm{g}}(\boldsymbol{x}) = 1$ ($q = 1$), $\mathbb{K}_\delta^{\mathrm{c}}(\boldsymbol{x}) = \mathbb{J}_\delta^{\mathsf{T}}(\boldsymbol{x})$ ($q = d = 3$), $\mathbb{K}_\delta^{\mathrm{d}}(\boldsymbol{x}) = \det(\mathbb{J}_\delta(\boldsymbol{x}))\mathbb{J}_\delta^{-1}(\boldsymbol{x})$ ($q = d$), and $\mathbb{K}_\delta^{\mathrm{b}}(\boldsymbol{x}) = \det(\mathbb{J}_\delta(\boldsymbol{x}))$ ($q = 1$), where $\mathbb{J}_\delta$ is the Jacobian matrix of $\boldsymbol{\varphi}_\delta$ at $\boldsymbol{x} \in D$. The mollification operator built using the field $\mathbb{K}_\delta^{\mathrm{x}}$ is denoted $\mathcal{K}_\delta^{\mathrm{x}}$ with $\mathrm{x} \in \{\mathrm{g}, \mathrm{c}, \mathrm{d}, \mathrm{b}\}$. In what follows, we just state the main properties of the mollification operator $\mathcal{K}_\delta$ (where we omit the superscript if the context is unambiguous), and we refer the reader to [15] for proofs.

Lemma 5.1 (Smoothness). *For all $f \in L^1(D;\mathbb{R}^q)$ and all $\delta \in (0,1]$, $\mathcal{K}_\delta(f) \in C^\infty(\overline{D};\mathbb{R}^q)$, i.e., $\mathcal{K}_\delta(f) \in C^\infty(D;\mathbb{R}^q)$ and $\mathcal{K}_\delta(f)$ as well as all its derivatives admit a continuous extension to $\overline{D}$.*

Let $p \in [1,\infty]$. Let us set $Z^{\mathrm{g},p}(D) = W^{1,p}(D) = \{f \in L^p(D) \mid \nabla f \in \boldsymbol{L}^p(D)\}$, $\boldsymbol{Z}^{\mathrm{c},p}(D) = \{\boldsymbol{g} \in \boldsymbol{L}^p(D) \mid \nabla\times\boldsymbol{g} \in \boldsymbol{L}^p(D)\}$ (for $d = 3$), and $\boldsymbol{Z}^{\mathrm{d},p}(D) = \{\boldsymbol{g} \in \boldsymbol{L}^p(D) \mid \nabla\cdot\boldsymbol{g} \in L^p(D)\}$.

Lemma 5.2 (Commuting). *The following holds true:*
(i) $\nabla\mathcal{K}_\delta^{\mathrm{g}}(f) = \mathcal{K}_\delta^{\mathrm{c}}(\nabla f)$ *for all* $f \in Z^{\mathrm{g},p}(D)$,
(ii) $\nabla\times\mathcal{K}_\delta^{\mathrm{c}}(\boldsymbol{g}) = \mathcal{K}_\delta^{\mathrm{d}}(\nabla\times\boldsymbol{g})$ *for all* $\boldsymbol{g} \in \boldsymbol{Z}^{\mathrm{c},p}(D)$ *(for $d = 3$)*,
(iii) $\nabla\cdot\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{g}) = \mathcal{K}_\delta^{\mathrm{b}}(\nabla\cdot\boldsymbol{g})$ *for all* $\boldsymbol{g} \in \boldsymbol{Z}^{\mathrm{d},p}(D)$,
that is to say, the following diagrams commute:

$$\begin{array}{ccccccc}
Z^{\mathrm{g},p}(D) & \xrightarrow{\nabla} & \boldsymbol{Z}^{\mathrm{c},p}(D) & \xrightarrow{\nabla\times} & \boldsymbol{Z}^{\mathrm{d},p}(D) & \xrightarrow{\nabla\cdot} & L^p(D) \\
\downarrow{\scriptstyle \mathcal{K}_\delta^{\mathrm{g}}} & & \downarrow{\scriptstyle \mathcal{K}_\delta^{\mathrm{c}}} & & \downarrow{\scriptstyle \mathcal{K}_\delta^{\mathrm{d}}} & & \downarrow{\scriptstyle \mathcal{K}_\delta^{\mathrm{b}}} \\
C^\infty(\overline{D}) & \xrightarrow{\nabla} & \boldsymbol{C}^\infty(\overline{D}) & \xrightarrow{\nabla\times} & \boldsymbol{C}^\infty(\overline{D}) & \xrightarrow{\nabla\cdot} & C^\infty(\overline{D}).
\end{array}$$

Theorem 5.3 (Convergence). *The following statements hold true:*

(i) *There are $c, \delta_0 > 0$, uniform, such that $\|\mathcal{K}_\delta(f)\|_{L^p(D;\mathbb{R}^q)} \le c\|f\|_{L^p(D;\mathbb{R}^q)}$ for all $f \in L^p(D;\mathbb{R}^q)$, all $\delta \in [0,\delta_0]$, and all $p \in [1,\infty]$. Moreover,*

$$\lim_{\delta\to 0} \|\mathcal{K}_\delta(f) - f\|_{L^p(D;\mathbb{R}^q)} = 0 \quad \textit{for all } f \in L^p(D;\mathbb{R}^q) \textit{ and all } p \in [1,\infty). \tag{5.3}$$

(ii) *There is c, uniform, such that for all $f \in W^{s,p}(D;\mathbb{R}^q)$, all $\delta \in [0,\delta_0]$, all $s \in (0,1]$, and all $p \in [1,\infty)$ ($p \in [1,\infty]$ if $s = 1$),*

$$\|\mathcal{K}_\delta(f) - f\|_{L^p(D;\mathbb{R}^q)} \le c\,\ell_D^{-s}\delta^s\|f\|_{W^{s,p}(D;\mathbb{R}^q)}.$$

Corollary 5.4 (Convergence of Derivatives). The following statements hold true:

(i) $\lim_{\delta\to 0}\|\nabla(\mathcal{K}_\delta^{\mathrm{g}}(f) - f)\|_{\boldsymbol{L}^p(D)} = 0$ for all $f \in Z^{\mathrm{g},p}(D)$, and if $\nabla f \in \boldsymbol{W}^{s,p}(D)$,

$$\|\nabla(\mathcal{K}_\delta^{\mathrm{g}}(f) - f)\|_{\boldsymbol{L}^p(D)} \le c\ell_D^{-s}\delta^s\|\nabla f\|_{\boldsymbol{W}^{s,p}(D)},$$

(ii) $\lim_{\delta\to 0}\|\nabla\times(\mathcal{K}_\delta^{\mathrm{c}}(\boldsymbol{g}) - \boldsymbol{g})\|_{\boldsymbol{L}^p(D)} = 0$ for all $\boldsymbol{g} \in \boldsymbol{Z}^{\mathrm{c},p}(D)$, and if $\nabla\times\boldsymbol{g} \in \boldsymbol{W}^{s,p}(D)$,

$$\|\nabla\times(\mathcal{K}_\delta^{\mathrm{c}}(\boldsymbol{g}) - \boldsymbol{g})\|_{\boldsymbol{L}^p(D)} \le c\ell_D^{-s}\delta^s\|\nabla\times\boldsymbol{g}\|_{\boldsymbol{W}^{s,p}(D)},$$

(iii) $\lim_{\delta\to 0}\|\nabla\cdot(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{g}) - \boldsymbol{g})\|_{L^p(D)} = 0$ for all $\boldsymbol{g} \in \boldsymbol{Z}^{\mathrm{d},p}(D)$, and if $\nabla\cdot\boldsymbol{g} \in W^{s,p}(D)$,

$$\|\nabla\cdot(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{g}) - \boldsymbol{g})\|_{L^p(D)} \le c\ell_D^{-s}\delta^s\|\nabla\cdot\boldsymbol{g}\|_{W^{s,p}(D)}.$$

In the above statements, convergence holds true for all $p \in [1,\infty)$, and convergence rates hold true with c uniform for all $\delta \in [0,\delta_0]$, all $s \in (0,1]$, and all $p \in [1,\infty)$ ($p \in [1,\infty]$ if $s = 1$).

Remark 5.5 (Convergence in D). Corollary 5.4 (i) strengthens the original result by Friedrichs where strong convergence of the gradient only occurs in compact subsets of D (see, e.g., [9, Theorem 9.2]). Note though that Corollary 5.4 (i) is valid for Lipschitz domains, whereas the original result by Friedrichs is valid for any open set.

Remark 5.6 (Density). Lemma 5.1, together with Lemma 5.2 and (5.3), implies that $C^\infty(\overline{D};\mathbb{R}^q)$ is dense in $Z^{\mathrm{x},p}(D)$ for all $\mathrm{x} \in \{\mathrm{g},\mathrm{c},\mathrm{d}\}$.

5.2 Inverse Inequalities on Faces

Let $F \in \mathcal{F}_h^\circ$ be an interface and let P_F be the finite element trace space defined in Section 3.1. In the following statement, we use a local length scale $\tilde{h}_F$ which is uniformly equivalent to the diameter h_F of F owing to the shape-regularity of the mesh sequence; the reason for this distinction is to provide a somewhat more precise geometric characterization of the relevant local length scale.

