SOME IMPLEMENTATIONS OF PROJECTION METHODS FOR NAVIER–STOKES EQUATIONS *

JEAN-LUC GUERMOND †

Abstract. This paper is concerned with the implementation of spatially discrete versions of Chorin–Temam's projection methods. The emphasis is put on the projection step, which enforces incompressibility. Three types of variational approximations are reviewed. In the first one, the projection step is solved as a div-grad problem with velocity test functions satisfying (at first glance) a paradoxical Dirichlet condition. In the second method, the projection step is still solved as a div-grad problem but the velocity test functions satisfy a boundary condition only for the normal component. In the third approach the projection step is solved in the form of a Poisson equation supplemented with a Neumann boundary condition. The first method is shown to be legitimate and economical for finite element approximations, whereas the second is shown to be useful for spectral approximations. The third one is probably the easiest to implement since it avoids the problem of the mass matrix occurring in the two others. Though the second and third approaches do not directly involve a inf-sup condition, this condition is pointed out to be necessary to establish convergence and rule out possible spurious pressure. Finally some links between these algorithms and some preconditioning techniques of the Uzawa operator are shown.

Résumé. Dans cet article on s'intéresse à quelques approximations spatiales des méthodes de projections du type Chorin–Temam. On s'intéresse plus particulièrement à l'étape de projection, qui sert à imposer l'incompressibilité. Trois types d'approximations variationnelles sont étudiées. Dans la première on résout l'étape de projection sous la forme d'un problème de Darcy avec des vitesses test satisfaisant une condition de Dirichlet (à première vue) paradoxale. Dans la seconde approche, le problème est encore résolu sous sa forme div-grad (*ie.* Darcy) mais les vitesses test satisfont cette fois-ci une condition à la frontière portant uniquement sur la composante normale. Dans la troisième méthode, l'étape de projection est résolue sous la forme d'une équation de Poisson avec une condition de Neumann. On montre que la première méthode est légitime pour des approximations par éléments finis, alors que la seconde a de l'intérêt aussi pour des approximations spectrales. La troisième méthode est probablement la plus aisée à mettre en œuvre puisque qu'elle permet d'éviter l'inversion d'une matrice de masse qui est obligatoire pour les deux autres. Bien que les deux dernières méthodes n'imposent pas directement de compatibilité entre les espaces de vitesse et de pression, on montre qu'une telle condition est nécessaire pour assurer la convergence de la méthode. Finalement on montre quelques liens entre ces algorithmes et certaines techniques de préconditionnement de l'opérateur d'Uzawa.

Key words. Projection method, Fractional step method, Navier–Stokes equations.

AMS(MOS) subject classifications. 35A40, 35Q10, 65J15

1. Introduction.

In this paper we consider discrete approximations of a class of fractional step techniques known as Chorin–Temam projection methods [8] [23]. These techniques have been proposed for approximating in time the unsteady incompressible Navier–Stokes equations. They are devised to turn around the coupling between the pressure and the velocity that is implied by the incompressibility constraint: $\mathrm{div} u = 0$. The basic idea consists in devising time marching procedures that uncouple viscous and

* Modél. Math. Anal. Numér. M2AN, **30** (1996) 637-667

† Laboratoire d'Informatique pour la Mécanique et les Sciences de l'Ingénieur, C.N.R.S., BP 133, 91403, Orsay, France (guermond@limsi.fr)

incompressibility effects. These techniques are very efficient and have probably "been the first numerical schemes enabling a cost-effective solution of three-dimensional time-dependent problems" (*cf.* Quartapelle [19, p. 177]). Their simplicity and sometimes surprising efficiency render them particularly attractive to the CFD community (see *e.g.* [2], [9], [11], [12], [25]). Although these techniques have long been used for calculating steady-state solutions to Navier–Stokes equations, they are now regaining their status as true time marching procedures for calculating time-dependent incompressible viscous flows. This renewed interest for time-dependent solutions to Navier–Stokes equations is prompted by the increasing capacities of computers and the success of large eddy theories which recognize that unsteadiness of large eddies should be well predicted whereas smaller scales can (reasonably) be filtered.

Since its initial appearance, the projection method has been implemented with various types of spatial approximations and the fractional step has been modified in order to improve the overall accuracy of the scheme. Though the stability of this method and its modified versions can generally be proven quite easily when space variables are continuous, the stability and convergence of their discrete counterparts are often overlooked in the literature.

For instance, the projection step may be put into two different forms. One possibility consists in solving a so-called div-grad or Darcy problem as follows

$$\begin{cases} u + \delta t \nabla \phi = \tilde{u} \\ \nabla \cdot u = 0 \\ u \cdot n_{|\partial\Omega} = 0. \end{cases} \tag{1.1}$$

The second possibility consists in obtaining from (1.1) a Poisson equation supplemented with a homogeneous Neumann boundary condition on the pressure

$$\begin{cases} \nabla^2 \phi = \dfrac{\nabla \cdot \tilde{u}}{\delta t} \\ \dfrac{\partial \phi}{\partial n}_{|\partial\Omega} = 0. \end{cases} \tag{1.2}$$

Although (1.1) and (1.2) are equivalent in some sense, their discrete counterparts are not in general. On the one hand, discrete variational approximations of the Darcy problem yield pressure equations of type $\mathcal{B}_h \mathcal{I}_h^{-1} \mathcal{B}_h^t \Phi = F$, where $\mathcal{I}_h$ is a mass matrix and $\mathcal{B}_h$ is a matrix associated with the divergence operator. In this case the homogeneous Neumann boundary condition on the pressure is not enforced, though it is implicitly accounted for by the velocity tests functions which have the normal component vanishing at the boundary. On the other hand, discrete variational approximations of (1.2) yield equations of type $\mathcal{D}_h \Phi = F$, where $\mathcal{D}_h$ is an approximation of the Laplace operator and the homogeneous Neumann boundary condition is explicitly, though weakly, enforced by the variational formulation. At this point, one may ask oneself which procedure is correct? If both are correct what are their respective range of application? It is shown in this paper that both approaches are correct, and each of them has its own advantages within its respective functional framework.

The other point that is discussed in this paper concerns the appropriate boundary condition that should be imposed on the end-of-step velocity. Thanks to theorem 2.1 (see below), it is clear on (1.1) that, as far as the spatially continuous problem is concerned, only the normal component of the end-of-step velocity should be constrained. However, when the problem is discretized in space, the answer is no longer clear: a full Dirichlet condition on the end-of-step velocity is sometimes advocated by some authors (*cf.* *e.g.* Gresho and Chan [12, part II]), whereas other authors impose a, more natural, condition on the normal component of the velocity (*cf.* Donea et al. [9], Azaiez et al. [1]). Which solution is correct? What are their respective advantages? It is the purpose of this paper to show that both solutions are suitable if applied in the correct functional frameworks. Actually, we show in this paper that the intermediate velocity and the final velocity should be approximated in two different spaces.

This paper is organized as follows. In §2, we review non incremental and incremental projection schemes in the space continuum. In §3, we analyze a discrete projection scheme in which the provisional velocity and the corrected one are approximated in the same space; that is, the end-of-step velocity satisfies a Dirichlet condition. This scheme is shown to be efficient for finite element approximations. A discrete projection scheme with an approximation space enforcing a boundary condition only on the normal component of the end-of-step velocity is analyzed in §4. This functional framework is shown to be useful for spectral approximations. In §5, the projection step is formulated as a Poisson problem supplemented with a Neumann boundary condition. This technique is probably the easiest to implement, for it turns around a mass matrix problem that plagues the two others. Some generalization and convergence results are presented in §6. In §7, we show that the three projection algorithms are, in some sense, equivalent to some known preconditioning techniques of the Uzawa operator for which the preconditioner is applied only once.

