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The LBB condition in fractional Sobolev spaces and applications

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The present work focuses on the approximation of the stationary Stokes equations by means of finite-element-like Galerkin methods. It is shown that, provided the velocity space and the pressure space are compatible in some sense, a Ladyzhenskaya–Babuška–Brezzi condition holds in the fractional Sobolev spaces $H^s(\Omega)$, $s \in [0, 1]$. This result is illustrated in two applications.

Keywords: Stokes operator; finite elements; Navier–Stokes equations; pressure estimates; negative norms; fractional Sobolev spaces.

1. Introduction

The present work focuses on the approximation of the stationary Stokes equations by means of finite-element-like Galerkin methods. It is shown that, provided the velocity space and the pressure space are compatible in some sense (see (3.4)), an inf–sup condition holds in the fractional Sobolev spaces $H^s(\Omega)$, $s \in [0, 1]$. This is a generalization of the Ladyzhenskaya–Babuška–Brezzi (LBB) condition. As an application of this fact, we construct an approximation theory of the stationary Stokes problem in $H^s(\Omega)$, $s \in [0, 1]$. In some sense, this work can be viewed as the H^s-counterpart of the far more sophisticated L^∞-approximation technique of Durán *et al.* (1988) and Girault *et al.* (2004). As an additional application, we deduce an estimate of the pressure in the $H^{-r}((0, T); H^{1-s}(\Omega))$-norm for the nonstationary Stokes equations. This bound is the Hilbertian counterpart of an $L^p((0, T); L^\ell(\Omega))$ estimate proved in Sohr & von Wahl (1986). This type of estimate is important for constructing weak solutions of the Navier–Stokes equations that are suitable in the sense of Scheffer (1977).

This paper is organized as follows. The rest of this section is devoted to introducing notation and recalling basic facts on H^s-spaces. In Section 2 it is proved that the gradient operator $\nabla\colon H^s(\Omega) \to \mathbf{H}^{s-1}(\Omega)$ has a closed range, $s \in [0, 1]$. This is done by constructing a left inverse of the gradient on the scale $\{\mathbf{H}^{s-1}(\Omega)\}_{s\in[0,1]}$. The discrete finite-element-like setting alluded to above is introduced in Section 3. The main result of this section is Theorem 3.2 that states the H^s-version of the LBB condition referred to above. Two applications are presented in Section 4: (i) it is shown how the techniques apply to the approximation theory of the stationary Stokes equations in $H^s(\Omega)$, $s \in [0, 1]$; (ii) Theorem 3.2 is applied to deduce an *a priori* bound on the approximate pressure of the time-dependent Stokes equations, and applications to the three-dimensional Navier–Stokes equations are discussed. Item (ii) is actually the main thrust that led the author to developing the present theory.

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2. Preliminaries

2.1 *Notation and conventions*

Let Ω be a connected, open, bounded domain in $\mathbb{R}^d$ ($d \in \{2, 3\}$ is the space dimension). The boundary of Ω is denoted by Γ. Unless specified otherwise, Ω is assumed to be Lipschitz.

Spaces of $\mathbb{R}^d$-valued functions acting on Ω are denoted in bold fonts. No notational distinction is made between $\mathbb{R}$-valued and $\mathbb{R}^d$-valued functions. Henceforth, c is a generic constant whose value may vary at each occurrence. Whenever E is a normed space, $\|\cdot\|_E$ denotes a norm in E.

For $0 < s < 1$, the space $H^s(\Omega) := [L^2(\Omega), H^1(\Omega)]_s$ is defined by the real method of interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, i.e. the so-called K-method of Lions & Peetre (1964) (see also Lions & Magenes, 1968 or Bramble & Zhang, 2000, Appendix A). We interpolate between $H^1(\Omega)$ and $H^2(\Omega)$ if $1 < s < 2$. We denote by $H^1_0(\Omega)$ the closure of $\mathscr{C}_0^\infty(\Omega)$ in $H^1(\Omega)$ and we set[1] $H^s_0(\Omega) := [L^2(\Omega), H^1_0(\Omega)]_s$. Let us recall that $H^s_0(\Omega)$ and $H^s(\Omega)$ coincide for $s \in \left[0, \frac{1}{2}\right)$ and their norms are equivalent (see e.g. Lions & Magenes, 1968, Theorem 11.1 or Grisvard, 1985, Corollary 1.4.4.5). Recall also that $\mathscr{C}_0^\infty(\Omega)$ is dense in $H^s_0(\Omega)$ for $s \in [0, 1]$. The space $H^{-s}(\Omega)$ is defined by duality with $H^s_0(\Omega)$ for $0 \leqslant s \leqslant 1$, i.e.

$$\|v\|_{H^{-s}} = \sup_{0 \neq w \in H^s_0(\Omega)} \frac{(v, w)}{\|w\|_{H^s}}.$$

It is a standard result that $H^{-s}(\Omega) = [L^2(\Omega), H^{-1}(\Omega)]_s$, i.e. $[L^2(\Omega), H^{-1}(\Omega)]_s = [L^2(\Omega), H^1_0(\Omega)]'_s$.

We define $L^2_{\int=0}(\Omega)$ $\big($respectively $H^s_{\int=0}(\Omega)\big)$ to be the space that is composed of those functions in $L^2(\Omega)$ (respectively $H^s(\Omega)$, $s \in [0, 1]$) that are of zero mean. Since we are going to interpolate between $L^2_{\int=0}(\Omega)$ and $H^1_{\int=0}(\Omega)$, we face the question of identifying the structure of $\left[L^2_{\int=0}(\Omega), H^1_{\int=0}(\Omega)\right]_s$. Upon setting $H^s_{\int=0}(\Omega) = \left\{v \in H^s(\Omega);\ \int_\Omega v = 0\right\}$, the answer to this question is given by the following lemma.

LEMMA 2.1 For all $s \in (0, 1)$, the following two spaces coincide with equivalent norms:

$$\left[L^2_{\int=0}(\Omega), H^1_{\int=0}(\Omega)\right]_s = H^s_{\int=0}(\Omega). \tag{2.1}$$

Proof. We use Lemma A1 whose proof is reported in the appendix. Using the notation of Lemma A1, we set $E_0 = L^2(\Omega)$, $E_1 = H^1(\Omega)$ and $Tv = v - \frac{1}{|\Omega|}\int_\Omega v$, where $|\Omega| := \int_\Omega 1\,\mathrm{d}x$. The operator T is a projection and is in $\mathscr{L}(L^2(\Omega); L^2(\Omega))$ and $\mathscr{L}(H^1(\Omega); H^1(\Omega))$. Moreover, since the condition $\int_\Omega v = 0$ is stable in $L^2(\Omega)$ and $H^1(\Omega)$, the range of T is closed in $L^2(\Omega)$ and $H^1(\Omega)$. □

To account for solenoidal vector fields, we set

$$\mathbf{V}^0 = \{v \in \mathbf{L}^2(\Omega);\ \nabla\cdot v = 0;\ v\cdot n|_\Gamma = 0\}, \tag{2.2}$$

$$\mathbf{V}^1 = \{v \in \mathbf{H}^1(\Omega);\ \nabla\cdot v = 0;\ v|_\Gamma = 0\}, \tag{2.3}$$

$$\mathbf{V}^2 = \{v \in \mathbf{V}^1;\ \Delta v \in \mathbf{L}^2(\Omega)\}. \tag{2.4}$$

We denote by $P\colon \mathbf{L}^2(\Omega) \to \mathbf{V}^0$ the $\mathbf{L}^2$-projection onto $\mathbf{V}^0$ (i.e. the so-called Leray projection).

