GUERMOND, J.-L.

Subgrid stabilization of Galerkin approximations of monotone operators[a]

[a]Presented at GAMM98, Bremmen, April 1998, published in *Zeitschrift für angewandte Mathematik und Mechanik*, ZAMM, **79** 1 (1999), 29-32

This paper presents a stabilized Galerkin technique for approximating monotone linear operators in a Hilbert space. The key idea consists in introducing an approximation space that is broken up into large scales and small scales so that the bilinear form associated with the problem satisfies a uniform inf-sup condition with respect to this decomposition. An optimal Galerkin approximation is obtained by introducing an artificial diffusion on the small scales.

1. The model problem

Let $V \subset L$ be two real Hilbert spaces so that V is dense in L and V is continuously embedded in L; we set $V \subset L \equiv L' \subset V'$. Let $a \in \mathcal{L}(V \times L, \mathbb{R})$; we assume a to be monotone; that is, $\forall u \in V$, $a(u,u) \geq 0$. We define the symmetric part of a: $a_s \in \mathcal{L}(V \times V, \mathbb{R})$ so that $a_s(u,v) = \frac{1}{2}(a(u,v) + a(v,u))$; this bilinear form is symmetric and monotone. Furthermore, we assume:

$$\exists c_1 > 0,\ \exists c_2 > 0 \qquad \inf_{u\in V} \sup_{v\in L} \frac{a(u,v)}{\|u\|_V \|v\|_L} \geq c_1, \qquad \inf_{v\in L} \sup_{u\in V} \frac{a(u,v)}{\|u\|_V \|v\|_L} \geq c_2. \tag{1}$$

Owing to a classical characterization of bijective linear operators in reflexive Banach spaces, this hypothesis is equivalent to assuming that the following problem is well-posed:

$$\text{For } f \in L, \text{ find } u \in V \text{ so that,} \qquad \forall v \in L, \qquad a(u,v) = (f,v)_L, \tag{2}$$

The goal of the present paper is to propose a stabilized Galerkin technique for approximating problem (2).

2. The discrete setting

The main difficulty in building a Galerkin approximation to problem (2) is that this problem is of Petrov-Galerkin type; that is, the solution space and the test space are different. As a consequence it is difficult, in general, to obtain a discrete counterpart of (1) when approximating L and V by means of a single finite dimensional space (unless the test space is finely tuned, see Baiocchi *et al.* (1993)). The main idea of the present work is to introduce two finite dimensional spaces (X_H) and (X_h) so that (1) is satisfied for the couple (X_H, X_h).

More precisely, we introduce two sequences of finite dimensional subspaces of V, (X_H) and (X_h), with $X_H \subset X_h$ and $H \to 0$ and $h \to 0$. We assume that there is a projection $P_H : X_h \longrightarrow X_H$ that is stable with respect to the norm of L uniformly with respect to the mesh parameters (H,h). We set $X_h^H = (1 - P_H)X_h$, and for all v_h in X_h we set $v_H = P_H v_h$ and $v_h^H = v_h - v_H$. The space X_H is referred to as the resolved scales space and X_h^H is referred to as the subgrid scales space. We assume also that X_h satisfies the following inverse stability property:

$$\exists \lambda(H) > 0, \qquad \forall v_h \in X_h, \qquad \|v_h\|_V \leq \lambda(H)^{-1} \|v_h\|_L. \tag{3}$$

We assume that there is W, a dense subspace of V, $k > 0$, and $c > 0$ so that

$$\forall v \in W, \qquad \inf_{w_H \in X_H} \|v - w_H\|_L + H\|v - w_H\|_V \leq cH^{k+1} \|v\|_W. \tag{4}$$

The three hypotheses above are standard. Now we introduce the key hypothesis of this paper: the triplet (X_H, X_h, a) is assumed to satisfy the following uniform inf-sup condition

$$\exists c_a > 0,\ \forall (H,h), \qquad \inf_{v_H \in X_H} \sup_{\phi_h \in X_h} \frac{a(v_H, \phi_h)}{\|v_H\|_V \|\phi_h\|_L} \geq c_a. \tag{5}$$

