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Approximation des équations de Navier–Stokes instationnaires par des méthodes de projection

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Remarques sur les méthodes de projection pour l'approximation des équations de Navier–Stokes[1]

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Summary. This paper shows that the convergence estimates on the pressure given in [5] and [6] for a series of projection methods are not correctly obtained since they are all based on an inequality which is not correct. As conclusion, the conjecture that projection methods yield convergence rates of order higher than $\mathcal{O}(\delta t^{1/2})$ for the pressure remains an open question[3].

Résumé. L'objet de cet article est de montrer que les estimations de convergence sur la pression pour les méthodes de projection décrites dans [5] et [6] ne sont pas obtenues correctement car elles sont toutes basées sur une inégalité fausse. La question de savoir si la méthode de projection a un taux de convergence pour la pression plus élevé que $\mathcal{O}(\delta t^{1/2})$ reste ouverte.

Key words. Navier–Stokes equations, projection methods, convergence estimates.

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[3]Since then, the author has found some answers to this question and proves (hopefully correctly!) estimates of order one in papers 2 and 3 of this report.

Chapitre 1

Remarques sur les méthodes de projection

1.1 Introduction

Les méthodes de projection introduites par Chorin [1] et Temam [9] pour approcher numériquement les solutions des équations de Navier–Stokes instationnaires sont largement utilisées pour leur simplicité et leur efficacité (au moins pour l'approximation de la vitesse). Cette classe de méthodes permet de découpler les approximations de la vitesse et de la pression à chaque pas de temps, évitant ainsi les difficultés inhérentes à la résolution du problème de Stokes. Afin d'illustrer ces méthodes, considérons le problème de Stokes instationnaire dans un domaine fluide Ω:

$$\left\{\begin{array}{l} \dfrac{\partial u}{\partial t} - \nu\nabla^2 u + \nabla p = f \\ \mathrm{div} u = 0 \\ u_{|\Gamma} = 0 \\ u(t=0,x) = v_0(x) \end{array}\right. \tag{1.1}$$

où u et p sont respectivement la vitesse et la pression. L'ouvert Ω est borné connexe régulier de $\mathbb{R}^d$, disons lipschitzien pour fixer les idées, et Γ est sa frontière. Les données f et u_0 sont supposées suffisamment régulières pour assurer l'existence et l'unicité d'une solution dans un bon cadre fonctionnel, *i.e.* on se place dans le cadre d'application du théorème de Lions (*cf.* Lions-Magenes [3, p. 257] ou Temam [8, p. 253]).

On s'intéresse par la suite à l'approximation de la solution de (1.1) sur un intervalle de temps fini $[0,T]$. Introduisons une partition de l'intervalle de temps: $t^k = k\delta t$ pour $0 \leq k \leq K$ où $\delta t = T/K$. L'idée de base des méthodes de projection repose sur une stratégie de prédiction-correction: elle consiste à construire deux suites d'approximation de la vitesse (u_k) et $(\tilde{u}_k)$ et une suite d'approximation de la pression (p_k) telles que, à chaque pas de temps $\tilde{u}_{k+1}$ soit une prédiction de $u(t^{k+1})$ et u_{k+1} une correction de

$\tilde{u}_{k+1}$. L'algorithme consistant d'ordre 1 le plus simple s'écrit:

$$(1.2)\qquad \begin{cases} \dfrac{\tilde{u}_{k+1}-u_k}{\delta t} - \nu\nabla^2\tilde{u}_{k+1} = f_{k+1} - \nabla p_k \\ \tilde{u}_{k+1|\Gamma} = 0 \end{cases}$$

et

$$(1.3)\qquad \begin{cases} \dfrac{u_{k+1}-\tilde{u}_{k+1}}{\delta t} + \nabla(p_{k+1}-p_k) = 0 \\ \mathrm{div} u_{k+1} = 0 \\ u_{k+1}.n_{|\Gamma} = 0 \end{cases}$$

La suite (u_k) est initialisée par $u_0 = v_0$ et la suite (p_k) initialisée en prenant p_0 arbitraire.

Remarquons que le second système est équivalent à $u_{k+1} = P_H\tilde{u}_{k+1}$ et $\nabla(p_{k+1} - p_k) = (\tilde{u}_{k+1} - P_H\tilde{u}_{k+1})/\delta t$ où H est le sous-espace de $L^2(\Omega)^d$ défini par:

$$(1.4)\qquad H = \{u \in L^2(\Omega)^d,\ \mathrm{div} u = 0,\ u.n = 0\},$$

et P_H désigne la projection orthogonale de $L^2(\Omega)^d$ sur H. La vitesse $\tilde{u}_{k+1}$ est une prédiction de $u(t^{k+1})$ qui satisfait la bonne condition au bord mais n'est pas à divergence nulle. Ce défaut est corrigé en projetant $\tilde{u}_{k+1}$ sur H (d'où le nom de la méthode), toutefois la projection u_{k+1} ne satisfait plus entièrement la condition au bord; seule la composante normale de la vitesse est nulle.

Le gros avantage technique de l'algorithme (1.2)–(1.3) est de ne faire appel qu'à des solveurs du problème de Helmholtz et du problème de Poisson avec condition de Neumann homogène. Cet algorithme contourne ainsi les difficultés liées à l'approximation numérique du problème de Stokes.

Sous des hypothèses de régularité suffisantes sur la solution (u,p) de (1.1), en l'occurence si:

$$(1.5)\qquad \int_0^T \left|\frac{\partial^2 u}{\partial t^2}\right|_0^2 dt < \infty$$

$$(1.6)\qquad \int_0^T \left|\frac{\partial \nabla p}{\partial t}\right|_0^2 dt < \infty$$

il est alors possible d'obtenir les estimations suivantes:

$$(1.7)\qquad \sup_{0\le k\le K} |u(t^k) - u_k|_0 \le c\delta t$$

$$(1.8)\qquad \left[\delta t \sum_1^K |u(t^k) - \tilde{u}_k|_1^2\right]^{1/2} \le \frac{c}{\sqrt{\nu}}\delta t$$

$$(1.9)\qquad \left[\delta t \sum_1^K |p(t^k) - p_k|_0^2\right]^{1/2} \le c\delta t^{1/2}$$

où c désigne une constante générique strictement positive. Noter que la convergence sur la pression est assez pauvre comparée à celle obtenue sur la vitesse. Lorsque le terme ∇p_k est absent de (1.2) (version originale de l'algorithme par Chorin [1] et Temam [9]), les estimations sur la vitesse sont en $\mathcal{O}(\delta t^{1/2})$ comme pour la pression.

On peut affaiblir l'hypothèse de régularité (1.5) en supposant uniquement:

$$(1.10)\qquad \int_0^T \left|\frac{\partial^2 u}{\partial t^2}\right|_{-1}^2 dt < \infty.$$

Dans ce cas les estimations (1.7), (1.8) et (1.9) sont encore correctes en remplaçant les constantes c par c/ν.

Dans deux articles récents, J. Shen propose de montrer qu'en conservant les hypothèses de régularité de la solution (1.6) et (1.10), l'estimation de convergence en $\mathcal{O}(\delta t^{1/2})$ pour la pression peut être améliorée en $\mathcal{O}(\delta t)$ [5, p. 72, Th. 2] et [6, p. 55, Th. 1]. Ce résultat serait en soit très intéressant s'il était correctement démontré. Or, l'auteur base toutes ses estimations pour la pression sur une inégalité qui n'est pas valable ici. Plus précisement, l'auteur introduit l'opérateur de Stokes $A = -P_H\nabla^2$, dont le domaine est le sous-espace de $H^2(\Omega)^d \cap H_0^1(\Omega)^d$ constitué des fonctions à divergence nulle, et il a besoin que $(u, A^{-1}u)$ réalise une norme équivalente à celle de $H^{-1}(\Omega)^d$ pour tout u dans H. En d'autre termes, l'auteur a besoin (cf. (2.1) dans [5] et (2.7) dans [6]) de l'existence de deux constantes $c_1 > 0$ et $c_2 > 0$ telles que

$$(1.11)\qquad \forall u \in H,\qquad c_1|u|_{-1}^2 \le (u, A^{-1}u) \le c_2|u|_{-1}^2.$$

L'existence de la constante c_2 est évidente; par contre la constante c_1, qui est fondamentale ici, n'existe pas (*i.e.* $c_1 = 0$) comme nous le montrons par la suite.

Le présent exposé est organisé comme suit. Dans une première partie on introduit l'opérateur de Stokes dans un cadre optimal et on rappelle les arguments défendus dans [5], [6]. Du cadre optimal présenté on déduit quelques arguments qui montrent que la convergence d'ordre 1 de la vitesse dans $H^1(\Omega)^d$ est insuffisante pour obtenir une convergence d'ordre 1 de la pression. Ceci semble indiquer qu'à moins d'obtenir une convergence de la vitesse $\tilde{u}_{k+1}$ dans $H^2(\Omega)^d$, on ne peut pas espérer une convergence sur la pression plus forte qu'en $\mathcal{O}(\delta t^{1/2})$. Dans une seconde partie on donne un résultat de densité qui permet de construire un contre-exemple à (1.11). Le contre-exemple en question et quelques conclusions sont présentés dans la dernière partie.

1.2 L'opérateur de Stokes

Afin d'éclairer l'exposé, on rappelle et on analyse l'idée poursuivie par J. Shen. Tout d'abord, pour définir correctement l'opérateur de Stokes, introduisons l'espace:

$$(2.1)\qquad H_0^{\mathrm{div}}(\Omega) = \{u \in L^2(\Omega)^d,\ \mathrm{div} u \in L^2(\Omega),\ u.n_{|\Gamma} = 0\}.$$

Muni de la norme $(|u|_0^2 + |\mathrm{div} u|_0^2)^{1/2}$, $H_0^{\mathrm{div}}(\Omega)$ est un espace de Hilbert. Soit $\mathcal{D}(\Omega)^d$ l'espace des fonctions indéfiniment différentiables à support compact dans Ω. On peut montrer que $\mathcal{D}(\Omega)^d$ est dense dans $H_0^{\mathrm{div}}(\Omega)$ (cf. Girault-Raviart [2, p. 29]). Le dual de $H_0^{\mathrm{div}}(\Omega)$ est donc un espace de distributions et on a les inclusions (avec densité et continuité):

$$(2.2)\qquad H_0^1(\Omega)^d \subset H_0^{\mathrm{div}}(\Omega) \subset L^2(\Omega)^d \equiv (L^2(\Omega)^d)' \subset H_0^{\mathrm{div}}(\Omega)' \subset H^{-1}(\Omega)^d.$$

D'autre part:

Lemme 1.2.1 *On a la décomposition*

$$H_0^{div}(\Omega)' = H \oplus \nabla(L^2(\Omega)/\mathbb{R}). \tag{2.3}$$

Démonstration. Il est clair que l'espace de droite est contenu dans $H_0^{\text{div}}(\Omega)'$. Réciproquement, soit $l \in H_0^{\text{div}}(\Omega)'$. On résout dans un premier temps le problème: trouver $u \in H$ tel que

$$\forall v \in H, \qquad (u,v) = < l, v >$$

où $< .,. >$ désigne la dualité entre $H_0^{\text{div}}(\Omega)'$ et $H_0^{\text{div}}(\Omega)$ et $(.,.)$ désigne le produit scalaire dans $L^2(\Omega)^d$. Comme $(.,.)$ est aussi un produit scalaire dans H, ce problème a une solution unique $u \in H$ et

$$|u|_0 \leq |l|_{H_0^{\text{div}}(\Omega)'}.$$

D'après la définition de u, la forme linéaire $l - u \in H_0^{\text{div}}(\Omega)'$ s'annule sur H. D'après les inclusions (2.2) on peut considérer $l - u$ comme une forme de $H^{-1}(\Omega)^d$ et d'après la définition de u cette forme s'annule sur V. D'après le théorème de De Rham [4], il existe un p dans $L^2(\Omega)$ défini à une constante près tel que $l - u = \nabla p$. De plus on a (cf. par exemple Girault-Raviart [2, p. 20]):

$$|p|_{L^2(\Omega)/\mathbb{R}} \leq c|l - u|_{-1}$$

En conclusion on a $l = u + \nabla p$ et la décomposition est unique. □

La décomposition (2.3) généralise celle, bien connue, de $L^2(\Omega)^d = H \oplus \nabla(H^1(\Omega)/\mathbb{R})$ (cf. Girault-Raviart [2, p. 29] ou Temam [8, p. 17] par exemple) et permet de définir une extension de la projection orthogonale P_H à $H_0^{\text{div}}(\Omega)'$; par la suite on note encore par abus de notation $P_H : H_0^{\text{div}}(\Omega)' \longrightarrow H$ l'extension en question.

Introduisons $V = \{u \in H_0^1(\Omega)^d;\ \text{div}u = 0\}$ et $D(A) = \{u \in V;\ \nabla^2 u \in H_0^{\text{div}}(\Omega)'\}$. Définissons l'opérateur de Stokes A tel que:

$$\begin{array}{rcl} A : D(A) & \longrightarrow & H \\ u & \longmapsto & -P_H \nabla^2 u \end{array} \tag{2.4}$$

On vérifie aisément que A est non borné, fermé, elliptique, auto-adjoint et inversible, d'inverse compact dans H. Remarquons de plus que si Ω est C^2 les théorèmes de régularité de la solution du problème de Stokes permettent de prendre $D(A) = H^2(\Omega)^d \cap V$ comme cela est fait dans [5], [6].

Pour étudier la convergence de l'algorithme (1.2)-(1.3) on définit les fonctions d'erreurs $e_k = u(t^k) - u_k$, $\tilde{e}_k = u(t^k) - \tilde{u}_k$ et $\delta_k = p(t^k) - p_k$. En soustrayant (1.2)+(1.3) de (1.1) on obtient le système:

$$\left\{ \begin{array}{rcl} \dfrac{e_{k+1} - e_k}{\delta t} + \nabla \delta_{k+1} & = & -R_k + \nu \nabla^2 \tilde{e}_{k+1} \\ \text{div}\left(\dfrac{e_{k+1} - e_k}{\delta t}\right) & = & 0 \\ \left(\dfrac{e_{k+1} - e_k}{\delta t}\right).n_{|\Gamma} & = & 0 \end{array} \right. \tag{2.5}$$

où R_k est le reste intégral de Taylor:

$$R_k(x) = \frac{1}{\delta t} \int_{t^k}^{t^{k+1}} (t - t^k) \frac{\partial^2 u(x,t)}{\partial t^2} dt. \tag{2.6}$$

L'idée poursuivie dans [5], [6] consiste à dire qu'il est possible de contrôler $|e_{k+1} - e_k|_{-1}/\delta t$ et $|\delta_{k+1}|_0$ par la norme de $-R_k + \nu \nabla^2 \tilde{e}_{k+1}$ dans $H^{-1}(\Omega)^d$. Pour ce faire, en remarquant que $(e_{k+1} - e_k)$ est dans H, l'auteur dualise la première équation par $A^{-1}(e_{k+1} - e_k)$, faisant ainsi disparaître le gradient de δ_{k+1} et applique l'inégalité (1.11) pour obtenir la majoration:

$$c_1 \frac{|e_{k+1} - e_k|_{-1}^2}{\delta t} \leq c| - R_k + \nu \nabla^2 \tilde{e}_{k+1}|_{-1} |e_{k+1} - e_k|_{-1} \tag{2.7}$$

d'où

$$\frac{|e_{k+1} - e_k|_{-1}}{\delta t} \leq c(|R_k|_{-1} + \nu |\tilde{e}_{k+1}|_1) \tag{2.8}$$

et par conséquent

$$|\delta_{k+1}|_0 \leq c|\nabla \delta_{k+1}|_{-1} \leq c'(|R_k|_{-1} + \nu |\tilde{e}_{k+1}|_1) \tag{2.9}$$

Cette inégalité "assure" la convergence de la pression en $\mathcal{O}(\delta t)$ car on peut effectivement contrôler $|R_k|_{-1}$ en $\mathcal{O}(\delta t)$ en demandant une régularité de u du type (1.10), et $|\tilde{e}_{k+1}|_1$ est contrôlé en $\mathcal{O}(\delta t)$ par (1.8).