[1] This definition is slightly different from what is usually done when s $= \frac{1}{2}$. What we hereafter denote by $H_0^{\frac{1}{2}}(\Omega)$ is usually denoted by $H_{00}^{\frac{1}{2}}(\Omega)$ elsewhere.

We denote by $-\Delta\colon \mathbf{D}(\Delta) := \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \to \mathbf{L}^2(\Omega)$ the unbounded vector-valued Laplace operator supplemented with homogeneous Dirichlet boundary conditions. We also introduce the Stokes operator $A\colon \mathbf{D}(A) := \mathbf{V}^2 \to \mathbf{V}^0$ by setting $A = -P\Delta|_{\mathbf{V}^2}$.

2.2 *The inf–sup condition*

Throughout this paper we assume that Ω is smooth enough so that the range of $\nabla\colon H^s_{\int=0}(\Omega) \to \mathbf{H}^{s-1}(\Omega)$, $s \in [0, 1]$, is closed. (We show at the end of this section that this condition holds if Ω is star-shaped with respect to a ball, see Theorem 2.4.) In other words, we assume that there is a constant $c > 0$ uniform with respect to s so that

$$\|\nabla q\|_{\mathbf{H}^{s-1}} \geqslant c\|q\|_{H^s} \quad \forall q \in H^s_{\int=0}(\Omega). \tag{2.5}$$

Using the characterization of the norm in $\mathbf{H}^{s-1}(\Omega) = [\mathbf{H}_0^{1-s}(\Omega)]'$, the above inequality can be equivalently rewritten as follows:

$$\sup_{0\neq v\in\mathbf{H}_0^{1-s}(\Omega)} \frac{\langle \nabla q, v\rangle}{\|v\|_{\mathbf{H}^{1-s}}} \geqslant c\|q\|_{H^s} \quad \forall q \in H^s_{\int=0}(\Omega). \tag{2.6}$$

The main objective of the paper is to prove the discrete counterparts of (2.6).

The property (2.5) (or equivalently (2.6)) is known to hold for $s = 0$ and $s = 1$ under the sole regularity assumption that Ω be Lipschitz. For $s = 1$, this is the so-called Poincaré–Friedrichs inequality. For $s = 0$, this is the well-known LBB condition, proofs of which can be found in Nečas (1967) or Bramble (2003) (see also Girault & Raviart, 1986). At this point it is tempting to think that (2.5) could be proved by interpolation between the following two inequalities: $\|\nabla q\|_{\mathbf{L}^2} \geqslant c\|q\|_{H^1}$ and $\|\nabla q\|_{\mathbf{H}^{-1}} \geqslant c\|q\|_{L^2}$. Unfortunately, such a theory does not exist to the best of the author's knowledge. In other words, (2.5) is a nontrivial inequality.

We conjecture that (2.5) holds if Ω is Lipschitz. One seemingly feasible way to prove this could be to revisit the proof in Bramble (2003) and make it work in the range $s \in [0, 1]$. This seems to be a nontrivial undertaking and we leave the matter for future investigation. We propose in the rest of this section an alternative approach to convince ourselves that the set of domains satisfying the hypothesis (2.5) is not empty.

We start by constructing a left inverse of the gradient operator, and to do so we follow Durán & Muschietti (2001). We assume that Ω is star-shaped with respect to a ball $B(x_0, \rho) \subset \Omega$ (i.e. for any $z \in B(x_0, \rho)$ and any $x \in \Omega$ the segment joining z and x is contained in Ω). Let $w\colon \Omega \to \mathbb{R}$ be a smooth function in $\mathscr{C}_0^\infty(B(x_0, \rho))$ such that $\int_\Omega w = 1$ and define the kernel

$$G(x, y) = \int_0^1 \frac{1}{s^{d+1}}(x - y)w\left(y + \frac{x - y}{s}\right) \mathrm{d}s. \tag{2.7}$$

Let $\psi \in \mathbf{H}_0^1(\Omega)$, then (see Durán & Muschietti, 2001, Theorem 2.1) the following holds:

$$-\int_\Omega G(x, y)\nabla\cdot\psi(y)\mathrm{d}y \in \mathbf{H}_0^1(\Omega). \tag{2.8}$$

This allows us to define the operator $L\colon \mathbf{H}^{-1}(\Omega) \to L^2_{\int=0}(\Omega)$ as follows: for any $f \in \mathbf{H}^{-1}(\Omega)$, $Lf \in L^2_{\int=0}(\Omega)$ is the unique function that solves

$$(Lf, \nabla\cdot\psi)_{L^2} = \left\langle f, -\int_\Omega G(\cdot, y)\nabla\cdot\psi(y)\mathrm{d}y\right\rangle \quad \forall \psi \in \mathbf{H}_0^1(\Omega), \tag{2.9}$$

where $(\cdot,\cdot)_{L^2}$ and $\langle\cdot,\cdot\rangle$ denote the L^2-scalar product and the $\mathbf{H}^{-1}(\Omega)$–$\mathbf{H}^1_0(\Omega)$ duality paring, respectively. Note that the above problem has a unique solution owing to $\nabla\cdot : \mathbf{H}^1_0(\Omega) \rightarrow L^2_{\int=0}(\Omega)$ being surjective.

LEMMA 2.2 If Ω is star-shaped with respect to a ball, the restriction of $L\colon \mathbf{H}^{-1}(\Omega) \rightarrow L^2_{\int=0}(\Omega)$ to $\mathbf{L}^2(\Omega)$ is a bounded operator $L\colon \mathbf{L}^2(\Omega) \rightarrow H^1_{\int=0}(\Omega)$, and L is a simultaneous left inverse of $\nabla\colon L^2_{\int=0}(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$ and $\nabla\colon H^1_{\int=0}(\Omega) \rightarrow \mathbf{L}^2(\Omega)$.

Proof.

(i) We first prove that L is the left inverse of $\nabla\colon L^2_{\int=0}(\Omega) \rightarrow \mathbf{H}^{-1}(\Omega)$. (Actually, the operator L constructed above is the adjoint of the right inverse of $\nabla\cdot : \mathbf{H}^1_0(\Omega) \rightarrow L^2_{\int=0}(\Omega)$ constructed in Durán & Muschietti, 2001.) Let q be a member of $L^2_{\int=0}(\Omega)$. Then, for all $\psi \in \mathbf{H}^1_0(\Omega)$,

$$
\begin{aligned}
(L\nabla q, \nabla\cdot\psi)_{L^2} &= \left\langle \nabla q, -\int_\Omega G(\cdot,y)\nabla\cdot\psi(y)\mathrm{d}y \right\rangle \\
&= \left(q, \nabla\cdot \int_\Omega G(\cdot,y)\nabla\cdot\psi(y)\mathrm{d}y \right)_{L^2} \\
&= (q, \nabla\cdot\psi)_{L^2} \quad \text{(see Durán \& Muschietti, 2001, Theorem 2.1)}.
\end{aligned}
$$

This implies that $L\nabla q = q$, which is the desired result. Note that this immediately implies that L is also a left inverse of $\nabla\colon H^1_{\int=0}(\Omega) \rightarrow \mathbf{L}^2(\Omega)$.