Lemma 5.7 (Verfürth's Inverse Inequality). *Let $F \in \mathcal{F}_h^\circ$ and let P_F be the finite element trace space. Furthermore, let $\Phi_F : L^2(F) \to H^{-1}(D_F)$ be the map such that $\Phi_F(r)(\varphi) := \int_F r\varphi\,\mathrm{d}s$ for all $\varphi \in H_0^1(D_F)$ and all $r \in L^2(F)$, where D_F is the interior of the set of the points of the two cells sharing F. Let $\tilde{h}_F := \frac{|D_F|}{|F|}$ and let $p \in (1,\infty)$. Then there exists a constant c, uniform with respect to h but depending on the shape-regularity of the mesh sequence and on the reference finite element, such that*

$$\|g\|_{L^2(F)} \le c\tilde{h}_F^{-\frac{1}{2}+d(\frac{1}{2}-\frac{1}{p})}\|\Phi_F(g)\|_{W_0^{1,p'}(D_F)'} \quad \textit{for all } g \in P_F,$$

where the dual space $W_0^{1,p'}(D_F)'$ is equipped with the norm

$$\|\Phi_F(g)\|_{W_0^{1,p'}(D_F)'} = \sup_{\varphi\in W_0^{1,p'}(D_F);\|\nabla\varphi\|_{\boldsymbol{L}^{p'}(D_F)}=1} |\Phi_F(g)(\varphi)|$$

and p' is the conjugate number of p (i.e., $\frac{1}{p} + \frac{1}{p'} = 1$).

Proof. The proof hinges on the use of suitable bubble functions introduced by Verfürth and on inverse inequalities proved by mapping to the reference cell $\widehat{K}$; see [29, Section 3.6] for the proof with $p = 2$. The adaptation for $p \neq 2$ is straightforward and is omitted for brevity. □

5.3 Localization of Weak Traces to Faces

The goal of this subsection is to give a weak meaning to the (normal or tangential) trace of some field satisfying some minimal regularity requirements on a given mesh cell $K \in \mathcal{T}_h$. The key point is that the trace is given a meaning on each face of K independently, and not just on the whole boundary of K. Let $K \in \mathcal{T}_h$ be a mesh cell and let $F \in \mathcal{F}_K$ be a face of K.

Let p and q be two real numbers such that

$$p > 2, \quad q > \frac{2d}{2+d}. \tag{5.4}$$

Note that $q > 1$ since $d \geq 2$. Let p' be the conjugate number of p, i.e., $\frac{1}{p} + \frac{1}{p'} = 1$ so that $p' \in (1, 2)$. Since $x \to \frac{xd}{x+d}$ is an increasing function, there is $\tilde{p} \in (2, p]$ such that $q \geq \frac{\tilde{p}d}{\tilde{p}+d}$; notice that these two conditions are equivalent to $2 < \tilde{p} \leq p$ and $\frac{1}{q'} \geq \frac{1}{\tilde{p}'} - \frac{1}{d}$, where q' is the conjugate number of q and $\tilde{p}'$ that of $\tilde{p}$.

Let us start by considering the normal component of fields defined in K. With the above real numbers p, q, and $\tilde{p}$ in hand, we consider the functional spaces

$$\begin{aligned} \boldsymbol{V}^{\mathrm{d}}(K) &:= \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \mid \nabla\cdot\boldsymbol{v} \in L^q(K)\}, \\ Y^{\mathrm{d}}(F) &:= W^{\frac{1}{\tilde{p}}, \tilde{p}'}(F), \end{aligned}$$

where the superscript refers to the fact that the normal trace is related to the divergence operator.

Lemma 5.8 (Lifting Operator). *There exists a constant c, uniform with respect to h (but depending on the shape-regularity of the mesh sequence and on the reference finite element) and a lifting operator $E_F^K : Y^{\mathrm{d}}(F) \to W^{1,\tilde{p}'}(K)$ such that the following holds true for any $\phi \in Y^{\mathrm{d}}(F)$:*

$$E_F^K(\phi)_{|\partial K \setminus F} = 0, \quad E_F^K(\phi)_{|F} = \phi$$

and

$$|E_F^K(\phi)|_{W^{1,p'}(K)} + h_K^{-1+d(\frac{1}{q}-\frac{1}{p})} \|E_F^K(\phi)\|_{L^{q'}(K)} \leq c h_K^{-\frac{1}{\tilde{p}}+d(\frac{1}{\tilde{p}}-\frac{1}{p})} \|\phi\|_{Y^{\mathrm{d}}(F)} \tag{5.5}$$

with the norm $\|\phi\|_{Y^{\mathrm{d}}(F)} = \|\phi\|_{L^{\tilde{p}'}(F)} + h_F^{\frac{1}{\tilde{p}}} |\phi|_{W^{\frac{1}{\tilde{p}},\tilde{p}'}(F)}$.

Proof. Following the ideas in, e.g., [1, Lemma 4.7] (see also [5, Corollary 3.3] for similar lifting operators in a Hilbert setting), the lifting operator E_F^K is constructed from a reference lifting operator $E_{\widehat{F}}^{\widehat{K}}$ where $\widehat{K}$ is the reference cell, $\widehat{F} = (\boldsymbol{T}_K)^{-1}(F)$, and $\boldsymbol{T}_K : \widehat{K} \to K$ the geometric map. The reference lifting operator is constructed by composing the zero-extension from $\widehat{F}$ to $\partial\widehat{K}$ with a bounded right-inverse of the trace map from $W^{1,\tilde{p}'}(\widehat{K})$ to $W^{\frac{1}{\tilde{p}},\tilde{p}'}(\partial\widehat{K})$. This construction is possible since the function equal to 1 on $\widehat{F}$ and 0 on $\partial\widehat{K}\setminus\widehat{F}$ is in $W^{\frac{1}{\tilde{p}},\tilde{p}'}(\partial\widehat{K})$ because $\frac{\tilde{p}'}{\tilde{p}} = \frac{1}{\tilde{p}-1} < 1$. The stability bound (5.5) follows from the transformation of Sobolev norms by pullbacks associated with the geometric maps on shape-regular mesh sequences and the fact that on the reference cell $\widehat{K}$, we have $|\widehat{\psi}|_{W^{1,p'}(\widehat{K})} + \|\widehat{\psi}\|_{L^{q'}(\widehat{K})} \leq \widehat{c}\|\widehat{\psi}\|_{W^{1,\tilde{p}'}(\widehat{K})}$ since $\tilde{p}' \geq p'$ and $\frac{1}{q'} \geq \frac{1}{\tilde{p}'} - \frac{1}{d}$. □

With the lifting operator E_F^K in hand, we can define the normal component of any field $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$ on the face F of K to be the linear form $(\boldsymbol{v}\cdot\boldsymbol{n}_K)_{|F} \in Y^{\mathrm{d}}(F)'$ such that

$$\langle(\boldsymbol{v}\cdot\boldsymbol{n}_K)_{|F}, \phi\rangle := \int_K \big(\boldsymbol{v}\cdot\nabla E_F^K(\phi) + (\nabla\cdot\boldsymbol{v}) E_F^K(\phi)\big)\,\mathrm{d}x \tag{5.6}$$

for all $\phi \in Y^{\mathrm{d}}(F)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $Y^{\mathrm{d}}(F)'$ and $Y^{\mathrm{d}}(F)$. Note that the right-hand side of (5.6) is well-defined owing to Hölder's inequality and (5.5).

Lemma 5.9 (Bound on Normal Component). *There exists a constant c, uniform with respect to h (but depending on the shape-regularity of the mesh sequence and on the reference finite element), so that the following estimate holds true for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{d}}(K)$:*

$$\|(\boldsymbol{v}\cdot\boldsymbol{n}_K)_{|F}\|_{Y^{\mathrm{d}}(F)'} \le c h_K^{-\frac{1}{\tilde{p}}+d(\frac{1}{\tilde{p}}-\frac{1}{p})}\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^{1+d(\frac{1}{p}-\frac{1}{q})}\|\nabla\cdot\boldsymbol{v}\|_{L^q(K)}\Big). \tag{5.7}$$

Moreover, we have

$$|\langle(\boldsymbol{v}\cdot\boldsymbol{n}_K)_{|F}, \phi_h\rangle| \le c\Big(h_{K_F}^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K_F)} + h_{K_F}^{1+d(\frac{1}{2}-\frac{1}{q})}\|\nabla\cdot\boldsymbol{v}\|_{L^q(K_F)}\Big) \times h_F^{-\frac{1}{2}}\|\phi_h\|_{L^2(F)} \tag{5.8}$$

for all $\phi_h \in P_F$ and all $F \in \mathcal{F}_h^{\partial}$, where K_F is the unique mesh cell having F as a face.

Proof. The bound (5.7) is a direct consequence of (5.6), Hölder's inequality, and Lemma 5.8. Moreover, the bound (5.8) follows from (5.7), the following inverse inequality on P_F:

$$\|\phi_h\|_{Y^{\mathrm{d}}(F)} \le c h_F^{(d-1)(\frac{1}{2}-\frac{1}{\tilde{p}})}\|\phi_h\|_{L^2(F)},$$

(note that $\frac{1}{2}-\frac{1}{\tilde{p}} = \frac{1}{\tilde{p}'}-\frac{1}{2}$) and the shape-regularity of the mesh sequence. □

Similar arguments can be deployed to define the tangential trace of vectors fields on a face of K. More specifically, let the real numbers p, q, and $\tilde{p}$ be as above, and consider the functional spaces

$$\boldsymbol{V}^{\mathrm{c}}(K) := \{\boldsymbol{v} \in \boldsymbol{L}^p(K) \mid \nabla\times\boldsymbol{v} \in \boldsymbol{L}^q(K)\}, \tag{5.9a}$$

$$\boldsymbol{Y}^{\mathrm{c}}(F) := \{\boldsymbol{\phi} \in \boldsymbol{W}^{\frac{1}{\tilde{p}},\tilde{p}'}(F) \mid \boldsymbol{\phi}\cdot\boldsymbol{n}_F = 0\}, \tag{5.9b}$$

where the superscript refers to the fact that the tangential trace is related to the curl operator.