Malheureusement la minoration (1.11) étant fausse, comme nous le montrons par la suite, l'obtention de (2.8) et (2.9) est donc incorrecte. D'autre part, à la lumière de ce qui est montré au début de cette section, il semble que (2.8) et (2.9) aient peu de chances d'être vraies sans exiger plus de régularité de la solution. En effet, d'après le lemme 1.2.1: $H_0^{\text{div}}(\Omega)' = H \oplus \nabla(L^2(\Omega)/\mathbb{R})$, c'est à dire que le couple $((e_{k+1} - e_k)/\delta t, \delta_{k+1})$ $\in H \times L^2(\Omega)/\mathbb{R}$ réalise la décomposition de $-R_k + \nu \nabla^2 \tilde{e}_{k+1}$ dans $H_0^{\text{div}}(\Omega)'$. Ainsi, la première égalité de (2.5) est "optimale" dans $H_0^{\text{div}}(\Omega)'$ et non pas dans $H^{-1}(\Omega)^d$. Par continuité des projections induites par la décomposition de $H_0^{\text{div}}(\Omega)'$ on obtient les estimations optimales:

Corollaire 1.2.1

$$\frac{|e_{k+1} - e_k|_0}{\delta t} \leq c(|R_k|_{H_0^{div}(\Omega)'} + |\nu \nabla^2 \tilde{e}_{k+1}|_{H_0^{div}(\Omega)'}) \tag{2.10}$$

$$|\delta_{k+1}|_0 \leq c(|R_k|_{H_0^{div}(\Omega)'} + |\nu \nabla^2 \tilde{e}_{k+1}|_{H_0^{div}(\Omega)'}) \tag{2.11}$$

En conséquence, pour obtenir une convergence sur la pression en $\mathcal{O}(\delta t)$ il faut contrôler la norme de $\nabla^2 \tilde{e}_{k+1}$ dans $H_0^{\text{div}}(\Omega)'$, c'est-à-dire il faut avoir une estimation sur la norme de $\tilde{e}_{k+1}$ dans $\nabla^{-2}(H_0^{\text{div}}(\Omega)')$, où ∇^{-2} désigne l'inverse de $\nabla^2 : H_0^1(\Omega)^d \rightarrow H^{-1}(\Omega)^d$. Un tel contrôle "optimal" semble relativement difficile à obtenir puisque, à la connaissance de l'auteur, il n'existe pas de propriété de monotonie connue du laplacien sur des espaces strictement compris entre $H^{-1}(\Omega)^d$ et $L^2(\Omega)^d$ d'une part ou entre $L^2(\Omega)^d$ et $H_0^1(\Omega)^d$ d'autre part. En d'autres termes, pour obtenir une bonne convergence de la pression (*i.e.* en $\mathcal{O}(\delta t)$) il faudrait une convergence de la vitesse $(\tilde{u}_k)$ dans $H^2(\Omega)^d$. Mais celle-ci semble peu probable.

Ces problèmes de convergence sur la pression se généralisent aux autres types de méthodes de projection (Navier–Stokes, ordre élevé, etc...), puisque le système (2.5) est, semble-t-il, un point passage obligatoire pour obtenir des estimations sur la pression.

Cette difficulté de convergence est probablement à mettre sur le compte du fait que la suite des pressions est solution du problème (1.3) qui impose implicitement la condition de Neumann arbitraire $\partial p_k/\partial n_{|\Gamma} = \ldots = \partial p_0/\partial n_{|\Gamma}$ (cf. Temam [10] pour d'autres arguments dans ce sens). Elle est peut-être aussi à mettre en parallèle avec, semble-t-il, l'impossibilité d'obtenir une régularité convenable de la pression lorsque le terme source n'est pas très régulier. En effet pour Ω borné connexe lipschitzien, et $f \in L^2(0,T;V')$, $u_0 \in H$, on peut montrer l'exitence de u dans $L^2(0,T;V)\cap C(0,T;H)$ et $du/dt \in L^2(0,T;V')$, mais l'existence de la pression n'est assurée qu'au sens des distributions dans $\mathcal{D}'(0,T;L^2(\Omega))$. La difficulté tient au fait essentiel que V' n'est pas un espace de distribution (cf. Simon [7] pour d'autres détails sur les particularités de cet espace).

1.3 Un résultat de densité

Avant d'exhiber un contre-exemple démontrant que (1.11) est faux, on établit un résultat de densité des fonctions indéfiniment différentiables à support compact dans Ω et à divergence nulle, dans un sous-espace des distributions de $H^{-1}(\Omega)^d$ à divergence nulle.

Par la suite on fait l'hypothèse que la frontière Γ de Ω a un nombre fini de composantes connexes: $\Gamma_0, \Gamma_1, \ldots, \Gamma_m$, où Γ_0 désigne la frontière de la composante connexe infinie de $\Omega' = \mathbb{R}^d \backslash \overline{\Omega}$.

Soit B le sous-espace fermé de $H^2(\Omega)/\mathbb{R}$ tel que:

(3.1) $$B = \{p \in H^2(\Omega)/\mathbb{R};\ \nabla p \in H_0^1(\Omega)^d\}$$

On vérifie aisément que pour tout p dans B, $\partial p/\partial n_{|\Gamma} = 0$ et $p_{|\Gamma_i} = a_i$ où les a_i sont des constantes réelles arbitraires pour $i = 0, \ldots, m$. On introduit aussi S et S_0 les sous-espaces de $H^{-1}(\Omega)^d$ tels que:

(3.2) $$S = \{u \in H^{-1}(\Omega)^d,\ \mathrm{div} u = 0\}$$

(3.3) $$S_0 = \{u \in S,\ \forall p \in B,\ (u, \nabla p) = 0\}.$$

S est l'espace des distributions solenoïdales de $H^{-1}(\Omega)^d$, et si on pouvait définir la trace normale sur Γ des distributions de S, les distributions de S appartenant à S_0 seraient celles pour lesquelles on aurait formellement:

(3.4) $$\forall i = 0, \ldots, m, \qquad (u.n, 1_{|\Gamma_i}) = 0$$

où $1_{|\Gamma_i}$ est la fonction indicatrice de Γ_i.

S_0 est un sous-espace fermé de S de codimension m. On peut aussi montrer que S_0 est l'image de $L^2(\Omega)^d$ par le rotationnel lorsque $d = 2$ ou 3.

Afin d'énoncer le résultat essentiel de cette section introduisons $\mathcal{V}$ le sous-espace de $\mathcal{D}(\Omega)^d$ composé des fonctions solenoïdales. On a alors le résultat de densité suivant:

Théorème 1.3.1 *$\mathcal{V}$ est dense dans S_0 équipé de la norme de $H^{-1}(\Omega)^d$.*

Démonstration. Soit f une forme linéaire continue sur S_0. Supposons que f s'annule sur $\mathcal{V}$, montrons alors que f s'annule sur S_0; un corollaire bien connu du théorème de Hahn–Banach permettra de conclure.

S_0 étant un sous-espace de $H^{-1}(\Omega)^d$, on étend f à $H^{-1}(\Omega)^d$ à l'aide du théorème de Hahn-Banach; soit $\tilde{f}$ une des extensions possibles. $H^{-1}(\Omega)^d$ étant réflexif on identifie $\tilde{f}$ à une fonction de $H_0^1(\Omega)^d$. Puisque f s'annule sur $\mathcal{V} \subset S_0$ il en est de même pour $\tilde{f}$; d'après le théorème de De Rham [4] il en résulte que $\tilde{f}$ est nécessairement le gradient d'une distribution, c'est à dire:

$$\exists p \in \mathcal{D}(\Omega)', \qquad \tilde{f} = \nabla p$$

Or $\tilde{f}$ est dans $H_0^1(\Omega)^d$ donc $\nabla p \in H_0^1(\Omega)^d$. D'après l'inégalité (cf. Girault-Raviart [2, p. 20]):

$$|p|_{L^2(\Omega)/\mathbb{R}} \leq c|\nabla p|_{-1}$$

et la continuité de l'injection de $H_0^1(\Omega)^d$ dans $H^{-1}(\Omega)^d$, on déduit $p \in L^2(\Omega)/\mathbb{R}$. C'est-à-dire $p \in H^2(\Omega)/\mathbb{R}$ et finalement $p \in B$, ce qui donne

$$\forall u \in S_0, \qquad (u, f) = (u, \tilde{f}) = (u, \nabla p) = 0,$$

d'où la conclusion. □

De ce théorème on déduit facilement la densité de V et H dans S_0 (voir Simon [7] pour d'autres conséquences de ce théorème de densité sur les équivalences entre solutions fortes et faibles des équations de Navier-Stokes).

1.4 Un contre-exemple - Conclusions

Pour démontrer que (1.11) est faux il suffit de démontrer la contraposée:

Théorème 1.4.1 *Il existe une suite (u_k) d'éléments de H telle que*

(4.1) $$|u_k|_{-1}^2 \geq k(u_k, A^{-1}u_k)$$

Démonstration. Soit $\Phi \in H^1(\Omega)/\mathbb{R}$ tel que

$$\begin{cases} \nabla^2 \Phi = 0 \\ \dfrac{\partial \Phi}{\partial n}_{|\Gamma_i} = g_i \end{cases}$$

avec $\int_{\Gamma_i} g_i = 0$ pour $i = 0, \ldots, m$. Pour chaque choix des (g_i) ce problème admet une solution unique. Par définition, $\nabla\Phi$ appartient à S, de plus on a

$$\forall q \in H^1(\Omega), \qquad \int_\Omega \nabla\Phi.\nabla q = \int_\Gamma \frac{\partial \Phi}{\partial n} q = \sum_{i=0}^{m} \int_{\Gamma_i} q g_i$$

En particulier pour tout $q \in B$ on a $q_{|\Gamma_i} = a_i$ (les a_i étant des constantes), d'où

$$\forall q \in B, \qquad (\nabla\Phi, \nabla q) = 0.$$

Posons $u = \nabla\Phi$, u appartient manifestement à S_0 d'après ce qui précède. D'après le théorème de densité 1.3.1 il existe une suite (u_k) d'éléments de $\mathcal{V} \subset H$ telle que $u_k \to u$ dans $H^{-1}(\Omega)^d$. Plus précisément, on peut choisir la suite de telle sorte que pour $k \geq 1$:

$$|u - u_k|_{-1} \leq \frac{|u|_{-1}}{k}$$

Soit $v_k \in D(A)$ et $p_k \in L^2(\Omega)/\mathbb{R}$ les solutions du problème de Stokes:

$$\begin{cases} -\nabla^2 v_k + \nabla p_k &= u_k \\ \text{div} v_k &= 0 \end{cases}$$

Par definition de l'opérateur de Stokes on a $v_k = A^{-1}u_k$. D'autre part on a:

$$\begin{cases} -\nabla^2 v_k + \nabla(p_k - \Phi) &= u_k - u \\ \text{div} v_k &= 0 \end{cases}$$

d'où les estimations:

$$\begin{cases} |p_k - \Phi|_{L^2(\Omega)/\mathbb{R}} \leq c|u_k - u|_{-1} \leq \frac{c}{k}|u|_{-1} \\ |v_k|_1 \leq c|u_k - u|_{-1} \leq \frac{c}{k}|u|_{-1} \end{cases}$$

La seconde série d'estimations conduit à

$$(u_k, A^{-1}u_k) = (u_k, v_k) \leq |u_k|_{-1}|v_k|_1 \leq \frac{c}{k}|u_k|_{-1}|u|_{-1} \leq \frac{c'}{k}|u_k|_{-1}^2$$

ce qui établit le résultat modulo des constantes près. □

Ce résultat est a priori un peu surprenant, il montre l'existence de potentiels (*ie.* les fonctions Φ) dont on peut approcher le gradient dans $H^{-1}(\Omega)^d$ par des fonctions de H, (ie. des fonctions à divergence nulle et de trace normale nulle au bord). L'intuition voudrait que le gradient en question satisfasse à la limite un problème de Neumann homogène dont l'unique solution est zéro modulo les constantes. En réalité l'approximation ayant lieu dans $H^{-1}(\Omega)^d$, la condition limite de Neumann n'a pas de sens. Autrement dit, les fonctions de H convergent vers $\nabla\Phi$ en développant une couche limite au bord.

En conclusion, la question concernant la possibilité d'un taux de convergence théorique de la pression d'ordre plus élevé que $\mathcal{O}(\delta t^{1/2})$ reste ouverte. Sous réserve que l'erreur sur la pression est effectivement contrôlée uniquement par (2.5), alors une convergence

d'ordre 1 ou plus de la vitesse $(\tilde{u}_k)$ dans $H^1(\Omega)^d$ ne peut conduire qu'à une convergence d'ordre 1/2 sur la pression. Un taux de convergence plus élevé sur la pression requière une convergence de la vitesse dans un espace plus régulier que $H^1(\Omega)^d$, c'est à dire dans $\nabla^{-2}(H_0^{\text{div}}(\Omega)')$. En fait, une convergence d'ordre 1 de $\nabla^2\tilde{e}_{k+1}$ en norme $L^2(\Omega)^d$ serait suffisante.

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Quelques estimations d'erreur pour l'approximation des équations de Navier-Stokes instationnaires par une méthode de projection[1]

27 Octobre, 1993

Jean-Luc Guermond[2]

Résumé. On formule la méthode de projection dans un cadre abstrait permettant de traiter simultanément les problèmes continus et discrets en espace. On donne des estimations de convergence en temps inconditionnelles d'ordre un et conditionnelles d'ordre deux. Les simulations numériques confirment que l'ordre un est l'ordre inconditionnel maximal contrairement à ce qui a été suggéré par certains auteurs.

Some error estimates for projection methods

Summary. After reformulating the projection method in an abstract framework, one proves uncondtionnal error estimates of order one and conditionnal error estimates of order two. Numerical simulations show that the order one is the maximal unconditional order of convergence.

Key words. Navier-Stokes equations, projection methods, convergence estimates.

Mathematics Subject Classification (1991): 35A40, 35Q10, 65J15

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Chapitre 2

Quelques estimations d'erreur pour une méthode de projection

2.1 Introduction

Dans cette Note on s'intéresse aux propriétés d'approximation d'une méthode de pas fractionnaire de Goda et Van Kan [3] [11] utilisée pour approcher en temps les solutions des équations de Navier–Stokes instationnaires. Cette technique est dérivée de la méthode de projection de Chorin–Temam [2] [9]. Sa simplicité et son efficacité la rendent particulièrement attractive pour traiter des applications complexes (*eg.* [3], [6], [8] ou [11]). L'idée originale de l'algorithme consiste à décomposer chaque étape de la marche en temps en sous-pas intermédiaires pour découpler les effets de la diffusion visqueuse de ceux de l'incompressibilité, la conséquence étant qu'à chaque pas de temps une équation de Stokes généralisée, difficile à inverser pratiquement, est remplacée par une séquence de deux sous-problèmes a priori plus simples.