From the definition of X_h^H we deduce $X_h = X_H \oplus X_h^H$; this decomposition can be formally interpreted as follows. For instance, assume that a is associated with a first order differential operator and assume that a finite element piecewise linear approximation of u is sought. The action of the differential operator on any piecewise linear function generates discontinuities at the interfaces of the finite elements. These discontinuities contain very small Fourier modes that cannot be captured when tested against piecewise linear test functions; as a consequence, the Galerkin technique is in general suboptimal for this class of problems. On the other hand, optimality can be recovered if subgrid scales test functions (*i.e.* those of X_h^H) are added to the conventional test space composed of piecewise linear functions (*i.e.* X_H; the resolved scales). The inf-sup condition (5) is the discrete counterpart of the first inequality in (1); it warrants the resolved scales of the approximate solution to be stable in the norm of V (*i.e.* to be free of spurious numerical wiggles).

Having introduced subgrid scales to control the resolved scales of the approximate solution, we are left with the problem of controlling also the subgrid scales. To this purpose, we introduce an artificial diffusion mechanism; that is, we define a bilinear form $b_h \in \mathcal{L}(X_h^H \times X_h^H, \mathrm{I\!R})$ that satisfies the following continuity and coercivity properties:

$$\forall (v_h^H, w_h^H) \in X_h^H \times X_h^H, \qquad \begin{cases} b_h(v_h^H, v_h^H) \geq \lambda(H) \|v_h^H\|_b^2, \\ b_h(v_h^H, w_h^H) \leq c_B \lambda(H) \|v_h^H\|_b \|w_h^H\|_b, \end{cases} \tag{6}$$

where the norm $\|\cdot\|_b$ is such that there are $c_{e1} > 0$ and $c_{e2} > 0$ so that:

$$\forall v_h^H \in X_h^H, \qquad c_{e1}\|v_h^H\|_V \leq \|v_h^H\|_b \leq c_{e2}\lambda(H)^{-1}\|v_h^H\|_L, \tag{7}$$

Example 1. *Denoting by $(\cdot,\cdot)_V$ the scalar product in V, the simplest choice is $b_h(v_h^H, w_h^H) = \lambda(H)(v_h^H, w_h^H)_V$.*

Example 2. *Let $X \subset V$ be a Hilbert space that is dense in V with continuous embedding and so that $\|v_h^H\|_X \leq c_{e2}\lambda(H)^{-1}\|v_h^H\|_L$ for all v_h^H in X_h^H. By assuming $X_h \subset X$ and by denoting $(\cdot,\cdot)_X$ the scalar product in X, an alternative choice is $b_h(v_h^H, w_h^H) = \lambda(H)(v_h^H, w_h^H)_X$.*

Example 3. *For the scalar transport equation $\beta{\cdot}\nabla u = f$ in Ω, (β being a smooth solenoidal vector field that fills Ω), we have $L = L^2(\Omega)$ and $V = \{v \in L^2(\Omega) \mid \beta{\cdot}\nabla v \in L^2(\Omega),\ v_{\Gamma^-} = 0\}$, where Γ^- is the inflow boundary. Since $H^1(\Omega) \subset V$, the following two definitions are possible for $b_h(v_h^H, w_h^H)$:*

$$\lambda(H)\int_\Omega v_h^H w_h^H + (\beta{\cdot}\nabla v_h^H)(\beta{\cdot}\nabla w_h^H), \quad \text{or} \quad \lambda(H)\int_\Omega v_h^H w_h^H + (\nabla v_h^H)\cdot(\nabla w_h^H). \tag{8}$$

Hereafter we are concerned with the following Galerkin problem: find u_h in X_h so that

$$\forall v_h \in X_h, \qquad a(u_h, v_h) + b_h(u_h^H, v_h^H) = (f, v_h). \tag{9}$$

Proposition 1. *Problem (9) has a unique solution.*

Theorem 2. *The solution, u_h, of (9) satisfies*

$$\begin{cases} a_s(u-u_h, u-u_h)^{1/2} \leq c \inf_{w_H \in X_H} \left[\lambda(H)^{-1/2}\|u-w_H\|_L + \lambda(H)^{1/2}\|u-w_H\|_V\right], \\ \|u-u_h\|_V + \|u_h^H\|_b \leq c \inf_{w_H \in X_H} \left[\lambda(H)^{-1}\|u-w_H\|_L + \|u-w_H\|_V\right]. \end{cases} \tag{10}$$