(ii) Proving that the range of $L\colon \mathbf{L}^2(\Omega) \rightarrow L^2_{\int=0}(\Omega)$ is a subset of $H^1_{\int=0}(\Omega)$ is slightly technical and consists of invoking dual arguments from Durán & Muschietti (2001). Let f be a member of $\mathbf{L}^2(\Omega)$ and let $p := Lf \in L^2_{\int=0}(\Omega)$. Then, for all $\psi \in \mathbf{H}^1_0(\Omega)$, we have

$$
\begin{aligned}
-(p, \nabla\cdot\psi)_{L^2} &= \int_\Omega f(x)\cdot\left(\int_\Omega G(x,y)\nabla\cdot\psi(y)\mathrm{d}y\right)\mathrm{d}x \\
&= \int_\Omega \nabla\cdot\psi(y)\left(\int_\Omega G(x,y)\cdot f(x)\mathrm{d}x\right)\mathrm{d}y = \left(\nabla\cdot\psi, \int_\Omega G(x,\cdot)\cdot f(x)\mathrm{d}x\right)_{L^2},
\end{aligned}
$$

where we have applied the Fubini–Tonelli theorem owing to the fact that $G(\cdot,y)$ is in $\mathbf{L}^1(\Omega)$ uniformly w.r.t. y (respectively $G(x,\cdot)$ is in $\mathbf{L}^1(\Omega)$ uniformly w.r.t. x) (see Durán & Muschietti, 2001, Lemma 2.1) and $f \in \mathbf{L}^2(\Omega) \subset \mathbf{L}^1(\Omega)$. This means that

$$
p = -\int_\Omega G(x,\cdot)\cdot f(x)\mathrm{d}x.
$$

Now, by proceeding as in Durán & Muschietti (2001), it can be shown that

$$
-\partial_{y_i} p(y) = \sum_{j=1}^{d}((Q_{ij}f_j)(y) + w_{ij}(y)f_j(y)),
$$

where, denoting by 1_Ω the characteristic function of Ω,

$$(Q_{ij}f_j)(y) = \lim_{\varepsilon\to 0}\int_{|x-y|>\varepsilon} 1_\Omega(x)\partial_{y_i}G_j(x,y)f_j(x)\mathrm{d}x,$$

$$w_{ij}(y) = \int_{\mathbb{R}^d}\frac{z_i z_j}{|z|^2}w(y+z)\mathrm{d}z.$$

The conclusion follows by showing that Q_{ij} is a Calderón–Zygmund operator by proceeding similarly to Durán & Muschietti (2001). The details are omitted for brevity. □

The following lemma relates the existence of a left inverse of an injective operator to the fact that the range of the operator in question is closed (see also Bacuta *et al.*, 2001, Lemma 2.3).

LEMMA 2.3 Let $E_1 \subset E_0$ and $F_1 \subset F_0$ be four Banach spaces with E_1 and F_1 continuously embedded and dense in E_0 and F_0, respectively. Let $T\colon E_j \to F_j$ be a bounded operator, $j = 0, 1$. Assume that T has a simultaneous left inverse on E_1 and E_0, i.e. there exists a bounded operator $L\colon F_j \to E_j$, $j \in \{0, 1\}$, such that $LT = I$ in E_1. Then the range of $T\colon E_\theta \to F_\theta$ is closed, uniformly in θ, in any interpolation pair (E_θ, F_θ), $E_1 \subset E_\theta \subset E_0$, $F_1 \subset F_\theta \subset F_0$.

Proof. By definition, (E_θ, F_θ) being an interpolation pair implies that $T\colon E_\theta \to F_\theta$ and $L\colon F_\theta \to E_\theta$ are bounded uniformly with respect to θ. Moreover, $LTv = v$ for all $v \in E_1 \subset E_\theta$. Since E_1 is dense in E_θ, this implies that $LT = I$ on E_θ. This, together with the boundedness of L, implies that, for all $v \in E_\theta$,

$$\|v\|_{E_\theta} = \|LTv\|_{E_\theta} \leqslant c\|Tv\|_{F_\theta},$$

where $c := \sup_\theta \|L\|_{\mathscr{L}(F_\theta,E_\theta)} < \infty$. That is, T is injective and its range is closed. □

We are now able to conclude.

THEOREM 2.4 If Ω is star-shaped with respect to a ball, then the operator $\nabla\colon H^s_{\int=0}(\Omega) \to \mathbf{H}^{s-1}(\Omega)$, $s \in [0, 1]$, is bounded and injective, and its range is closed uniformly w.r.t. s (i.e. (2.5) holds).

Proof. Using the notation of Lemmas 2.2 and 2.3, set $E_1 = H^1_{\int=0}(\Omega)$, $E_0 = L^2_{\int=0}(\Omega)$, $F_1 = \mathbf{L}^2(\Omega)$, $F_0 = \mathbf{H}^{-1}(\Omega)$, $T = \nabla$ and L as defined in (2.9). Then conclude by applying Lemmas 2.2 and 2.3 and using the identification (2.1). □

3. The discrete setting

We introduce a discrete approximation setting in this section. Our goal is to prove a counterpart of (2.6) within this setting. The main result is Theorem 3.2.

3.1 *Preliminaries*

We assume that we have at hand two families of finite-dimensional spaces, $\{\mathbf{X}_h\}_{h>0}$, $\{M_h\}_{h>0}$, such that $\mathbf{X}_h \subset \mathbf{H}^1_0(\Omega)$ and $M_h \subset L^2_{\int=0}(\Omega)$. To avoid irrelevant technicalities, we assume that $M_h \subset H^1_{\int=0}(\Omega)$.

To characterize the approximation properties of the spaces $\{\mathbf{X}_h\}_{h>0}$, we assume that there is a linear mapping $\mathscr{C}_h : \mathbf{L}^2(\Omega) \to \mathbf{X}_h$ and a constant $c > 0$ uniform in h such that, for all $s \in [0, 1]$,

$$\|\mathscr{C}_h v\|_{\mathbf{H}^s} \leqslant c\|v\|_{\mathbf{H}^s} \quad \forall\, v \in \mathbf{H}_0^s(\Omega), \tag{3.1}$$

$$\|v - \mathscr{C}_h v\|_{\mathbf{L}^2} \leqslant ch^s\|v\|_{\mathbf{H}^s} \quad \forall\, v \in \mathbf{H}_0^s(\Omega). \tag{3.2}$$

One can think of $\mathscr{C}_h$ as the Scott–Zhang (1990) operator in the case of finite elements (it could also be the Clément interpolation operator if the space dimension is two; Clément, 1975).

We moreover assume that the following inverse inequality holds: there is a c uniform in h such that, for all $s \in [0, 1]$,

$$\|v_h\|_{\mathbf{H}^s} \leqslant ch^{-s}\|v_h\|_{\mathbf{L}^2} \quad \forall\, v_h \in \mathbf{X}_h. \tag{3.3}$$

The above hypotheses are usually satisfied when $\mathbf{X}_h$ and M_h are constructed by using finite elements based on quasi-uniform mesh families (Girault & Raviart, 1986).

3.2 *Compatibility between $\mathbf{X}_h$ and M_h*

Let $\pi_h\colon \mathbf{L}^2(\Omega) \to \mathbf{X}_h$ be the L^2-projection onto $\mathbf{X}_h$. One key hypothesis on which the present work is based is the following: $\mathbf{X}_h$ and M_h are compatible in the sense that there is a $c > 0$ independent of h such that

$$\|\pi_h \nabla q_h\|_{\mathbf{L}^2} \geqslant c\|\nabla q_h\|_{L^2} \quad \forall\, q_h \in M_h. \tag{3.4}$$

Owing to the Poincaré–Friedrichs inequality, the above inequality can also be equivalently rewritten as follows:

$$\sup_{0\neq v_h\in\mathbf{X}_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{L}^2}} \geqslant c\|q_h\|_{H^1} \quad \forall\, q_h \in M_h. \tag{3.5}$$

The hypothesis (3.4) has been shown in Guermond (2006, Lemma 2.2) to hold for various pairs of finite-element spaces, e.g. the MINI finite element and the Hood–Taylor finite element.

