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An interpretation of the Navier–Stokes-alpha model as a frame-indifferent Leray regularization

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Abstract

We show in this note that the Leray regularization of the Navier–Stokes equations is almost frame-indifferent but for one second-order term. Neglecting this term, we readily recover the Navier–Stokes-alpha equations introduced by Holm et al. [Adv. Math. 137 (1998) 1; Phys. Fluids 11 (8) (1999) 2343]. Hence, as an alternative to the exposition in [Adv. Math. 137 (1998) 1; Phys. Fluids 11 (8) (1999) 2343], we propose to interpret the Navier–Stokes-alpha model as a frame-indifferent perturbation of the Leray regularization.
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1. Introduction

In recent years, Holm et al. [1,2] introduced a new turbulence model, the so-called Navier–Stokes-alpha (NS-α) model, which falls into the family of large eddy simulation (LES) models. LES models [3] are based on the idea that the whole range of flow scales in turbulence may not be important in many engineering applications. This observation has led modelers to devise new artifacts for modeling the interaction between the numerically unreachable small scales and the large ones so that turbulent flow can be simulated without having to accurately approximate all the fine scales present in the flow. One attractive feature of the NS-α model is that it is founded on rigorous mathematical arguments, which is not always true for other LES models. However, as originally presented by the authors, the model is “derived by applying time averaging procedures to the Hamilton’s principle for an ideal incompressible fluid flow”, and “by using the Euler–Poincaré variational framework”, interpreting the “Euler–Poincaré equations as the Lagrangian version of Lie–Poisson Hamilton systems”. We show in this note that

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a simpler interpretation of the model can be advanced by invoking the Leray regularization of the Navier–Stokes equations.

Leray, in his ground-breaking paper [4], introduced the notion of weak solutions of the Navier–Stokes equations, the so-called "*solutions turbulentes*", and proved existence of such solutions. He also recognized that his mathematical theory of the Navier–Stokes equations in 3D was incomplete since uniqueness could not be proved. Since 2000, the uniqueness question is one of the seven outstanding mathematical problems selected by the Clay Institute for its 1M$ prizes. Leray interpreted the difficulty by suggesting that unbounded vorticity bursts at very small scales could intermittently occur in fluid flows, leaving doubt on the determinism of the equations. To somehow overcome the occurrence of such vorticity bursts in turbulent solutions, Leray proposed to regularize the Navier–Stokes equations, replacing the advective velocity by a smooth velocity, and thereby was able to prove existence and uniqueness of solutions to the regularized system.

Following Leray's ideas, our goal in this note is to propose a new interpretation of the NS-α model by showing that this model is actually a perturbation of the Leray regularization that transforms the frame-dependent Leray regularization into a frame-indifferent model. The present note is organized into three sections. In Section 2 we recall basic facts about the Leray regularization and filtering operators. We also suggest that the Leray regularization can be viewed as a LES model. In Section 3, we present our study on frame indifference and conclude that the regularized equations are not frame-indifferent due to the presence of a small second-order term. We show in Section 4 that neglecting this term from the Leray regularized equations allows us to recover, in a simple manner, the NS-α model. We finally comment on the fact that NS-α constitutes a mathematically rigorous model for LES.

2. Leray regularization

2.1. Preliminaries

In this note, we consider periodic solutions of the Navier–Stokes equations in the domain $\Omega = (0, 2\pi)^3$

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \text{ is periodic}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{2.1}$$

where $\mathbf{u}_0$ is the initial velocity and $\mathbf{f}$ a source term that is assumed to be independent of time for the sake of simplicity.

Determining a class of functions in which this problem has a unique solution in time still remains an open question. The issue is that even if reasonably smooth initial data are prescribed, it has not been proven yet (or disproved) that the resulting solutions are smooth for arbitrarily long times. In particular, no a priori estimate has yet been derived that guarantees that the enstrophy $\int_\Omega \|\nabla \times \mathbf{u}\|^2 \, dx$ remains finite for all times; in other words, it is still unknown whether the vorticity can blow up intermittently in some regions of the domain. Hence, we cannot exclude a priori the possible occurrence of rare (intermittent) vorticity bursts driving the energy deep down to scales much shorter than the standard Kolmogorov scale. In the present setting, the vortex stretching mechanism generated by the nonlinear term of the Navier–Stokes equations may have such powerful effects that one is not guaranteed that the linear viscosity is strong enough to stop the energy cascade everywhere in the flowfield.