Corollary 3. *If u, solution of (2), is in W, the solution, u_h, of (9) satisfies*

$$\begin{cases} a_s(u-u_h, u-u_h)^{1/2} \leq c \left[\dfrac{H}{\lambda(H)} + \dfrac{\lambda(H)}{H}\right]^{1/2} H^{k+1/2}\|u\|_W, \\ \|u-u_h\|_V + \|u_h^H\|_b \leq c \left[1 + \dfrac{H}{\lambda(H)}\right] H^k \|u\|_W. \end{cases} \tag{11}$$

Remark 1. *The proofs of proposition 1, theorem 2, and corollary 3 can be found in Guermond (1998b).*

Remark 2. *If $\lambda(H) \sim H$, the error estimate (11) is optimal in V. In practice, if X_H is a finite element space we have $\lambda(H) \sim H$. If a_s si L-coercive, (11) is not optimal in L for a factor $H^{1/2}$ is missing. Actually, it can be shown, by proceeding like in Zhou (1997) or Guermond (1998a), that optimality can be recovered if the mesh underlying the approximation space X_h satisfies special geometrical properties.*

Remark 3. *The estimate (11) is identical to the one that could be obtained by applying the counterpart of the streamline diffusion method to the present problem (see Johnson et al. (1984) or Brooks and Hughes (1982)).*

Remark 4. *The results above hold also if the condition (5) is weakened as follows:*

$$\forall u_h \in X_h, \qquad \sup_{v_h \in X_h} \frac{a(u_H, v_h)}{\|v_h\|_L} \geq c_a \|u_H\|_V - c_\delta [a_s(u_h, u_h)^{1/2} + \|u_h^H\|_V]. \tag{12}$$

3. A singular perturbation problem

We retain in this section the notations and the hypotheses of the previous section. Let $X \subset V$ be a Hilbert space that is dense in V with continuous embedding. Let $d \in \mathcal{L}(X \times X, \mathbb{R})$ and assume that $a + d$ is X coercive: $\|v\|_X^2 \leq a(v,v) + d(v,v)$. For $0 \leq \epsilon \leq 1/2$, we consider the problem:

$$\text{For } f \in L, \text{ find } u \in X, \quad \forall v \in X, \quad a(u,v) + \epsilon d(u,v) = (f,v). \tag{13}$$

Assume that $X_h \subset X$ and for all $v_h \in X_h$, $\|v_h\|_X \leq \lambda(H)^{-1}\|v_h\|_L$. This hypothesis means that X and V are associated to differential operators of the same order. The discrete problem consists in finding $u_h \in X_h$ so that

$$\forall v_h \in X_h, \qquad a(u_h, v_h) + \epsilon d(u_h, v_h) + b_h(u_h^H, v_h^H) = (f, v_h). \tag{14}$$

The following result is proved in Guermond (1998b):

Theorem 4. *The solution to (14) satisfies*

$$\begin{aligned} a_s(u-u_h, u-u_h)^{1/2} + \epsilon^{1/2}\|u-u_h\|_X \quad &\leq \quad c \inf_{w_H \in X_H} \Big[\lambda(H)^{1/2}\|u - w_H\|_V \\ &\quad + \epsilon^{1/2}\|u-w_H\|_X + \lambda(H)^{-1/2}\|u-w_H\|_L \Big], \end{aligned} \tag{15}$$

$$\|u-u_h\|_V \quad \leq \quad c \inf_{w_H \in X_H} \left[\|u-w_H\|_X + \lambda(H)^{-1}\|u-w_H\|_L \right]. \tag{16}$$

4. Examples

Let Ω be an open subset of $\mathbb{R}^d$. Introduce the sequence of matrix valued functions $(A^k)_{k=1,d}$, $A^k : \Omega \longrightarrow \mathcal{M}_m(\mathbb{R})$, for m in $\mathbb{N} - \{0\}$. For any smooth function $u : \Omega \longrightarrow \mathbb{R}^m$, we define $\beta{\cdot}\nabla\, u : \Omega \longrightarrow \mathbb{R}^m$ by $(\beta{\cdot}\nabla\, u)_i = \sum_{k=1}^d \sum_{j=1}^m A_{ij}^k \partial_{x_k} u_j$ for $1 \leq i \leq m$. For any smooth function $v : \Omega \longrightarrow \mathbb{R}^m$ we define $v \cdot (\beta{\cdot}\nabla\, u) = \sum_{i=1}^m v_i (\beta{\cdot}\nabla\, u)_i$, and we set $|u|_{1,\beta} = \left[\int_\Omega (\beta{\cdot}\nabla\, u) \cdot (\beta{\cdot}\nabla\, u)\right]^{1/2}$. Now, we are interested in bilinear forms of the following type: $\int_\Omega v \cdot (\beta{\cdot}\nabla\, u)$.