Lemma 5.10 (Lifting Operator). *There exist a constant c, uniform with respect to h (but depending on the shape-regularity of the mesh sequence and on the reference finite element) and a lifting operator $E_F^K : \boldsymbol{Y}^{\mathrm{c}}(F) \to \boldsymbol{W}^{1,\tilde{p}'}(K)$ such that the following holds true for any $\boldsymbol{\phi} \in \boldsymbol{Y}^{\mathrm{c}}(F)$:*

$$E_F^K(\boldsymbol{\phi})_{|\partial K\setminus F} = \boldsymbol{0}, \quad E_F^K(\boldsymbol{\phi})_{|F} = \boldsymbol{\phi},$$

and

$$|E_F^K(\boldsymbol{\phi})|_{\boldsymbol{W}^{1,p'}(K)} + h_K^{-1+d(\frac{1}{q}-\frac{1}{p})}\|E_F^K(\boldsymbol{\phi})\|_{\boldsymbol{L}^{q'}(K)} \le c h_K^{-\frac{1}{\tilde{p}}+d(\frac{1}{\tilde{p}}-\frac{1}{p})}\|\boldsymbol{\phi}\|_{\boldsymbol{Y}^{\mathrm{c}}(F)}, \tag{5.10}$$

with the norm $\|\boldsymbol{\phi}\|_{\boldsymbol{Y}^{\mathrm{c}}(F)} = \|\boldsymbol{\phi}\|_{\boldsymbol{L}^{\tilde{p}'}(F)} + h_F^{\frac{1}{\tilde{p}}}|\boldsymbol{\phi}|_{\boldsymbol{W}^{\frac{1}{\tilde{p}},\tilde{p}'}(F)}$.

With this lifting operator in hand, we can define the tangential component of any field $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$ on the face F of K to be the antilinear form $(\boldsymbol{v}\times\boldsymbol{n}_K)_{|F} \in \boldsymbol{Y}^{\mathrm{c}}(F)'$ such that

$$\langle(\boldsymbol{v}\times\boldsymbol{n}_K)_{|F}, \boldsymbol{\phi}\rangle := \int_K \big(\boldsymbol{v}\cdot\nabla\times E_F^K(\overline{\boldsymbol{\phi}}) - (\nabla\times\boldsymbol{v})\cdot E_F^K(\overline{\boldsymbol{\phi}})\big)\,\mathrm{d}x \tag{5.11}$$

for all $\boldsymbol{\phi} \in \boldsymbol{Y}^{\mathrm{c}}(F)$, where $\langle\cdot,\cdot\rangle$ denotes the duality pairing between $\boldsymbol{Y}^{\mathrm{c}}(F)'$ and $\boldsymbol{Y}^{\mathrm{c}}(F)$. Note that the right-hand side of (5.11) is well-defined owing to Hölder's inequality and (5.10).

Lemma 5.11 (Bound on Tangential Component). *There exists a constant c, uniform with respect to h (but depending on the shape-regularity of the mesh sequence and on the reference finite element), so that the following estimate holds true for all $\boldsymbol{v} \in \boldsymbol{V}^{\mathrm{c}}(K)$:*

$$\|(\boldsymbol{v}\times\boldsymbol{n}_K)_{|F}\|_{Y^{\mathrm{c}}(F)'} \le c h_K^{-\frac{1}{\tilde{p}}+d(\frac{1}{\tilde{p}}-\frac{1}{p})}\Big(\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K)} + h_K^{1+d(\frac{1}{p}-\frac{1}{q})}\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K)}\Big).$$

Moreover, we have

$$|\langle(\boldsymbol{v}\times\boldsymbol{n}_K)_{|F}, \boldsymbol{\phi}_h\rangle| \le c\Big(h_{K_F}^{d(\frac{1}{2}-\frac{1}{p})}\|\boldsymbol{v}\|_{\boldsymbol{L}^p(K_F)} + h_{K_F}^{1+d(\frac{1}{2}-\frac{1}{q})}\|\nabla\times\boldsymbol{v}\|_{\boldsymbol{L}^q(K_F)}\Big) \times h_F^{-\frac{1}{2}}\|\boldsymbol{\phi}_h\|_{L^2(F)}$$

for all $\boldsymbol{\phi}_h \in \boldsymbol{P}_F$ such that $\boldsymbol{\phi}\cdot\boldsymbol{n}_F = 0$ and all $F \in \mathcal{F}_h^{\partial}$, where K_F is the unique mesh cell having F as a face.

6 Application to the Diffusion Equation

In this section, we show how the trimmed error estimate from Lemma 4.6 and the mollified error estimate from Lemma 4.8 can be applied to the approximation of the diffusion equation using the boundary penalty method described in Section 3.2. The discrete spaces are $W_h = V_h = P^{\mathrm{g}}(\mathcal{T}_h)$, and the space $V_\flat := H_0^1(D) + V_h$ is equipped with the norm $\|\cdot\|_{V_\flat}$ that extends to $V_\flat$ the norm $\|\cdot\|_{V_h}$ originally defined by (3.7) on V_h. The discrete forms $a_h(\cdot,\cdot)$ and $\ell_h(\cdot)$ are defined by (3.4). The constants in the error estimates derived in this section depend on the shape-regularity of the mesh sequence and on the reference finite element.

6.1 Trimmed Error Estimate

We define the trimming operator $T : P^{\mathrm{g}}(\mathcal{T}_h) \to P_0^{\mathrm{g}}(\mathcal{T}_h)$ as follows. For all $w_h \in P^{\mathrm{g}}(\mathcal{T}_h)$, $T(w_h)_{|K}$ is defined, for all $K \in \mathcal{T}_h$, by zeroing out all the degrees of freedom of w_h that are attached to vertices, edges, and faces located at the boundary ∂D. This type of construction has been analyzed recently in [16] in the more general context of quasi-interpolation operators in canonical finite element spaces with prescribed boundary conditions. Let $\mathcal{T}_h^\partial$ be the collection of the mesh cells touching the boundary; note that $w_h - T(w_h)$ vanishes on all the mesh cells in $\mathcal{T}_h \setminus \mathcal{T}_h^\partial$ but does not on the mesh cells in $\mathcal{T}_h^\partial$. For all $K \in \mathcal{T}_h^\partial$, one can prove that the following bounds hold true for all $w_h \in P^{\mathrm{g}}(\mathcal{T}_h)$ with c uniform with respect to h: If $\partial K \cap \partial D$ is composed of one or more boundary faces, then

$$h_K\|\nabla(w_h - T(w_h))\|_{\boldsymbol{L}^2(K)} + \|w_h - T(w_h)\|_{L^2(K)} \le c h_K^{\frac{1}{2}} \|w_h\|_{L^2(\partial K\cap\partial D)}, \tag{6.1}$$

whereas if $\partial K \cap \partial D$ is a manifold of dimension $d' < d-1$, then

$$h_K\|\nabla(w_h - T(w_h))\|_{\boldsymbol{L}^2(K)} + \|w_h - T(w_h)\|_{L^2(K)} \le c h_K^{\frac{1}{2}} \|w_h\|_{L^2(F)} \quad \text{for all } F \in \mathcal{F}_K^\partial, \tag{6.2}$$

where $\mathcal{F}_K^\partial := \{F \in \mathcal{F}_h^\partial \mid \partial K \cap \partial D \subseteq F\}$ is the collection of the boundary faces containing the manifold $\partial K \cap \partial D$. We introduce the contrast factor

$$\xi_\kappa := \max_{K\in\mathcal{T}_h^\partial} \frac{\kappa_K}{\max_{F\in\mathcal{F}_K^\partial} \kappa_{K_F}}, \tag{6.3}$$

where we recall that, for all $F \in \mathcal{F}_K^\partial \subset \mathcal{F}_h^\partial$, K_F is the unique mesh cell having F as a boundary face. Finally, let us set

$$V_{\mathrm{s}} := \{v \in H_0^1(D) \mid \nabla\cdot(\kappa\nabla v) \in L^q(D)\},$$

with $q \in (2_*, 2]$, $2_* = \frac{2d}{2+d}$, and let us equip the space $V_\sharp := V_{\mathrm{s}} + V_h$ with the norm

$$\|v\|_{V_\sharp} := \Big(\|v\|_{V_\flat}^2 + \sum_{K\in\mathcal{T}_h^\partial} \kappa_K^{-1} h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla\cdot(\kappa\nabla v)\|_{L^q(K)}^2\Big)^{\frac{1}{2}}. \tag{6.4}$$

For simplicity, we assume that the trace space P_F contains the traces of the normal derivatives of functions in P_K (this is obviously the case if P_K is the polynomial space $\mathbb{P}_{k,d}$).

Lemma 6.1 (Trimmed Error Estimate). *The assumptions of Lemma* 4.6 *hold true with the trimming operator* $T : P^{\mathrm{g}}(\mathcal{T}_h) \to P_0^{\mathrm{g}}(\mathcal{T}_h)$ *defined above, where the constants* $\omega_{V_\sharp,V_h}^{\mathrm{tri}}$ *and* $\varpi_{V_\sharp,V_h}^{\mathrm{tri}}$ *are proportional to* $\xi_\kappa^{1/2}$ *with the contrast factor* ξ_κ *defined by* (6.3).