If such blow-ups were to occur, the time-evolution of the solution would certainly not be unique and the deterministic Newtonian postulate would then be lost. Moreover, finite time singularities would imply that arbitrarily small-scale structures would develop in the flow, thus violating the axiom of continuum mechanics, which assumes scale separation between individual atomic evolution and collective hydrodynamics motion. Hence, although the uniqueness question may seem to be an irrelevant issue to many, it is "actually intimately tied up with the efficiency of the Navier–Stokes equations as a model for fluid turbulence" [5, p. xii].

2.2. Regularized equations

In the breakthrough paper [4] published as early as 1934, Leray proposed to regularize the Navier–Stokes equations in order to circumvent the uniqueness problem. Considering a sequence of mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying:

$$\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3), \qquad \mathrm{supp}(\phi_\varepsilon) \subset \mathcal{B}(0,\varepsilon), \qquad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}x = 1 \tag{2.2}$$

and the convolution product $\phi_\varepsilon * \mathbf{v}$ such that

$$\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y}) \cdot \phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}y, \tag{2.3}$$

Leray suggested to regularize the Navier–Stokes equations as follows:

$$\partial_t \mathbf{u} + (\phi_\varepsilon * \mathbf{u}) \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \phi_\varepsilon * \mathbf{f}, \qquad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u}\ \text{is periodic}, \quad \mathbf{u}|_{t=0} = \phi_\varepsilon * \mathbf{u}_0. \tag{2.4}$$

Although we use in these equations the same Latin letters as in (2.1) to denote velocity and pressure, we want to emphasize that the solution of (2.4) is different from that of (2.1). Upon introducing the now standard spaces of solenoidal (divergence-free) vector fields:

$$\mathbf{V} = \{\mathbf{v} \in (H^1(\Omega))^3, \nabla \cdot \mathbf{v} = 0\ \text{and}\ \mathbf{v}\ \text{is periodic}\}, \qquad \mathbf{H} = \{\mathbf{v} \in (L^2(\Omega))^3, \nabla \cdot \mathbf{v} = 0\ \text{and}\ \mathbf{v}\ \text{is periodic}\}, \tag{2.5}$$

Leray proved the following theorem:

Theorem 2.1 (Leray [4]). *For all* $\mathbf{u}_0 \in \mathbf{H}, \mathbf{f} \in \mathbf{H}$, *and* $\varepsilon > 0$, (2.4) *has a unique* $\mathcal{C}^\infty$ *solution. This solution is bounded in* $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ *and one subsequence converges weakly in* $L^2(0,T;\mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon \to 0$.

The striking conclusion of this theorem is that the solution of (2.4) is unique, which means that moderate smoothing of the advection velocity (and if necessary, of the data $\mathbf{u}_0$ and $\mathbf{f}$) is sufficient to guarantee uniqueness of a $\mathcal{C}^\infty$ solution. In other words, moderate smoothing ascertains that the Kolmogorov energy cascade is stopped everywhere in the domain and for all times.

2.3. Filtering operators

Another interesting feature of the Leray regularization is that it can be viewed as a rigorous justification for a particular, possibly crude, LES strategy. To make this point clear, let us introduce the operator $\mathbf{v} \to \bar{\mathbf{v}} := \phi_\varepsilon * \mathbf{v}$, as defined in (2.3). It is clear that, owing to the periodic boundary conditions, this operator commutes with time and space derivatives. It is commonly called a filtering operator in reference to the fact that it is a mapping between $L^1(\Omega)$ and $\mathcal{C}^\infty(\Omega)$.

Another filtering operator that we shall also consider in the present paper can be defined as follows:

$$\bar{\mathbf{v}} := (1 - \varepsilon^2 \nabla^2)^{-1} \mathbf{v}, \tag{2.6}$$

where for a given function $\mathbf{v}$, $\bar{\mathbf{v}}$ is the solution of the Helmholtz equation $\bar{\mathbf{v}} - \varepsilon^2 \nabla^2 \bar{\mathbf{v}} = \mathbf{v}$ supplemented with periodic boundary conditions. This operator belongs to the class of differential filters described in Germano [6,7] and possesses smoothing properties similar to those of the mollifying operator. It is an isomorphism between $L^q(\Omega)$, $1 < q < +\infty$, and $W^{2,q}(\Omega)$ that commutes with time and space derivatives. We shall henceforth refer to this operator as the Helmholtz filter.