Bien que l'idée soit simple en principe, l'implémentation pratique de l'étape incompressible dans le cadre d'une approximation spatiale ne semble pas claire et certaines questions sur l'ordre de convergence en temps de la méthode ne sont pas encore clairement résolues [4] [8]. On propose dans cette Note quelques réponses à ces questions.

2.2 Le problème de Stokes abstrait

Afin de traiter simultanément les problèmes de Stokes continus et discrets en espace on se place dans un cadre abstrait. Soient X, L et M trois espaces de Hilbert réels tels que $X \subset L$ avec inclusion dense et continue. On fait les identifications $X \subset L \equiv L' \subset X'$ et $M \equiv M'$. Pour un espace de Hilbert donné E, on note respectivement $(.,.)_E$ et $|.|_E$ le produit scalaire et la norme de E.

On introduit maintenant une forme bilinéaire continue $b : X \times M \longrightarrow \mathbb{R}$ et par l'intermédiaire du théorème de Riesz on lui associe $B : X \longrightarrow M$ et son transposé

$B^t : M \longrightarrow X'$ tels que pour tout (u,p) dans $X \times M$ on ait $(Bu,p)_M = b(u,p)$ et $< u, B^t p >_{X',X} = b(u,p)$, où $< .,. >_{X',X}$ est le produit de dualité entre X and X'.

Le noyau de B jouant un rôle important par la suite on définit $V = \ker(B)$ qu'on équipe de la norme et topologie induite par X. On note H le complété de V dans L qu'on équipe de la norme et topologie de L. Les espaces V et H sont des Hilbert et $V \subset H$ avec densité et continuité. On suppose enfin que B est surjectif. Rappellons alors (voir Brezzi [1], ou Girault-Raviart [5, p. 58-59] par exemple):

Lemme 2.2.1 *Les propositions suivantes sont équivalentes:*
(i) $B : X \longrightarrow M$ *est surjectif.*
(ii) $B^t : M \longrightarrow X'$ *est injectif et* $B^t(M)$ *fermé dans dans* X'.
(iii) $B : V^\perp \longrightarrow M$ *bijectif et* $|Bv|_M \geq \beta |v|_X$, *pour tout* $v \in V^\perp$.
(iv) $B^t : M \longrightarrow V^o$ *est bijectif et* $|B^t p|_{X'} \geq \beta |p|_M$, *pour tout* $p \in M$.
où $V^\perp$ *est l'orthogonal de* V *dans* X *et* V^o *le pôlaire de* V *in* X'.

On introduit aussi $M^1 = \{p \in M,\ B^t p \in L\}$ où la condition $B^t p \in L$ signifie que Bp peut s'identifier par prolongement continu à un élément de L. Une conséquence du lemme 1.2.1, (iv) est que $|p|_{M^1} = |B^t p|_M$ est une norme dans M^1; plus précisement:

Corollaire 2.2.1 M^1 *équipé de la norme* $|.|_{M^1}$ *est un Hilbert et* $L = H \oplus B^t(M^1)$.

Cette décomposition de L est l'analogue de $L^2(\Omega) = H \oplus \text{grad}(H^1(\Omega)/\mathbb{R})$.

Introduisons enfin la forme bilinéaire continue $a : X \times X \longrightarrow \mathbb{R}$ qu'on suppose X-elliptique, *i.e.* $a(u,u) \geq \alpha |u|_X^2$ pour tout u dans X. A l'aide du théorème de Riesz on lui associe $A : X \longrightarrow X'$ tel que pour tout $(u,v) \in X \times X$, $a(u,v) =< Au, v >_{X',X}$.

Dans le cadre fonctionel ainsi défini on reformule le problème de Stokes comme suit: pour $f \in L^2(0,T,X')$ et $u_0 \in H$ chercher $u \in L^2(0,T;X)$ et $p \in L^2(0,T;M)$ tels que:

$$(2.1) \qquad \begin{cases} \frac{\partial u}{\partial t} + Au + B^t p = f & \text{p.p. } t \in [0,T] \\ Bu = 0 \\ u(t=0) = v_0 \end{cases}$$

Ce problème est bien posé cf. [7, p. 257, thm 4.1].

Il est maintenant possible de réécrire dans ce cadre la version modifée par Goda [3] de l'algorithme de projection de Chorin-Temam. Dans cette perspective on introduit $t^k = k\delta t$ pour $0 \leq k \leq K$ où $\delta t = T/K$ et les suites approchées $(u_k) \in H$, $(\tilde{u}_k) \in X$ et $(p_k) \in M$ telles que

$$(2.2) \qquad \frac{\tilde{u}_{k+1} - u_k}{\delta t} + A\tilde{u}_{k+1} = f_{k+1} - B^t p_k$$

et

$$(2.3) \qquad \begin{cases} \frac{u_{k+1} - \tilde{u}_{k+1}}{\delta t} + B^t(p_{k+1} - p_k) = 0 \\ Bu_{k+1} = 0 \end{cases}$$

où on pose $u_0 = v_0$ et $p_0 = p(t=0)$ en supposant $p \in C^0(0,T;M)$.

Remarque 1. Le problem (2.2) est bien posé grâce à la X-ellipticité de A. Le problème (2.3) est lui aussi bien posé grâce au corollaire 1.2.1: en fait le couple $(u_{k+1}, \delta t(p_{k+1} - p_k))$ est la décomposition de $\tilde{u}_{k+1}$ dans $H \oplus B^t(M^1)$; *i.e.* $u_{k+1} = P_H \tilde{u}_{k+1}$ où P_H est la projection orthogonale de L sur H. Pour cette raison on appelle (2.3) l'étape de projection.

Remarque 2. En pratique lorsque X et M sont des espaces d'approximation conformes de $H_0^1(\Omega)$ et $L^2(\Omega)/\mathbb{R}$, le problème (2.3) est équivalent à

$$(2.4) \qquad \begin{cases} \forall v_h \in X_h \subset H_0^1(\Omega), & \frac{1}{\delta t}(u_{k+1} - \tilde{u}_{k+1}, v_h) - (\text{div} v_h, p_{k+1} - p_k) = 0 \\ \forall q_h \in M_h \subset L^2(\Omega)/\mathbb{R}, & (\text{div} u_{k+1}, q_h) = 0 \end{cases}$$

où les fonctions tests de X_h satisfont $v_{h|\partial\Omega} = 0$. En pratique, en choisissant des bases de X_h et M_h et en notant I_h la matrice de masse, B_h la matrice associée à l'opérateur divergence et U_k, P_k les vecteurs vitesse et pression inconnus on aboutit au système linéaire

$$(2.5) \qquad B_h I_h^{-1} B^t (P_{k+1} - P_k) = \frac{B_h \tilde{U}_{k+1}}{\delta t}$$

et non pas à un système approchant un problème de Poisson variationel avec condition de Neumann homogène tel que suggéré par le problème continu en espace [6], [8], [10].

2.3 Estimations de convergence

Une première mesure des qualités approximantes des suites (u_k), $(\tilde{u}_k)$ et (p_k) est donnée par

Théorème 2.3.1 *Si la solution de (2.1) est telle que* $u_{tt} \in L^2(0,T;L)$ *et* $p_t \in L^2(0,T;M^1)$ *et si* $|u_0 - v_0|_L \leq c\delta t$ *et* $|p_0 - p(t=0)|_{M_1} \leq c$ *on a alors les estimations d'erreurs:*

$$(3.1) \qquad \sup_{0 \leq t^k \leq T} \left[|e_k|_L^2 + \delta t^2 |\delta_k|_{M^1}^2 + \alpha \delta t \sum_{l=1}^{k} |\tilde{e}_l|_X^2 \right] \leq c\delta t^2$$

où on a défini les fonctions d'erreur $e_k = u(t^k) - u_k$, $\tilde{e}_k = u(t^k) - \tilde{u}_k$ et $\delta_k = p(t^k) - \delta_k$.

Remarque 1. Ce résultat établi dans [8] pour le problème de Stokes continu en espace n'était pas encore été établi dans le cadre discret.

Remarque 2. Le théorème ci-dessus ne donne pas de résultat fort pour la convergence en pression en norme du gradient; c'est dû à la condition de Neumann homogène implicitement imposée sur la pression par (2.3). Pour établir une convergence en norme $l^2(M)$ on soustrait (2.2)+(2.3) de (2.1) pour obtenir:

$$(3.2) \qquad \frac{e_{k+1} - e_k}{\delta t} + A\tilde{e}_{k+1} + B^t \delta_{k+1} = R_k$$

où R_k est le reste intégral de Taylor. Puisque (3.1) donne un estimation de $A\tilde{e}_{k+1}$ dans $l^2(X')$ et que R_k est facilement majoré, on obtient une estimation en pression en majorant le terme $(e_{k+1} - e_k)/\delta t$ dans $L \subset X'$. La majoration souhaitée est donnée par:

Théorème 2.3.2 *Si la solution de (2.1) est telle que* $u_{ttt} \in L^2(0,T;L)$ *et* $p_{tt} \in L^2(0,T;M^1)$ *et si les initialisations vérifient* $|u_0 - v_0|_L \leq c\delta t^2$ *et* $|p_0 - p(t=0)|_{M_1} \leq c\delta t$ *alors:*

$$(3.3) \qquad \sup_{0\leq t^k<T}\left[|e_{k+1}-e_k|_L^2 + \delta t^2|\delta_{k+1}-\delta_k|_{M^1}^2 + \alpha\delta t\sum_{l=1}^{k}|\tilde{e}_{l+1}-\tilde{e}_l|_X^2\right] \leq c\delta t^4$$

Pour montrer ce résultat on soustrait deux étapes successives de prédiction et de projection, on se ramène ainsi exactement au cadre du théorème 1.3.1.

Corollaire 2.3.1 *Sous les hypothèses du théorème 1.3.2 on a l'estimation en pression:*

$$(3.4) \qquad \left[\delta t\sum_{0}^{K}|\delta_k|_M^2\right]^{1/2} \leq c\delta t$$

Cette estimation résulte de (3.2), de l'inégalité (iv) du lemme 1.2.1 et de (3.1) et (3.3).

2.4 Les ordres supérieurs

Afin d'améliorer l'ordre de convergence global Van Kan [11] propose de remplacer (2.2) par

$$(4.1) \qquad \frac{\tilde{u}_{k+1}-u_k}{\delta t} + \frac{1}{2}A(\tilde{u}_{k+1}+\tilde{u}_k) = f_{k+1/2} - B^t p_{k-1/2}$$

On montre exactement comme précédemment que l'algorithme ainsi modifié est inconditionnellement d'ordre un, mais en supposant $A \in \mathcal{L}(L,L)$, ce qui est le cas pour les problèmes d'approximation spatiale, on a

Théorème 2.4.1 *Soit* $Q = \sup_{v\in L}|Av|_L/|v|_L$. *Si* u_{ttt} *et* Au_{tt} *sont dans* $L^2(0,T;L)$, $p_{tt} \in L^2(0,T;M^1)$ *et* $\max(|e_0|_L, \delta t^{1/2}|\tilde{e}_0|_X, \delta t|p(t^{1/2}) - p^{-1/2}|_{M^1}) \leq c\delta t^2$ *et si* $\delta t \leq 1/2Q$ *alors*

$$(4.2) \qquad \sup_{0\leq t^k\leq T}[|e_k|_L + |\tilde{e}_k|_X + [\delta t\sum_{l=1}^{k}|\delta_l|_M^2]^{1/2}] \leq c(Q)\delta t^2$$

En pratique on observe bien l'ordre 2 pour δt suffisamment petit et l'ordre 1 sinon. De plus, on obtient uniquement l'ordre 1 si les problèmes (4.1) et (2.3) sont résolus approximativement avec des résidus qui ne sont pas très nettement inférieurs à $\min(\delta t^2, 1/Q)$. Ceci suggère que l'ordre 1 est l'ordre inconditionnel maximal de l'algorithme et non pas les ordres 3/2 ou 2 comme avancé par certains auteurs. On a bien sûre des résultats équivalents avec une approximation d'Euler rétrograde d'ordre deux.

$$(4.3) \qquad \frac{3\tilde{u}_{k+1} - \tilde{u}_k - 3u_k + u_{k-1}}{2\delta t} + A\tilde{u}_{k+1} = f_{k+1} - B^t p_k$$

et

$$(4.4) \qquad \begin{cases} \dfrac{3u_{k+1} - u_k - 3\tilde{u}_{k+1} + \tilde{u}_k}{2\delta t} + B^t(p_{k+1} - p_k) = 0 \\ Bu_{k+1} = 0 \end{cases}$$

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Some error estimates for the approximation of the unsteady Navier–Stokes equations by means of projection methods[1]

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Summary. After reformulating the projection methods in an abstract framework which is suitable for treating either continuous in space or discrete Stokes problems, one proves unconditional error estimates of order one and conditional error estimates of order two. Numerical simulations confirm that the order one is the maximal unconditional order of convergence.

Résumé. On formule les méthodes de projection dans un cadre abstrait permettant de traiter simultanément les problèmes continus et discrets en espace. On donne des estimations de convergence en temps inconditionnelles d'ordre un et conditionnelles d'ordre deux. Les simulations numériques confirment que l'ordre un est l'ordre inconditionnel maximal contrairement à ce qui est parfois suggéré dans la littérature.

Key words. Projection method, Fractional step method, Navier–Stokes equations.

AMS(MOS) subject classification. 35A40, 35Q10, 65J15

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Chapter 3

Some error estimates for projection methods

3.1. Introduction

This paper is concerned with some approximation properties of a class of fractional step methods of Chorin and Temam [4] [17] and its extensions by Goda and Van Kan [6] [19]. These techniques, also known as projection methods, have been proposed for solving the unsteady incompressible Navier–Stokes equations. The simplicity and sometimes surprising efficiency of projection methods render them particularly attractive to CFD practitioners [1] [6] [7] [11] [13] [19]. The original idea behind this class of methods is to split the time marching procedure into intermediate steps in order to uncouple incompressibility effects and viscous effects; the consequence of this splitting being that at each time step an otherwise hard-to-solve Stokes problem is replaced by two simpler problems.

Though the basic idea is simple in principle, a global understanding of the approximation properties of this method and of its practical implementation are marred by lingering questions concerning the appropriate boundary condition that should be imposed either on the end-of-step velocity or on the pressure. Furthermore the possibility of building schemes that are second order accurate in time or possibly of higher order seems to remain an open question [8] [14].

The paper is organized as follows. In a first part, the continuous Stokes problem is introduced and the non incremental and incremental versions of the projection method are reviewed. In a second part, the unsteady Stokes problem together with some projection algorithms is set in an abstract variational framework which is suitable for treating either the semi-discrete problem (discrete in time but continuous in space) or the fully discrete one (both time and space are discretized). The abstract framework is particularly suited for showing that no explicit boundary condition on the pressure is actually required for solving the equation of the incompressibility step. It is shown that the discrete projection step that naturally comes in is not a discrete representation of a Poisson equation possibly supplemented by a homogeneous Neumann boundary condition as is sometimes assumed by some authors. Convergence estimates for Goda's scheme are given in a third part. It is shown that this scheme is first order accurate both in terms of velocity and pressure. Convergence estimates of $\mathcal{O}(\delta t^{1/2})$ are also given for Chorin–Temam's scheme. These results may be seen, in some sense, as a generalization of those given in Shen [14], though the estimates in pressure in [14] were not correctly demonstrated (see [8] [15]). In a fourth part, schemes of higher order like Van Kan's are studied. Conditional error estimates of order two are given. One also gives some arguments indicating that it is not possible to build projection schemes that are both unconditionally stable and of order higher than one. This order barrier comes from the underlying Neumann boundary condition that is more or less implicitly satisfied by the pressure, even though it does not appear explicitly in the numerical schemes. In the last part, one gives some numerical examples and compare theoretical and numerical results. One numerical test illustrates the fact that Van Kan's scheme is indeed conditionally second order accurate but on the whole the maximal unconditional order of accuracy is one.