Proof. (1) Let us verify that (4.9) holds true. Let $(v_h, w_h) \in V_h \times V_h$. Since $T(w_h) \in H_0^1(D)$, we infer that

$$a(u, T(w_h)) - a_h(v_h, T(w_h)) = \int_D \kappa\nabla(u - v_h)\cdot\nabla T(w_h)\,\mathrm{d}x \le \|u - v_h\|_{V_\flat} \|\kappa^{\frac{1}{2}}\nabla T(w_h)\|_{\boldsymbol{L}^2(D)}.$$

Since $\|u - v_h\|_{V_\flat} \le \|u - v_h\|_{V_\sharp}$, we just have to prove that $\|\kappa^{\frac{1}{2}}\nabla T(w_h)\|_{\boldsymbol{L}^2(D)} \le c\|w_h\|_{V_h}$. We have $T(w_h) = w_h$ on all $K \in \mathcal{T}_h \setminus \mathcal{T}_h^\partial$ so that we only need to bound $\|\kappa^{\frac{1}{2}}\nabla T(w_h)\|_{\boldsymbol{L}^2(K)}$ for all $K \in \mathcal{T}_h^\partial$. In this case, the triangle inequality implies that

$$\|\kappa^{\frac{1}{2}}\nabla T(w_h)\|_{\boldsymbol{L}^2(K)} \le \|\kappa^{\frac{1}{2}}\nabla w_h\|_{\boldsymbol{L}^2(K)} + \kappa_K^{\frac{1}{2}}\|\nabla\hat{w}_h\|_{\boldsymbol{L}^2(K)},$$

where we have set $\hat{w}_h := w_h - T(w_h)$. If $\partial K \cap \partial D$ is composed of one or more boundary faces, we use the approximation property (6.1) together with the shape-regularity of the mesh sequence to infer that

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{\boldsymbol{L}^2(K)} \le c\kappa_K^{\frac{1}{2}} h_K^{-\frac{1}{2}} \|w_h\|_{L^2(\partial K\cap\partial D)} \le c' \Big(\sum_{\substack{F\in\mathcal{F}_h^{\partial} \\ F\subseteq \partial K\cap\partial D}} \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(F)}^2 \Big)^{\frac{1}{2}}.$$

Instead, if $\partial K \cap \partial D$ is a manifold of dimension $d' < d - 1$, we use the approximation property (6.2) together with the shape-regularity of the mesh sequence to infer that, for all $F \in \mathcal{F}_K^{\partial}$,

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{\boldsymbol{L}^2(K)} \le c\Big(\frac{\kappa_K}{\kappa_{K_F}}\Big)^{\frac{1}{2}} \kappa_{K_F}^{\frac{1}{2}} h_F^{-\frac{1}{2}} \|w_h\|_{L^2(F)} \le c'\Big(\frac{\kappa_K}{\kappa_{K_F}}\Big)^{\frac{1}{2}} \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(F)}.$$

It is at this point that the contrast factor ξ_κ comes into play. The reason is that K is not connected to ∂D by any of its faces, and (6.2) gives an estimate of $\|\nabla\hat{w}_h\|_{\boldsymbol{L}^2(K)}$ that involves a boundary face F that cannot be a face of K. There is necessarily a mismatch between κ_K and the coefficient κ_{K_F} involved in (3.6). We now take a boundary face in $\mathcal{F}_K^{\partial}$, say F_*, such that κ_{K_F} is maximal so as to make the above upper bound as small as possible. We obtain

$$\kappa_K^{\frac{1}{2}} \|\nabla \hat{w}_h\|_{\boldsymbol{L}^2(K)} \le c'\Big(\frac{\kappa_K}{\max_{F\in\mathcal{F}_K^{\partial}} \kappa_{K_F}}\Big)^{\frac{1}{2}} \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(F_*)} \le c'\Big(\max_{K\in\mathcal{T}_h^{\partial}} \frac{\kappa_K}{\max_{F\in\mathcal{F}_K^{\partial}} \kappa_{K_F}}\Big)^{\frac{1}{2}} \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(F_*)}.$$

Recalling the definition (6.3) of the contrast factor ξ_κ, we infer that $\kappa_K^{\frac{1}{2}}\|\nabla\hat{w}_h\|_{\boldsymbol{L}^2(K)} \le c'\xi_\kappa^{\frac{1}{2}} \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(F_*)}$ with $F_* \in \mathcal{F}_K^{\partial} \subset \mathcal{F}_h^{\partial}$. It is now straightforward to complete the proof of (4.9).

(2) Let us verify (4.10). Let $(v_h, w_h) \in V_h \times V_h$ and let us set $e_h := u - v_h$ and (as above) $\hat{w}_h := w_h - T(w_h)$. A direct calculation shows that

$$\begin{aligned}\ell_h(w_h) - \ell(T(w_h)) - a_h(v_h, (I-T)(w_h)) &= \int_D f\hat{w}_h\,\mathrm{d}x - \int_D \kappa\nabla v_h\cdot\nabla\hat{w}_h\,\mathrm{d}x + \int_{\partial D} (\boldsymbol{n}\cdot\kappa\nabla v_h)\hat{w}_h\,\mathrm{d}s - \int_{\partial D} \eta_h v_h \hat{w}_h\,\mathrm{d}s \\ &= \sum_{K\in\mathcal{T}_h^{\partial}} \int_K -\nabla\cdot(\kappa\nabla e_h)\hat{w}_h\,\mathrm{d}x - \sum_{F\in\mathcal{F}_h^{\circ\partial}} \int_F [\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F \hat{w}_h\,\mathrm{d}s - \int_{\partial D} \eta_h v_h\hat{w}_h\,\mathrm{d}s,\end{aligned}$$

where $\mathcal{F}_h^{\circ\partial}$ is the collection of the mesh interfaces that touch the boundary (note that $\hat{w}_h$ vanishes on all the remaining interfaces in $\mathcal{F}_h^{\circ}$). The Cauchy–Schwarz inequality leads to

$$\|\ell_h - \ell\circ T - a_h(v_h, (I-T)(\cdot))\|_{V_h'} \le c\mathfrak{T}_1\mathfrak{T}_2$$

with $\mathfrak{T}_1$ and $\mathfrak{T}_2$ defined by

$$\mathfrak{T}_1 := \Big(\sum_{K\in\mathcal{T}_h^{\partial}} \kappa_K^{-1} h_K^{2+2d(\frac{1}{2}-\frac{1}{q})} \|\nabla\cdot(\kappa\nabla e_h)\|_{L^q(K)}^2 + \sum_{F\in\mathcal{F}_h^{\circ\partial}} \kappa_{K_F}^{-1}\tilde{h}_F \|[\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F\|_{L^2(F)}^2 + \|\rho_h^{\frac{1}{2}} e_h\|_{L^2(\partial D)}^2 \Big)^{\frac{1}{2}},$$

$$\mathfrak{T}_2 := \Big(\sum_{K\in\mathcal{T}_h^{\partial}} \kappa_K h_K^{-2+2d(\frac{1}{q}-\frac{1}{2})} \|\hat{w}_h\|_{L^{q'}(K)}^2 + \sum_{F\in\mathcal{F}_h^{\circ\partial}} \kappa_{K_F}\tilde{h}_F^{-1} \|\hat{w}_h\|_{L^2(F)}^2 + \|\rho_h^{\frac{1}{2}} w_h\|_{L^2(\partial D)}^2 \Big)^{\frac{1}{2}},$$

where, for all $F \in \mathcal{F}_h^{\circ\partial}$, K_F is the mesh cell sharing F and having the larger value of κ_K (the choice of K_F is irrelevant if both cells give the same value), $\tilde{h}_F$ is defined in Lemma 5.7, and where q' is the conjugate number of q. Moreover, in the last term defining $\mathfrak{T}_1$ and $\mathfrak{T}_2$, we have exploited the fact that u and $T(w_h)$ have zero trace at the boundary ∂D so that $\|\rho_h^{1/2} v_h\|_{L^2(\partial D)} = \|\rho_h^{1/2} e_h\|_{L^2(\partial D)}$ and $\|\rho_h^{1/2}\hat{w}_h\|_{L^2(\partial D)} = \|\eta_h^{1/2} w_h\|_{L^2(\partial D)}$.

(2a) Bound on $\mathfrak{T}_1$. We need to bound $\|[\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F\|_{L^2(F)}$ and to this purpose we use Lemma 5.7. This is possible since, by assumption, κ is piecewise constant on the mesh $\mathcal{T}_h$ and, therefore, $[\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F \in P_F$. We infer that

$$\tilde{h}_F^{\frac{1}{2}} \|[\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F\|_{L^2(F)} \le c\tilde{h}_F^{d(\frac{1}{2}-\frac{1}{q})} \sup_{\substack{\varphi\in W_0^{1,q'}(D_F) \\ \|\nabla\varphi\|_{\boldsymbol{L}^{q'}(D_F)}=1}} \int_F [\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F \varphi\,\mathrm{d}s.$$

Let $\varphi \in W_0^{1,q'}(D_F)$ be such that $\|\nabla\varphi\|_{\boldsymbol{L}^{q'}(D_F)} = 1$. By the definition of the jump and using the divergence formula, we have

$$\begin{aligned}\int_F [\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F\varphi\,\mathrm{d}s &= \int_F (\kappa\nabla v_h)_{|K_l}\cdot\boldsymbol{n}_{K_l}\varphi\,\mathrm{d}s + \int_F (\kappa\nabla v_h)_{|K_r}\cdot\boldsymbol{n}_{K_r}\varphi\,\mathrm{d}s \\ &= \sum_{K\in\mathfrak{T}_F}\int_K \nabla\cdot(\varphi\kappa\nabla v_h)\,\mathrm{d}x = \sum_{K\in\mathfrak{T}_F}\int_K (\varphi\nabla\cdot(\kappa\nabla v_h) + \kappa\nabla v_h\cdot\nabla\varphi)\,\mathrm{d}x,\end{aligned}$$

where K_l, K_r are the two mesh cells sharing the interface F and where we have set $\mathfrak{T}_F = \{K_l, K_r\}$. Moreover, since $q' \ge 2$, the zero-extension of φ to D is in $H_0^1(D)$, and this implies that

$$\sum_{K\in\mathfrak{T}_F}\int_K (\varphi\nabla\cdot(\kappa\nabla u) + \kappa\nabla u\cdot\nabla\varphi)\,\mathrm{d}x = 0.$$