2. Preliminaries.

2.1. The continuous unsteady Stokes problem. In this paper we consider numerical approximations with respect to time and space of the time-dependent Navier–Stokes equations formulated in the primitive variables, namely velocity and pressure. However, to simplify the presentation and since we are mainly concerned with the parabolic aspect of the problem, we restrict ourselves to the time-dependent Stokes problem

$$
\begin{cases}
\dfrac{\partial u}{\partial t} - \nabla^2 u + \nabla p = f \\
\nabla \cdot u = 0 \\
u_{|\partial\Omega} = 0 \\
u_{|t=0} = v_0,
\end{cases}
\tag{2.1}
$$

where $f(t)$ is a body force, and the boundary condition on the velocity is set to zero for sake of simplicity. The fluid domain Ω is open connected and bounded in $\mathrm{I\!R}^d$ ($d = 2$ or 3 in practical applications). The domain boundary $\partial\Omega$ is assumed to be smooth; say $\partial\Omega$ is Lipschitz and Ω is locally on one side of its boundary.

In the following we work within the classical framework of the Sobolev spaces. The set of real functions infinitely differentiable with compact support in Ω is denoted by $\mathrm{D}(\Omega)$, and the set of distributions on Ω is denoted by $\mathrm{D}'(\Omega)$. As usual, $\mathrm{L}^2(\Omega)$ denotes the space of real-valued functions the squares of which are summable in Ω. The inner product in $\mathrm{L}^2(\Omega)$ is denoted by $(\,\cdot\,,\,\cdot\,)$ and $|\cdot|_0$ is the associated norm; we identify $\mathrm{L}^2(\Omega)$ with its dual. $\mathrm{H}^m(\Omega)$, $m \geq 0$, is the set of distributions the successive derivatives of which, up to order m, can be identified with square summable functions. The space $\mathrm{H}^m(\Omega)$, equipped with the norm $|u|_m^2 = (\sum_{|\alpha|=0}^m |D^\alpha u|_0^2)^{1/2}$, expressed in the multi-index notation, is a Hilbert space [18]. We define $\mathrm{H}_0^m(\Omega)$ as the completion of $\mathrm{D}(\Omega)$ in $\mathrm{H}^m(\Omega)$, and we denote $\mathrm{H}^{-m}(\Omega)$ the dual of $\mathrm{H}_0^m(\Omega)$.

The incompressibility condition on the velocity leads to consider solenoidal vector fields. For this reason, we define $\mathrm{N}(\Omega) = \{v \in \mathrm{D}(\Omega),\ \nabla\cdot v = 0\}$, and we denote by H and V the completions of $\mathrm{N}(\Omega)$ in $\mathrm{L}^2(\Omega)^d$ and $\mathrm{H}_0^1(\Omega)^d$, respectively. Spaces H and V are characterized by:

$$\mathrm{H} = \{v \in \mathrm{L}^2(\Omega)^d,\ \nabla\cdot v = 0,\ v\cdot n_{|\partial\Omega} = 0\}, \tag{2.2}$$

$$\mathrm{V} = \{v \in \mathrm{H}^1(\Omega)^d,\ \nabla\cdot v = 0,\ v_{|\partial\Omega} = 0\}. \tag{2.3}$$

See for instance Temam [22, pp. 15–18] for a proof. In the following, the space H plays an important role by means of

THEOREM 2.1. *Under the hypotheses on Ω stated above, we have the orthogonal decomposition*

$$\mathrm{L}^2(\Omega)^d = \mathrm{H} \oplus \nabla(\mathrm{H}^1(\Omega)/\mathrm{I\!R}). \tag{2.4}$$

Proof. See for instance Girault and Raviart [10]. □

If one assumes that $f \in \mathrm{L}^2(0,T;\mathrm{H}^{-1}(\Omega)^d)$ and $u_0 \in \mathrm{H}$, then it is well-known that (2.1) is well-posed (look for $u \in \mathrm{L}^2(0,T;\mathrm{V}) \cap \mathrm{C}^0(0,T;\mathrm{H})$ and restrict the time evolution problem to $\mathrm{L}^2(0,T;\mathrm{V}')$ where V' is the dual of V and apply Lions's theorem [18, p. 257]). Furthermore, one may verify that $p \in \mathrm{L}^2(0,T;\mathrm{L}^2(\Omega)/\mathrm{I\!R})$. It is hereafter assumed that the data are regular enough and satisfy all the compatibility conditions that are needed for a smooth solution to exit.

We now turn the attention to the time approximation of (2.1) by means of projection methods. To make the presentation self-contained, we begin by recalling the main features of some projection schemes.

2.2. The non-incremental and incremental projection schemes. Projection methods have been introduced by Chorin [8] and Temam [23]. They are time marching procedures based on a fractional step technique that may be viewed also as a predictor-corrector strategy aiming at uncoupling viscous diffusion and incompressibility effects. The time interval $[0,T]$ on which the solution is sought is partitioned into $K+1$ time steps that are hereafter denoted by $t^k = k\delta t$ for $0 \le k \le K$, where $\delta t = T/K$. In the algorithm originally devised by Chorin and Temam, each time step is decomposed into two substeps as follows. For each time step, solve first

$$\begin{cases} \dfrac{\tilde{u}^{k+1} - u^k}{\delta t} - \nabla^2 \tilde{u}^{k+1} = f^{k+1} \\ \tilde{u}^{k+1}_{|\partial\Omega} = 0, \end{cases} \tag{2.5}$$

then project the provisional velocity $\tilde{u}^{k+1}$ onto H; in other words, solve

$$\begin{cases} \dfrac{u^{k+1} - \tilde{u}^{k+1}}{\delta t} + \nabla p^{k+1} = 0 \\ \nabla \cdot u^{k+1} = 0 \\ u^{k+1} \cdot n_{|\partial\Omega} = 0. \end{cases} \tag{2.6}$$

The series (u^k) is initialized by $u^0 = v_0$. The velocity $\tilde{u}^{k+1}$ is a prediction of $u(t^{k+1})$, and u^{k+1} is a correction of $\tilde{u}^{k+1}$.

One possible improvement of the algorithm above consists in predicting a better value of the provisional velocity $\tilde{u}^{k+1}$ by putting the gradient of the pressure that has been calculated at the time step t^k in the right-hand side of (2.5). This algorithm, hereafter referred to as the incremental form of the projection technique [11], consists in the following. Initialize the series (u^k) and (p^k) respectively by $u^0 = v_0$ and $p^0 = p_{|t=0}$, assuming that $p \in C^0(0,T;\mathrm{L}^2(\Omega)/\mathbb{R})$. For each time step solve

$$\begin{cases} \dfrac{\tilde{u}^{k+1} - u^k}{\delta t} - \nabla^2 \tilde{u}^{k+1} = f^{k+1} - \nabla p^k \\ \tilde{u}^{k+1}_{h|\partial\Omega} = 0, \end{cases} \tag{2.7}$$

and project $\tilde{u}^{k+1}$ onto H

$$\begin{cases} \dfrac{u^{k+1} - \tilde{u}^{k+1}}{\delta t} + \nabla (p^{k+1} - p^k) = 0 \\ \nabla \cdot u^{k+1} = 0 \\ u^{k+1} \cdot n_{|\partial\Omega} = 0. \end{cases} \tag{2.8}$$

Note that this algorithm assumes more regularity than the non incremental one since it requires an additional condition, *ie.* $p^0 = p_{|t=0}$, that was not specified in the original Stokes problem (2.1) so that some regularity on p as $t \to 0$ needs to be assured.