3.2. The continuous Stokes problem

3.2.1. The unsteady Stokes problem

Let Ω be an open connected bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3 in practical applications) with a smooth boundary $\partial\Omega$; say $\partial\Omega$ is Lipschtizian and Ω is locally on one side of its boundary. In the following, the set of real functions infinitely differentiable and of compact support in Ω is denoted by $\mathrm{D}(\Omega)$. The set of distributions on Ω is denoted by $\mathrm{D}'(\Omega)$.

As usual, $\mathrm{L}^2(\Omega)$ denotes the space of real-valued functions, the squares of which are summable in Ω. We denote the inner product in $\mathrm{L}^2(\Omega)$ by $(\cdot\,,\cdot)$ and let $|\cdot|_0$ be its norm. $\mathrm{H}^m(\Omega)$, $m \geq 0$, is the set of distributions the successive derivatives of which, up to order m, can be identified with square summable functions. The space $\mathrm{H}^m(\Omega)$, equipped with the norm $|u|_m^2 = (\sum_{|\alpha|=0}^m |D^\alpha u|_0^2)^{1/2}$, expressed in the multi-index notation, is a Hilbert space. Now we define $\mathrm{H}_0^m(\Omega)$ as the completion of $\mathrm{D}(\Omega)$ in $\mathrm{H}^m(\Omega)$ and we denote $\mathrm{H}^{-m}(\Omega)$ the dual of $\mathrm{H}_0^m(\Omega)$. The duality product is still denoted by $(\cdot\,,\cdot)$ since we identify $\mathrm{L}^2(\Omega)$ with its dual. Spaces of vector-valued functions are hereafter denoted with boldface type, though no distinction is made in the notation of inner products and norms.

The analysis of the incompressible Navier–Stokes equations leads to consider solenoidal velocity fields; hence we define $\mathbf{N}(\Omega) = \{v \in \mathbf{D}(\Omega),\ \nabla.v = 0\}$ and we denote by $\mathbf{H}$ and $\mathbf{V}$ the completions of $\mathbf{N}(\Omega)$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}_0^1(\Omega)$ respectively. We have

THEOREM 3.2.1. *Spaces $\mathbf{H}$ and $\mathbf{V}$ are characterized by:*

$$\mathbf{H} = \{v \in \mathbf{L}^2(\Omega),\ \nabla.v = 0,\ v.n_{|\partial\Omega} = 0\} \tag{2.1}$$

$$\mathbf{V} = \{v \in \mathbf{H}^1(\Omega),\ \nabla.v = 0,\ v_{|\partial\Omega} = 0\} \tag{2.2}$$

Proof. See for instance Temam [16, p.15–18]. □

Although the nonlinear terms of the Navier–Stokes equations have tremendous effects on the stability properties of time approximations, they do not modify the parabolic character of the equations. Hence in order to simplify the presentation we restrict ourselves to the time-dependent Stokes problem, that is, we look for a velocity field and a pressure field such that

$$\begin{cases} \dfrac{\partial u}{\partial t} - \nu\nabla^2 u + \nabla p = f \\ \nabla.u = 0 \\ u_{|\partial\Omega} = 0 \\ u_{|t=0} = v_0 \end{cases} \tag{2.3}$$

where $f(t)$ is a body force and the boundary condition on the velocity is set to zero for sake of simplicity. The time interval $[0, T]$ is assumed to be finite. The problem (2.3) is meaningless unless we specify the regularity of u and p that we should expect, that is we clearly specify the functional framework. It is usually convenient to look for u in $\mathrm{L}^2(0, T; \mathbf{H}_0^1(\Omega))$ and p in $\mathrm{D}'(0, T; \mathrm{L}^2(\Omega)/\mathbb{R})$. If one assume that $f \in \mathrm{L}^2(0, T; \mathbf{H}^{-1}(\Omega))$ and $u_0 \in \mathbf{H}$, then thanks to Lions's theorem [12, p. 257] one may show that (2.3) is well-posed (look for $u \in \mathrm{L}^2(0, T; \mathbf{V})$ and restrict the time evolution problem to $\mathrm{L}^2(0, T; \mathbf{V}')$ where $\mathbf{V}'$ is the dual of $\mathbf{V}$). Furthermore, one may verify that $p \in \mathrm{L}^2(0, T; \mathrm{L}^2(\Omega)/\mathbb{R})$.

3.2.2. The non-incremental and incremental projection schemes

In order to define a time approximation of (2.3) on the interval $[0, T]$ we introduce a partition: $t^k = k\delta t$ for $0 \leq k \leq K$ where $\delta t = T/K$. The principle of projection methods [4] [17] is based on a predictor-corrector strategy: it consists in building two series of approximate velocities (u_k) and $(\tilde{u}_k)$ and one series of approximate pressure (p_k) such that at each time step $\tilde{u}_{k+1}$ is a prediction of $u(t^{k+1})$ and u_{k+1} is a correction of $\tilde{u}_{k+1}$. The algorithm reads as follows:

$$\begin{cases} \dfrac{\tilde{u}_{k+1} - u_k}{\delta t} - \nu\nabla^2\tilde{u}_{k+1} = f_{k+1} \\ \tilde{u}_{k+1|\partial\Omega} = 0 \end{cases} \tag{2.4}$$

and

$$\begin{cases} \dfrac{u_{k+1} - \tilde{u}_{k+1}}{\delta t} + \nabla p_{k+1} = 0 \\ \nabla.u_{k+1} = 0 \\ u_{k+1}.n_{|\partial\Omega} = 0 \end{cases} \tag{2.5}$$

The series (u_k) is initialized by $u_0 = v_0$.

Actually, the algorithm (2.4)–(2.5) may not be the best for simulating time-dependent Stokes (and Navier–Stokes) problems since, as it will be shown farther, its unconditional accuracy order is $\mathcal{O}(\delta t^{1/2})$. This lack of precision can be cured by putting ∇p_k in the right-hand side of (2.4), thus increasing the accuracy of the predicted velocity $\tilde{u}_{k+1}$. This algorithm, hereafter referred to as Goda's version [6] or the incremental form of the projection technique can be put as follows

$$(2.6)\qquad \begin{cases} \dfrac{\tilde{u}_{k+1}-u_k}{\delta t} - \nu\nabla^2\tilde{u}_{k+1} = f_{k+1} - \nabla p_k \\ \tilde{u}_{k+1|\partial\Omega} = 0 \end{cases}$$

and

$$(2.7)\qquad \begin{cases} \dfrac{u_{k+1}-\tilde{u}_{k+1}}{\delta t} + \nabla(p_{k+1}-p_k) = 0 \\ \nabla.u_{k+1} = 0 \\ u_{k+1}.n_{|\partial\Omega} = 0 \end{cases}$$

The series (u_k) is initialized by $u_0 = v_0$ and the series (p_k) is initialized by $p_0 = p_{|t=0}$ assuming that $p \in C(0,T;\mathrm{L}^2(\Omega)/\mathbb{R})$. Note that this algorithm assumes more regularity than the non incremental one since it requires some regularity on $p_{|t=0}$.

Note that the systems (2.5) and (2.7) are equivalent to $u_{k+1} = P_H\tilde{u}_{k+1}$ and either $\nabla p_{k+1} = (\tilde{u}_{k+1} - P_H\tilde{u}_{k+1})/\delta t$ or $\nabla(p_{k+1}-p_k) = (\tilde{u}_{k+1} - P_H\tilde{u}_{k+1})/\delta t$, where P_H is the orthogonal projection of $\mathbf{L}^2(\Omega)$ onto $\mathbf{H}$. In both cases, the velocity $\tilde{u}_{k+1}$ is a prediction of $u(t^{k+1})$ that satisfies the correct boundary condition but is not divergence free. This defect is corrected by projecting $\tilde{u}_{k+1}$ onto $\mathbf{H}$ (this step has given its name to the method). However, the end-of-step velocity, u_{k+1}, does not satisfy exactly the correct boundary condition, the tangential components of u_{k+1} being not necessarily zero.

The main advantage of the fractional step techniques (2.4)-(2.5) or (2.6)-(2.7) over classical coupled approaches is that it only requires solving Helmholtz problems and computing the projection P_H, whereas classical coupled techniques usually involve a Uzawa operator that is non-local and ill-conditioned: $\nabla.(id - \sigma\nabla^2)^{-1}\nabla$ where σ is proportional to $\delta t/\nu$.

Note that a possible alternative to computing P_H may be inferred from (2.5) or (2.7). By denoting either $\phi_{k+1} = p_{k+1}$ or $\phi_{k+1} = p_{k+1} - p_k$, and by taking the divergence of the first equation of (2.5) or (2.7) one is led to solve the following Poisson problem supplemented with homogeneous Neumann boundary condition:

$$(2.8)\qquad \begin{cases} \nabla^2\phi_{k+1} = \dfrac{\nabla.\tilde{u}_{k+1}}{\delta t} \\ \dfrac{\partial\phi_{k+1}}{\partial n} = 0 \end{cases}$$

One immediate consequence is that $\partial\phi_{k+1}/\partial n = \ldots = \partial\phi_0/\partial n$; as it will be shown later, the presence of this boundary condition, not necessarily satisfied by the exact pressure, spoils in some sense the possibility of building algorithms that have accuracy order higher than one. Furthermore (2.8) is misleading in the sense that when it comes to discretize the problem in space one might think that a good approximation of P_H could be obtained by approximating (2.8). Indeed this idea is biased and has led many CFD practitioners to speculate about the adequate pressure boundary condition that should be implemented for solving the projection step. It will be shown later that, even though the homogeneous Neumann boundary condition is always more or less implicitly implied, the operator P_H can be approximated without imposing any boundary condition on the pressure. In order to understand this point it is convenient to set the Stokes problem in an abstract framework.

3.3. The abstract Stokes problem

3.3.1. Preliminaries

Let X, L and M be three real Hilbert spaces so that $X \subset L$, the embedding being continuous and X being dense in L. In the following we make the identifications $X \subset L \equiv L' \subset X'$ and $M \equiv M'$. For a given Hilbert space E, we respectively denote by $(.,.)_E$ and $|.|_E$ the inner product and norm of E.

Remark 3.3.1. For the continuous Stokes problem we set $X = \mathbf{H}^1_0(\Omega)$, $M = \mathrm{L}^2(\Omega)/\mathbb{R}$ and when the problem is discretized in space we set $X = X_h$ and $M = M_h$ where X_h and M_h are convergent stable approximations of $\mathbf{H}^1_0(\Omega)$ and $M = \mathrm{L}^2(\Omega)/\mathbb{R}$. Note that in this case X and M are finite dimensional vector spaces, hence the three vector spaces X, L and X' are identical. However, though their topologies are also identical, one makes a clear difference between their norms.

We now introduce the continuous bilinear form $b : X \times M \longrightarrow \mathbb{R}$, and by means of Riesz's theorem we associate with b the continuous linear operator $B : X \longrightarrow M$ so that for every couple (u,p) in $X \times M$ we have $< Bu, p >_{X',X} = b(u,p)$, where $< .,. >_{X',X}$ denotes the duality products between X and X'. The null space of B playing an important rôle in the following we set $V = \ker(B)$ and we equip V with the topology induced by that of X. We also denote by H the completion of V in L, *i.e.*, $H = \overline{V}^L$ and we equip H with the topology induced by that of L. Spaces V and H are Hilbert spaces and the embedding of V into H is continuous and dense.

We also assume that B is onto. Some consequences of this hypothesis are summerized by the following well-known result.

LEMMA 3.3.1. *The following propositions are equivalent:*

(i) $B : X \longrightarrow M$ *is onto.*

(ii) $B^t : M \longrightarrow X'$ *is into and* $B^t(M)$ *closed in* X'.

(iii) $B : V^\perp \longrightarrow M$ *is one to one and* $|Bv|_M \geq \beta|v|_X$, *for all* $v \in V^\perp$.

(iv) $B^t : M \longrightarrow V^o$ *is one to one and* $|B^tp|_{X'} \geq \beta|p|_M$, *for all* $p \in M$.

Where $V^\perp$ *is the orthogonal of* V *in* X *and* V^o *is the orthogonal (polar set) of* V *in* X'.

Proof. cf. Brezis [2] p. 29-30, Brezzi [3], or Girault-Raviart [5] p. 58-59. □

The constant β involved in (iii) and (iv) is usually referred to as the "inf-sup" constant.

For later results it is convenient to introduce M^1 the subspace of M so that $M^1 = \{p \in M,\ B^tp \in L\}$ where the condition $B^tp \in L$ is understood in the sense that Bp can be identified by continuous extension with an element of L. One consequence of lemma 1.3.1, (iv) is that $|p|_{M^1} = |B^tp|_M$ is a norm in M^1; furthermore we have:

COROLLARY 3.3.1. *(i)* M^1 *equipped with the norm* $|.|_{M^1}$ *is a Hilbert space.*

(ii) We have the decomposition:

$$(3.1)\qquad L = H \oplus B^t(M^1)$$

Proof. (i) is evident.

(ii) Let $l \in L$, since H is closed in L we can define P_H the orthogonal projection onto H. Then we have $(l - P_H l, h) = 0$ for all h in H, in other words we also have $(l - P_H l, v) = 0$ for all v in V, that is to say $(l - P_H l) \in V^o$. From (iv) of lemma 1.3.1 we infer that there is a unique $p \in M$ so that $B^t p = l - P_H l$. Since $B^t p = l - P_H l \in L$ we have also $p \in M^1$. □

Remark 3.3.2. Note that the above decomposition of L is the analog of the classical decomposition: $\mathbf{L}^2(\Omega) = \mathbf{H} \oplus \nabla(H^1(\Omega)/\mathbb{R})$. Once again, when M has a finite dimension, M and M^1 coincide as vector spaces but they are equipped with distinct norms.

Let us also introduce the continuous bilinear form $a : X \times X \longrightarrow \mathbb{R}$, and assume that a is X-elliptic, that is:

$$\exists \alpha > 0,\ \forall u \in X \qquad a(u,u) \geq \alpha |u|_X^2 \tag{3.2}$$

Indeed, wellposedness of the forthcoming abstract Stokes problem would only require that a was V-elliptic, however the projection methods that are described below require a to be X-elliptic for stability reasons. By means of Riesz's theorem we associate with a the linear continuous operator $A : X \longrightarrow X'$ so that for all $(u, v) \in X \times X$, $a(u,v) =< Au, v >_{X',X}$. The X-ellipticity of a implies that $A : X \longrightarrow X'$ is one to one.