It is shown by Guermond (2006, Lemma 2.1) that (3.4) implies that the pair $(\mathbf{X}_h, M_h)$ satisfies the so-called LBB condition; that is to say, there is a constant c independent of h such that

$$\sup_{0\neq v_h\in\mathbf{X}_h} \frac{(q_h, \nabla\cdot v_h)}{\|v_h\|_{\mathbf{H}^1}} \geqslant c\|q_h\|_{L^2} \quad \forall\, q_h \in M_h. \tag{3.6}$$

Note that (3.5) and (3.6) are the discrete counterparts of (2.6) for $s = 0$ and $s = 1$. One of the goals of the present paper is to prove that (3.4) implies that similar inequalities hold for the entire range $s \in [0, 1]$.

3.3 *The LBB condition in H^s*

We start with a perturbation lemma *à la* Verfürth (1984).

LEMMA 3.1 Under the (smoothness) assumption (2.5) on Ω and assuming that (3.1) and (3.2) hold, there is a c uniform in h such that, for all $s \in [0, 1]$,

$$\sup_{0\neq v_h\in\mathbf{X}_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{H}^{1-s}}} \geqslant c\|q_h\|_{H^s} - c'h^{1-s}\|\nabla q_h\|_{\mathbf{L}^2} \quad \forall\, q_h \in M_h. \tag{3.7}$$

Proof. Let $q_h \neq 0$ be a nonzero member of M_h. Then, using successively (3.1), (2.5) and (3.2), we infer that

$$\sup_{0\neq v_h\in\mathbf{X}_h}\frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{H}^{1-s}}} \geqslant \sup_{0\neq w\in\mathbf{H}_0^{1-s}(\Omega)}\frac{(\nabla q_h, \mathscr{C}_h w)}{\|\mathscr{C}_h w\|_{\mathbf{H}^{1-s}}} \geqslant c \sup_{0\neq w\in\mathbf{H}_0^{1-s}(\Omega)}\frac{(\nabla q_h, \mathscr{C}_h w)}{\|w\|_{\mathbf{H}^{1-s}}}$$

$$\geqslant c \sup_{0\neq w\in\mathbf{H}_0^{1-s}(\Omega)}\frac{(\nabla q_h, w)}{\|w\|_{\mathbf{H}^{1-s}}} - c \sup_{0\neq w\in\mathbf{H}_0^{1-s}(\Omega)}\frac{(\nabla q_h, \mathscr{C}_h w - w)}{\|w\|_{\mathbf{H}^{1-s}}}$$

$$\geqslant c'\|q_h\|_{H^s} - c\|\nabla q_h\|_{\mathbf{L}^2} \sup_{0\neq w\in\mathbf{H}_0^{1-s}(\Omega)}\frac{\|w - \mathscr{C}_h w\|_{\mathbf{L}^2}}{\|w\|_{\mathbf{H}^{1-s}}}$$

$$\geqslant c\|q_h\|_{H^s} - c' h^{1-s}\|\nabla q_h\|_{\mathbf{L}^2}.$$

This completes the proof. □

We are now in a position to state the discrete counterpart of (2.6), which is the main result of this section.

THEOREM 3.2 Under the (smoothness) assumption (2.5) on Ω and assuming that (3.1)–(3.4) hold, there is a c uniform in h such that, for all $s \in [0, 1]$,

$$\sup_{0\neq v_h\in\mathbf{X}_h}\frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{H}^{1-s}}} \geqslant c\|q_h\|_{H^s} \quad \forall q_h \in M_h. \tag{3.8}$$

Proof. Let q_h be a nonzero member of M_h. Then, using the compatibility hypothesis (3.4) together with the inverse inequality (3.3), we infer that

$$\sup_{0\neq v_h\in\mathbf{X}_h}\frac{(\nabla q_h, v_h)}{\|v_h\|_{\mathbf{H}^{1-s}}} \geqslant \frac{(\nabla q_h, \pi_h\nabla q_h)}{\|\pi_h\nabla q_h\|_{\mathbf{H}^{1-s}}} \geqslant \frac{\|\pi_h\nabla q_h\|_{\mathbf{L}^2}^2}{\|\pi_h\nabla q_h\|_{\mathbf{H}^{1-s}}}$$

$$\geqslant c\frac{\|\pi_h\nabla q_h\|_{\mathbf{L}^2}^2}{h^{s-1}\|\pi_h\nabla q_h\|_{\mathbf{L}^2}} \geqslant ch^{1-s}\|\pi_h\nabla q_h\|_{\mathbf{L}^2}$$

$$\geqslant ch^{1-s}\|\nabla q_h\|_{\mathbf{L}^2}.$$

Then use Lemma 3.1 to conclude. □

REMARK 3.3 L^p-versions of Theorem 3.2 can be found in Ern & Guermond (2004, Section 4.2).

4. Applications

In this section we present two applications of the above analysis: the H^s-stability for the Stokes problem and an *a priori* estimate of the pressure for the nonstationary Stokes equations. The first application is quite straightforward, whereas the second is slightly more sophisticated and has far-reaching consequences for the analysis of the three-dimensional Navier–Stokes equations.

We assume that Ω is smooth enough so that there is a $c > 0$ such that

$$\forall v \in \mathbf{V}^2, \quad \|v\|_{\mathbf{H}^2} + \|(1 - P)\Delta v\|_{\mathbf{L}^2} \leqslant c\|Av\|_{\mathbf{L}^2}. \tag{4.1}$$

Ω being convex or Ω being of class $\mathscr{C}^{1,1}$ are known to be sufficient conditions for (4.1) to hold in two and three space dimensions ($d = 2, 3$) (cf. e.g. Grisvard, 1985; Dauge, 1989, Theorem 6.3).

4.1 *H^s-approximation for the Stokes problem*

We define the discrete Laplace operator $\Delta_h \colon \mathbf{X}_h \to \mathbf{X}_h$ as follows:

$$(\Delta_h x_h, y_h) = -(\nabla x_h, \nabla y_h) \quad \forall x_h, y_h \in \mathbf{X}_h.$$

We set

$$\mathbf{V}_h = \{v_h \in \mathbf{X}_h;\ (v_h, \nabla q_h) = 0 \quad \forall q_h \in M_h\}. \tag{4.2}$$

$\mathbf{V}_h$ is composed of the fields of $\mathbf{X}_h$ that are discretely divergence free. This allows us to define the discrete Stokes operator $A_h \colon \mathbf{V}_h \to \mathbf{V}_h$ as follows: for all $u_h \in \mathbf{V}_h$, $A_h u_h$ is the element of $\mathbf{V}_h$ such that

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h) \quad \forall v_h \in \mathbf{V}_h. \tag{4.3}$$

Then we have the following discrete counterpart of (4.1).