Steps (2.6) and (2.8) are called projection steps since, according to theorem 2.1, they are equivalent to $u^{k+1} = \mathrm{P_H}\tilde{u}^{k+1}$ and either $\nabla p^{k+1} = (\tilde{u}^{k+1} - \mathrm{P_H}\tilde{u}^{k+1})/\delta t$ or $\nabla(p^{k+1} - p^k) = (\tilde{u}^{k+1} - \mathrm{P_H}\tilde{u}^{k+1})/\delta t$, where $\mathrm{P_H}$ is the orthogonal projection of $\mathrm{L}^2(\Omega)^d$

onto H. In both cases, the velocity $\tilde{u}^{k+1}$ is a prediction of $u(t^{k+1})$ that satisfies the correct boundary condition but is not divergence free. This defect is corrected by projecting $\tilde{u}^{k+1}$ onto H (this step has given its name to the method). However, the end-of-step velocity, u^{k+1}, does not satisfy exactly the correct boundary condition since its tangential component is not necessarily zero.

Note that fractional step techniques (2.5)–(2.6) and (2.7)–(2.8) uncouple viscous diffusion and incompressibility. In practice these techniques require solving Helmholtz problems and performing projections onto H, whereas classical coupled techniques usually involve a Uzawa operator: $\nabla \cdot (I_d - \sigma\nabla^2)^{-1}\nabla$ where σ is proportional to δt and I_d is the identity. This operator, called after Uzawa's algorithm in which it is implicitly used (see for instance Temam [22, p. 138]), is non-local and ill-conditioned as the time step tend to zero (see also §7 for other details).

When it comes to analyzing the convergence of projection algorithms (2.5)–(2.6) and (2.7)–(2.8), it is sufficient to restrict the analysis to that of the incremental algorithm, for the error equations of the non-incremental one can be put into an incremental form as follows

$$\begin{cases} \dfrac{\tilde{e}^{k+1} - e^k}{\delta t} - \nabla^2 \tilde{e}^{k+1} = R^{k+1} - \nabla p(t^{k+1}) \\ \tilde{e}^{k+1}_{|\partial\Omega} = 0, \end{cases}$$

and

$$\begin{cases} \dfrac{e^{k+1} - \tilde{e}^{k+1}}{\delta t} + \nabla(\delta^{k+1} - p(t^{k+1})) = 0 \\ \nabla \cdot e^{k+1} = 0 \\ e^{k+1} \cdot n_{|\partial\Omega} = 0. \end{cases}$$

Where R^{k+1} is the integral Taylor residual, and we have defined the error functions $e^k = u(t^k) - u^k$, $\tilde{e}^k = u(t^k) - \tilde{u}^k$ and $\delta^k = p(t^k) - p^k$. As a consequence, in the following we only consider the incremental form of the projection algorithm.

Global accuracy of projection schemes can be further improved by replacing the one-step backward Euler scheme in (2.7) by a Crank–Nicolson approximation as Van Kan did [25] or by a two-step backward Euler approximation. Stability and convergence of some of these modified scheme is studied in [13], [15], [16], [20], [21], and [25], but these considerations are out of the scope of the present paper. The objective of the work presented herein is to bring some answers to questions concerning spatially discrete approximations of the projection step (2.6) or (2.8).

2.3. The spatially discrete unsteady Stokes problem. Let X_h and M_h be convergent, internal, and stable approximations of $\mathrm{H}^1_0(\Omega)^d$ and $\mathrm{L}^2(\Omega)/\mathbb{R}$. It is hereafter assumed that X_h and M_h are finite dimensional vector spaces. We define X'_h the dual of X_h; X'_h is identical to X_h in terms of vector space but is equipped with the dual norm induced be the scalar product of $\mathrm{L}^2(\Omega)^d$. We identify M_h with its dual space, for the natural norm of M_h is that of $\mathrm{L}^2(\Omega)$.

We now introduce the continuous bilinear form $b_h : X_h \times M_h \longrightarrow \mathrm{I\!R}$ so that $b_h(u_h, p_h) = -(\mathrm{div} u_h, p_h)$, and we associate with b_h the continuous linear operator $B_h : X_h \longrightarrow M_h$ and its transpose $B_h^t : M_h \longrightarrow X_h'$ so that for every couple (u_h, p_h) in $X_h \times M_h$ we have $(B_h u_h, p_h) = b_h(u_h, p_h)$ and $(u_h, B_h^t p_h) = b_h(u_h, p_h)$. We assume that $B_h : X \longrightarrow M$ is onto. An important consequence of the surjectivity of B_h is summarized by the following well-known result which is a corollary of Banach's closed range theorem.

LEMMA 2.1. *Let E and F be two Hilbert spaces, and $T \in \mathcal{L}(E, F)$. The following propositions are equivalent:*

(i) *$T : E \longrightarrow F$ is onto.*

(ii) *$T^t : F' \longrightarrow \ker(T)^o$ is one to one*[1] *and there is $\beta > 0$, so that $|T^t p|_{E'} \geq \beta |p|_E$, for all $p \in E$.*

For a proof, the reader is referred to Brezzi [5] or Girault–Raviart [10, pp. 58–59]. As a result, there is $\beta_h > 0$ so that

$$\forall q_h \in M_h, \qquad |B_h^t q_h|_{X_h'} \geq \beta_h |q_h|_0. \tag{2.9}$$

The constant β_h is sometimes referred to as the inf-sup constant or the LBB constant (LBB being for Ladyzhenskaya–Babŭska–Brezzi). A large choice of discrete spaces X_h and M_h satisfying such a condition is available in the literature. A review of compatible spaces in the framework of finite elements may be found in Girault–Raviart [10]. For spectral approximation see, for instance, Bernardi–Maday [4].

The null space of B_h playing an important rôle in the following, we set $V_h = \ker(B_h)$, and we equip V_h with the norm induced by that of X_h. We also define by $H_h = V_h$ in term of vector space and we equip H_h with the norm of $\mathrm{L}^2(\Omega)^d$ (in some sense H_h plays the role of the completion of V_h in $\mathrm{L}^2(\Omega)^d$)

Let us also introduce the continuous bilinear form $a_h : X_h \times X_h \longrightarrow \mathrm{I\!R}$ so that $a_h(u_h, v_h) = (\nabla u_h, \nabla v_h)$, and recall that, thanks to the Poincaré inequality in $\mathrm{H}_0^1(\Omega)^d$, a_h is X_h-elliptic; that is,

$$\exists \alpha > 0,\ \forall u_h \in X_h, \qquad a_h(u_h, u_h) \geq \alpha |u_h|_1^2. \tag{2.10}$$

We associate with a_h the linear continuous operator $A_h : X_h \longrightarrow X_h'$ so that for all $(u_h, v_h) \in X_h \times X$, $a_h(u, v) = (A_h u_h, v_h)$.