In the functional framework defined above, the time-dependent Stokes problem can be reformulated as follows. For $f \in \mathrm{L}^2(0,T,X')$ and $u_0 \in H$ find $u \in \mathrm{L}^2(0,T;X)$ and $p \in \mathrm{L}^2(0,T;M)$ so that:

$$\begin{cases} \frac{\partial u}{\partial t} + Au + B^t p = f \\ Bu = 0 \\ u_{|t=0} = v_0 \end{cases} \tag{3.3}$$

Remark 3.3.3. The reader who is not accustomed to the abstract formulation (3.3) may think of it as the matrix form of a variational problem; though, this matrix form is intrinsic in the sense that it does not depend on any particular bases of X and M. At this stage some clarification concerning the interpretation of the abstract operators A, B and B^t could be helpful. For the continuous Stokes problem, the bilinear form b in question is defined by $(\nabla.u, p)$ and B is the restriction of $\nabla.$ to $\mathbf{H}_0^1(\Omega)$. In the framework of spatial approximations of Stokes problem, B can sometimes be identified with the restriction of $\nabla.$ to X (see for instance Fortin's finite element: conformal Bubble/P_2 for velocity and non-conformal P_1 for pressure). However, in general B is not the restriction of the divergence operator to X; the reader may convince himself by considering the approximation of Stokes problem by the following conformal finite elements (Bubble/P_1, P_1) or (P_2, P_1). This point is clearer for B^t if we assume that X is a set of conformal finite elements and the pressure is approximated by P_k finite elements. In general for an approximate pressure p_h, ∇p_h is discontinuous; hence, $\nabla(M) \not\subset X'$, where recall that in finite dimension X' is equal to X in terms of vector space, though it is equipped with the dual norm. Hence, in general B^t can not be identified with the restriction of ∇ to M. Likewise, note that in the framework of finite dimensional Stokes problems A can not in general be identified with the restriction of ∇^2 to X.

Thanks to Lions's theorem [12, p. 257], problem (3.3) is wellposed. We hereafter assume that the solution to (3.3) is such that u_t is in $\mathrm{L}^2(0,T;L)$ and p is in $\mathrm{L}^2(0,T;M^1)$. We turn now the attention to the approximation of this problem by means of projection schemes.

3.3.2. The projection schemes

In the following we restrict ourselves to the incremental algorithm (2.6)-(2.7) and reformulate it within the abstract functional framework defined above. However, most of what is said below also applies to the non-incremental formulation. Introduce a partition of the time interval $[0,T]$: $t^k = k\delta t$ for $0 \leq k \leq K$ where $\delta t = T/K$ and define two series of approximate velocities $(u_k) \in H$ and $(\tilde{u}_k) \in X$ and one series of approximate pressure $(p_k) \in M$ so that

$$\left\{ \frac{\tilde{u}_{k+1} - u_k}{\delta t} + A\tilde{u}_{k+1} = f_{k+1} - B^t p_k \right. \tag{3.4}$$

and

$$\begin{cases} \frac{u_{k+1} - \tilde{u}_{k+1}}{\delta t} + B^t(p_{k+1} - p_k) = 0 \\ Bu_{k+1} = 0 \end{cases} \tag{3.5}$$

The series (u_k) is initialized by $u_0 = v_0$ and assuming that $p \in C(0,T;M)$ the series (p_k) is initialized by $p_0 = p_{|t=0}$.

Remark 3.3.4. The problem (3.4) is wellposed since A is X-elliptic. The problem (3.5) is also wellposed thanks to corollary 1.3.1: indeed the couple $(u_{k+1}, \delta t(p_{k+1} - p_k))$ is the decomposition of $\tilde{u}_{k+1}$ in $H \oplus B^t(M^1)$; *i.e.*, $u_{k+1} = P_H \tilde{u}_{k+1}$ where P_H is the orthogonal projection of L onto H.

Remark 3.3.5. When $X = X_h$ and $M = M_h$ are internal approximation spaces of $\mathbf{H}_0^1(\Omega)$ and $\mathrm{L}^2(\Omega)/\mathbb{R}$, (3.5) does not involve any boundary condition on $p_{k+1} - p_k$. Indeed, in this context, problem (3.5) is equivalently reformulated as follows:

$$\begin{cases} \forall v_h \in X_h \subset \mathbf{H}_0^1(\Omega), & \frac{1}{\delta t}(u_{k+1} - \tilde{u}_{k+1}, v_h) - (\nabla.v_h, p_{k+1} - p_k) = 0 \\ \forall q_h \in M_h \subset \mathrm{L}^2(\Omega)/\mathbb{R}, & (\nabla.u_{k+1}, q_h) = 0 \end{cases} \tag{3.6}$$

Note that in (3.6) the test functions of X_h satisfy $v_{h|\partial\Omega} = 0$ though it might seem more natural to assume only $v_h.n_{|\partial\Omega} = 0$. Note first that (3.6) is the exact transcription of (3.5); hence, as stated in remark 1, (3.6) is wellposed. Second, if we had chosen test functions satisfying $v_h.n_{|\partial\Omega} = 0$, then the associated abstract problems (3.5) and (3.6) could not have been set in the same spaces and the demonstrations of convergence in §4 would have been somewhat unnecessarily complicated. In practice, after choosing bases of X_h and M_h and by denoting I_h the mass matrix, B_h the matrix associated with the divergence operator, and U_k, P_k the vectors of velocity and pressure unknowns one is led to solve the following linear system

$$B_h I_h^{-1} B_h^t (P_{k+1} - P_k) = \frac{B_h \tilde{U}_{k+1}}{\delta t}. \tag{3.7}$$

Note that the linear operator above is not to be considered as the approximation of a variational Poisson problem supplemented with a homogeneous Neumann boundary condition (2.8) as it is sometimes suggested [7], [14], [18]. Indeed, the Poisson formulation, though being a possible alternative, is not that which naturally comes in as far as the abstract formulation is concerned. This point has also been brought to light by Van Kan [19] (remark at bottom of p. 874).

3.4. Error estimates

3.4.1. Error in velocity for the incremental formulation

A first measure of the ability of the solution to (3.4)–(3.5) to approximate that of (3.3) is given by

THEOREM 3.4.1. *If the solution to (3.3) is such that* $u_{tt} \in L^2(0,T;L)$ *and* $p_t \in L^2(0,T;M^1)$ *and if the series* (u_k) *and* (p_k) *are initialized so that* $|u_0 - v_0|_L \leq c\delta t$ *and* $|p_0 - p_{|t=0}|_{M^1} \leq c$ *then we have the following error estimate:*

$$\sup_{0\leq t^k \leq T}\left[|e_k|_L^2 + \delta t^2|\delta_k|_{M^1}^2 + 2\alpha\delta t\sum_{l=1}^{k}|\tilde{e}_l|_X^2\right] \leq c\delta t^2 \tag{4.1}$$

Where we have defined the error functions $e_k = u(t^k) - u_k$, $\tilde{e}_k = u(t^k) - \tilde{u}_k$ *and* $\delta_k = p(t^k) - p_k$.

Proof. Let us first put together the equations that control the error functions. First, introduce the consistency error induced by the approximation of the time derivative by finite differences. Thanks to the assumed regularity of u we have

$$u_t(t^{k+1}) = \frac{u(t^{k+1}) - u(t^k)}{\delta t} + \frac{1}{\delta t}\int_{t^k}^{t^{k+1}}(s - t^k)u_{ss}(s)\,ds \tag{4.2}$$

Let R_{k+1} be the integral Taylor residual of the equation above.

Second, by subtracting (3.4) from (3.3) and using (4.2) we derive the equation that controls $\tilde{e}_{k+1}$:

$$\frac{\tilde{e}_{k+1} - e_k}{\delta t} + A\tilde{e}_{k+1} = -R_{k+1} - B^t[\delta_k + p(t^{k+1}) - p(t^k)]$$

where $\tilde{e}_{k+1}$ is in X. The equation that controls e_k and δ_k is directly obtained from (3.5) by using the definitions of the error functions:

$$\frac{e_{k+1} - \tilde{e}_{k+1}}{\delta t} + B^t[\delta_{k+1} - (\delta_k + p(t^{k+1}) - p(t^k))] = 0$$

where e_k is in H. For sake of legibility let us set $\phi_k = \delta_k + p(t^{k+1}) - p(t^k)$, then the equations above can be put in the following close form that will be used repeatedly later in other demonstrations:

$$\left\{ \frac{\tilde{e}_{k+1} - e_k}{\delta t} + A\tilde{e}_{k+1} = -R_{k+1} - B^t\phi_k \right. \tag{4.3}$$

and

$$\left\{ \begin{array}{l} \frac{e_{k+1} - \tilde{e}_{k+1}}{\delta t} + B^t(\delta_{k+1} - \phi_k) = 0 \\ Be_{k+1} = 0 \end{array} \right. \tag{4.4}$$

By taking the inner product of (4.3) by $2\delta t\tilde{e}_{k+1}$ and using the X-ellipticity of A together with the classical relation $2(a, a-b) = |a|^2 + |a-b|^2 - |b|^2$ we obtain:

$$|\tilde{e}_{k+1}|_L^2 + |\tilde{e}_{k+1} - e_k|_L^2 - |e_k|_L^2 + 2\alpha\delta t|\tilde{e}_{k+1}|_X^2 \leq -2\delta t(\tilde{e}_{k+1}, R_{k+1} + B^t\phi_k)$$

Using the generalized Cauchy–Schwartz inequality: $2(a,b) \leq 2\gamma|a|^2 + |b|^2/2\gamma$ for all $\gamma > 0$, the right hand side is bounded above as follows:

$$\begin{aligned} -2\delta t(\tilde{e}_{k+1}, R_{k+1} + B^t\phi_k) &\leq \frac{\delta t}{2}|\tilde{e}_{k+1}|_L^2 + 2\delta t|R_{k+1}|_L^2 - 2\delta t(1-\delta t)(\tilde{e}_{k+1}, B^t\phi_k) \\ &\quad -2\delta t^2(\tilde{e}_{k+1}, B^t\phi_k) \\ &\leq \delta t|\tilde{e}_{k+1}|_L^2 + 2\delta t|R_{k+1}|_L^2 - 2\delta t(1-\delta t)(\tilde{e}_{k+1}, B^t\phi_k) \\ &\quad +2\delta t^3|\phi_k|_{M^1}^2 \end{aligned}$$

The integral residual, is bounded above by

$$|R_{k+1}|_L^2 \leq \frac{\delta t}{3}\int_{t^k}^{t^{k+1}}|u_{ss}|_L^2\,ds.$$

Hence, after dropping a positive term, we finally have

$$\begin{aligned} (1-\delta t)|\tilde{e}_{k+1}|_L^2 + 2\alpha\delta t|\tilde{e}_{k+1}|_X^2 &\leq |e_k|_L^2 - 2\delta t(1-\delta t)(\tilde{e}_{k+1}, B^t\phi_k) \\ &\quad +2\delta t^3|\phi_k|_{M^1}^2 + \delta t^2\int_{t^k}^{t^{k+1}}|u_{ss}|_L^2\,ds. \end{aligned} \tag{4.5}$$

Now, in order to get a control on the term $2\delta t(\tilde{e}_{k+1}, B^t\phi_k)$, we take the inner product of the first equation of (4.4) by $2\delta t^2B^t\phi_k$ and obtain:

$$-2\delta t(\tilde{e}_{k+1}, B^t\phi_k) = \delta t^2|\phi_k|_{M^1}^2 + |e_{k+1} - \tilde{e}_{k+1}|_L^2 - \delta t^2|\delta_{k+1}|_{M^1} \tag{4.6}$$

where we have taken into account the fact that $e_{k+1} \in H = B^t(M^1)^\perp$.

A control on e_{k+1} is obtained by taking the inner product of (4.4) by $2\delta te_{k+1}$:

$$|e_{k+1}|_L^2 + |e_{k+1} - \tilde{e}_{k+1}|_L^2 - |\tilde{e}_{k+1}|_L^2 = 0 \tag{4.7}$$

By summing (4.5) + $(1-\delta t)$(4.6) + $(1-\delta t)$(4.7) we obtain:

$$\begin{aligned} (1-\delta t)|e_{k+1}|_L^2 + (1-\delta t)\delta t^2|\delta_{k+1}|_{M^1}^2 + 2\alpha\delta t|\tilde{e}_{k+1}|_X^2 &\leq |e_k|_L^2 + (1+\delta t)\delta t^2|\phi_k|_{M^1}^2 \\ &\quad +\delta t^2\int_{t^k}^{t^{k+1}}|u_{ss}|_L^2\,ds. \end{aligned} \tag{4.8}$$

Furthermore, if the pressure has the regularity with respect to time that is assumed in the statement of the theorem, then ϕ_k is equal to δ_k up to the term $p(t^{k+1}) - p(t^k)$ which is of order δt. More precisely we have:

$$|\phi_k|_{M^1}^2 = |\delta_k|_{M^1}^2 + 2\left(B^t\delta_k, B^t\int_{t^k}^{t^{k+1}}p_s(s)\,ds\right) + \left|\int_{t^k}^{t^{k+1}}p_s(s)\,ds\right|_{M^1}^2$$

that is to say

$$|\phi_k|^2_{M^1} \leq (1+\delta t)|\delta_k|^2_{M^1} + (1+\delta t)\int_{t^k}^{t^{k+1}} |p_s|^2_{M^1}\, ds$$

Putting the above inequality into (4.8) and taking the sum for k from 0 to N, ($0 \leq N \leq K-1$) and adding some convenient positive terms in the right hand side of the inequality we obtain:

$$\begin{aligned} |e_{N+1}|^2_L + \delta t^2|\delta_{N+1}|^2_{M^1} + 2\alpha\delta t\sum_0^N |\tilde{e}_{k+1}|^2_X \;\leq\;& 2\delta t(1+\delta t)\sum_0^{N+1}\left[|e_k|^2_L + \delta t^2|\delta_k|^2_{M^1}\right] \\ &+|e_0|^2_L + \delta t^2|\delta_0|^2_{M^1} + \delta t^2\int_0^T |u_{ss}|^2_L\, ds \\ &+\delta t^2(1+\delta t)^2\int_0^T |p_s|^2_{M^1}\, ds \end{aligned}$$

Then, if δt is small enough, the final bound is a consequence of the discrete Gronwall lemma 1.4.1 (see below). □

LEMMA 3.4.1. *(Discrete Gronwall lemma) Let (θ_k) be a series of positive real numbers. Assume there is $\gamma > 0$ and $\lambda > 0$ such that for all $N \geq 0$, the following inequality $\theta_{N+1} \leq \gamma\sum_0^N \theta_k + \lambda$ holds, then θ_{N+1} is bounded above as follows:*

$$\forall N \geq 0, \qquad \theta_{N+1} \leq \gamma e^{N\gamma}\theta_0 + \lambda e^{(N+1)\gamma}. \tag{4.9}$$

Proof. By induction on N. □

Remark 3.4.1. The bound (4.1) was obtained in Shen [14] for the continuous Stokes problem. It was also more or less obtained by Van Kan [19] for a matrix form of a Stokes problem discretized by finite differences.

Remark 3.4.2. The theorem above does not yield any strong convergence result on the pressure; more precisely it only shows that the error in pressure is bounded above by a constant in the M^1 norm.