LEMMA 4.1 Under the smoothness assumption (4.1) on Ω and assuming that (3.1)–(3.4) hold, there is a $c > 0$ uniform in h so that uniformly

$$\|\Delta_h v_h\|_{\mathbf{L}^2} \leqslant c \|A_h v_h\|_{\mathbf{L}^2} \quad \forall v_h \in \mathbf{V}_h. \tag{4.4}$$

Proof. The proof is standard and can be found in, for example, Heywood & Rannacher (1982, Corollary 4.4) or Guermond & Pasciak (2007, Lemma 4.1). We nevertheless reproduce it here for completeness. Let v_h be a member of $\mathbf{V}_h$. Let $(v, p) \in \mathbf{H}^1_0(\Omega) \times L^2_{\int=0}(\Omega)$ be the solution of the Stokes problem with the data $A_h v_h$, i.e.

$$\begin{aligned} (\nabla v, \nabla l) - (p, \nabla \cdot l) &= (A_h v_h, l) \quad \forall l \in \mathbf{H}^1_0(\Omega), \\ (\nabla \cdot v, q) &= 0 \quad \forall q \in L^2_{\int=0}(\Omega). \end{aligned}$$

Let $(w_h, r_h) \in \mathbf{X}_h \times M_h$ be the solution to

$$\begin{aligned} (\nabla w_h, \nabla l_h) - (r_h, \nabla \cdot l_h) &= (A_h v_h, l_h) \quad \forall l_h \in \mathbf{X}_h, \\ (\nabla \cdot w_h, q_h) &= 0 \quad \forall q_h \in M_h. \end{aligned}$$

Clearly, $w_h \in \mathbf{V}_h$ and actually $w_h = v_h$. This means that v_h is the Galerkin approximation to v. The theory of mixed problems together with the smoothness assumptions (4.1) and (3.6) implies that

$$\|v - v_h\|_{\mathbf{H}^1} \leqslant ch(\|v\|_{\mathbf{H}^2} + \|p\|_{H^1}) \leqslant ch\|A_h v_h\|_{\mathbf{L}^2}.$$

We then have, for $x_h \in \mathbf{X}_h$,

$$\begin{aligned} |(\nabla v_h, \nabla x_h)| &\leqslant |(\nabla(v_h - v), \nabla x_h)| + |(\Delta v, x_h)| \\ &\leqslant c(h\|x_h\|_{\mathbf{H}^1} + \|x_h\|_{\mathbf{L}^2})\|A_h v_h\|_{\mathbf{L}^2} \leqslant c\|x_h\|_{\mathbf{L}^2}\|A_h v_h\|_{\mathbf{L}^2}. \end{aligned}$$

Thus

$$\|\varDelta_h v_h\|_{\mathbf{L}^2} = \sup_{0\neq x_h\in\mathbf{X}_h} \frac{(\nabla v_h, \nabla x_h)}{\|x_h\|_{\mathbf{L}^2}} \leqslant c\|A_h v_h\|_{\mathbf{L}^2},$$

which completes the proof of the lemma. □

Finally, we assume that the family of approximation spaces $(\mathbf{X}_h)_{h>0}$ is such that π_h is uniformly $\mathbf{H}^1$-stable, i.e. there is a c independent of h such that

$$\|\pi_h v\|_{\mathbf{H}^1} \leqslant c\|v\|_{\mathbf{H}^1}, \tag{4.5}$$

for all v in $\mathbf{H}^1_0(\Omega)$. When the spaces $(\mathbf{X}_h)_{h>0}$ are finite element based, this assumption is known to hold under quite weak regularity requirements on the underlying mesh family (Bramble *et al.*, 2002).

Let us define the mappings $R\colon \mathbf{H}^{-1}(\Omega) \to \mathbf{H}^1_0(\Omega)$ and $S\colon \mathbf{H}^{-1}(\Omega) \to L^2_{\int=0}(\Omega)$ such that

$$\begin{cases} -\varDelta R(f) + \nabla S(f) = f, \\ \nabla\cdot R(f) = 0, \quad R(f)|_\Gamma = 0. \end{cases} \tag{4.6}$$

We now define the approximate mappings $R_h\colon \mathbf{H}^{-1}(\Omega) \to \mathbf{X}_h$ and $S_h\colon \mathbf{H}^{-1}(\Omega) \to M_h$ such that, for all $f \in \mathbf{H}^{-1}(\Omega)$, $R_h(f)$ and $S_h(f)$ solve

$$\begin{cases} (\nabla R_h(f), \nabla v_h) - (S_h(f), \nabla\cdot v_h) = \langle f, v_h\rangle & \forall v_h \in \mathbf{X}_h, \\ (q_h, \nabla\cdot R_h(f)) = 0 & \forall q_h \in M_h. \end{cases} \tag{4.7}$$

It is well known that this yields a stable and convergent approximation method (see e.g. Girault & Raviart, 1986; Brezzi & Fortin, 1991). In particular, the following stability estimate holds:

$$\|R_h(f)\|_{\mathbf{H}^1} + \|S_h(f)\|_{L^2} \leqslant c\|f\|_{\mathbf{H}^{-1}}. \tag{4.8}$$

A more general result is stated in the following theorem that together with Corollary 4.3 is the main result of this section.

THEOREM 4.2 Under the hypotheses of Lemma 4.1, there is a c uniform in h so that, for all $s \in [0, 1]$ and for all $f \in \mathbf{H}^{-s}(\Omega)$, the following holds:

$$\|\varDelta_h R_h(f)\|_{\mathbf{H}^{-s}} + \|S_h(f)\|_{H^{1-s}} \leqslant c\|f\|_{\mathbf{H}^{-s}}. \tag{4.9}$$

Proof.

(i) Bound on $\|\varDelta_h R_h(f)\|_{\mathbf{L}^2}$. Assume that $\|f\|_{\mathbf{L}^2}$ is bounded. Using (4.4), we infer that

$$\begin{aligned}\|\varDelta_h R_h(f)\|_{\mathbf{L}^2} &\leqslant c\|A_h R_h(f)\|_{\mathbf{L}^2} \leqslant c \sup_{0\neq v_h\in\mathbf{V}_h} \frac{(A_h R_h(f), v_h)}{\|v_h\|_{\mathbf{L}^2}} \\ &\leqslant c \sup_{0\neq v_h\in\mathbf{V}_h} \frac{(\nabla R_h(f), \nabla v_h)}{\|v_h\|_{\mathbf{L}^2}} \leqslant c \sup_{0\neq v_h\in\mathbf{V}_h} \frac{\langle f, v_h\rangle}{\|v_h\|_{\mathbf{L}^2}} \\ &\leqslant c\|f\|_{\mathbf{L}^2}.\end{aligned}$$

(ii) Bound on $\|\Delta_h R_h(f)\|_{\mathbf{H}^{-1}}$. Using the $\mathbf{H}^1$-stability of π_h, we obtain

$$\|\Delta_h R_h(f)\|_{\mathbf{H}^{-1}} = \sup_{0\neq v\in\mathbf{H}_0^1(\Omega)} \frac{(\Delta_h R_h(f), v)}{\|v\|_{\mathbf{H}^1}} = \sup_{0\neq v\in\mathbf{H}_0^1(\Omega)} \frac{(\nabla R_h(f), \nabla(\pi_h v))}{\|v\|_{\mathbf{H}^1}}$$

$$= \sup_{0\neq v\in\mathbf{H}_0^1(\Omega)} \frac{\langle f, \pi_h v\rangle + (S_h(f), \nabla\cdot(\pi_h v))}{\|v\|_{\mathbf{H}^1}}$$

$$\leqslant c(\|f\|_{\mathbf{H}^{-1}} + \|S_h(f)\|_{L^2}).$$

Then, using the stability estimate (4.8), we deduce that

$$\|\Delta_h R_h(f)\|_{\mathbf{H}^{-1}} \leqslant c\|f\|_{\mathbf{H}^{-1}}.$$

(iii) Interpolation. We now apply the real method of interpolation (Lions & Peetre, 1964; Lions & Magenes, 1968) to the mapping $T\colon \mathbf{H}^{-1}(\Omega) \ni f \mapsto \Delta_h R_h(f) \in \mathbf{H}^{-1}(\Omega)$ and $T\colon \mathbf{L}^2(\Omega) \ni f \mapsto \Delta_h R_h(f) \in \mathbf{L}^2(\Omega)$. This gives