Combining these two relations, we infer that

$$\int_F [\![\kappa\nabla v_h]\!]\cdot\boldsymbol{n}_F\varphi\,\mathrm{d}s = \sum_{K\in\mathfrak{T}_F}\int_K (\varphi\nabla\cdot(\kappa\nabla e_h) + \kappa\nabla e_h\cdot\nabla\varphi)\,\mathrm{d}x.$$

Invoking Hölder's inequality and the Poincaré–Steklov inequality in $W_0^{1,q'}(D_F)$, which implies that $\|\varphi\|_{L^{q'}(K)} \le ch_K\|\nabla\varphi\|_{\boldsymbol{L}^{q'}(D_F)} = ch_K$ for all $K \in \mathfrak{T}_F$, we infer that

$$\left|\int_K \varphi\nabla\cdot(\kappa\nabla e_h)\,\mathrm{d}x\right| \le \|\varphi\|_{L^{q'}(K)}\|\nabla\cdot(\kappa\nabla e_h)\|_{L^q(K)} \le ch_K\|\nabla\cdot(\kappa\nabla e_h)\|_{L^q(K)}.$$

Moreover, invoking Hölder's inequality together with $\|\nabla\varphi\|_{\boldsymbol{L}^{q'}(K)} \le \|\nabla\varphi\|_{\boldsymbol{L}^{q'}(D_F)} = 1$ for all $K \in \mathfrak{T}_F$, $q \le 2$, and the shape-regularity of the mesh sequence, we infer that

$$\left|\int_K \kappa\nabla e_h\cdot\nabla\varphi\,\mathrm{d}x\right| \le \|\nabla\varphi\|_{\boldsymbol{L}^{q'}(K)}\|\kappa\nabla e_h\|_{\boldsymbol{L}^q(K)} \le ch_K^{d(\frac{1}{q}-\frac{1}{2})}\|\kappa\nabla e_h\|_{\boldsymbol{L}^2(K)}.$$

Putting the above bounds together and since K_F has been chosen so that $\kappa_{K_F} = \max_{K\in\mathfrak{T}_F}\kappa_K$, we conclude that $\mathfrak{T}_1 \le c\|e_h\|_{V_\sharp}$.

(2b) Bound on $\mathfrak{T}_2$. Applying a inverse inequality from $L^{q'}(K)$ to $L^2(K)$ for all $K \in \mathfrak{T}_h^\partial$, we infer that

$$h_K^{-1+d(\frac{1}{q}-\frac{1}{2})}\|\hat{w}_h\|_{L^{q'}(K)} = h_K^{-1+d(\frac{1}{2}-\frac{1}{q'})}\|\hat{w}_h\|_{L^{q'}(K)} \le ch_K^{-1}\|\hat{w}_h\|_{L^2(K)}.$$

Moreover, applying an inverse trace inequality on K_F for all $F \in \mathcal{F}_h^{\circ\partial}$, and invoking the shape-regularity of the mesh sequence, we infer that

$$\tilde{h}_F^{-\frac{1}{2}}\|\hat{w}_h\|_{L^2(F)} \le ch_{K_F}^{-1}\|\hat{w}_h\|_{L^2(K_F)}.$$

Finally, using the approximation property (6.2) on all $K \in \mathfrak{T}_h^\partial$ and recalling the definition of the contrast factor ξ_κ, we conclude that

$$\mathfrak{T}_2 \le c\xi_\kappa^{\frac{1}{2}}\|w_h\|_{V_h}.$$

This completes the proof of (4.10). □

6.2 Mollified Error Estimate

We are going to assume in this subsection that there is a real number $r > 0$ so that the exact solution u is in $H^{1+r}(D)$. Let $k \ge 1$ be the degree of the underlying finite elements. Let us set $t := \min(r, k)$. If $2t \ge d$, let p be any real number larger than 2. If $2t < d$, let us set $p = \frac{2d}{d-2t}$; clearly $p > 2$ since $t > 0$. Let us now consider some real number q such that $q > \frac{2d}{2+d}$. We define the functional space

$$V_{\mathrm{s}} := \{v \in H^1(D) \mid \boldsymbol{\sigma}(v) \in \boldsymbol{L}^p(D),\ \nabla\cdot\boldsymbol{\sigma}(v) \in L^q(D)\}, \tag{6.5}$$

with the shorthand notation $\boldsymbol{\sigma}(v) := -\kappa\nabla v$ for all $v \in H^1(D)$. Notice that the pair (p, q) satisfies the requirements in (5.4).

Lemma 6.2 (Exact Solution). *If $u \in H^{1+r}(D)$, $r > 0$, and if the source term f is in $L^q(D)$ with $q > \frac{2d}{2+d}$, then u is in V_S as defined by* (6.5).

Proof. Owing to the Sobolev Embedding Theorem (see e.g., [9, Section 9.3]), we infer that $\boldsymbol{H}^t(D) \hookrightarrow \boldsymbol{L}^p(D)$ (indeed, if $2t < d$, then we have $\boldsymbol{H}^t(D) \hookrightarrow \boldsymbol{L}^s(D)$ for all $s \in [2, \frac{2d}{d-2t}] = [2, p]$, whereas if $2t \geq d$, then we have $\boldsymbol{H}^t(D) \hookrightarrow \boldsymbol{H}^{\frac{d}{2}}(D) \hookrightarrow \boldsymbol{L}^s(D)$ for all $s \in [2, \infty)$, and choosing $s = p$ again yields $\boldsymbol{H}^t(D) \hookrightarrow \boldsymbol{L}^p(D)$). Since $r \geq t$, we infer that $\boldsymbol{H}^r(D) \hookrightarrow \boldsymbol{H}^t(D)$, so that the above argument implies that $\nabla u \in \boldsymbol{L}^p(D)$, and since κ is piecewise constant and $\boldsymbol{\sigma}(u) = -\kappa\nabla u$, we have $\boldsymbol{\sigma}(u) \in \boldsymbol{L}^p(D)$. Moreover, since $\nabla\cdot\boldsymbol{\sigma}(u) = f$ and $f \in L^q(D)$ with $q > \frac{2d}{2+d}$ by assumption, we have $\nabla\cdot\boldsymbol{\sigma}(u) \in L^q(D)$. In conclusion, $u \in V_S$. □

We are now ready to perform the error analysis. We consider the setting of Section 4.3 and we want to apply Lemma 4.8. We set $V_\sharp := V_S + V_h$ that we equip with the norm

$$\|v\|_{V_\sharp}^2 := \|v\|_{V_\flat}^2 + \sum_{K\in\overline{\mathcal{T}}_h^\partial} \kappa_K^{-1}\Big(h_K^{d(\frac{1}{2}-\frac{1}{p})}\|\kappa\nabla v\|_{\boldsymbol{L}^p(K)} + h_K^{1+d(\frac{1}{2}-\frac{1}{q})}\|\nabla\cdot(\kappa\nabla v)\|_{L^q(K)}\Big)^2,$$

where $\overline{\mathcal{T}}_h^\partial$ is the collection of all the mesh cells having a boundary face, i.e., $\overline{\mathcal{T}}_h^\partial := \bigcup_{F\in\mathcal{F}_h^\partial}\{K_F\}$. Compared with the norm defined by (6.4) used for the trimmed error estimate, we observe that there is now an additional term measuring $\kappa\nabla v$ in the $\boldsymbol{L}^p$-norm, but the summation is now restricted to the smaller set $\overline{\mathcal{T}}_h^\partial \subsetneq \mathcal{T}_h^\partial$. Notice also that (4.4) holds true with $c_\flat = 1$. We define the following bilinear form on $V_\sharp \times V_h$:

$$a_\sharp(v, w_h) := \int_D \kappa\nabla v\cdot\nabla w \,dx - \sum_{F\in\mathcal{F}_h^\partial} \langle(\boldsymbol{\sigma}(v)_{|K_F}\cdot\boldsymbol{n})_{|F}, w_h\rangle, \tag{6.6}$$

recalling that for all $F \in \mathcal{F}_h^\partial$, $\boldsymbol{n}_{K_F} = \boldsymbol{n}$ (the unit outward normal to D), and the action of the linear form $\langle(\boldsymbol{\sigma}(v)_{|K_F}\cdot\boldsymbol{n})_{|F}, \cdot\rangle$ has been defined in (5.6) for all $F \in \mathcal{F}_h^\partial$.

Lemma 6.3 (Mollified Error Estimate). *The assumptions of Lemma* 4.8 *hold true for the bilinear form $a_\sharp$ defined by* (6.6) *and the stabilization bilinear form s_h defined by* (3.5b). *Moreover, the constant $\omega_{V_\sharp,W_h}^{\mathrm{mol}}$ involved in* (4.12) *and the constant σ_{V_h,W_h} involved in* (4.14) *are independent of the contrast in κ.*

Proof. (1) Proof of (4.12). This is a direct consequence of (5.8), the Cauchy–Schwarz inequality, the choice (3.6) of the penalty parameter ρ_h, and the fact that $\|\rho_h^{\frac{1}{2}} w_h\|_{L^2(\partial D)} \leq \|w_h\|_{V_h}$ for all $w_h \in V_h$.