In the functional framework defined above, the spatially discrete version of the time-dependent Stokes problem can be formulated as follows. For $f_h \in \mathrm{L}^2(0, T, X_h')$ and $v_{0,h} \in H_h$, find $u_h \in \mathrm{L}^2(0, T; X_h) \cap \mathrm{C}^0(0, T; H_h)$ and $p_h \in \mathrm{L}^2(0, T; M_h)$ so that

$$\begin{cases} \dfrac{\mathrm{d}u_h}{\mathrm{d}t} + A_h u + B_h^t p = f_h \\ B_h u_h = 0 \\ u_{h|t=0} = v_{0,h}, \end{cases} \tag{2.11}$$

[1] Recall that the polar set of a space $V \subset E$ is defined by $V^o = \{e' \in E',\ \langle e', v \rangle = 0,\ \forall v \in V\}$

where f_h and v_h are suitable approximations of f and v_0 in X_h. The data, f and v_0, are assumed to be as smooth as needed, and in the rest of the paper we focus on time approximations of (2.11). This problem has a unique solution (u_h, p_h), and this solution is stable (in the appropriate norms) with respect to the data.

Since X_h and M_h are convergent and stable internal approximations of $\mathrm{H}_0^1(\Omega)^d$ and $\mathrm{L}^2(\Omega)/\mathbb{R}$ respectively, the solution to (2.11) converges in an appropriate sense to that of the continuous unsteady Stokes problem (2.1). Our main concern now consists in approximating the time derivative in (2.11). In what follows, we are exclusively concerned with time approximations of problem (2.11) by means of projection techniques similar to (2.5)–(2.6) or (2.7)–(2.8).

3. A full Dirichlet boundary condition on the end-of-step velocity.

3.1. The functional framework. In this section we build a discrete projection algorithm in which we take the provisional velocity $\tilde{u}_h^k$ and the end-of-step velocity u_h^k in the same approximation space.

In order to build an analogy between the discrete framework and its continuum counterpart, we introduce a subspace of M_h that is the analogue of $\mathrm{H}^1(\Omega) \subset \mathrm{L}^2(\Omega)$. For this purpose we define the positive bilinear form $(p, q)_{M_h^1} = (B_h^t p, B_h^t q)$. According to (ii) in lemma 2.1, it is clear that $(\cdot, \cdot)_{M_h^1}$ is a scalar product, and $|p|_{M_h^1} = |B_h^t p|_0$ is a norm. We now define M_h^1 so that $M_h^1 = M_h$ in terms of vector space, but we equip M_h^1 with the norm $|\cdot|_{M_h^1}$. Furthermore, we define $Y_h = X_h$ in term of vector space and we equip Y_h with the norm of $\mathrm{L}^2(\Omega)^d$. The introduction M_h^1 and Y_h is justified by

Corollary 3.1. *We have the stable, orthogonal decomposition:*

$$Y_h = H_h \oplus B_h^t(M_h^1). \tag{3.1}$$

Proof. Let $l_h \in Y_h$ and define $P_{H_h} : Y_h \longrightarrow H_h$ the orthogonal projection onto H_h. We have $(l_h - P_{H_h} l_h, v_h) = 0$ for all v_h in H_h, in other words we also have $(l_h - P_{H_h} l_h, v_h) = 0$ for all v_h in V_h; that is to say, $(l_h - P_{H_h} l_h) \in V_h^o$. From (ii) of lemma 2.1, we infer that there is a unique $p_h \in M_h$ so that $B_h^t p_h = l_h - P_{H_h} l_h$. Furthermore, it is clear that $(P_{H_h} l_h, B_h^t p_h) = 0$; as a result, $|P_{H_h} l_h|_0^2 = (l_h, P_{H_h} l_h)$ and $|B_h^t p_h|_0^2 = (l_h, B_h^t p_h)$, from which we infer that the decomposition is stable: $|P_{H_h} l_h|_0 \leq |l_h|_0$ and $|p_h|_{M_h^1} \leq |l_h|_0$. □

Note that the above decomposition of Y_h is the spatially discrete counterpart of the classical decomposition: $\mathrm{L}^2(\Omega)^d = \mathrm{H} \oplus \nabla(\mathrm{H}^1(\Omega)/\mathbb{R})$.

3.2. The discrete projection algorithm. With the functional framework introduced above, the logical implementation of the viscous step (2.7) consists in looking for $\tilde{u}_h^{k+1}$ in X_h so that

$$\frac{\tilde{u}_h^{k+1} - u_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} = f_h^{k+1} - B_h^t p_h^k. \tag{3.2}$$

This problem is well posed thanks to the X_h-ellipticity of A_h. Note that $\tilde{u}_h^{k+1}$, being approximated in X_h, satisfies the Dirichlet condition $\tilde{u}_{h|\partial\Omega}^{k+1} = 0$. We now turn the attention to the discrete projection step.

The projection step of the incremental algorithm can be implemented as follows. Find u_h^{k+1} in Y_h and $p_h^{k+1} - p_h^k$ in M_h so that

$$\left\{ \begin{array}{l} \dfrac{u_h^{k+1} - \tilde{u}_h^{k+1}}{\delta t} + B_h^t(p_h^{k+1} - p_h^k) = 0 \\ B_h u_h^{k+1} = 0. \end{array} \right. \tag{3.3}$$

According to corollary 3.1 this problem is well posed. Actually, the couple $(u_h^{k+1}, \delta t(p_h^{k+1} - p_h^k))$ is the decomposition of $\tilde{u}_h^{k+1}$ in $H_h \oplus B_h^t(M_h^1)$; that is, $u_h^{k+1} = P_{H_h}\tilde{u}_h^{k+1}$. Note, however, that this way of setting the discrete projection step may not seem the most appropriate since the velocity test functions involved satisfy a Dirichlet boundary condition. In order to emphasize this point we can reformulate (3.3) as follows: find u_h^{k+1} in X_h and p_h^{k+1} in M_h so that

$$\left\{ \begin{array}{ll} \forall v_h \in X_h \subset \mathrm{H}_0^1(\Omega)^d, & \dfrac{1}{\delta t}(u_h^{k+1} - \tilde{u}_h^{k+1}, v_h) - (\nabla \cdot v_h, p_h^{k+1} - p_h^k) = 0 \\ \forall q_h \in M_h \subset \mathrm{L}^2(\Omega)/\mathrm{I\!R}, & (\nabla \cdot u_h^{k+1}, q_h) = 0. \end{array} \right. \tag{3.4}$$

X_h being an internal approximation to $\mathrm{H}_0^1(\Omega)^d$, u_h^{k+1} satisfies $u^{k+1}_{|\partial\Omega} = 0$ and the velocity test functions satisfy $v_{h|\partial\Omega} = 0$, whereas it might seem more appropriate to enforce only $u_h^{k+1} \cdot n_{|\partial\Omega} = 0$ and $v_h \cdot n_{|\partial\Omega} = 0$ as suggested by the continuous projection step (2.6) or (2.8). We show in the following that this choice has some consequence on the condition number of the pressure operator involved the linear system (3.4).

3.3. The condition number issue. If the velocity u_h^{k+1} is eliminated from (3.3), the projection step reduces to solving the following pressure problem

$$B_h B_h^t(p_h^{k+1} - p_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \tag{3.5}$$

We now turn the attention to the influence of the end-of-step boundary condition on the condition number of the pressure operator $B_h B_h^t$. We analyze this influence for finite element approximations and for spectral approximations. It is shown that the full Dirichlet boundary condition on the end-of-step velocity is optimal, in term of condition number of $B_h B_h^t$, for finite element approximations, whereas it is not for spectral approximations.