Then, using either filter defined above, the regularization of the Navier–Stokes equations (2.4) proposed by Leray can be rewritten as follows:

$$\partial_t \mathbf{u} + \bar{\mathbf{u}} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \bar{\mathbf{f}}, \qquad \nabla \cdot \mathbf{u} = 0, \quad \mathbf{u} \text{ is periodic}, \quad \mathbf{u}|_{t=0} = \bar{\mathbf{u}}_0. \tag{2.7}$$

2.4. LES modeling

To interpret (2.7) as a LES modeling, we now compare this system with the filtered Navier–Stokes equations. By applying either one of the two filters defined above to the Navier–Stokes equations (2.1) and by using the fact that both filters commute with time and space derivatives, we obtain the filtered system:

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla \bar{p} - \nu \nabla^2 \bar{\mathbf{u}} = \bar{\mathbf{f}} - \nabla \cdot \mathbb{T}, \qquad \nabla \cdot \bar{\mathbf{u}} = 0, \qquad \bar{\mathbf{u}} \text{ is periodic}, \quad \bar{\mathbf{u}}|_{t=0} = \bar{\mathbf{u}}_0, \tag{2.8}$$

where $\mathbb{T} = \overline{\mathbf{u} \otimes \mathbf{u}} - \bar{\mathbf{u}} \otimes \bar{\mathbf{u}}$ is the so-called subgrid-scale tensor or Reynolds stress tensor. The goal of LES is to model $\mathbb{T}$ with respect to the filtered solution only, the so-called closure problem [3].

Now, returning to (2.7), we rewrite the momentum equation as:

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} + \nabla p - \nu \nabla^2 \mathbf{u} = \bar{\mathbf{f}} - \{\bar{\mathbf{u}} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u}\} \tag{2.9}$$

and upon introducing the tensor $\mathbb{T}_L$ such that

$$\nabla \cdot \mathbb{T}_L = \bar{\mathbf{u}} \cdot \nabla \mathbf{u} - \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (\mathbf{u} \otimes \bar{\mathbf{u}} - \mathbf{u} \otimes \mathbf{u}), \tag{2.10}$$

it is reasonable to think of (2.7) as a LES model where the subgrid-scale tensor $\mathbb{T}$ is now approximated by $\mathbb{T}_L$ (notice that the variables in (2.8) are denoted by $(\bar{\mathbf{u}}, \bar{p})$ while they are simply denoted by $(\mathbf{u}, p)$ in (2.7) and (2.9)).

Note that this tensor is formally small since in the case of the Helmholtz filter, we have

$$\mathbb{T}_L = \varepsilon^2 \mathbf{u} \otimes \nabla^2 \bar{\mathbf{u}}.$$

We also remark that the model is asymmetric which guarantees that the contribution of the nonlinear term to the global energy balance is zero, a property sometimes overlooked in standard LES models, as pointed out by Layton in [8].

3. Frame indifference

In this section we show that (2.7) is not frame indifferent and we identify the term that is responsible for this lack of frame invariance. We restrict ourselves to the Helmholtz filter in the present section and the following.

3.1. General results

The concept of frame indifference (also called frame invariance) for differential equations describing physical phenomena is of paramount importance to ensure that solutions of a system of equations, when solved with respect to two arbitrary frames of reference, could still be exactly superimposed when passing from one frame to the other.