For the sake of simplicity we assume that Ω is a polyhedron in $\mathbb{R}^d$. Let $\mathcal{T}_H$ be a regular quasi-uniform triangulation of Ω composed of affine simplexes (T_H). We denote by $\hat{T}$ the reference simplex and by $F_H : T_H \longrightarrow \hat{T}$ the affine mapping that maps T_H onto $\hat{T}$.

To build a P_1 interpolation space we define X_H as follows

$$X_H = \{v_H \in H^1(\Omega)^m \mid v_{H|T_H} \in P_1(T_H)^m, \ \forall T_H \in \mathcal{T}_H\}. \tag{17}$$

Let $\hat{\psi}$ be in $H_0^1(\hat{T})$ with $0 \leq \hat{\psi} \leq 1$; $\hat{\psi}$ is hereafter referred to as the bubble function. (*cf. e.g.* Arnold–Brezzi–Fortin (1984)). By denoting $\psi_h = \hat{\psi}(F_H)$, we define $X_h^T(T_H) = [\text{span}(\psi_h)]^m$ for all T_H in $\mathcal{T}_H$, and we set

$$X_h^H = \oplus_{T_H} X_h^H(T_H). \tag{18}$$

An other possibility that we shall also consider consists in defining X_H as being the P_2 finite element space (conformal in $H^1(\Omega)^m$) associated with the triangulation $\mathcal{T}_H$:

$$X_H = \{v_H \in H^1(\Omega)^m \mid v_{H|T_H} \in P_2(T_H)^m, \ \forall T_H \in \mathcal{T}_H\}. \tag{19}$$

Let $\hat{\psi}_1, \ldots, \hat{\psi}_{d+1}$ be a family a $d{+}1$ linearly independent, real-valued functions in $H_0^1(\hat{T})$. To guaranty that a discrete inf-sup condition holds, we need to enforce additional hypotheses on the subgrid scale functions. Let $\hat{a}_1, \ldots, \hat{a}_{d+1}$ be the nodes of the reference simplex $\hat{T}$. Let R_{ij} be the symmetry of $\hat{T}$ such that $R_{ij}(\hat{a}_i) = \hat{a}_j$ and $R_{ij}(\hat{a}_l) = \hat{a}_l$ if $l \notin \{i,j\}$. Now, we assume that the functions $(\hat{\psi}_i)_{i=1,\ldots,d+1}$ satisfy the following symmetry properties

$$\left\{ \begin{array}{l} \hat{\psi}_i(R_{ij}) = \hat{\psi}_j, \\ \hat{\psi}_i(R_{jl}) = \hat{\psi}_i, \ \text{if } i \notin \{j,l\}. \end{array} \right. \tag{20}$$

We denote $\psi_{i,h} = \hat{\psi}_i(F_H)$ for $1 \leq i \leq d+1$, we set $X_h^H(T_H) = [\mathrm{span}(\psi_{1,h}, \ldots, \psi_{d+1,h})]^m$, and we finally define

$$X_h^H = \oplus_{T_H} X_h^H(T_H). \tag{21}$$

It is shown in Guermond (1998a) that for the P_1 and P_2 interpolation spaces defined above, the decomposition $X_h = X_H \oplus X_h^H$ is L^2-stable. Moreover, by reasoning as in Guermond (1998a), the following results can be proved:

Theorem 5. *If β is piecewise constant on each simplex T_H of $\mathcal{T}_H$, there is $c_\beta > 0$ independent of (H, h), so that*

$$\inf_{u_H \in X_H} \sup_{v_h \in X_h} \frac{\int_\Omega v_h \cdot (\beta \cdot \nabla u_H)}{|u_H|_{1,\beta} \|v_h\|_0} \geq c_\beta. \tag{22}$$