3.4.2. Error in pressure for the incremental formulation

In order to obtain some new bounds on δ_k we add (4.3) to (4.4) and obtain:

$$\frac{e_{k+1}-e_k}{\delta t} + A\tilde{e}_{k+1} + B^t\delta_{k+1} = -R_{k+1} \tag{4.10}$$

Since $A\tilde{e}_{k+1}$ converges in some sense in X' and R_{k+1} can be easily bounded above, a convergence estimate on δ_k can be obtained if we can bound above the term $(e_{k+1} - e_k)/\delta t$ in $L \subset X'$. This is achieved by means of the following result:

THEOREM 3.4.2. *If the solution of (3.3) is such that u_{tt} and $u_{ttt} \in L^2(0,T;L)$, p_t and $p_{tt} \in L^2(0,T;M^1)$, and if the series (u_k) and (p_k) are initialized so that $|e_0|_L \leq c\delta t^2$ and $|\delta_0|_{M^1} \leq c\delta t$ then we have the following error estimate:*

$$\sup_{0\leq t^k<T}\left[|e_{k+1}-e_k|^2_L + \delta t^2|\delta_{k+1}-\delta_k|^2_{M^1} + 2\alpha\delta t\sum_{l=1}^k |\tilde{e}_{l+1}-\tilde{e}_l|^2_X\right] \leq c\delta t^4 \tag{4.11}$$

Proof. First, note that the hypotheses on the initial values u_0 and p_0 yield the error estimates:

$$|e_1 - e_0| \leq c\delta t^2, \text{ and } |\delta_1 - \delta_0| \leq c\delta t^2. \tag{4.12}$$

Second, by subtracting step k from step $k+1$ of (4.3) and (4.4) we derive

$$\left\{ \frac{\tilde{e}_{k+1}-\tilde{e}_k-(e_k-e_{k-1})}{\delta t} + A(\tilde{e}_{k+1}-\tilde{e}_k) = -R_{k+1} + R_k - B^t\psi_k \right. \tag{4.13}$$

and

$$\left\{ \begin{array}{l} \dfrac{e_{k+1}-e_k-(\tilde{e}_{k+1}-\tilde{e}_k)}{\delta t} + B^t(\delta_{k+1}-\delta_k-\psi_k) = 0 \\ B(e_{k+1}-e_k) = 0 \end{array} \right. \tag{4.14}$$

where we have set $\psi_k = \delta_k - \delta_{k-1} + p(t^{k+1}) - 2p(t^k) + p(t^{k-1})$ for sake of legibility. Note that the two equations above have exactly the same structure as those which control e_{k+1}, $\tilde{e}_{k+1}$ and δ_{k+1} in the proof of theorem 1.4.1, namely eqs. (4.3) (4.4). Hence, by proceeding exactly as in the proof in question, the error bound (4.11) is an easy consequence of (4.12) and the following bounds:

$$\begin{aligned} |R_k - R_{k-1}|^2_L &= \left|u_t(t^{k+1}) - u_t(t^k) - \frac{u(t^{k+1}) - 2u(t^k) + u(t^{k-1})}{\delta t}\right|^2_L \\ &\leq \left|\int_{t^k}^{t^{k+1}} (t^{k+1}-s)u_{sss}(s)\, ds - \frac{1}{2\delta t}\int_{t^k}^{t^{k+1}} (t^{k+1}-s)^2 u_{sss}(s)\, ds \right. \\ &\quad \left. +\frac{1}{2\delta t}\int_{t^{k-1}}^{t^k} (t^{k-1}-s)^2 u_{sss}(s)\, ds\right|^2_L \\ &\leq \delta t^3 \int_{t^{k-1}}^{t^{k+1}} |u_{sss}|^2_L\, ds \end{aligned}$$

and

$$\begin{aligned} |p(t^{k+1}) - 2p(t^k) + p(t^{k-1})|^2_{M^1} &= \left|\int_{t^k}^{t^{k+1}} (t^{k+1}-s)p_{ss}(s)\, ds - \int_{t^{k-1}}^{t^k} (t^{k-1}-s)p_{ss}(s)\, ds\right|^2_{M^1} \\ &\leq \frac{4}{3}\delta t^3 \int_{t^{k-1}}^{t^{k+1}} |p_{ss}|^2_{M^1}\, ds. \;\square \end{aligned}$$

Remark 3.4.3. The estimate (4.11) should not be surprising, it only shows that $(u_{k+1} - u_k)/\delta t$ is a first order approximation of $\partial u(t^{k+1})/\partial t$ as one should expect.

It is now an easy matter to derive

COROLLARY 3.4.1. *Under the hypotheses of theorem 1.4.1 we have the following error estimate on the pressure:*

$$\left[\delta t\sum_0^K |\delta_k|^2_M\right]^{1/2} \leq c\delta t \tag{4.15}$$

Proof. From (4.10) and the "inf-sup" inequality (iv) of lemma 1.3.1 we derive

$$\beta|\delta_{k+1}|_M \leq |B^t\delta_{k+1}|_{X'} \leq \left|\frac{e_{k+1}-e_k}{\delta t}\right|_L + |A|_{X,X'}|\tilde{e}_{k+1}|_X + |R_{k+1}|_L$$

The desired error estimate is a consequence of the inequality above and theorems 1.4.1 and 1.4.2. □

Remark 3.4.4. The convergence estimate on the pressure seems to be new, though it was given but not quite correctly demonstrated in [14] for the continuous Stokes problem (see also [8] [9] and [15]). In conclusion p_{k+1} is as good an approximation of $p(t^{k+1})$ as $(u_{k+1}-u_k)/\delta t$ is an approximation of $\partial u(t^{k+1})/\partial t$.

3.4.3. Estimates for the non-incremental formulation

The technique developed above directly applies to the non-incremental formulation of the projection method.

THEOREM 3.4.3. *Assume the solution of (3.3) is such that* $u_{tt} \in L^2(0,T;L)$ *and* $p \in L^\infty(0,T;M^1)$. *If* $|e_0|_L \leq c\delta t$, *we have the error estimates:*

$$\sup_{0\leq t^k\leq T}\left[|e_k|_L^2 + 2\alpha\delta t\sum_{l=1}^{k}|\tilde{e}_l|_X^2 + \delta t^2\sum_{l=1}^{k}|\delta_l|_{M^1}^2\right] \leq c\delta t \tag{4.16}$$

Furthermore, if $u_{ttt} \in L^2(0,T;L)$, $p_t \in L^2(0,T;M^1)$ *and* $|e_0|_L \leq c\delta t^2$

$$\sup_{0\leq t^k<T}\left[|e_{k+1}-e_k|_L^2 + 2\alpha\delta t\sum_{l=1}^{k}|\tilde{e}_{l+1}-\tilde{e}_l|_X^2 + +\delta t^2|\sum_{l=1}^{k}\delta_{l+1}-\delta_l|_{M^1}^2\right] \leq c\delta t^3 \tag{4.17}$$

and

$$\left[\delta t\sum_{1}^{K}|\delta_k|_M^2\right]^{1/2} \leq c\delta t^{1/2} \tag{4.18}$$

Proof. The equations that control the errors are easily found to be

$$\left\{\frac{\tilde{e}_{k+1}-e_k}{\delta t} + A\tilde{e}_{k+1} = -R_{k+1} - B^t\phi_k\right.$$

and

$$\left\{\begin{array}{l}\dfrac{e_{k+1}-\tilde{e}_{k+1}}{\delta t} + B^t(\delta_{k+1}-\phi_k) = 0\\ Be_{k+1} = 0\end{array}\right.$$

where we have set $\phi_k = p(t^{k+1})$. These equations have exactly the same structure as that of (4.3)-(4.4); hence we deduce a stability inequality which is identical to (4.8)

$$\begin{aligned}(1-\delta t)|e_{k+1}|_L^2 + (1-\delta t)\delta t^2|\delta_{k+1}|_{M^1}^2 + 2\alpha\delta t|\tilde{e}_{k+1}|_X^2 &\leq |e_k|_L^2 + (1+\delta t)\delta t^2|\phi_k|_{M^1}^2\\ &\quad + \delta t^2\int_{t^k}^{t^{k+1}}|u_{ss}|_L^2\,ds\\ &\leq |e_k|_L^2 + c\delta t^2 + \delta t^2\int_{t^k}^{t^{k+1}}|u_{ss}|_L^2\,ds.\end{aligned}$$

The desired bound (4.16) is obtained by applying Gronwall lemma 1.4.1. The other bounds (4.17) and (4.18) are obtained by proceeding as in theorem 1.4.2 and corollary 1.4.1. □

In practice, one generally does not observe the $\mathcal{O}(\delta t^{1/2})$ convergence that is predicted but one rather obtains a $\mathcal{O}(\delta t)$ convergence. This surprising experimental result can be interpreted if one recalls that in finite dimension the operator A is bounded in L; hence we can obtain conditional bounds involving the norm of A in $\mathcal{L}(L,L)$.

THEOREM 3.4.4. *Let* $Q = \sup_{v\in L}|Av|_L/|v|_L$, *and assume that the hypotheses of theorem 1.4.3 are satisfied and if* $\delta t \leq 1/Q$ *then*

$$\sup_{0\leq t^k\leq T}\left[|e_k|_L + |\tilde{e}_k|_X + [\delta t\sum_{l=1}^{k}|\delta_l|_M^2]^{1/2}\right] \leq c(Q)\delta t \tag{4.19}$$

Proof. Proceed as in proofs of theorems 1.5.2, 1.5.4. □

In conclusion both the incremental and the non-incremental projection schemes are of first order in practice; however, for the incremental one the bounds are unconditional whereas for the non-incremental one they are conditional. The incremental scheme should be preferred to the non-incremental one as far as sensitivity (ie. stability) to small perturbations is concerned.

3.5. Higher order schemes

3.5.1. A conjecture

Many attempts have been made in order to improve the order of accuracy of projection schemes. To the author's knowledge these attempts have not yet yielded any scheme both unconditionally stable and with accuracy order strictly larger than one; though, as shown farther, conditionally stable and second order accurate schemes are available. After having spent a certain amount of time in the search of such a method, the author has been led to the following conjecture: *It is not possible to build unconditionally stable projection schemes with accuracy order strictly larger than one.* In order to shore up this idea some arguments are now given.

As suggested by Gresho and Chan [7], one (likely to be the only) way to improve the accuracy order of projection schemes is to improve that of the prediction step. For instance, using a Crank-Nicolson approximation of the viscous diffusion (other types of unconditionally stable and second order accurate approximations are possible), such a step may be put into the form:

$$\frac{\tilde{u}_{k+1}-u_k}{\delta t} + \frac{1}{2}A(\tilde{u}_{k+1}+\tilde{u}_k) = f_{k+1/2} - B^t\psi_{k-1/2}, \tag{5.1}$$

where $\psi_{k-1/2}$ is a second order accurate prediction of $p(t^{k+1/2})$. Such a prediction can be obtained as follows:

$$\psi_{k-1/2} = p_{k-1/2} + \delta t p'_{k-1/2}. \tag{5.2}$$

where $p'_{k-1/2}$ is a first order approximation of $\partial p(t^{k-1/2})/\partial t$. It is indeed possible to obtain such an approximation by means of the projection scheme presented in the previous sections. Assuming enough regularity of the solution to (3.3), $p'_{k-1/2}$ is obtained by applying (3.4)–(3.5) to

$$\text{(5.3)} \qquad \begin{cases} \frac{\partial^2 u}{\partial t^2} + A\frac{\partial u}{\partial t} + B^t\frac{\partial p}{\partial t} = \frac{\partial f}{\partial t} \\ B\frac{\partial u}{\partial t} = 0 \\ \frac{\partial u}{\partial t}_{|t=0} = w_0 \end{cases}$$

The equations that control the errors of this scheme are easily found to be

$$\text{(5.4)} \qquad \left\{ \frac{\tilde{e}_{k+1} - e_k}{\delta t} + \frac{1}{2}A(\tilde{e}_{k+1} + \tilde{e}_k) = -R_{k+1/2} - B^t\gamma_{k-1/2} \right.$$

and

$$\text{(5.5)} \qquad \begin{cases} \frac{e_{k+1} - \tilde{e}_{k+1}}{\delta t} + B^t(\delta_{k+1/2} - \gamma_{k-1/2}) = 0 \\ Be_{k+1} = 0, \end{cases}$$

where for sake of legibility we have set $\gamma_{k-1/2} = p(t^{k+1/2}) - p_{k-1/2} - \delta t p'_{k-1/2}$. For sake of simplicity we also assume that A is self-adjoint; though it is not a necessary hypothesis, it notably simplifies the subsequent calculations. By using the self-adjointness of A and the technique developed in theorem 1.4.1, we obtain the following stability inequality

$$\begin{aligned}(1-\delta t)|e_{k+1}|_L^2 + (1-\delta t)\delta t^2|\delta_{k+1}|_{M^1}^2 + \alpha\delta t|\tilde{e}_{k+1}|_X^2 \;&\leq\; |e_k|_L^2 + (1+\delta t)\delta t^2|\gamma_{k-1/2}|_{M^1}^2 \\ &+\alpha\delta t|\tilde{e}_k|_X^2 + c\delta t^2\int_{t^k}^{t^{k+1}}\left[|u_{sss}|_L^2 + |Au_{ss}|_L^2\right] ds.\end{aligned}$$

Then, we would obtain second order accuracy if we could show that $|\gamma_{k-1/2}|_{M^1}$ is equal to $|\delta_{k-1/2}|_{M^1}$ plus second order terms. Actually we have

$$\begin{aligned}|\gamma_{k-1/2}|_{M^1} &= |p(t^{1+1/2}) - p_{k-1/2} - \delta t p'_{k-1/2}|_{M^1} \\ &= |p(t^{1+1/2}) - p(t^{k-1/2}) - \delta t p'(t^{k-1/2}) - \delta_{k-1/2} - \delta t\delta'_{k-1/2}|_{M^1} \\ &\leq |\delta_{k-1/2}|_{M^1} + \delta t|\delta'_{k-1/2}|_{M^1} + c\delta t^{3/2}\left[\int_{t^{k-1/2}}^{t^{k+1/2}} |p_{ss}|_{M^1}^2\, ds\right]^{1/2}.\end{aligned}$$

As expected, the integral residual is of $\mathcal{O}(\delta t^2)$, but the term $\delta t|\delta'_{k-1/2}|_{M^1}$ is not of $\mathcal{O}(\delta t^2)$ though $\delta t|\delta'_{k-1/2}|_M$ is of $\mathcal{O}(\delta t^2)$. The only thing we can say for sure is that it is of $\mathcal{O}(\delta t)$ since, according to theorem 1.4.1, $|\delta'_{k-1/2}|_{M^1}$ is uniformly bounded by a constant. Hence, by using Gronwall lemma 1.4.1, we obtain unconditional bounds of order one for this scheme; *i.e.* it is one order lower than expected. Unconditional second order is not achievable since $\delta'_{k-1/2}$ does not converge in the M^1 norm. This point is clearer for the continuous Stokes problem; in this context the M^1 norm is that of $H^1(\Omega)/\mathbb{R}$, *i.e.* it is the gradient norm. If $p'_{k-1/2}$ converged in $H^1(\Omega)/\mathbb{R}$ then $\partial p'_{k-1/2}/\partial n$ would also converge in some sense to the exact value $\partial p'(t^{k-1/2})/\partial n$. Such a convergence is impossible since the projection scheme (2.7) imposes $\partial p'_{k-1/2}/\partial n = \ldots = \partial p'_{-1/2}/\partial n$, a condition which is not necessarily satisfied by the exact solution.

In conclusion, the projection step (2.7) or (3.5), by inducing an implicit Neumann boundary condition on the pressure, does not allow convergence of the pressure in M^1 and consequently spoils any possibility to build higher order schemes.

Another conclusion that can be drawn from this example is that it is possible to build conditional second order accurate schemes. For instance, if we assume $B^t \in \mathcal{L}(M, L)$ then $|\delta'_{k-1/2}|_{M^1} \leq c|\delta'_{k-1/2}|_M$ which means that the algorithm is second order accurate. Actually, it is the purpose of the following subsection to show that if we only put $\psi_{k-1/2} = p_{k-1/2}$ we maintain second order accuracy.