(2) Proof of (4.13a). Let $v_h, w_h \in V_h$. Let $F \in \mathcal{F}_h^\partial$ and let K_F be the mesh cell having F as a boundary face. Since the restriction of $\boldsymbol{\sigma}(v_h)$ to K_F is smooth and since the restriction of $E_F^{K_F}(w_h)$ is nonzero only on the face F of K_F, we have

$$\begin{aligned}\langle(\boldsymbol{\sigma}(v_h)_{|K_F}\cdot\boldsymbol{n})_{|F}, w_h\rangle &= \int_{K_F} (\boldsymbol{\sigma}(v_h)\cdot E_F^{K_F}(w_h) + (\nabla\cdot\boldsymbol{\sigma}(v_h))E_F^{K_F}(w_h))\,dx \\ &= \int_{\partial K_F} (\boldsymbol{\sigma}(v_h)_{|K_F}\cdot\boldsymbol{n}_{K_F})E_F^{K_F}(w_h)\,ds \\ &= \int_F (\boldsymbol{\sigma}(v_h)_{|K_F}\cdot\boldsymbol{n})w_h\,ds,\end{aligned}$$

where we have used the divergence formula in K_F and where we have dropped the restriction to K_F in the integral over K_F to alleviate the notation. Summing over all the boundary faces and recalling the definition (3.5) of $\tilde{a}_h$, we conclude that (4.13a) holds true.

(3) Proof of (4.13b). Let $w_h \in V_h$ and let $v \in V_S$. Let $\mathcal{K}_\delta^{\mathrm{d}} : \boldsymbol{L}^1(D) \to \boldsymbol{C}^\infty(\overline{D})$ and $\mathcal{K}_\delta^{\mathrm{b}} : L^1(D) \to C^\infty(\overline{D})$ be the mollification operators introduced in Section 5.1. Recall the following key commuting property:

$$\nabla\cdot(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\tau})) = \mathcal{K}_\delta^{\mathrm{b}}(\nabla\cdot\boldsymbol{\tau}) \tag{6.7}$$

for all $\boldsymbol{\tau} \in \boldsymbol{L}^1(D)$ such that $\nabla\cdot\boldsymbol{\tau} \in L^1(D)$. It is important to realize that this property can be applied to $\boldsymbol{\sigma}(v)$ since $\nabla\cdot\boldsymbol{\sigma}(v) \in L^1(D)$ by the definition of V_S. Let us consider the mollified bilinear form

$$n_{\sharp\delta}(v, w_h) := \sum_{F\in\mathcal{F}_h^\partial} \langle(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))_{|K_F}\cdot\boldsymbol{n})_{|F}, w_h\rangle.$$

Owing to the commuting property (6.7), we infer that

$$\langle (\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))_{|K_F}\cdot\boldsymbol{n})_{|F}, w_h\rangle = \int_{K_F} \big(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))\cdot E_F^{K_F}(w_h) + \mathcal{K}_\delta^{\mathrm{b}}(\nabla\cdot\boldsymbol{\sigma}(v))E_F^{K_F}(w_h)\big)\,\mathrm{d}x.$$

By letting $\delta\downarrow 0$, Theorem 5.3 implies that

$$\begin{aligned}\lim_{\delta\downarrow 0}\int_{K_F} \big(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))\cdot E_F^{K_F}(w_h) + \mathcal{K}_\delta^{\mathrm{b}}((\nabla\cdot\boldsymbol{\sigma}(v)))E_F^{K_F}(w_h)\big)\,\mathrm{d}x &= \int_{K_F}\big(\boldsymbol{\sigma}(v)\cdot E_F^{K_F}(w_h) + (\nabla\cdot\boldsymbol{\sigma}(v))E_F^{K_F}(w_h)\big)\,\mathrm{d}x \\ &= \langle(\boldsymbol{\sigma}(v)_{|K_F}\cdot\boldsymbol{n})_{|F}, w_h\rangle.\end{aligned}$$

Summing over the mesh boundary faces, we infer that

$$n_{\sharp\delta}(v, w_h) \to \int_D \kappa\nabla v\cdot\nabla w_h\,\mathrm{d}x - a_\sharp(v, w_h) \quad \text{as } \delta\downarrow 0.$$

Moreover, since the mollified function $\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))$ is smooth, by repeating the calculation done in step (2), we also have

$$n_{\sharp\delta}(v, w_h) = \sum_{F\in\mathcal{F}_h^\partial}\int_F (\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))\cdot\boldsymbol{n})w_h\,\mathrm{d}s.$$

Since $[\![\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))]\!]\cdot\boldsymbol{n}_F = [\![w_h]\!] = 0$ for all $F\in\mathcal{F}_h^\circ$, we obtain

$$n_{\sharp\delta}(v, w_h) = \sum_{K\in\mathcal{T}_h}\int_{\partial K}\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))_{|K}\cdot\boldsymbol{n}_K w_h\,\mathrm{d}s = \int_D\big(\mathcal{K}_\delta^{\mathrm{d}}(\boldsymbol{\sigma}(v))\cdot\nabla w_h + \mathcal{K}_\delta^{\mathrm{b}}(\nabla\cdot\boldsymbol{\sigma}(v))w_h\big)\,\mathrm{d}x,$$

where we used the divergence formula in each mesh cell K and the commuting property (6.7). Letting $\delta\downarrow 0$ and invoking again Theorem 5.3 shows that

$$n_{\sharp\delta}(v, w_h) \to \int_D\big(\boldsymbol{\sigma}(v)\cdot\nabla w_h + (\nabla\cdot\boldsymbol{\sigma}(v))w_h\big)\,\mathrm{d}x = -\int_D\big(\kappa\nabla v\cdot\nabla w_h + (\nabla\cdot(\kappa\nabla v))w_h\big)\,\mathrm{d}x.$$

The proof of (4.13b) follows by identifying the two limits of $n_{\sharp\delta}(v, w_h)$ and since $\int_D(\nabla\cdot(\kappa\nabla u))w_h\,\mathrm{d}x = \ell_h(w_h)$.

(4) Proof of (4.14). The Cauchy–Schwarz inequality implies that

$$\|s_h(v_h,\cdot)\|_{V_h'} \le \eta_0\|\rho_h^{\frac{1}{2}}v_h\|_{L^2(\partial D)}$$

for all $v_h\in V_h$, and (4.14) follows since any function v in $V = H_0^1(D)$ has a zero trace on ∂D. □

7 Application to the Time-Harmonic Maxwell's Equations

In this section, we show how the trimmed error estimate from Lemma 4.6 and the mollified error estimate from Lemma 4.8 can be applied to the approximation of the time-harmonic Maxwell's equations using the boundary penalty method described in Section 3.3. The discrete space is $\boldsymbol{V}_h = \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h)$, and the space $\boldsymbol{V}_\flat := \boldsymbol{H}_0(\mathrm{curl}; D) + \boldsymbol{V}_h$ can be equipped with the norm $\|\cdot\|_{\boldsymbol{V}_\flat}$ that extends to $\boldsymbol{V}_\flat$ the norm $\|\cdot\|_{\boldsymbol{V}_h}$ originally defined by (3.10) on $\boldsymbol{V}_h$; notice in particular that functions in $\boldsymbol{V}_\flat$ have a well-defined tangential trace on ∂D. Indeed, any function $\boldsymbol{b}\in\boldsymbol{V}_\flat$ can be written as $\boldsymbol{b} = \boldsymbol{b}_0 + \boldsymbol{b}_h$ with $\boldsymbol{b}_0\in\boldsymbol{H}_0(\mathrm{curl}; D)$ and $\boldsymbol{b}_h\in\boldsymbol{V}_h$, and we have $\gamma^{\mathrm{c}}(\boldsymbol{b}) = \boldsymbol{b}_{h|\partial D}\times\boldsymbol{n}$ so that

$$\|\rho_h^{\frac{1}{2}}\boldsymbol{b}\|_{\boldsymbol{L}^2(\partial D)} = \|\rho_h^{\frac{1}{2}}\boldsymbol{b}_h\|_{\boldsymbol{L}^2(\partial D)}.$$

Finally, the discrete forms $a_h(\cdot,\cdot)$ and $\ell_h(\cdot)$ are defined by (3.8). The constants in the error estimates derived in this section depend on the shape-regularity of the mesh sequence and on the reference finite element. These constants can also depend on the local ratios $\frac{\mu_{\sharp,K}}{\mu_{r,K}}$ and $\frac{\kappa_{\sharp,K}}{\kappa_{r,K}}$ for all $K\in\mathcal{T}_h$; for simplicity, we will not track these dependencies in what follows. Notice that these ratios are equal to 1 when the coefficients $\tilde{\mu}$ and κ are real.