$$\|\Delta_h R_h(f)\|_{\mathbf{H}^{-s}} \leqslant c\|f\|_{\mathbf{H}^{-s}}.$$

(iv) Estimate of the pressure. The estimate of the pressure is obtained by using Theorem 3.2:

$$\|S_h(f)\|_{H^{1-s}} \leqslant c \sup_{0\neq v_h\in\mathbf{X}_h} \frac{(\nabla S_h(f), v_h)}{\|v_h\|_{\mathbf{H}^s}}$$

$$= c \sup_{0\neq v_h\in\mathbf{X}_h} \frac{(\nabla R_h(f), \nabla v_h) - \langle f, v_h\rangle}{\|v_h\|_{\mathbf{H}^s}}$$

$$= c \sup_{0\neq v_h\in\mathbf{X}_h} \frac{(-\Delta_h R_h(f), v_h) - \langle f, v_h\rangle}{\|v_h\|_{\mathbf{H}^s}}$$

$$\leqslant c(\|\Delta_h R_h(f)\|_{\mathbf{H}^{-s}} + \|f\|_{\mathbf{H}^{-s}}) \leqslant c\|f\|_{\mathbf{H}^{-s}}.$$

This completes the proof. □

Let f be a given function in $\mathbf{H}^{-s}(\Omega)$. We now make some change of notation by setting $u := R(f)$, $p := S(f)$, $u_h := R_h(f)$ and $p_h := S_h(f)$. The following corollary gives an estimate of the way the pair (u_h, p_h) approximates (u, p).

COROLLARY 4.3 There is a c uniform in h so that, for all $s \in [0, 1]$,

$$\|\Delta u - \Delta_h u_h\|_{\mathbf{H}^{-s}} + \|p - p_h\|_{H^{1-s}}$$

$$\leqslant c\left(\inf_{v_h\in\mathbf{V}_h} \|\Delta u - \Delta_h v_h\|_{\mathbf{H}^{-s}} + \inf_{q_h\in\mathbf{M}_h} \|p - q_h\|_{H^{1-s}}\right). \tag{4.10}$$

Proof. Let $v_h \in \mathbf{V}_h$ and $q_h \in M_h$ be two arbitrary discrete functions and let us set $g := -\varDelta_h v_h + \nabla q_h$. It is clear that $R_h(g) = v_h$ and $S_h(g) = q_h$. Then Theorem 4.2 implies that

$$\begin{aligned}\|\varDelta_h R_h(f-g)\|_{\mathbf{H}^{-s}} + \|S_h(f-g)\|_{H^{1-s}} &\leqslant c\|f-g\|_{\mathbf{H}^{-s}}\\ &\leqslant c(\|-\varDelta u + \nabla p + \varDelta_h v_h - \nabla q_h\|_{\mathbf{H}^{-s}})\\ &\leqslant c(\|-\varDelta u + \varDelta_h v_h\|_{\mathbf{H}^{-s}} + \|p - q_h\|_{\mathbf{H}^{1-s}}).\end{aligned}$$

Then, using the triangle inequality and the above estimate, we infer that

$$\begin{aligned}\|\varDelta_h u_h - \varDelta u\|_{\mathbf{H}^{-s}} + \|p_h - p\|_{H^{1-s}} &= \|\varDelta_h R_h(f) - \varDelta u\|_{\mathbf{H}^{-s}} + \|S_h(f) - p\|_{H^{1-s}}\\ &\leqslant \|\varDelta_h R_h(f-g)\|_{\mathbf{H}^{-s}} + \|S_h(f-g)\|_{H^{1-s}}\\ &\quad + \|\varDelta_h v_h - \varDelta u\|_{\mathbf{H}^{-s}} + \|q_h - p\|_{H^{1-s}}\\ &\leqslant c(\|-\varDelta u + \varDelta_h v_h\|_{\mathbf{H}^{-s}} + \|p - q_h\|_{\mathbf{H}^{1-s}});\end{aligned}$$

then conclude by taking the infimum on v_h and q_h. □

REMARK 4.4 Note in passing that (4.9) gives an estimate for the velocity $R_h(f)$ in $\mathbf{H}^{2-s}(\Omega)$ when $s \in \left(\frac{1}{2}, 1\right]$. It is shown in Guermond & Pasciak (2007, Lemma 2.2) that, under the assumptions on the discrete setting stated above, there is a positive nonincreasing function c_l and a positive nondecreasing function c_u, both uniform in h, such that, for all $s \in \left(-\frac{3}{2}, \frac{3}{2}\right)$,

$$c_l(|s|)\|v_h\|_{\mathbf{H}_0^s} \leqslant ((-\varDelta_h)^s v_h, v_h)^{\frac{1}{2}} \leqslant c_u(|s|)\|v_h\|_{\mathbf{H}_0^s} \quad \forall v_h \in \mathbf{X}_h,$$

where $\mathbf{H}_0^s(\Omega) := [\mathbf{H}^1, \mathbf{H}^2]_s \cap \mathbf{H}_0^1(\Omega)$ for $s \in \left[1, \frac{3}{2}\right)$ and $\mathbf{H}_0^{-s}(\Omega)$ is the dual of $\mathbf{H}_0^s(\Omega)$. Applying these two bounds to $\varDelta_h R_h(f)$ with $s \in \left(\frac{1}{2}, 1\right]$, we obtain

$$\|\varDelta_h R_h(f)\|_{\mathbf{H}^{-s}} \geqslant c((-\varDelta_h)^{2-s} R_h(f), R_h(f))^{\frac{1}{2}} \geqslant c'\|R_h(f)\|_{\mathbf{H}^{2-s}},$$

and the conclusion follows readily.

4.2 *Application to the nonstationary Stokes equations*

As an application of Theorem 3.2 we show in this section how to derive an *a priori* estimate of the pressure for the Galerkin approximation of the nonstationary Stokes equations.

Let $(0, T)$ be a time interval (possibly arbitrarily large). Let $u_0 \in \mathbf{V}^0$, $p \in (1, +\infty)$, $q \in (1, +\infty)$ and $f \in L^p(0, T; \mathbf{L}^q(\Omega))$, and consider the time-dependent Stokes equations

$$\begin{cases} \partial_t u - \varDelta u + \nabla \mathsf{p} = f & \text{in } \Omega_T,\\ \nabla \cdot u = 0 & \text{in } \Omega_T,\\ u|_\Gamma = 0, & u|_{t=0} = u_0, \end{cases} \tag{4.11}$$

where $\Omega_T = \Omega \times (0, T)$ (note that p is the pressure and p is an exponent). It is well known that this problem has a unique weak solution in appropriate functional spaces. In particular, if $u_0 = 0$, $p = q$

and Ω is of class $\mathscr{C}^2$, it is proved in Solonnikov (1976) that the following bound holds:

$$\|\nabla \mathsf{p}\|_{L^p(\Omega_T)} + \|\partial_t u\|_{L^p(\Omega_T)} + \|\Delta u\|_{L^p(\Omega_T)} \leqslant c\|f\|_{L^p(\Omega_T)}. \tag{4.12}$$

Still assuming that Ω is of class $\mathscr{C}^2$, this estimate has been significantly generalized in Sohr & von Wahl (1986) to account for different exponents p and q:

$$\|\nabla \mathsf{p}\|_{L^p(0,T;\mathbf{L}^q)} + \|\partial_t u\|_{L^p(0,T;\mathbf{L}^q)} + \|\Delta u\|_{L^p(0,T;\mathbf{L}^q)} \leqslant c\|f\|_{L^p(0,T;\mathbf{L}^q)}, \tag{4.13}$$

$$\|\mathsf{p}\|_{L^p(0,T;\mathbf{L}^\ell)} \leqslant c\|f\|_{L^p(0,T;\mathbf{L}^q)}, \tag{4.14}$$

where $\frac{1}{\ell} := \frac{1}{q} - \frac{1}{d}$. These estimates are important for constructing weak solutions of the Navier–Stokes equations that are suitable in the sense of Scheffer (1977).