In the following we assume that X_h and M_h are composed of finite elements based on a uniformly regular triangulation $\mathcal{T}_h$ of Ω, the characteristic mesh size of which is denoted by h. We also assume that these two spaces are uniformly compatible in the sense that the inf-sup constant β_h is independent of h. Recall that for uniformly regular triangulation we have the inverse inequality

LEMMA 3.1. *(cf. Girault–Raviart [10, p. 103]) There is a constant $c > 0$ independent of h so that*

$$\forall v_h \in X, \qquad |\nabla v_h|_0 \le c h^{-1} |v_h|_0. \tag{3.6}$$

We now give an upper bound on the condition number $\kappa(B_h B_h^t)$

THEOREM 3.1. *There is a constant $c > 0$ so that*

$$\kappa(B_h B_h^t) \leq \frac{c}{\beta_h^2 h^2}. \tag{3.7}$$

Proof. Let p_h be in M_h and define g_h in M_h so that $g_h = B_h B_h^t p_h$. By setting $u_h = -B_h^t p_h$ we have

$$\begin{cases} u_h + B_h^t p_h = 0 \\ B_h u_h = g_h. \end{cases}$$

(a) From this system we deduce the bound $|u_h|_0^2 = -(B_h u_h, p_h) \leq |g_h|_0 |p_h|_0$. Furthermore, (ii) in lemma 2.1 yields $|u_h|_0 = |B_h^t p_h|_0 \geq \beta_h |p_h|_0$. As a result, we have $\beta_h^2 |p_h|_0 \leq |g_h|_0$, from which we deduce that the smallest eigenvalue of $B_h B_h^t$ is bounded from below by β_h^2.
(b) From the system above we also deduce $|u_h|_0^2 = (\text{div} u_h, p_h)$, which together with the inverse inequality (3.6), yields $h|u_h|_0 \leq c|p_h|_0$. Furthermore, we have $|g_h|_0^2 = -(\text{div} u_h, g_h)$, which for the same reason as above yields $h|g_h|_0 \leq c|u_h|_0$. By combining the two bounds above, we obtain that the largest eigenvalue of $B_h B_h^t$ is bounded from above by c/h^2.
(c) The operator $B_h B_h^t$ being symmetric, its condition number is equal to the ratio of its largest to its smallest eigenvalues. □

This bound on $\kappa(B_h B_h^t)$ is likely to be optimal since the underlying partial differential equation is a Poisson problem supplemented with a Neumann boundary condition, and it is known that approximating such a problem by finite elements yields an operator the condition number of which is equivalent to $1/h^2$. Hence, the important conclusion of this section is that although approximating the projection step by means of test functions satisfying $v_{h|\partial\Omega} = 0$ does not seem natural, it nevertheless remains optimal in the framework of finite element approximations since it yields a pressure operator with an optimal condition number.

Now assume that $\Omega =]-1, +1[^2$, and let $N \geq K > 0$ be two integers. Let $X_h = \mathbb{P}_N(\Omega) \cap \mathrm{H}_0^1(\Omega)^d$ be the space of the polynomial velocities on Ω vanishing on $\partial\Omega$ and the partial degree of which are less than or equal to N. Likewise, we define $M_h = \mathrm{P}_K(\Omega) \cap \mathrm{L}^2(\Omega)/\mathbb{R}$ the space of the pressures on Ω that are polynomials with partial degree less than or equal to K. We assume that M_h is compatible with X_h, and we denote by β_{NK} the corresponding inf-sup constant (cf. Bernardi–Maday [4] for a review on such approximations). The counterpart of lemma 2.1 for spectral approximations is the following.

LEMMA 3.2. *(cf. Canuto–Quarteroni [7] lemma 2.1, p. 73) There is a constant $c > 0$, independent of N, so that*

$$\forall v_n \in P_N([-1, +1]), \qquad |\nabla v_n|_0 \leq cN^2 |v_n|_0. \tag{3.8}$$

We now give an upper bound on the condition number of the pressure operator.

THEOREM 3.2. *There is $c > 0$ so that*

$$\kappa(B_h B_h^t) \leq \frac{cN^4}{\beta_{NK}^2}. \tag{3.9}$$

Proof. Proceed as for the proof of theorem 3.1 □

In general the exponent 4 is optimal for spectral Poisson problems supplemented with either Dirichlet or Neumann boundary conditions. It may be further reduced to 3 through the action of a mass matrix if the base functions are conveniently weighted (cf. Bernardi–Maday [4, Lemma 5.5, ch. III]). Note that the present functional framework is intrinsic (*ie.* no particular base is chosen); that is, we deal with operators instead of matrices. In this context, the intrinsic counterpart of the mass matrix is the identity operator. This reason explains why we obtained the exponent 4 instead of 3.

The bound (3.9) would be optimal if β_{NK} were bounded from below by a constant when N and K tend to infinity. Unfortunately, in general β_{NK} tends to zero. For instance, for $K = N - 2$ (*ie.* the $(\mathbb{P}_N^0, \mathrm{P}_{N-2})$ approximation) we have (see Bernardi-Maday [4, p. 147])

$$\frac{c}{\sqrt{N}} \leq \beta_{N,N-2} \leq \frac{c'}{\sqrt{N}}. \tag{3.10}$$

As a result, $\kappa(B_h B_h^t) \leq cN^5$ which is no longer optimal. This default to the optimality is even worse if the pressure is taken in $\mathrm{P}_N(\Omega)$, for in this case (once the spurious modes are discarded) the inf-sup constant behaves like $1/N$.

In conclusion, enforcing $v_{|\partial\Omega} = 0$ on the test functions for the projection step may not be optimal in the framework of spectral approximations. This default to the optimality comes from the inf-sup constant that vanish as the number of modes increases. It is shown in the next section that if the boundary condition on the velocity test functions is relaxed (*ie.* $v \cdot n_{|\partial\Omega} = 0$), then the corresponding inf-sup constant can be uniformly bounded from below by a strictly positive constant.

4. The normal trace of the end-of-step velocity enforced.

4.1. The functional framework. In order to take into account the boundary condition $v \cdot n_{|\partial\Omega} = 0$, we introduce $\mathrm{H}_0^{\mathrm{div}}(\Omega) = \{v \in \mathrm{L}^2(\Omega)^d,\ \mathrm{div} v \in \mathrm{L}^2(\Omega),\ v \cdot n_{|\partial\Omega} = 0\}$. Equipped with the norm $(|v|_0^2 + |\mathrm{div} v|_0^2)^{1/2}$, $\mathrm{H}_0^{\mathrm{div}}(\Omega)$ is a Hilbert space, and we have $\mathrm{H}_0^1(\Omega)^d \subset \mathrm{H}_0^{\mathrm{div}}(\Omega) \subset \mathrm{L}^2(\Omega)^d \equiv \mathrm{L}^2(\Omega)^{d'} \subset \mathrm{H}_0^{\mathrm{div}}(\Omega)' \subset \mathrm{H}^{-1}(\Omega)^d$, where the embeddings are dense and continuous.