7.1 Trimmed Error Estimate

We define the trimming operator $T : \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h) \to \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathrm{curl}; D) = \{\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h) \mid \boldsymbol{b}_{h|\partial D}\times\boldsymbol{n} = \boldsymbol{0}\}$ such that, for all $\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h)$, $T(\boldsymbol{b}_h)_{|K}$, for all $K \in \mathcal{T}_h$, is defined by zeroing out all the degrees of freedom of the tangential component of $\boldsymbol{b}_h$ at the boundary. Note that the trimming operator couples the Cartesian components of $\boldsymbol{b}_h$ if the faces composing the boundary ∂D are not orthogonal to the coordinate axes. We have $T(\boldsymbol{b}_h) = \boldsymbol{b}_h$ on all $K \in \mathcal{T}_h \setminus \mathcal{T}_h^{\partial}$, whereas for all $K \in \mathcal{T}_h^{\partial}$, one can prove the following bounds for all $\boldsymbol{b}_h \in \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h)$ with c uniform with respect to h: If $\partial K \cap \partial D$ is composed of one or more boundary faces, then

$$h_K\|\nabla(\boldsymbol{b}_h - T(\boldsymbol{b}_h))\|_{\boldsymbol{L}^2(K)} + \|\boldsymbol{b}_h - T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(K)} \le c\, h_K^{\frac{1}{2}}\|\boldsymbol{b}_h\times\boldsymbol{n}\|_{\boldsymbol{L}^2(\partial K\cap\partial D)},$$

whereas if $\partial K \cap \partial D$ is a manifold of dimension $d' < d - 1$, then

$$h_K\|\nabla(\boldsymbol{b}_h - T(\boldsymbol{b}_h))\|_{\boldsymbol{L}^2(K)} + \|\boldsymbol{b}_h - T(\boldsymbol{b}_h)\|_{L^2(K)} \le c\, h_K^{\frac{1}{2}}\|\boldsymbol{b}_h\times\boldsymbol{n}\|_{\boldsymbol{L}^2(F)} \quad \text{for all } F \in \mathcal{F}_K^{\partial},$$

where we recall that $\mathcal{F}_K^{\partial}$ is the collection of the boundary faces containing the manifold $\partial K \cap \partial D$. We introduce the contrast factor ξ_{κ_r} for the parameter κ_r which is defined similarly to (6.3) by replacing κ by κ_r. We also define the local magnetic Reynolds numbers $\zeta_{\mu\kappa,F} := \mu_{r,K_F} h_{K_F}^2/\kappa_{r,K_F}$ for all $F \in \mathcal{F}_h^{\partial}$. Finally, let us set

$$\boldsymbol{V}_{\mathrm{S}} := \{\boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{curl}; D) \mid \nabla\times(\kappa\nabla\times\boldsymbol{v}) \in \boldsymbol{L}^2(D)\},$$

and let us equip the space $\boldsymbol{V}_\sharp := \boldsymbol{V}_{\mathrm{S}} + \boldsymbol{V}_h$ with the norm

$$\|\boldsymbol{b}\|_{\boldsymbol{V}_\sharp} := \Big(\|\boldsymbol{b}\|_{\boldsymbol{V}_\flat}^2 + \sum_{K\in\mathcal{T}_h^{\partial}} \kappa_{r,K}^{-1} h_K^2 \|\nabla\times(\kappa\nabla\times\boldsymbol{b})\|_{\boldsymbol{L}^2(K)}^2\Big)^{\frac{1}{2}}. \tag{7.1}$$

For simplicity, we assume that the trace space $\boldsymbol{P}_F$ contains the traces of the tangential derivatives of functions in $\boldsymbol{P}_K$ (this is obviously the case if $\boldsymbol{P}_K$ is the polynomial space $\mathbb{P}_{k,d}(K;\mathbb{R}^3)$).

Lemma 7.1 (Trimmed Error Estimate). *The assumptions of Lemma 4.6 hold true with the trimming operator* $T : \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h) \to \boldsymbol{P}^{\mathrm{g}}(\mathcal{T}_h) \cap \boldsymbol{H}_0(\mathrm{curl}; D)$ *defined above, where the constants* $\omega_{\boldsymbol{V}_\sharp,\boldsymbol{V}_h}^{\mathrm{tri}}$ *and* $\varpi_{\boldsymbol{V}_\sharp,\boldsymbol{V}_h}^{\mathrm{tri}}$ *are proportional to* $\xi_{\kappa_r}^{1/2}$ *and to* $\max(1, \zeta_{\mu\kappa}^{1/2})$*, where* ξ_{κ_r} *is the contrast factor for* κ_r *and* $\zeta_{\mu\kappa} := \max_{F\in\mathcal{F}_h^{\partial}} \zeta_{\mu\kappa,F}$ *where* $\zeta_{\mu\kappa,F}$ *is the local magnetic Reynolds number associated with the boundary face* F.

Proof. We only highlight the differences with respect to the proof of Lemma 6.1.

(1) Verification of (4.9). Let $(\boldsymbol{v}_h, \boldsymbol{b}_h) \in \boldsymbol{V}_h \times \boldsymbol{V}_h$. Since $T(\boldsymbol{b}_h) \in \boldsymbol{H}_0(\mathrm{curl}; D)$, we infer that

$$\begin{aligned} a(\boldsymbol{A}, T(\boldsymbol{b}_h)) - a_h(\boldsymbol{v}_h, T(\boldsymbol{b}_h)) &= \int_D (\tilde{\mu}(\boldsymbol{A} - \boldsymbol{v}_h)\cdot\overline{T(\boldsymbol{b}_h)} + \kappa\nabla\times(\boldsymbol{A} - \boldsymbol{v}_h)\cdot\nabla\times\overline{T(\boldsymbol{b}_h)})\,\mathrm{d}x \\ &\le \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{V}_\flat}(\|\mu_r^{\frac{1}{2}}\, T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(D)} + \|\kappa_r^{\frac{1}{2}}\,\nabla\times T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(D)}). \end{aligned}$$

Since $\|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{V}_\flat} \le \|\boldsymbol{A} - \boldsymbol{v}_h\|_{\boldsymbol{V}_\sharp}$, we just have to prove that $\|\mu_r^{\frac{1}{2}}\, T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(D)} + \|\kappa_r^{\frac{1}{2}}\,\nabla\times T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(D)} \le c\|\boldsymbol{w}_h\|_{\boldsymbol{V}_h}$. We have $T(\boldsymbol{b}_h) = \boldsymbol{b}_h$ on all $K \in \mathcal{T}_h \setminus \mathcal{T}_h^{\partial}$, so that we only need to bound $T(\boldsymbol{b}_h)$ on all $K \in \mathcal{T}_h^{\partial}$. Reasoning as in the proof of Lemma 6.1 and estimating the approximation properties of $\nabla\times T(\boldsymbol{b}_h)$ by those of $\nabla T(\boldsymbol{b}_h)$, we infer that, for all $K \in \mathcal{T}_h^{\partial}$, if $\partial K \cap \partial D$ is composed of one or more boundary faces, then

$$\|\kappa_r^{\frac{1}{2}}\,\nabla\times T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(K)} \le \|\kappa_r^{\frac{1}{2}}\,\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2(K)} + c\|\rho_h^{\frac{1}{2}}\,(\boldsymbol{b}_h\times\boldsymbol{n})\|_{\boldsymbol{L}^2(\partial K\cap\partial D)},$$

whereas if the manifold $\partial K \cap \partial D$ is of dimension $d' < d - 1$, then

$$\|\kappa_r^{\frac{1}{2}}\,\nabla\times T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(K)} \le \|\kappa_r^{\frac{1}{2}}\,\nabla\times\boldsymbol{b}_h\|_{\boldsymbol{L}^2(K)} + c\xi_{\kappa_r}^{\frac{1}{2}}\|\rho_h^{\frac{1}{2}}\,(\boldsymbol{b}_h\times\boldsymbol{n})\|_{\boldsymbol{L}^2(F)},$$

where F is a boundary face in $\mathcal{F}_K^{\partial}$ such that κ_{r,K_F} is maximal. The reasoning to bound $\|\mu_r^{\frac{1}{2}}\, T(\boldsymbol{b}_h)\|_{\boldsymbol{L}^2(K)}$ for all $K \in \mathcal{T}_h^{\partial}$ is similar and leads to the additional dependency on the factor $\max(1, \zeta_{\mu\kappa})$.

(2) Verification of (4.10). Let $(\boldsymbol{v}_h, \boldsymbol{b}_h) \in \boldsymbol{V}_h \times \boldsymbol{V}_h$ and let us set $\boldsymbol{e}_h := \boldsymbol{A} - \boldsymbol{v}_h$ and $\boldsymbol{d}_h := \boldsymbol{b}_h - T(\boldsymbol{b}_h)$. A direct calculation shows that

$$\begin{aligned}\ell_h(\boldsymbol{b}_h) - \ell(T(\boldsymbol{b}_h)) - a_h(\boldsymbol{v}_h, (I-T)(\boldsymbol{b}_h)) &= \int_D \boldsymbol{f}\cdot\overline{\boldsymbol{d}}_h \,\mathrm{d}x - \int_D (\tilde{\mu}\boldsymbol{v}_h\cdot\overline{\boldsymbol{d}}_h + \kappa\nabla\times\boldsymbol{v}_h\cdot\nabla\times\overline{\boldsymbol{d}}_h)\,\mathrm{d}x - \int_{\partial D}(\boldsymbol{n}\times(\kappa\nabla\times\boldsymbol{v}_h))\cdot\overline{\boldsymbol{d}}_h\,\mathrm{d}s \\ &\quad - \int_{\partial D} \eta_h(\boldsymbol{v}_h\times\boldsymbol{n})\cdot(\overline{\boldsymbol{d}}_h\times\boldsymbol{n})\,\mathrm{d}s \\ &= \sum_{K\in\mathcal{T}_h^\partial}\int_K (\tilde{\mu}\boldsymbol{e}_h\cdot\overline{\boldsymbol{d}}_h + \nabla\times(\kappa\nabla\times\boldsymbol{e}_h)\cdot\overline{\boldsymbol{d}}_h)\,\mathrm{d}x + \sum_{F\in\mathcal{F}_h^{\circ\partial}}\int_F \boldsymbol{n}_F\times[\![\kappa\nabla\times\boldsymbol{v}_h]\!]\cdot\overline{\boldsymbol{d}}_h\,\mathrm{d}s \\ &\quad - \int_{\partial D}\eta_h(\boldsymbol{v}_h\times\boldsymbol{n})\cdot(\overline{\boldsymbol{d}}_h\times\boldsymbol{n})\,\mathrm{d}s,\end{aligned}$$

where we recall that $\mathcal{F}_h^{\circ\partial}$ is the collection of the mesh interfaces that touch the boundary. The Cauchy–Schwarz inequality leads to $\|\ell_h - \ell\circ T - a_h(\boldsymbol{v}_h, (I-T)(\cdot))\|_{\boldsymbol{V}_h'} \le c\mathfrak{T}_1\mathfrak{T}_2$ with $\mathfrak{T}_1$ and $\mathfrak{T}_2$ defined by