In the case of the Navier–Stokes equations, a necessary and sufficient condition for frame indifference (e.g. see [9]) is that the momentum equation be expressed in the form

$$\partial_t \mathbf{u} + \mathbf{u} \cdot \nabla \mathbf{u} = \nabla \cdot (q\mathbf{I} + \mathbf{T}) + \mathbf{f}, \tag{3.1}$$

where q is a scalar function, $\mathbf{I}$ the identity tensor, and $\mathbf{T}$ a frame-indifferent tensor. Moreover, introducing the following notation

$$\mathbf{D} = \tfrac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^{\mathrm{T}}) \quad \text{(deformation tensor)}, \qquad \mathbf{\Omega} = \tfrac{1}{2}(\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}) \quad \text{(vorticity tensor)},$$
$$\mathring{\mathbf{D}} = \partial_t\mathbf{D} + \mathbf{u}\cdot\nabla\mathbf{D} + \mathbf{D}\mathbf{\Omega} - \mathbf{\Omega}\mathbf{D} \quad \text{(Jaumann derivative)},$$

it can be shown that the tensor $\mathbf{T}$ is frame-indifferent if and only if it depends only on the deformation tensor and the Jaumann derivative. From these results, it is clear that the Navier–Stokes equations are frame-indifferent with q given by the pressure p and $\mathbf{T}$ depending on the deformation tensor $\mathbf{D}$ only. In the same manner, the filtered momentum equation in (2.8) is frame-indifferent if the Reynolds stress tensor $\mathbb{T}$, or any given model of $\mathbb{T}$, is shown to be dependent on the deformation tensor and the Jaumann derivative only. We now analyze the regularized equations (2.7) and arrive at the conclusion that the momentum equation is not frame-indifferent with respect to either the solution $\mathbf{u}$ or the filtered velocity $\bar{\mathbf{u}}$. We perform the analysis in the case of the Helmholtz filter (2.6) for simplicity.

3.2. Analysis with respect to the solution of the regularized equations

We analyze first the frame invariance of (2.7) with respect to $\mathbf{u}$. Looking at (2.9), we conclude that the regularized equation is frame-indifferent with respect to $\mathbf{u}$ if the term $\bar{\mathbf{u}}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u}$ is also frame-indifferent once written in terms of $\mathbf{u}$ only. From the definition of the Helmholtz filter and its inverse, we infer

$$\bar{\mathbf{u}} = \mathbf{u} + \varepsilon^2\nabla^2(1-\varepsilon^2\nabla^2)^{-1}\mathbf{u}, \tag{3.2}$$

so that

$$\bar{\mathbf{u}}\cdot\nabla\mathbf{u} - \mathbf{u}\cdot\nabla\mathbf{u} = \varepsilon^2(\nabla^2(1-\varepsilon^2\nabla^2)^{-1}\mathbf{u})\cdot\nabla\mathbf{u}.$$

Unfortunately, it is not difficult to show that the resulting system is not frame-indifferent by simple rotation of the frame of reference. We then conclude that (2.7) is not frame-indifferent with respect to $\mathbf{u}$.

3.3. Analysis with respect to the filtered velocity

Here we study the frame indifference of Eqs. (2.7) with respect to the filtered velocity $\bar{\mathbf{u}}$. Using (3.2) again, we rewrite (2.7) in terms of $\bar{\mathbf{u}}$ only

$$\begin{aligned}&\partial_t\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} = \bar{\mathbf{f}} - \nabla p + \nu\nabla^2\bar{\mathbf{u}} + \varepsilon^2[\partial_t\nabla^2\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla(\nabla^2\bar{\mathbf{u}}) - \nu\nabla^2\nabla^2\bar{\mathbf{u}}],\\ &\nabla\cdot\bar{\mathbf{u}} = 0, \quad \bar{\mathbf{u}}\ \text{is periodic}, \quad \bar{\mathbf{u}}|_{t=0} = \bar{\mathbf{u}}_0.\end{aligned} \tag{3.3}$$

Then, from the definition of $\mathring{\mathbf{D}}$, we readily obtain:

$$2\nabla\cdot\mathring{\mathbf{D}} = \partial_t\nabla^2\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla(\nabla^2\bar{\mathbf{u}}) + \nabla(\mathbf{D}:\mathbf{D}) + (\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}.$$

Multiplying this equation by ε^2 and subtracting it from the momentum equation in (3.3), we infer

$$\partial_t\bar{\mathbf{u}} + \bar{\mathbf{u}}\cdot\nabla\bar{\mathbf{u}} = \bar{\mathbf{f}} + \nabla\cdot(-(p+\varepsilon^2\mathbf{D}:\mathbf{D})\mathbf{I} + 2\nu(1-\varepsilon^2\nabla^2)\mathbf{D} + 2\varepsilon^2\mathring{\mathbf{D}}) - \varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}.$$

Note that this way of writing the momentum equation is unique up to gradients, for $(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}$, $\nabla\cdot\nabla^2\mathbf{D}$, and $\nabla\cdot\mathring{\mathbf{D}}$ are linearly independent.