3.5.2. Crank–Nicolson approximation

In order to improve the overall convergence of Goda's scheme [6], Van Kan [19] proposes to replace the implicit Euler step of order one in (3.4) by a Crank–Nicolson approximation:

$$\text{(5.6)} \qquad \frac{\tilde{u}_{k+1} - u_k}{\delta t} + \frac{1}{2}A(\tilde{u}_{k+1} + \tilde{u}_k) = f_{k+1/2} - B^t p_{k-1/2},$$

the projection step being

$$\text{(5.7)} \qquad \begin{cases} \frac{u_{k+1} - \tilde{u}_{k+1}}{\delta t} + B^t(p_{k+1/2} - p_{k-1/2}) = 0 \\ Bu_{k+1} = 0. \end{cases}$$

For sake of simplicity we assume that A is self-adjoint; though it is not a necessary hypothesis it notably simplifies the calculations. By using the techniques developed in §4 together with the self-adjointness of A, one derives the following

THEOREM 3.5.1. *Assume the solution of (3.3) is such that u_{tt}, u_{ttt}, Au_t and Au_{tt} are in $L^2(0,T;L)$, p_t and $p_{tt} \in L^2(0,T;M^1)$. If the series (u_k), $(\tilde{u}_k)$ and (p_k) are initialized so that $|e_0|_L \leq c\delta t$, $|\tilde{e}_0|_L \leq c\delta t^{1/2}$ and $|\delta_{-1/2}|_{M^1} \leq c$ then we have the following error estimates:*

$$\text{(5.8)} \qquad \sup_{0\leq t^k\leq T}\left[|e_k|_L^2 + \frac{\alpha\delta t}{2}\sum_{l=1}^{k}|\tilde{e}_l + \tilde{e}_{l-1}|_X^2 + \frac{\alpha\delta t}{2}|\tilde{e}_k|_X^2 + \delta t^2|\delta_{k-1/2}|_{M^1}^2\right] \leq c\delta t^2$$

Furthermore, if $|e_0|_L \leq c\delta t^2$, $|\tilde{e}_0|_L \leq c\delta t^{3/2}$ and $|\delta_{-1/2}|_{M^1} \leq c\delta t$ then we have

$$\text{(5.9)} \qquad \begin{aligned}\sup_{0\leq t^k<T}\Bigg[&|e_{k+1} - e_k|_L^2 + \frac{\alpha\delta t}{2}\sum_{l=1}^{k}|\tilde{e}_{l+1} - \tilde{e}_{l-1}|_X^2 + \\ &\frac{\alpha\delta t}{2}|\tilde{e}_{k+1} - \tilde{e}_k|_X^2 + \delta t^2|\delta_{k+1/2} - \delta_{k-1/2}|_{M^1}^2\Bigg] \leq c\delta t^4\end{aligned}$$

and

$$\text{(5.10)} \qquad \left[\delta t\sum_{0}^{K}|\delta_{k-1/2}|_M^2\right]^{1/2} \leq c\delta t$$

Proof. Proceed as in proofs of theorems 4.1 and 4.2 using the fact that the integral Taylor residual satisfies

$$|R_{k+1/2}|_L^2 \leq c\delta t^3 \int_{t^k}^{t^{k+1}} \left[|u_{ttt}|_L^2 + |Au_{tt}|_L^2\right] ds. \ \square$$

Remark 3.5.1. The bound (5.8) shows that this scheme is unconditionally of order one for the velocity u_{k+1} in the $l^\infty(L)$ norm and for the mean velocity $(\tilde{u}_{k+1} + \tilde{u}_k)/2$ in the $l^2(X)$ norm. The difficulty to have a nice convergence in $l^2(X)$ for the velocity $\tilde{u}_{k+1}$ reflects the fact that Crank–Nicolson scheme is only marginally stable on its own. It is shown further that second order Euler scheme yields a better result in this respect and for this reason should be preferred to the Crank–Nicolson approximation.

Now if we assume that $A \in \mathcal{L}(L, L)$, which is the case when we deal with discrete approximations of Stokes problem we can prove second order accuracy.

THEOREM 3.5.2. *Let $Q = \sup_{v \in L} |Av|_L/|v|_L$. If u_{tt}, u_{ttt}, Au_t and Au_{tt} are in $L^2(0,T;L)$, p_t and $p_{tt} \in L^2(0,T;M^1)$ and $\max(|e_0|_L, \delta t^{1/2}|\tilde{e}_0|_X, \delta t|\delta_{-1/2}|_{M^1}) \leq c\delta t^2$ and if $\delta t \leq 1/Q$ then*

$$\sup_{0 \leq t^k \leq T} \left[|e_k|_L + |\tilde{e}_k|_X + [\delta t \sum_{l=1}^{k} |\delta_{l-1/2}|_M^2]^{1/2}\right] \leq c(Q)\delta t^2 \qquad (5.11)$$

Proof. Summing the prediction and projection steps of the equations controlling the errors one gets

$$\frac{e_{k+1} - e_k}{\delta t} + \frac{1}{2}A(\tilde{e}_{k+1} + \tilde{e}_k) + B^t\delta_{k+1/2} = R_{k+1/2} \qquad (5.12)$$

which can be put into the equivalent form

$$e_{k+1} + B^t(\delta t\delta_{k+1/2}) = e_k - \frac{\delta t}{2}A(\tilde{e}_{k+1} + \tilde{e}_k) + \delta t R_{k+1/2}$$

where we should recall that e_{k+1} is in H; hence, the decomposition $H \oplus B^t(M^1)$ being orthogonal in L one deduces

$$|e_{k+1}|_L \leq |e_k|_L + \frac{\delta t}{2}|A(\tilde{e}_{k+1} + \tilde{e}_k)|_L + \delta t|R_{k+1/2}|_L.$$

It is at this very point that the conditional stability of A plays its rôle. This condition enables us to assume the existence of a constant $c(Q)$ so that we have the bound

$$|A(\tilde{e}_{k+1} + \tilde{e}_k)|_L \leq c(Q)(|\tilde{e}_{k+1}|_L + |\tilde{e}_k|_L). \qquad (5.13)$$

The constant $c(Q)$ is necessarily bounded above by Q, though numerical results suggest that in some circumstances $c(Q)$ may be of order one. In the following, $c(Q)$ is a generic constant that depends on Q. Furthermore, the projection steps yields

$$\begin{aligned}
|\tilde{e}_{k+1}|_L &= |e_{k+1} + \delta t B^t(p_{k-1/2} - p_{k+1/2})|_L \\
&\leq |e_{k+1}|_L + \delta t|\delta_{k+1/2} - \delta_{k-1/2}|_{M^1} + \delta t\left|\int_{t^{k-1/2}}^{t^{k+1/2}} p_s(s)\, ds\right|_{M^1} \\
&\leq |e_{k+1}|_L + c\delta t^2 + c\delta t \int_{t^{k-1/2}}^{t^{k+1/2}} |p_s|_{M^1}\, ds
\end{aligned}$$

Note that, thanks to (5.9), we have used the unconditional bound $|\delta_{k+1/2} - \delta_{k-1/2}|_{M^1} \leq c\delta t$. The same inequality is derived for $\tilde{e}_k$. Note also that since p_t and $p_{tt} \in L^2(0,T;M^1)$, p_t is in $C([0,T];M^1)$, that is the integral residual in the inequality above is bounded by $c\delta t^2$. Putting these inequalities together we obtain the following stability inequality

$$(1 - \frac{c(Q)\delta t}{2})|e_{k+1}|_L \leq (1 + \frac{c(Q)\delta t}{2})|e_k|_L + c\delta t^3 + c\delta t^{5/2}\left[\int_{t^k}^{t^{k+1}} \left[|u_{ttt}|_L^2 + |Au_{tt}|_L^2\right] dt\right]^{1/2}$$

If $c(Q)\delta t/2 \leq 1/2$ (the constant 1/2 is arbitrary, it only should be smaller than one) then there is another constant $c(Q)$ so that

$$\sup_{0 \leq t^k < T} |e_{k+1}|_L \leq c(Q)\delta t^2.$$

Putting this bound in the inequality that has already been obtained for $|\tilde{e}_{k+1}|_L$ we derive

$$\sup_{0 \leq t^k < T} |\tilde{e}_{k+1}|_L \leq c(Q)\delta t^2.$$

Using the X-ellipticity and the L-stability of A we derive a similar bound for $|\tilde{e}_{k+1}|_X$

$$\sup_{0 \leq t^k < T} |\tilde{e}_{k+1}|_X \leq c(Q)\delta t^2.$$

For the pressure estimate we put (5.12) into the equivalent form

$$\frac{e_{k+1} - e_k}{\delta t} + B^t\delta_{k+1/2} = -\frac{1}{2}A(\tilde{e}_{k+1} + \tilde{e}_k) + R_{k+1/2}$$

where we should recall that $(e_{k+1} - e_k)/\delta t$ is in H; hence, the decomposition $H \oplus B^t(M^1)$ being orthogonal in L one deduces

$$\beta|\delta_{k+1/2}|_M \leq |\delta_{k+1/2}|_{M^1} \leq \frac{1}{2}|A(\tilde{e}_{k+1} + \tilde{e}_k)|_L + |R_{k+1/2}|_L.$$

that is to say

$$|\delta_{k+1/2}|_M^2 \leq c|\delta_{k+1/2}|_{M^1}^2 \leq c(Q)\delta t^4 + c\delta t^3 \int_{t^k}^{t^{k+1}} \left[|u_{ttt}|_L^2 + |Au_{tt}|_L^2\right] ds$$

which yields the desired result. $\square$

3.5.3. Implicit Euler of second order

We have similar results if we replace the Crank–Nicolson approximation by an implicit Euler step of second order:

$$\frac{3\tilde{u}_{k+1} - \tilde{u}_k - 3u_k + u_{k-1}}{2\delta t} + A\tilde{u}_{k+1} = f_{k+1} - B^t p_k \qquad (5.14)$$

and

$$(5.15)\qquad \begin{cases} \dfrac{3u_{k+1}-u_k-3\tilde{u}_{k+1}+\tilde{u}_k}{2\delta t}+B^t(p_{k+1}-p_k)=0 \\ Bu_{k+1}=0 \end{cases}$$

For sake of simplicity we assume once more that A is self-adjoint; though it is not a necessary hypothesis, it simplifies the calculations. By using the techniques developed in §4 together with the self-adjointness of A, one derives the following

THEOREM 3.5.3. *Assume the solution of (3.3) is such that* u_{tt}, u_{ttt} *are in* $L^2(0,T;L)$, p_t *and* $p_{tt}\in L^2(0,T;M^1)$. *If the series* (u_k), $(\tilde{u}_k)$ *and* (p_k) *are initialized so that* $|e_0|_L\le c\delta t$, $|\tilde{e}_0|_L\le c\delta t^{1/2}$ *and* $|\delta_0|_{M^1}\le c$ *then we have the following error estimates:*

$$(5.16)\qquad \sup_{0\le t^k\le T}\left[\frac{1}{4}|3e_{k+1}-e_k|_L^2+2\alpha\delta t\sum_{l=0}^{k}|\tilde{e}_l|_X^2+\frac{\alpha\delta t}{2}|\tilde{e}_k|_X^2+\delta t^2|\delta_k|_{M^1}^2\right]\le c\delta t^2$$

Furthermore, if $|e_0|_L\le c\delta t^2$, $|\tilde{e}_0|_L\le c\delta t^{3/2}$ *and* $|\delta_0|_{M^1}\le c\delta t$ *then we have*

$$\sup_{0\le t^k<T}\left[\frac{1}{4}|3e_{k+1}-4e_k+e_{k-1}|_L^2+2\alpha\delta t\sum_{l=0}^{k-1}|\tilde{e}_{l+1}-\tilde{e}_l|_X^2+\right.$$

$$(5.17)\qquad \left.\frac{\alpha\delta t}{2}|\tilde{e}_{k+1}-\tilde{e}_k|_X^2+\delta t^2|\delta_{k+1}-\delta_k|_{M^1}^2\right]\le c\delta t^4$$

and

$$(5.18)\qquad \left[\delta t\sum_{0}^{K}|\delta_k|_M^2\right]^{1/2}\le c\delta t$$

Proof. Proceed as in proofs of theorems 4.1 and 4.2 by formally replacing $(3e_{k+1}-e_k)/2$ and $(3\tilde{e}_{k+1}-\tilde{e}_k)/2$ by e_{k+1} and $\tilde{e}_{k+1}$ respectively. Use also the equality

$$(3\tilde{e}_{k+1}-\tilde{e}_k,A\tilde{e}_{k+1})=2(\tilde{e}_{k+1},A\tilde{e}_{k+1})+(\tilde{e}_{k+1}-\tilde{e}_k,A\tilde{e}_{k+1})$$

which yields

$$(3\tilde{e}_{k+1}-\tilde{e}_k,A\tilde{e}_{k+1})\ge 2\alpha|\tilde{e}_{k+1}|_X^2+\frac{\alpha}{2}|\tilde{e}_{k+1}|_X^2+\frac{\alpha}{2}|\tilde{e}_{k+1}-\tilde{e}_k|_X^2-\frac{\alpha}{2}|\tilde{e}_k|_X^2,$$

since A is assumed to be self-adjoint. □

Now if we assume that $A\in\mathcal{L}(L,L)$, we can prove second order accuracy.

THEOREM 3.5.4. *Let* $Q=\sup_{v\in L}|Av|_L/|v|_L$. *If* u_{ttt} *and* Au_{tt} *are in* $L^2(0,T;L)$, p_t *and* $p_{tt}\in L^2(0,T;M^1)$ *and* $\max(|e_0|_L,\delta t^{1/2}|\tilde{e}_0|_X,\delta t|\delta_0|_{M^1})\le c\delta t^2$ *and if* $\delta t\le 1/Q$ *then*

$$(5.19)\qquad \sup_{0\le t^k\le T}\left[|e_k|_L+|\tilde{e}_k|_X+[\delta t\sum_{l=1}^{k}|\delta_l|_M^2]^{1/2}\right]\le c(Q)\delta t^2$$

Proof. By proceeding exactly as in the demonstration of theorem 1.5.2 we obtain the following stability inequality:

$$\begin{bmatrix}|3e_{k+1}-e_k|_L\\ |\tilde{e}_{k+1}|_L\end{bmatrix}\le\frac{1}{3-c(Q)\delta t}\begin{bmatrix}3 & c(Q)\delta t\\ 1 & 1\end{bmatrix}\begin{bmatrix}|3e_k-e_{k-1}|_L\\ |\tilde{e}_k|_L\end{bmatrix}+\begin{bmatrix}\mathcal{O}(\delta t^3)\\ \mathcal{O}(\delta t^2)\end{bmatrix}.$$

The eigenvalues of the the amplification matrix are $1+c(Q)\delta t$ and $1/3+c(Q)\delta t$ and the desired bounds follow. □

3.6. Numerical examples

In order to illustrate the theoretical results given above, we now present some numerical tests.