$$\mathfrak{T}_1 = \bigg(\sum_{K\in\mathcal{T}_h^\partial}\mu_{r,K}\|\boldsymbol{e}_h\|^2_{\boldsymbol{L}^2(K)} + \kappa_{r,K}^{-1}h_K^2\|\nabla\times(\kappa\nabla\times\boldsymbol{e}_h)\|^2_{\boldsymbol{L}^2(K)} + \sum_{F\in\mathcal{F}_h^{\circ\partial}}\kappa_{r,K_F}^{-1}\tilde{h}_F\|[\![\kappa\nabla\times\boldsymbol{v}_h]\!]\times\boldsymbol{n}_F\|^2_{\boldsymbol{L}^2(F)} + \|\rho_h^{\frac12}(\boldsymbol{v}_h\times\boldsymbol{n})\|^2_{\boldsymbol{L}^2(\partial D)}\bigg)^{\frac12}$$

and

$$\mathfrak{T}_2 = \bigg(\sum_{K\in\mathcal{T}_h^\partial}(\mu_{r,K} + \kappa_{r,K}h_K^{-2})\|\boldsymbol{d}_h\|^2_{\boldsymbol{L}^2(K)} + \sum_{F\in\mathcal{F}_h^{\circ\partial}}\kappa_{r,K_F}\tilde{h}_F^{-1}\|\boldsymbol{d}_h\|^2_{\boldsymbol{L}^2(F)} + \|\rho_h^{\frac12}(\boldsymbol{b}_h\times\boldsymbol{n})\|^2_{\boldsymbol{L}^2(\partial D)}\bigg)^{\frac12},$$

where, for all $F \in \mathcal{F}_h^{\circ\partial}$, K_F is the mesh cell sharing F and having the larger value of $\kappa_{r,K}$ (the choice of K_F is irrelevant if both cells give the same value), and $\tilde{h}_F$ is defined in Lemma 5.7.

(2a) Bound on $\mathfrak{T}_1$. The bound on the terms composing the summation over $K \in \mathcal{T}_h^\partial$ is straightforward. To bound $\|[\![\kappa\nabla\times\boldsymbol{v}_h]\!]\times\boldsymbol{n}_F\|_{\boldsymbol{L}^2(F)}$ for all $F \in \mathcal{F}_h^{\circ\partial}$, we use Lemma 5.7 (with $p = 2$). This is possible since, by assumption, κ is piecewise constant on the mesh $\mathcal{T}_h$ and, therefore, $[\![\kappa\nabla\times\boldsymbol{v}_h]\!]\times\boldsymbol{n}_F \in \boldsymbol{P}_F$. Finally,

$$\|\rho_h^{\frac12}(\boldsymbol{v}_h\times\boldsymbol{n})\|_{\boldsymbol{L}^2(F)} = \|\rho_h^{\frac12}(\boldsymbol{e}_h\times\boldsymbol{n})\|_{\boldsymbol{L}^2(F)}$$

for all $F \in \mathcal{F}_h^\partial$, since the exact solution $\boldsymbol{A}$ has a zero tangential trace on ∂D.

(2b) Bound on $\mathfrak{T}_2$. Reasoning as in the proof of Lemma 6.1, we infer that

$$|\mathfrak{T}_2| \le c\xi_{\kappa_r}^{\frac12}\max(1, \zeta_{\mu\kappa}^{\frac12})\|\boldsymbol{b}_h\|_{\boldsymbol{V}_h}.$$ □

7.2 Mollified Error Estimate

We are going to assume in this subsection that there is a real number $r > 0$ so that the exact solution $\boldsymbol{A}$ is such that $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$. Let $k \ge 1$ be the degree of the underlying finite elements. We define p and t as in Section 6.2 and we set $q = 2$. Let us define the functional space

$$\boldsymbol{V}_{\mathrm{s}} := \{\boldsymbol{b} \in \boldsymbol{H}_0(\mathrm{curl}; D) \mid \kappa\nabla\times\boldsymbol{b} \in \boldsymbol{L}^p(D),\ \nabla\times(\kappa\nabla\times\boldsymbol{b}) \in \boldsymbol{L}^2(D)\}. \tag{7.2}$$

Lemma 7.2 (Exact Solution). *If $\boldsymbol{A} \in \boldsymbol{H}_0(\mathrm{curl}; D)$, with $\kappa\nabla\times\boldsymbol{A} \in \boldsymbol{H}^r(D)$, $r > 0$, then $\boldsymbol{A}$ is in $\boldsymbol{V}_{\mathrm{s}}$ as defined by* (7.2).

Let us equip the space $\boldsymbol{V}_\sharp := \boldsymbol{V}_{\mathrm{s}} + \boldsymbol{V}_h$ with the norm

$$\|\boldsymbol{b}\|^2_{\boldsymbol{V}_\sharp} := \|\boldsymbol{b}\|^2_{\boldsymbol{V}_\flat} + \sum_{K\in\overline{\mathcal{T}}_h^\partial}\kappa_{r,K}^{-1}\Big(h_K^{d(\frac12-\frac1p)}\|\kappa\nabla\times\boldsymbol{b}\|_{\boldsymbol{L}^p(K)} + h_K\|\nabla\times(\kappa\nabla\times\boldsymbol{b})\|_{\boldsymbol{L}^2(K)}\Big)^2.$$

Compared with the norm defined by (7.1) used for the trimmed error estimate, we observe that there is now an additional term measuring $\kappa\nabla\times\boldsymbol{b}$ in the $\boldsymbol{L}^p$-norm, but the summation is now restricted to the smaller set $\overline{\mathcal{T}}_h^\partial \subsetneq \mathcal{T}_h^\partial$. Let us define the following sesquilinear form on $\boldsymbol{V}_\sharp \times \boldsymbol{V}_h$:

$$a_\sharp(\boldsymbol{v}, \boldsymbol{b}_h) := \int_D (\tilde{\mu}\boldsymbol{v}\cdot\overline{\boldsymbol{b}}_h + \kappa\nabla\times\boldsymbol{v}\cdot\nabla\times\overline{\boldsymbol{b}}_h)\,\mathrm{d}x + \sum_{F\in\mathcal{F}_h^\partial}\langle((\kappa\nabla\times\boldsymbol{v})_{|K_F}\times\boldsymbol{n})_{|F}, \Pi_F(\boldsymbol{b}_h)\rangle, \tag{7.3}$$

where Π_F is the ℓ^2-orthogonal projection onto the hyperplane tangent to F, i.e., $\Pi_F(\boldsymbol{b}_h) = \boldsymbol{b}_h - (\boldsymbol{b}_h, \boldsymbol{n})_{\ell^2}\boldsymbol{n}$. Notice that $\Pi_F(\boldsymbol{b}_h)$ is indeed a member of the space $\boldsymbol{Y}^{\mathrm{c}}(F)$ defined by (5.9b) since $\Pi_F(\boldsymbol{b}_h)\cdot\boldsymbol{n} = 0$, and that $\|\Pi_F(\boldsymbol{b}_h)\|_{\ell^2} = \|\boldsymbol{b}_h\times\boldsymbol{n}\|_{\ell^2}$.

Lemma 7.3 (Mollified Error Estimate). *The assumptions of Lemma* 4.8 *hold true for the bilinear form* $a_\sharp$ *defined by* (7.3) *and the stabilization bilinear form* s_h *defined by* (3.8b). *Moreover, the constant* $\omega^{\mathrm{mol}}_{V_\sharp,W_h}$ *involved in* (4.12) *and the constant* σ_{V_h,W_h} *involved in* (4.14) *are independent of the contrast in* κ.

Proof. We only highlight the differences with respect to the proof of Lemma 6.3.

(1) Verification of (4.12). This is a direct consequence of (5.8), the Cauchy–Schwarz inequality, the choice (3.9) of the penalty parameter ρ_h, and the fact that

$$\|\rho_h^{\frac{1}{2}}\boldsymbol{w}_h\|_{L^2(\partial D)} \le \|\boldsymbol{w}_h\|_{\boldsymbol{V}_h}$$

for all $\boldsymbol{w}_h \in \boldsymbol{V}_h$.

(2) Proof of (4.13a). The argument is the same as in the proof of Lemma 6.3.

(3) Proof of (4.13b). Let $\mathcal{K}^{\mathrm{c}}_\delta : \boldsymbol{L}^1(D) \to \boldsymbol{C}^\infty(\overline{D})$ and $\mathcal{K}^{\mathrm{d}}_\delta : \boldsymbol{L}^1(D) \to C^\infty(\overline{D})$ be the mollification operators introduced in Section 5.1. The proof of (4.13b) now relies on the following key commuting property:

$$\nabla\times(\mathcal{K}^{\mathrm{c}}_\delta(\boldsymbol{\tau})) = \mathcal{K}^{\mathrm{d}}_\delta(\nabla\times\boldsymbol{\tau}),$$

which holds true for all $\boldsymbol{\tau} \in \boldsymbol{L}^1(D)$ such that $\nabla\times\boldsymbol{\tau} \in \boldsymbol{L}^1(D)$. The rest of the argument follows the same lines as in the proof of Lemma 6.3.

(3) Verification of (4.14). The Cauchy–Schwarz inequality implies that $\|s_h(\boldsymbol{b}_h,\cdot)\|_{\boldsymbol{V}'_h} \le \eta_0\|\rho_h^{\frac{1}{2}}\boldsymbol{b}_h\|_{\boldsymbol{L}^2(\partial D)}$ for all $\boldsymbol{b}_h \in \boldsymbol{V}_h$, and (4.14) follows since any function $\boldsymbol{v}$ in $V = \boldsymbol{H}_0(\mathrm{curl}; D)$ has a zero tangential trace on ∂D. □

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