By identifying $-(p+\varepsilon^2\mathbf{D}:\mathbf{D})$ with q and $2\nu(1-\varepsilon^2\nabla^2)\mathbf{D} + 2\varepsilon^2\mathring{\mathbf{D}}$ with $\mathbf{T}$ in (3.1), we conclude that (2.4) is not frame-indifferent with respect to $\bar{\mathbf{u}}$ because of the presence of the term $\varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}$, whose expression, we recall, is *unique* up to gradients. Note, however, that the default to frame invariance is small since the faulty term is formally of second-order with respect to ε. As a result, a reasonable strategy to obtain a frame-indifferent model is to neglect a priori this term from the momentum equation in (2.7). We present the resulting model in the following section.

4. NS-α model

4.1. From Leray regularization to the NS-α model

In this section we consider the regularized system (2.7) from which we subtract the term $\varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}$; that is to say

$$\partial_t\mathbf{u} + \bar{\mathbf{u}}\cdot\nabla\mathbf{u} - \varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}} + \nabla p - \nu\nabla^2\mathbf{u} = \bar{\mathbf{f}}, \qquad \nabla\cdot\bar{\mathbf{u}} = 0, \quad \mathbf{u}\text{ is periodic}, \quad \mathbf{u}|_{t=0} = \bar{\mathbf{u}}_0. \tag{4.1}$$

Although we still use the same notations, the solution to the above problem is different from that of (2.7), as (4.1) is a perturbation of (2.7). Starting from the definition of the Helmholtz filter, we can derive the following equalities:

$$\begin{aligned}\varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}} &= (\nabla\bar{\mathbf{u}})^{\mathrm{T}}(\bar{\mathbf{u}} - \mathbf{u}) = \tfrac{1}{2}\nabla\bar{\mathbf{u}}^2 - (\nabla\bar{\mathbf{u}})^{\mathrm{T}}\mathbf{u} = \tfrac{1}{2}\nabla\bar{\mathbf{u}}^2 - (\nabla\bar{\mathbf{u}})^{\mathrm{T}}\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}\bar{\mathbf{u}} + (\nabla\mathbf{u})^{\mathrm{T}}\bar{\mathbf{u}}\\ &= -\nabla(\bar{\mathbf{u}}\cdot\mathbf{u} - \tfrac{1}{2}\bar{\mathbf{u}}^2) + (\nabla\mathbf{u})^{\mathrm{T}}\bar{\mathbf{u}}.\end{aligned}$$

By inserting this equality into (4.1), we arrive at

$$\partial_t\mathbf{u} + \bar{\mathbf{u}}\cdot\nabla\mathbf{u} - (\nabla\mathbf{u})^{\mathrm{T}}\bar{\mathbf{u}} + \nabla p' - \nu\nabla^2\mathbf{u} = \bar{\mathbf{f}}, \qquad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\text{ is periodic}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{4.2}$$

where we have set $p' = p + \bar{\mathbf{u}}\cdot\mathbf{u} - (1/2)\bar{\mathbf{u}}^2$. We observe that (4.2) is exactly the so-called NS-α model, which was introduced in [1,2]. Mathematical aspects of this model are analyzed in [10,11]. Note that its name refers to the notational choice $\alpha = \varepsilon$ that the authors consistently used in the series of papers [1,2,10].

In [2], it is pointed out that the inviscid counterpart of (4.2) is "the equation for the geodesic motion on the diffeomorphism group with respect to the metric given by the kinetic energy Lagrangian, ... which is right invariant under the action of the diffeomorphism group". Our interpretation of the NS-α model as a frame-indifferent $\mathcal{O}(\varepsilon^2)$ perturbation of the Leray regularization should provide an alternate interpretation more accessible to practitioners.