The present work is part of a research programme aiming at characterizing suitable weak solutions of the Navier–Stokes equations in three space dimensions. To understand the importance of suitable weak solutions, recall that at present the best partial regularity result for the Navier–Stokes equations asserts that the one-dimensional Hausdorff measure of the set of singularities of a suitable weak solution is zero (this is the so-called Caffarelli–Kohn–Nirenberg theorem; Caffarelli *et al.*, 1982; Lin, 1998). This result is not known to hold for weak solutions (i.e. suitable weak solutions are *a priori* smoother than weak solutions). It is not known if suitable weak solutions are unique (a positive answer would close the Navier–Stokes debate). It is not known if there are weak solutions that are not suitable.

One goal of the research programme mentioned above is to prove that finite-element-based Faedo–Galerkin approximations to the three-dimensional Navier–Stokes equations converge (up to subsequences) to suitable weak solutions. This property has been proved to hold in the three-dimensional torus, i.e. with periodic boundary conditions (Guermond, 2006). (At present this result is not known to hold for Fourier-based Faedo–Galerkin approximations.) To eventually prove that the result is also true with Dirichlet boundary conditions, it is important to reproduce discrete counterparts of the estimates (4.13) and (4.14) using the discrete (finite-element-like) setting introduced above.

For this purpose and to avoid using the non-Hilbertian $L^p(\mathbf{L}^q)$-framework, we define fractional Sobolev spaces in time. Let H be a Hilbert space with norm $\|\cdot\|_H$. Consider δ with $1 \leqslant \delta < \infty$, and define $L^\delta(\mathbb{R}; H) = \{\psi : \mathbb{R} \ni t \mapsto \psi(t) \in H;\ \int_{-\infty}^{+\infty} \|\psi(t)\|_H^\delta \,\mathrm{d}t < \infty\}$. For all $\psi \in L^1(\mathbb{R}; H)$, denote by $\hat{\psi}(k) = \int_{-\infty}^{+\infty} \psi(t) \mathrm{e}^{-2\mathrm{i}\pi kt} \,\mathrm{d}t$ for all $k \in \mathbb{R}$. This notion of Fourier transform is then extended to the space of tempered distributions on $\mathbb{R}$ with values in H, say $\mathscr{S}'(\mathbb{R}; H)$. Then, following Lions & Magenes (1968, p. 21), we define

$$H^\gamma(\mathbb{R}; H) = \left\{ v \in \mathscr{S}'(\mathbb{R}; H);\ \int_{-\infty}^{+\infty} (1 + |k|)^{2\gamma} \|\hat{v}\|_H^2 \,\mathrm{d}k < +\infty \right\}. \tag{4.15}$$

We then define the space $H^\gamma((0,T); H)$ to be composed of those tempered distributions in $\mathscr{S}'((0,T);$ $H)$ that can be extended to $\mathscr{S}'(\mathbb{R}; H)$ and whose extension is in $H^\gamma(\mathbb{R}; H)$. The norm in $H^\gamma((0,T); H)$ is the quotient norm, i.e.

$$\|v\|_{H^\gamma((0,T);H)} = \inf_{\substack{\tilde{v}=u \\ \text{a.e. on } (0,T)}} \|\tilde{v}\|_{H^\gamma(\mathbb{R};H)}. \tag{4.16}$$

We henceforth assume the following:

$$q \in (1,2) \quad \text{and} \quad p \in (1,2). \tag{4.17}$$

Then, upon setting $s = s(q) := d\left(\frac{1}{q} - \frac{1}{2}\right)$ and $\overline{r} := \frac{1}{p} - \frac{1}{2}$, standard embedding inequalities imply

$$f \in L^p(0, T; \mathbf{L}^q(\Omega)) \subset H^{-r}((0, T); \mathbf{H}^{-s}(\Omega)) \quad \forall r > \overline{r}. \tag{4.18}$$

Our goal is to reformulate (4.13) and (4.14) using the fractional Sobolev spaces $H^{-r}((0, T); \mathbf{H}^{-s}(\Omega))$.

To avoid unimportant technicalities, we assume $u_0 = 0$. The approximate counterpart of (4.11) is as follows:

$$\begin{cases} \partial_t u_h - \varDelta_h u_h + B_h \mathsf{p}_h = \pi_h f, \quad \text{for a.e. } t \in (0, T), \\ B_h^{\mathrm{T}} u_h = 0, \\ u_h|_{t=0} = 0, \end{cases} \tag{4.19}$$

where $B_h := \pi_h \nabla|_{M_h}$. This discrete problem has a unique solution (this is a system of linear ordinary differential equations). The following stability estimates are proved in Guermond & Pasciak (2007).

PROPOSITION 4.5 There is a c independent of h so that, for all $r > \overline{r} := \frac{1}{p} - \frac{1}{2}$,

$$\|\varDelta_h u_h\|_{H^{-r}((0,T);\mathbf{H}^{-s})} \leqslant c. \tag{4.20}$$

Moreover, if q is such that $s(q) < \frac{1}{2}$, then

$$\|\partial_t u_h\|_{H^{-r}((0,T);\mathbf{H}^{-s})} \leqslant c. \tag{4.21}$$

As an immediate consequence of (4.20) and (4.21), we deduce that

$$\|B_h \mathsf{p}_h\|_{H^{-r}((0,T);\mathbf{H}^{-s})} \leqslant c \tag{4.22}$$

whenever $s(q) < \frac{1}{2}$.

REMARK 4.6 Observe that (4.20)–(4.22) are the discrete counterparts of (4.13) in the Hilbert space $H^{-r}((0, T); \mathbf{H}^{-s}(\Omega))$, where the members of the pairs (s, q) and $(\overline{r}, p)$ are in correspondence through the continuous embeddings $\mathbf{H}^s(\Omega) \subset \mathbf{L}^q(\Omega)$ and $H^{\overline{r}}(0, T) \subset L^p(0, T)$, where $\frac{1}{q} = \frac{1}{2} - \frac{s}{d}$ and $\frac{1}{p} = \frac{1}{2} - \overline{r}$.

Now we have to derive a discrete counterpart for (4.14). It is clear that (4.14) is just an application of Sobolev's embedding, and one could imagine using a similar argument to deduce an estimate for the discrete pressure. Unfortunately, the embedding argument cannot be applied in (4.22) for two reasons: $\mathbf{H}^{-s}(\Omega)$ does not embed in any Lebesgue space and B_h is a discrete operator. Actually, Theorem 3.2 is the key argument that will do the job (and the primary motivation for the present paper).