Corollary 6. *If β is in $C^1(\overline{\Omega}; \mathcal{M}_m(\mathrm{IR})^d)$, there are $c_\beta > 0$ and $c_\delta \geq 0$, both independent of (H, h), so that*

$$\forall u_H \in X_H, \qquad \sup_{v_h \in X_h} \frac{\int_\Omega v_h \cdot (\beta \cdot \nabla u_H)}{\|v_h\|_0} \geq c_\beta |u_H|_{1,\beta} - c_\delta \|u_H\|_0. \tag{23}$$

Corollary 7. *Set $a(u, v) = a_0(u, v) + \int_\Omega v \cdot (\beta \cdot \nabla u)$ where $a_0 \in \mathcal{L}(L^2(\Omega)^m \times L^2(\Omega)^m; \mathrm{IR})$, and define $\|v\|_{1,\beta} = (\|v\|_0^2 + |v|_{1,\beta}^2)^{1/2}$. If β is in $C^1(\overline{\Omega}; \mathcal{M}_m(\mathrm{IR})^d)$, β and a_0 are such that a is $L^2(\Omega)^m$-coercive, then there is $c_\beta > 0$, independent of (H, h), so that*

$$\forall u_H \in X_H, \qquad \inf_{u_H \in X_H} \sup_{v_h \in X_h} \frac{a(u_H, v_h)}{\|u_H\|_{1,\beta} \|v_h\|_0} \geq c_\beta. \tag{24}$$

5. Conclusion

It has been shown in Brezzi et al. (1992) that for solving convection dominated advection-diffusion equations Galerkin-Least-Square techniques can, at least formally, be recast into the classical Galerkin framework provided some suitably shaped bubble functions are added to the approximation spaces. Furthermore, it has been proven in Baiocchi–Brezzi–Franca (1993) that there always exists a set of internal bubble functions that reproduce the effects of Galerkin-Least-Square techniques; the exact shape of the bubble functions, though, is in general unknown. The main result of the present paper is that classical (isotropic) bubble functions always stabilize the Galerkin technique provided an artificial viscosity mechanism is added to control the subgrid scales (*i.e.* the projection of the approximate solution onto the bubble functions). If a is associated to a differential operator, the funding principles of the proposed technique can be rephrased as follows: introduce subgrid scales to capture the discontinuities generated by the differential operator and control the subgrid scales by an artificial viscosity.

Acknowledgements

The present work has been partly supported by ASCI (UPR-CNRS 9029), Orsay.

6. References

1 D.N. Arnold, F. Brezzi, M. Fortin, A stable finite element for the Stokes equations, *Calcolo* **21** (1984) 337–344.
2 C. Baiocchi, F. Brezzi, L.P. Franca, Virtual bubbles and Galerkin-least-square type methods (Ga.L.S), *Comput. Methods Appl. Mech. Engrg.* **105** (1993) 125–141.
3 F. Brezzi, M.O. Bristeau, L. Franca, M. Mallet, and G. Rogé, A relationship between stabilized finite element methods and the Galerkin method with bubble functions, *Comput. Methods Appl. Mech. Engrg.* **96** (1992) 117–129.
4 A.N. Brooks, T.J.R. Hughes, Streamline Upwind/Petrov–Galerkin formulations for convective dominated flows with particular emphasis on the incompressible Navier–Stokes equations, *Comput. Methods Appl. Mech. Engrg.* **32** (1982) 199–259.
5 Jean-Luc Guermond, Stabilization of Galerkin approximations of transport equations by subgrid modeling, Part I, Submitted to (M2AN): *Modél. Math. Anal. Num.* (1998a), and LIMSI report 98-01.
6 Jean-Luc Guermond, Subgrid stabilization of Galerkin approximations of monotone operators, Part II, Submitted to *Numer. Math.* (1998b), and LIMSI report 98-03.
7 C. Johnson, U. Nävert, J. Pitkäranta, Finite element methods for linear hyperbolic equations, *Comput. Methods Appl. Mech. Engrg.* **45** (1984) 285–312.
8 G. Zhou, How accurate is the streamline diffusion finite element method?, *Math. Comp.* **66** (1997) 31–44.

Address: Dr. Guermond J.-L., LIMSI, BP 133, 91403 Orsay, France; e-mail : guermond@limsi.fr