Note that the way we propose to deduce the NS-α model from the Leray regularization is neither more nor less accurate than that proposed in [2, Section 3]. The term we propose to neglect, $\varepsilon^2(\nabla\bar{\mathbf{u}})^{\mathrm{T}}\nabla^2\bar{\mathbf{u}}$, is of the same order as those neglected in the Taylor expansion

$$\mathbf{u}^{\boldsymbol{\sigma}}(\mathbf{x} + \boldsymbol{\sigma}(\mathbf{x}, t)) \approx \mathbf{u}(\mathbf{x}, t) + \boldsymbol{\sigma}(\mathbf{x}, t)\cdot\nabla\mathbf{u}(\mathbf{x}, t),$$

introduced in [2, Section 3] to justify the model and where $\boldsymbol{\sigma}(\mathbf{x}, t)$ is a random vector field of $\mathcal{O}(\varepsilon)$ and $\mathbf{u}^{\boldsymbol{\sigma}}$ is a $\mathcal{O}(\varepsilon)$ perturbation of $\mathbf{u}$.

Another interpretation of the NS-α model can also be proposed starting from the Navier–Stokes equations in which the advection term is expressed in terms of the vorticity $\nabla\times\mathbf{u}$:

$$\partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\mathbf{u} + \nabla\pi - \nu\nabla^2\mathbf{u} = \mathbf{f}, \qquad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\text{ is periodic}, \quad \mathbf{u}|_{t=0} = \mathbf{u}_0, \tag{4.3}$$

where $\pi = p + (1/2)\mathbf{u}^2$. In the same spirit as that of Leray, we then suggest to filter the advection velocity and data so that (4.3) becomes

$$\partial_t\mathbf{u} + (\nabla\times\mathbf{u})\times\bar{\mathbf{u}} + \nabla\pi - \nu\nabla^2\mathbf{u} = \bar{\mathbf{f}}, \qquad \nabla\cdot\mathbf{u} = 0, \quad \mathbf{u}\text{ is periodic}, \quad \mathbf{u}|_{t=0} = \bar{\mathbf{u}}_0. \tag{4.4}$$

Finally, using the identity

$$(\nabla \times \mathbf{u}) \times \bar{\mathbf{u}} = \bar{\mathbf{u}} \cdot \nabla \mathbf{u} - (\nabla \mathbf{u})^{\mathrm{T}} \bar{\mathbf{u}},$$

we again obtain (4.2), but this time without dropping any term. This is another way to confirm that the NS-α model can be viewed as a by-product of the Leray regularization. This observation was also made in a set of remarks in [10, Section 2].

4.2. NS-α model as a LES model

In the same spirit as in [2, Section 4], system (4.1) can be alternatively written in terms of the filtered velocity $\bar{\mathbf{u}}$, as follows:

$$\partial_t \bar{\mathbf{u}} + \bar{\mathbf{u}} \cdot \nabla \bar{\mathbf{u}} + \nabla p' - \nu \nabla^2 \bar{\mathbf{u}} = \bar{\mathbf{f}} + 2\varepsilon^2 \nabla \cdot (\mathring{\mathbf{D}} - \nu \nabla^2 \mathbf{D}), \quad \nabla \cdot \bar{\mathbf{u}} = 0, \quad \bar{\mathbf{u}} \text{ is periodic}, \quad \bar{\mathbf{u}}|_{t=0} = \bar{\mathbf{u}}_0. \tag{4.5}$$

Hence, we can view the NS-α model as a LES model[1] where the subgrid tensor $\mathbb{T}$ is, in this case, modeled by

$$\mathbb{T}_\varepsilon = 2\varepsilon^2 (\nu \nabla^2 \mathbf{D} - \mathring{\mathbf{D}}).$$

The first term amounts to introducing some hyper-viscosity (i.e. a bi-Laplacian), while the second one introduces dispersion effects. Note, however, that for computational purposes, the equivalent set of Eqs. (2.6)–(2.10), (3.1)–(3.3), (4.1)–(4.4) is probably more interesting, since it does not explicitly involve fourth-order derivatives.

We conclude the note by emphasizing that this model is one of the very few LES models that can be rigorously justified. It yields a unique solution, is frame-indifferent, and complies with the main objective of LES to restrict the inertial range of the energy cascade to wavenumbers smaller than those associated with the small length scale ε, thereby making the solution computable [10].

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[1] One reviewer kindly pointed to our attention the ERCOFTAC papers [12,13] where this point of view is also defended.

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