Let Y_h be a finite dimensional, internal and convergent approximation of $\mathrm{H}_0^{\mathrm{div}}(\Omega)$. Though it may seem natural to equip Y_h with the norm of $\mathrm{H}_0^{\mathrm{div}}(\Omega)$, we equip it with the norm of $\mathrm{L}^2(\Omega)^d$ for it will be sufficient for our purposes as shown below. A discrete divergence operator $C_h : Y_h \longrightarrow M_h \equiv M_h'$ and its transpose $C_h^t : M_h \longrightarrow Y_h'$ are defined so that for all (v_h, q_h) in $Y_h \times M_h$ we have $(C_h v_h, q_h) = -(\mathrm{div} v_h, p_h) = (v_h, C_h^t p_h)$. We assume also that $C_h : Y_h \longrightarrow M_h$ is onto in the sense that C_h^t satisfies the following inf-sup condition

$$\exists \beta_h' > 0, \forall q_h \in M_h \qquad |C_h^t q_h|_0 \geq \beta_h' |q_h|_0. \tag{4.1}$$

Now, in order to build some discrete counterpart to H, define $H_h = \ker(C_h)$ and equip H_h with the L^2 norm. Introduce also the norm $|q_h|_{M_h^1} = |C_h^t q_h|_0$, and define M_h^1 so that $M_h^1 = M_h$ in terms of vector space, and equip M_h^1 with the $|\cdot|_{M_h^1}$ norm. We are now in measure of establishing

Corollary 4.1. *We have the decomposition:*

$$Y_h = H_h \oplus C_h^t(M_h^1). \tag{4.2}$$

This decomposition is orthogonal and stable with respect to the L^2 *norm.*

It is also assumed that $X_h \subset Y_h$. We denote by $i_h : X_h \longrightarrow Y_h$ the natural injection of X_h into Y_h and $i_h^t : Y_h' \equiv Y_h \longrightarrow X_h'$ its transpose. Note that i_h^t can be identified to the L^2 projection of Y_h onto X_h; in particular, we have

$$|i_h^t v_h|_0 \leq |v_h|_0. \tag{4.3}$$

The relationship between B_h and C_h is brought to light by

Proposition 4.1. *C_h is an extension of B_h and $i_h^t C_h^t = B_h^t$.*

Proof. (a) For all (v_h, q_h) in $X_h \times M_h$, we have $(C_h i_h v_h, q_h) = -(\mathrm{div} v_h, q_h) = (B_h v_h, q_h)$ since $X_h \subset Y_h$; that is to say, $C_h i_h v_h = B_h v_h$ for all $v_h \in X_h$.
(b) From (a) we have $C_h i_h = B_h$, from which we easily infer $i_h^t C_h^t = B_h^t$. ∎

4.2. The discrete projection algorithm. In the framework defined above the viscous step reads as follows: find $\tilde{u}_h^k \in X$ so that

$$\frac{\tilde{u}_h^{k+1} - i_h^t u_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} = f_h^{k+1} - B_h^t p_h^k. \tag{4.4}$$

This problem is well posed thanks to the ellipticity of A_h. Note that u_h^k must be projected onto X_h' since it naturally belongs to Y_h.

The discrete projection step reads: find u_h^{k+1} in Y_h and p_h^{k+1} in M_h so that

$$\begin{cases} \dfrac{u_h^{k+1} - i_h \tilde{u}_h^{k+1}}{\delta t} + C_h^t (p_h^{k+1} - p_h^k) = 0 \\ C_h u_h^{k+1} = 0. \end{cases} \tag{4.5}$$

Corollary 4.1 implies that this linear system has a unique solution.

Note that the solvability of the viscous and projection steps (4.4)–(4.5) do not require B_h^t to be into. In other words, the pressure is uniquely defined (in a seemingly stable manner) by the projection step (4.5). At this point it may come to one's mind that requiring B_h to be onto is an un-necessary stringent condition. As a result, one may think of choosing X_h and M_h with no reference to any inf-sup condition so that B_h^t may possibly have spurious modes. However, when it comes to studying the convergence in time of the global scheme, it is shown below (see proof of theorem 6.2) that the discrete pressure of interest is that which is rid of the possible spurious modes of B_h^t, and the global stability constant on the pressure is not that of $C_h^t : M_h \longrightarrow Y_h$ but that of $B_h^t : M_h \longrightarrow X_h'$.

In order to illustrate the definitions above, let us give an example in the framework of spectral approximations (this example is borrowed from Azaiez–Bernardi–Grundmann [1]). Define $\Omega =]-1,+1[^2$, and let $X_h = \mathbb{P}_N(\Omega) \cap \mathrm{H}_0^1(\Omega)^d$ be the space of the polynomial velocities on Ω vanishing on $\partial\Omega$ and the partial degree of which are less than or equal to N. Likewise, define $Y_h = \mathbb{P}_N(\Omega) \cap \mathrm{H}_0^{\mathrm{div}}(\Omega)$ the space of the velocities which are polynomials with partial degree less than or equal to N and the normal component of which vanish on $\partial\Omega$. When the pressure space is $\mathrm{P}_N(\Omega)/\mathbb{R}$ and the velocity space is Y_h, it can be shown that the gradient operator has three spurious pressure modes (*cf.* [1, Lemma 4.1]): $\mathrm{span}\langle L_N(x), L_N(y), L_N(x)L_N(y)\rangle$, where L_N is the Legendre polynomial of order N. We get rid of these unwelcome modes by defining N_h as the orthogonal of the modes in question in $P_N(\Omega)/\mathbb{R}$. Thus defined, N_h and Y_h are compatible and there is a inf-sup constant, β', independent of N and strictly positive (*cf.* [1, Lemma 4.2]) so that

$$\inf_{q_h \in N_h} \sup_{v_h \in Y_h} \frac{(\mathrm{div} v_h, q_h)}{|v_h|_0 |q_h|_0} \geq \beta'.$$

In this spectral framework, the condition number of the pressure operator $C_h C_h^t$ satisfies the optimal bound

$$\kappa(C_h C_h^t) \leq c N^4. \tag{4.6}$$

Furthermore, it is shown in Bernardi–Maday [4, Proposition 5.2, p. 135] that the discrete gradient $B_h^t : N_h \longrightarrow X_h'$ has still four spurious modes: $S_h = \mathrm{span}\langle L_N'(x)L_N'(y)$, $L_N'(x)yL_N'(y)$, $xL_N'(x)L_N'(y)$, $xL_N'(x)yL_N'(y)\rangle$. This unwelcome modes should be discarded by defining M_h as the orthogonal of S_h in N_h. Hence, not only the pressure space should be rid of the three modes $\mathrm{span}\langle L_N(x), L_N(y), L_N(x)L_N(y)\rangle$ so that C_h^t is into, but it should also be rid of the four spurious modes of S_h. In practice, the regularization of the pressure can be done as a post processing. Furthermore, it can be shown (*cf.* [4, p. 135]) that the resulting stability constant of B_h^t, β_h, is equivalent to $1/N$.

In conclusion, introducing velocity test functions satisfying the boundary condition $v \cdot n_{|\partial\Omega} = 0$ may be justified for some spectral approximations, for it yields a pressure operator with an optimal condition number. Note, however, that this procedure is less economical than the previous one since it involves two approximation spaces for the velocity and it involves two gradient operators (*ie.* two matrices), namely B_h^t and C_h^t. It is not clear whether the present approach should be preferred to the previous one as far as finite element approximations are concerned (see also Quartapelle [19, pp. 191–201] for other details on this technique). Furthermore, although the algorithm (4.4)–(4.5) does not explicitly require B_h^t to be into, the pressure of interest (*ie.* the one on which we have some stability) is that which is rid of the spurious modes of B_h^t (if such modes exist).