3.6.1. Preliminaries

Let $\Omega=]0,1[\times]0,1[$ and define on Ω a staggered mesh-and-cell (MAC) introduced by Harlow and Welch [10]. Let δx and δy be the mesh size in the x direction and y direction respectively. For sake of simplicity we hereafter set $\delta x=\delta y=h$. One nice feature of the MAC approximation is that it is both easily interpretable in terms of finite differences and fits into a variational framework. In some sense, The MAC approximation reconciles the weak formulation with the so called strong one. Let A_h be the vector Laplace operator, B_h be the divergence operator, and consider the following discrete problem: find (u_h,p_h) so that:

$$(6.1)\qquad \begin{cases}\frac{\partial u_h}{\partial t}+A_hu+B_h^tp_h=f_h\\ B_hu_h=g_h\\ u_{h|t=0}=v_{h0}\\ u_{h|\partial\Omega_h}=a_h\end{cases}$$

Actually, in order to test the projection schemes we take the problem the other way around and we define an analytical solution (u_h,p_h) from which we calculate the right-hand side of (6.1). The exact solution that has been chosen reads:

$$\begin{aligned} u_{xh} &= \sqrt{3}\sin(\sqrt{2}\pi x+\pi t)\cos(\sqrt{3}\pi y+\pi t)\\ u_{yh} &= \sqrt{2}\cos(\sqrt{2}\pi x+\pi t)\sin(\sqrt{3}\pi y+\pi t)\\ p_h &= \sqrt{6\pi}\sin(2\pi x-\sqrt{5}\pi y+0.7\pi t)\sin(\sqrt{5}\pi y+0.3\pi t)\end{aligned}$$

3.6.2. Tests on Goda's scheme

In a first series of tests we have tested the scheme (3.4)-(3.5) for several time steps on a time period $T=1$. In figure 1.1 are reported the errors on the velocity, the pressure and the pressure gradient on a log / log diagram for a 33×33 grid, that is $h=1/32$. The error on the velocity is measured in the $l^\infty\times l^2$ norm defined as follows:

$$|e|_{l^\infty\times l^2}=\sup_{0\le t^k\le T}|e^k|_{l^2(\Omega_h)}.$$

The norm on the pressure is measured in the $l^2\times l^2$ norm:

$$|\delta|_{l^2\times l^2}=\left(\delta t\sum_{l=1}^{K}|\delta^k|_{l^2(\Omega_h)}^2\right)^{1/2}.$$

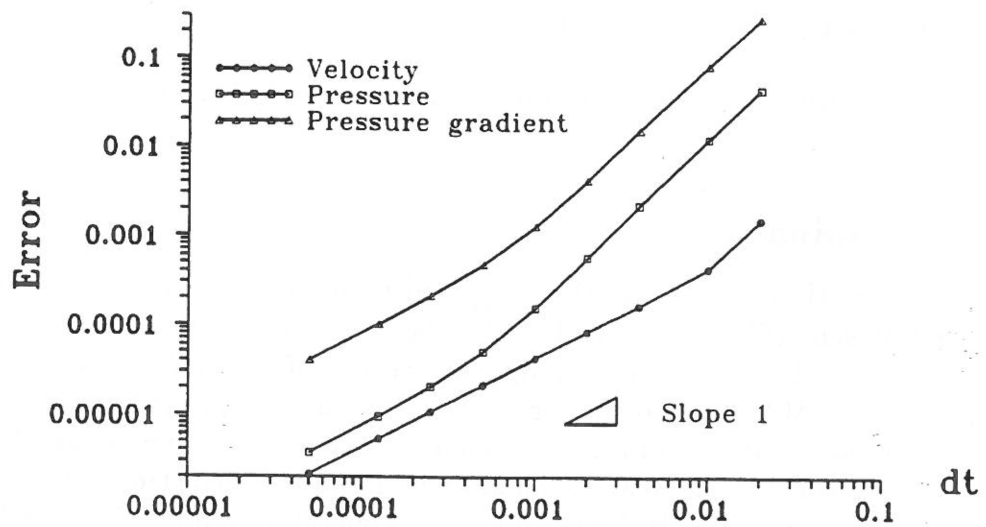

Figure 3.1: Error on velocity, pressure and pressure gradient with respect to time steps for the projection scheme (3.4)–(3.5). Note the first order slope.

Likewise, the error on the pressure gradient is measured in the $l^\infty \times l^2$ norm. As expected the algorithm is first order accurate; the slope of the error on the velocity is equal to one. Note, however, that the error on the pressure reaches the asymptotic first order when $\delta t \leq 5.10^{-3}$. Note also that the pressure gradient converges; this is due to the fact that in finite dimension B_h is bounded, ie. $B_h \in \mathcal{L}(M_h, L_h)$. Though, the convergence is not as good as that of the pressure; the error on the pressure gradient is one magnitude order higher than that on the pressure. Of course, the error is concentrated near $\partial\Omega_h$.

3.6.3. Tests on Van Kan's scheme

In a second series of tests we have tried Van Kan's scheme (5.6)–(5.7) on a 33×33 grid for $T = 1$. The errors on the velocity in the $l^\infty \times l^2$ norm and that on the pressure in the $l^2 \times l^2$ norm are reported in figure 1.2. In this figure is also reported the $l^2 \times l^2$ norm of $(A_h e_h, e_h)$, which is a measure of the convergence of the velocity in the $\mathbf{H}^1$ norm.

The second order slope is reached for the three quantities when $\delta t \leq 0.01$. For time steps above this value, the three curves leave the second order slope and seem to switch to a first order slope. An interpretation to this phenomenon that may be proposed is that since the error satisfies the bound

$$e \leq \min(c(Q)\delta t^2, c\delta t),$$

the error must fit the second order slope when δt is small enough, whereas if δt is too large the second order error is larger than that of first order and the curve switches back to the first order slope.

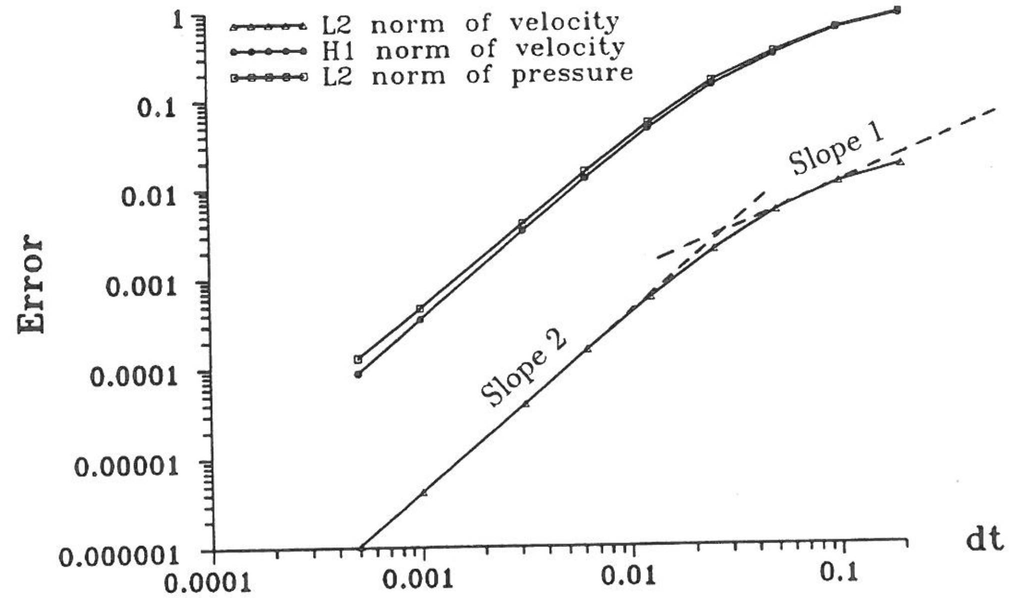

Figure 3.2: Error on velocity and pressure with respect to time steps for the projection scheme (5.14)–(5.15). Note the second order slope for $\delta t \leq 0.01$ and the first order slope for $0.05\delta \leq t \leq 0.1$.

3.6.4. Tests on the conditional accuracy

In order to test the sensitivity of the method to the mesh size, the program has been run on several types of meshes. In figure 1.3 (a) are reported the error on the velocity in the $l^2 \times l^2$ norm for 4 mesh sizes: $h = 1/32$, $h = 1/64$, $h = 1/128$, and $h = 1/248$. The computations have been carried out in double precision on a IBM RS 6000 320H workstation. Note that the second order accuracy is sustained on the whole range of time steps for every grids. There is no stability restriction on δt; this seems to indicate that the constant $c(Q)$ in (5.11) is of order one.

The same computations have also been carried out in single precision on a Cray C 90. The results are reported in figure 1.3 (b). This time, the results seem to depend on h; the second order accuracy is destabilized if either h or δt is too small and in these cases the first order accuracy prevails. This surprising behavior can be explained as follows.

The problems (5.6) and (5.7) are solved iteratively by multigrid techniques and the subroutines in charge of these problems are assigned to solve them in less than 20 multigrid cycles up to a precision criterion which is of order $\delta t \min(\delta t^2, h^2)$, though a threshold of order $\min(\delta t^2, h^2)$ should be sufficient were the method be unconditionally of second order.

In case (a), the computations being carried out in double precision, the theshold is always attained before the maximum number of multigrid cycles is reached and the scheme behaves as if (5.6) and (5.7) were solved exactly. In case (b), the simple precision operations on the Cray C 90 are somewhat less accurate than that in double precision on the workstation; hence, the stringent threshold $\delta t \min(\delta t^2, h^2)$ is not always reached in the assigned maximum number of multigrid cycles. The residual error, though

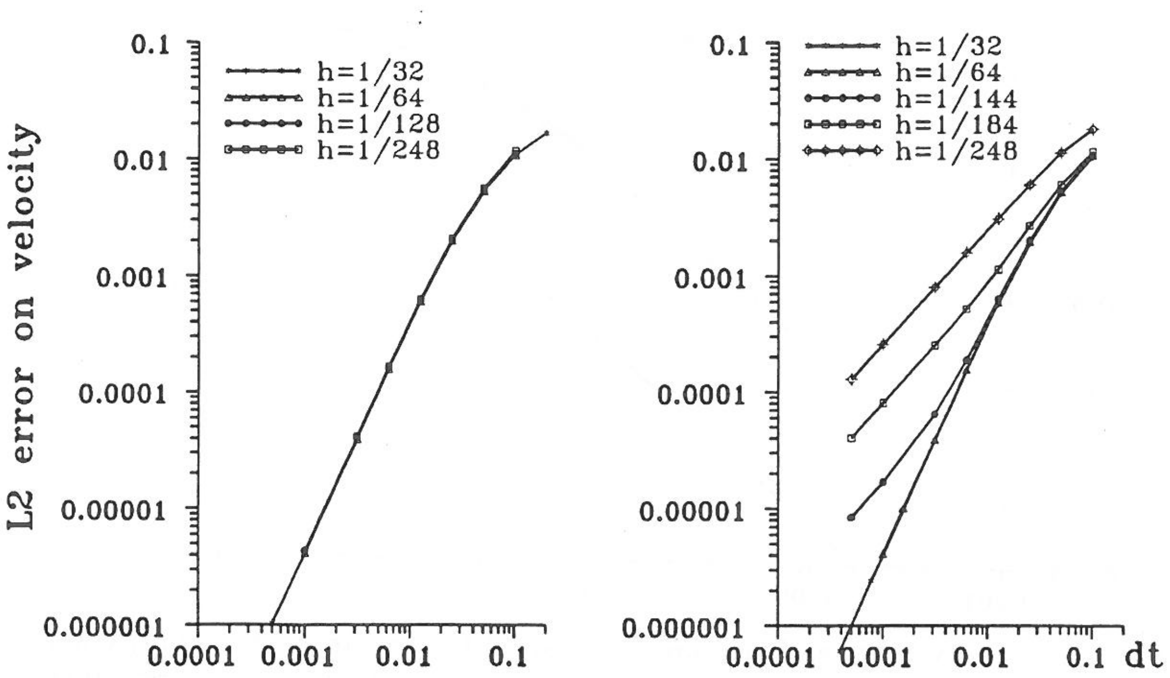

Figure 3.3: Error on velocity as a function of time step for the projection scheme (5.14)–(5.15) on several grids and two different computers. In figure (a), the computations are carried out in double precision on a RS 6000 workstation, the second order is sustained on the whole range of time steps for every grids. In figure (b), the computations are carried out in single precision on a Cray C 90, the second order accuracy is destabilized if either h or δt is too small; in these cases the first order accuracy prevails.

being always smaller than $\min(\delta t^2, h^2)$, is blown up by the Laplace operator so that the constant $c(Q)$ that applies in (5.13) is $c(Q) = 1/h^2$. Consequently, the scheme can not sustain the global second order accuracy. The smaller h and δt, the more dramatic this effect. Note, however, that the first order accuracy being unconditional, the error is always bounded by $c\delta t$ even if the second order is destabilized. The results plotted in figure 1.3 (b) show that for $h = 1/32$ and $h = 1/64$, the scheme behaves like in (a); that is, the scheme switches from first order to second order for $0.01 \leq \delta t \leq 0.1$ and is fully second order for $\delta t \leq 0.01$. For $h = 1/144$, the error switches from first order to second order when $0.01 \leq \delta t \leq 0.1$ as expected, but the second order is destabilized for $\delta t \leq 0.005$ and the curve switches back to the first order slope. This phenomenon is amplified for $h = 1/184$. For $h = 1/248$, the second order is never reached and the error is of first order on the whole range of time steps. Of course when the authorized maximum number of multigrid steps is slightly increased in case (b), one immediately recovers the results of case (a).

In conclusion, this test nicely demonstrates that the second order accuracy is only conditional since it is destabilized by errors smaller than $\min(\delta t^2, h^2)$. Furthermore, when the second order accuracy is destabilized, one always recover the first order accuracy. This hints that the first order is the maximal unconditional order of accuracy of this type of scheme. As a final recommendation, it should be advised to users of

second order methods to either solve exactly the prediction and projection problems or to solve them iteratively down to a threshold of order $h^2 \min(\delta t^2, h^2)$. In either case they should bear in mind that the second order is only conditional. Finally, the incremental form of the projection technique should be preferred to the non-incremental one and the implicit Euler scheme of second order should be preferred to Crank–Nicolson's.

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Auteurs (Authors) : Jean-Luc Guermond

Titre : Approximation des équations de Navier–Stokes instationnaires par des méthodes de projection.

Title : Approximation of the time-dependent Navier–Stokes equations by projection methods.

Nombre de pages (Number of pages) : 47

Résumé : *Ce rapport contient trois articles. Dans le premier, on montre que les estimations de convergence sur la pression pour les méthodes de projection décrites dans Shen SIAM J. Numer. Anal., 29, 1, 57–77 et Shen Numer. Math., 62, 57–77 ne sont pas obtenues correctement car elles sont toutes basées sur une inégalité fausse. On donne un contre exemple. Dans les deux autres articles, on formule les méthodes de projection dans un cadre abstrait permettant de traiter simultanément les problèmes continus et discrets en espace. On donne des estimations de convergence en temps inconditionnelles d'ordre un et conditionnelles d'ordre deux. Les simulations numériques confirment que l'ordre un est l'ordre inconditionnel maximal contrairement à ce qui est parfois suggéré dans la littérature. Le second article est un resumé en français du troisième.*

Motes clés : Ecoulements incompressibles, équations de Navier–Stokes, méthodes de projection, méthodes de pas fractionnaires, estimations de convergence.

Abstract : *This report contains three papers. In the first one it is shown that that the convergence estimates on the pressure for projection methods presented in Shen, SIAM J. Numer. Anal., 29, 1, 57–77 and Shen, Numer. Math., 62, 57–77 are not correctly obtained since they are based on an inequality which is not correct. A counter-example is given. In the second and third paper, incremental and non-incremental projection methods are reformulated in an abstract framework which is suitable for treating either continuous in space or discrete Stokes problems. One proves unconditional error estimates of order one and conditional error estimates of order two. Numerical simulations confirm that the order one is the maximal unconditional order of convergence. The second paper, written in french, is a summary of the third one.*

Key words : Incompressible flows, Navier-Stokes equations, projection methods, fractional step methods, convergence estimates.

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