COROLLARY 4.7 If q is such that $s(q) < \frac{1}{2}$ then, for all $r > \overline{r} := \frac{1}{p} - \frac{1}{2}$,

$$\|\mathsf{p}_h\|_{H^{-r}((0,T);\mathbf{H}^{1-s})} \leqslant c. \tag{4.23}$$

Proof. Clearly, we have $\pi_h \nabla \mathsf{p}_h = \pi_h f - \partial_t u_h + \varDelta_h u_h$. Then applying Theorem 3.2 we infer that

$$\|\mathsf{p}_h\|_{H^{1-s}} \leqslant \sup_{0 \neq v_h \in \mathbf{X}_h} \frac{(\pi_h f - \partial_t u_h + \varDelta_h u_h, v_h)}{\|v_h\|_{\mathbf{H}^s}}.$$

Conclude using Proposition 4.5. □

The application of this estimate to the aforementioned research programme for the construction of suitable weak solutions to the three-dimensional Navier–Stokes equations is reported in Guermond (2007).

REMARK 4.8 The careful reader may object at this point that when applied to the Navier–Stokes equations in three space dimensions, the restriction $s(q) < \frac{1}{2}$ in Proposition 4.5 makes the bound (4.21) somewhat useless. Indeed, the above analysis applies to the Navier–Stokes equations with $f = g - u_h{\cdot}\nabla u_h$, where g is a given smooth source and $u_h{\cdot}\nabla u_h$ is the nonlinear advection term. Since a standard uniform estimate in $L^\infty((0,T);\mathbf{L}^2(\Omega)) \cap L^2((0,T);\mathbf{H}^1_0(\Omega))$ holds on u_h, we find that $f \in L^p(0,T;\mathbf{L}^q(\Omega))$, where p and q satisfy the equality $\frac{2}{p} + \frac{3}{q} = 4$ and $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \frac{3}{2}$. The restriction on q yields $\frac{1}{2} \leqslant s = 3\left(\frac{1}{q} - \frac{1}{2}\right) \leqslant \frac{3}{2}$, which is contradictory to the assumption $s < \frac{1}{2}$. This objection is overcome as follows. As shown in Guermond & Pasciak (2007), it is possible to exploit the *a priori* bound $\|u_h\|_{L^2((0,T);\mathbf{H}^1)} \leqslant c$ to deduce the bound $\|\partial_t u_h\|_{H^{-\frac{2}{5}-3\varepsilon}\left((0,T);\mathbf{H}^{-\frac{1}{2}+\varepsilon}\right)} \leqslant c$ (valid for all $\varepsilon \in \left[0, \frac{1}{4}\right]$), which is slightly sharper than (4.21). Then, taking $p = 1$ (i.e. $q = \frac{3}{2}$ and $s = \frac{1}{2}$), we infer from (4.20) that $\|\Delta_h u_h\|_{H^{-\frac{1}{2}-\varepsilon}\left((0,T);\mathbf{H}^{-\frac{1}{2}}\right)}$ is bounded, and repeating the argument in the proof of Corollary 4.7, we deduce that $\|p_h\|_{H^{-\frac{1}{2}-\varepsilon}\left((0,T);H^{\frac{1}{2}}\right)}$ is bounded (see Guermond & Pasciak, 2007 and Guermond, 2007 for the details). Note again that Theorem 3.2 is the key argument.

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Appendix

The purpose of this appendix is to prove Lemma A1 and give an illustration of this lemma that is slightly less trivial than proving Lemma 2.1.

LEMMA A1 Let $E_1 \subset E_0$ be two Banach spaces with E_1 continuously embedded in E_0. Let $T\colon E_j \to E_j$ be a bounded operator with closed range and assume that T is a projection, $j \in \{0, 1\}$. Denote by K_0 and K_1 the ranges of $T|_{E_0}$ and $T|_{E_1}$, respectively. Then the following two spaces coincide with equivalent norms:

$$[K_0, K_1]_s = [E_0, E_1]_s \cap K_0 \quad \forall\, s \in (0, 1).$$

Proof. (1) For all $v \in K_j$, we set $\|v\|_{K_j} := \|v\|_{E_j}$; this makes sense since K_j is closed in E_j, $j \in \{0, 1\}$. (2) Let $s \in (0, 1)$. We now prove that $[K_0, K_1]_s \subset [E_0, E_1]_s \cap K_0$ with continuous injection. Let u be a member of $[K_0, K_1]_s \subset K_0 + K_1 \subset E_0 + E_1$. The K-functional associated with the norm in $[K_0, K_1]_s$ satisfies the following:

$$K(t, u, K_0, K_1) := \inf_{v \in K_1} (\|v\|^2_{K_0} + t^2 \|u - v\|^2_{K_1})^{\frac{1}{2}}$$

$$\geqslant \inf_{v \in E_1} (\|v\|^2_{E_0} + t^2 \|u - v\|^2_{E_1})^{\frac{1}{2}} := K(t, u, E_0, E_1).$$

As a result, $u \in [E_0, E_1]_s$ and the embedding $[K_0, K_1]_s \subset [E_0, E_1]_s$ is continuous. Moreover, clearly $[K_0, K_1]_s \subset K_0$, i.e. $[K_0, K_1]_s \subset [E_0, E_1]_s \cap K_0$.

(3) Let us prove the converse. Let u be a member of $[E_0, E_1]_s \cap K_0$. Then, owing to the fact that T is a projection, i.e. $Tu = u$, and T is bounded on E_1 and E_0, the K-functional associated with the norm in $[E_0, E_1]_s$ satisfies the following:

$$K(t, u, E_0, E_1) := \inf_{v \in E_1} (\|v\|^2_{E_0} + t^2 \|u - v\|^2_{E_1})^{\frac{1}{2}}$$

$$\geqslant c \inf_{v \in E_1} (\|Tv\|^2_{E_0} + t^2 \|u - Tv\|^2_{E_1})^{\frac{1}{2}}$$

$$\geqslant c \inf_{w \in K_1} (\|w\|^2_{K_0} + t^2 \|u - w\|^2_{K_1})^{\frac{1}{2}}.$$

In other words, $u \in [K_0, K_1]_s$ and the injection $[E_0, E_1]_s \cap K_0 \subset [K_0, K_1]_s$ is continuous. □

Let us now assume the following smoothness hypothesis on the domain Ω: there is a $c > 0$ so that

$$\|v\|_{\mathbf{H}^2} \leqslant c \|\Delta v\|_{\mathbf{L}^2} \quad \forall v \in \mathbf{D}(\Delta). \tag{A.1}$$

This property is known to hold in arbitrary space dimension if Ω is convex or is of class $\mathscr{C}^1$. We finally give the following illustration of Lemma A1.

LEMMA A2 Provided the elliptic regularity (A.1) holds, the following two spaces coincide with equivalent norms:

$$[H^1_0(\Omega), H^2(\Omega) \cap H^1_0(\Omega)]_s = H^{1+s}(\Omega) \cap H^1_0(\Omega) \quad \forall s \in (0, 1).$$

This result seems to be part of the folklore in numerical analysis, but the only proof the author is aware of is that of Bacuta *et al.* (2001) that is somewhat involved and restricted to two space dimensions (without the elliptic regularity assumption (A.1) though).

Proof. Let us define the mapping $T : H^2(\Omega) \to H^2(\Omega) \cap H^1_0(\Omega)$ such that $\Delta T v = \Delta v$, i.e. Tv solves a Poisson equation with homogeneous Dirichlet boundary condition. Note that T is obviously bounded in $H^1(\Omega)$ and it is also bounded in H^2 owing to the elliptic regularity (A.1). T is clearly a projection. The range of $T|_{H^1(\Omega)}$ is $H^1_0(\Omega)$ and is clearly closed in $H^1(\Omega)$. The range of $T|_{H^2(\Omega)}$ is $H^2(\Omega) \cap H^1_0(\Omega)$ and is clearly closed in $H^2(\Omega)$. Then Lemma A2 is a simple consequence of Lemma A1. □