5. The projection step as a Poisson problem.

5.1. Motivation. Assume as in the previous sections that the projection step is formulated in the form of a Darcy problem with velocity test functions satisfying either a Dirichlet condition or a boundary condition only for the normal component. In practical implementations we have to choose particular bases of X_h (or Y_h) and M_h. Each of these choices yields a mass matrix $\mathcal{I}_h$ and a matrix $\mathcal{B}_h$ associated with the divergence operator B_h. For a velocity field u_h in X_h (or Y_h) and a pressure field p_h in M_h, we denote by U_h and P_h the vectors of the components of u_h and p_h in the bases in question. In this context, the projection step (3.3) (or (4.5)) yields the following linear system in terms of the pressure unknowns

$$\mathcal{B}_h \mathcal{I}_h^{-1} \mathcal{B}_h^t (P_{k+1} - P_k) = \frac{\mathcal{B}_h \tilde{U}_{k+1}}{\delta t}. \tag{5.1}$$

Though this approach is quite natural, for it is based on the definition of the projection operator (*cf.* theorem 2.1), the presence of the inverse of the mass matrix may hamper its practicability in some circumstances. For instance, for finite element approximations the mass matrix is not diagonal, and the direct solution of the pressure problem may not be feasible, especially when a large number of unknowns is involved. In practice, alternative approaches consist in lumping the mass matrix (*cf.* Gresho and Chan [12, part II] or Quartapelle [19, pp. 191–201]). Though this technique may work, no stability result has yet been proven.

It is the purpose of this section to show that the mass matrix problem may be circumvented if the projection step is recast in the form of a Poisson problem (1.2), as advocated in Temam [24].

5.2. The functional framework. As in the previous sections we denote by X_h the Hilbert space in which the provisional velocities are approximated. Recall that X_h is an internal approximation of $\mathrm{H}_0^1(\Omega)^d$. Thanks to the Poincaré-Wirtinger inequality in $\mathrm{H}^1(\Omega)/\mathbb{R}$, we equip $\mathrm{H}^1(\Omega)/\mathbb{R}$ with the equivalent norm $|\nabla p|_0$. We now introduce M_h^1 a stable and internal approximation of $\mathrm{H}^1(\Omega)/\mathbb{R}$ and we equip it with the norm of $\mathrm{H}^1(\Omega)/\mathbb{R}$ defined above. To make a clear difference between the H^1 norm in M_h^1, the L^2 norm, and the dual norm, we introduce M_h and $M_h^{1\prime}$. These spaces are identical to M_h^1 in terms of vector space and they are respectively equipped with the L^2 norm and the dual norm induced by the L^2 scalar product.

Note that we now impose $M_h \subset \mathrm{H}^1(\Omega)/\mathbb{R}$, whereas in the previous sections we only needed $M_h \subset \mathrm{L}^2(\Omega)/\mathbb{R}$. In the following, the definitions of $A_h : X_h \longrightarrow X_h'$, and $B_h : X_h \longrightarrow M_h$ remain unchanged.

In order to build a discrete Poisson problem for the pressure, we introduce the continuous bilinear form $d_h : M_h^1 \times M_h^1 \longrightarrow \mathbb{R}$ so that

$$\forall (p_h, q_h) \in M_h^1 \times M_h^1, \qquad d_h(p_h, q) = (\nabla p_h, \nabla q_h). \tag{5.2}$$

This bilinear form is obviously M_h^1-elliptic. We associate with d_h the linear continuous operator $D_h : M_h^1 \longrightarrow M_h^{1\prime}$ so that $d_h(p_h, q_h) = (D_h p_h, q_h)$ for all (p_h, q_h) in $M_h^1 \times M_h^1$. The ellipticity of d_h implies that $D_h : M_h^1 \longrightarrow M_h^{1\prime}$ is one to one.

We introduce also the vector space $Y_h = X_h + \nabla M_h^1$ and we equip it with the norm of $\mathrm{L}^2(\Omega)^d$. It is clear that $X_h \subset Y_h$. We respectively denote by $i_h : X_h \longrightarrow Y_h$ and by $i_h^t : Y_h' \longrightarrow X_h'$ the natural injection of X_h into Y_h and its transpose. Note that we have the following stability inequality

$$\forall v_h \in Y_h, \qquad |i_h^t v_h|_0 \leq |v_h|_0. \tag{5.3}$$

5.3. The discrete projection algorithm. In a way much similar to that in the other sections, the viscous step reduces to finding $\tilde{u}_h^{k+1}$ in X_h so that

$$\frac{\tilde{u}_h^{k+1} - i_h^t u_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} = f_h^{k+1} - B_h^t p_h^k. \tag{5.4}$$

The ellipticity of A_h guarantees that this problem is well posed. Note that for reasons explained further, u_h^k must be projected onto X_h' for u_h^k naturally belongs to $Y_h = X_h + \nabla M_h^1$ (see (5.6)).

We now solve the projection step (2.8) as a Poisson problem supplemented with a homogeneous Neumann condition

$$D_h(p_h^{k+1} - p_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \tag{5.5}$$

Given the ellipticity of D_h, this discrete problem has a unique solution. Note that this problem is very classical, and the CFD and applied mathematics communities have spent a lot of energy devising efficient numerical algorithms permitting to solve it. In some sense this problem is more attractive than its div-grad counterparts (3.3) or (4.5) since its matrix formulation does not involve the inverse of the mass matrix.

The last step of the algorithm consists in correcting the velocity. This is done by setting

$$u_h^{k+1} = \tilde{u}_h^{k+1} - \delta t \nabla (p_h^{k+1} - p_h^k). \tag{5.6}$$

Note that u_h^{k+1} belongs to $X_h + \nabla M_h^1$ which is a subset of $\mathrm{L}^2(\Omega)^d$. Strictly speaking, this velocity is not divergence free since there is no reason for $\mathrm{div}\tilde{u}_h^{k+1}$ to be equal to $\nabla^2(p_h^{k+1} - p_h^k)$. For instance, if P_1 finite elements are used for approximating the pressure, the Laplacian of the pressure increment is a $\mathrm{H}^{-1}(\Omega)$ measure, whereas the divergence of the provisional velocity is in $\mathrm{L}^2(\Omega)$; hence, the divergence of u_h^{k+1} is a $\mathrm{H}^{-1}(\Omega)$ measure. However, we have

PROPOSITION 5.1. *u_h^{k+1} is weakly divergence free in the sense that*

$$\forall q_h \in M_h, \qquad (u_h^{k+1}, \nabla q_h) = 0, \tag{5.7}$$

and its divergence and normal trace converge to zero (in some weak sense) as $h \to 0$ if $\tilde{u}_h^{k+1}$ converges in $\mathrm{H}_0^1(\Omega)^d$ when $h \to 0$.

Proof. (a) By definition, (5.5) is equivalent to

$$\forall q_h \in M_h, \qquad \delta t(\nabla(p_h^{k+1} - p_h^k), \nabla q_h) + (\mathrm{div}\tilde{u}_h^{k+1}, q_h) = 0.$$

Furthermore, by taking the inner product of (5.6) by ∇q_h we obtain

$$\forall q \in M_h, \qquad (u_h^{k+1}, \nabla q_h) = -(\mathrm{div}\tilde{u}_h^{k+1}, q_h) - \delta t(\nabla(p_h^{k+1} - p_h^k), \nabla q_h),$$

from which (5.7) is easily deduced.

(b) Assume for sake of simplicity that Ω is a polygon and $\tilde{u}_h^{k+1}$ converges in $\mathrm{H}_0^1(\Omega)^d$ to some function $\tilde{u}$, then the series of functions $(p_h^{k+1} - p_h^k)$ converges in $\mathrm{H}^1(\Omega)/\mathbb{R}$ to the solution of the Poisson problem $\nabla^2\phi = \mathrm{div}\tilde{u}/\delta t$ and $\partial\phi/\partial n_{|\partial\Omega} = 0$. As a result, u_h^{k+1} converges to $u = \tilde{u} - \delta t\nabla\phi$ in $\mathrm{L}^2(\Omega)^d$; the limit function u is divergence free and its normal trace is zero. ◻

Indeed, the scheme (5.4)–(5.5) can be put formally in a form quite similar to that of the two other techniques described in the sections above. A discrete divergence operator $C_h : Y_h \longrightarrow M_h$ can be defined by

$$\forall v_h \in Y_h,\ \forall q_h \in M_h, \qquad (C_h v_h, q_h) = (v_h, \nabla q_h). \tag{5.8}$$

By setting $H_h = \ker(C_h)$, we clearly have

$$Y_h = H_h \oplus C_h^t(M_h^1). \tag{5.9}$$

Furthermore, the relation between C_h and B_h is brought to light by

PROPOSITION 5.2. *C_h is an extension of B_h and $i_h^t C_h^t = B_h^t$.*

The particular choice we have made on Y_h implies that

PROPOSITION 5.3. *C_h^t is the restriction of ∇ to M_h.*

Proof. For all (v_h, q_h) in $Y_h \times M_h$, we have $(C_h^t q_h, v_h) = (\nabla q_h, v_h)$; that is to say $(C_h^t q_h - \nabla q_h, v_h) = 0$. But ∇q_h is in Y_h by definition and $C_h^t q$ is in Y_h' ($= Y_h$ in terms of vector space), hence $C_h^t q_h = \nabla q_h$. ◻

COROLLARY 5.1. *The projection step (5.4)–(5.5) is equivalent to the problem: look for u_h^{k+1} in Y_h and p_h^{k+1} in M_h so that*

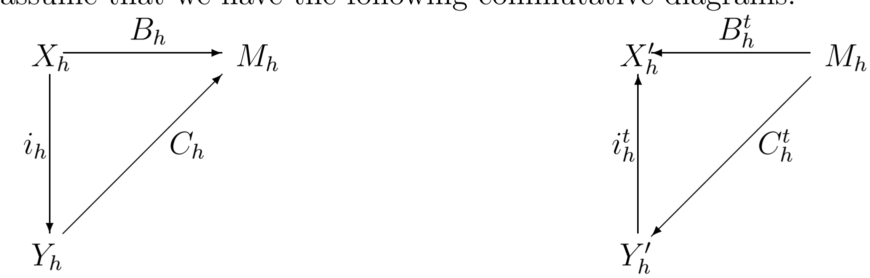
$$\begin{cases} \dfrac{u_h^{k+1} - i_h\tilde{u}_h^{k+1}}{\delta t} + C_h^t(p_h^{k+1} - p_h^k) = 0 \\ C_h u_h^{k+1} = 0. \end{cases} \tag{5.10}$$

Remark that the algorithm described above does not explicitly involve any compatibility condition between X_h and M_h; that is, no inf-sup condition is required to ensure wellposedness of any of the fractional steps (5.4), (5.5), or (5.6). For instance, $(\mathbb{P}_1, \mathrm{P}_1)$ finite elements would be perfectly suited to the algorithm above. Actually a compatibility condition shows up when we are interested in the stability and the convergence of the scheme (see proof of theorem 6.2). After all, we are interested in approximating the velocity and pressure which are solution to (2.11), but uniqueness and stability on the pressure in (2.11) is ensured only if B_h^t has no spurious modes. Hence, if such modes exist, they have eventually to be discarded. If for some reason one is interested in solving the problem by means of continuous finite elements of degree one, a possible choice consists in using ($\mathbb{P}_1$–iso–$\mathbb{P}_2$,P_1) finite elements (*cf.* Bercovier–Pironneau [3]).

6. Generalization and error bounds.

6.1. Generalization. We show in this section that the three projection algorithms described above can be put into a unified framework.

The definitions of the spaces X_h and M_h together that of the operators A_h and B_h are unchanged. We define Y_h a finite dimensional subspace of $\mathrm{L}^2(\Omega)^d$ and endow Y_h with the norm of $\mathrm{L}^2(\Omega)^d$; we assume also that $X_h \subset Y_h$ (in terms of vector space) and we denote by i_h the continuous injection of X_h into Y_h; the transpose of i_h is the L^2 projection of Y_h onto X_h. Note that Y_h is an internal approximation of $\mathrm{L}^2(\Omega)^d$, for X_h is an approximation of $\mathrm{H}^1_0(\Omega)^d$ and $\mathrm{H}^1_0(\Omega)^d$ is dense in $\mathrm{L}^2(\Omega)^d$. In addition, we assume that there is an operator $C_h : Y_h \longrightarrow M_h$ so that C_h is an extension of B_h; in other words we assume that we have the following commutative diagrams:



Since B_h is onto and C_h is an extension of B_h, C_h is necessarily onto. One consequence of this (together with (ii) of lemma 2.1) is that $|C^t_h q|_0$ is a norm; this norm is hereafter denoted by $|q|_{M^1_h} = |C^t_h q|_0$ and we denote by M^1_h the vector space M_h equipped with this norm. The null space of C_h is denoted by H_h. The definitions above enable us to build a discrete counterpart of the aforementioned orthogonal decomposition $\mathrm{L}^2(\Omega)^d = \mathrm{H} \oplus \nabla(\mathrm{H}^1(\Omega))$:

$$Y_h = H_h \oplus C^t_h(M^1_h). \tag{6.1}$$

The three algorithms described above can be put into the following unified form. The diffusion step reads: find $\tilde{u}^k_h \in X$ so that

$$\frac{\tilde{u}^{k+1}_h - i^t_h u^k_h}{\delta t} + A_h \tilde{u}^{k+1}_h = f^{k+1}_h - B^t_h p^k_h. \tag{6.2}$$

The projection step consists in looking for u^{k+1}_h in Y_h and p^{k+1}_h in M_h so that

$$\begin{cases} \dfrac{u^{k+1}_h - i_h \tilde{u}^{k+1}_h}{\delta t} + C^t_h(p^{k+1}_h - p^k_h) = 0 \\ C_h u^{k+1}_h = 0. \end{cases} \tag{6.3}$$

The decomposition (6.1) implies that this linear system has a unique solution. This step is a projection step in the sense that u^{k+1}_h is the projection of $\tilde{u}^{k+1}_h$ onto H_h.

We now turn our attention to the convergence issue.

6.2. The convergence issue. We prove in this section that the solution to the projection algorithm composed of the viscous step (6.2) and the projection step (6.3) converges in some sense to the solution of (2.1). However, since a complete proof of convergence is out of the scope of the present paper we only give the main ideas. For complete proofs of convergence in the context of the non-linear Navier–Stokes equations, the reader is referred to Guermond–Quartapelle [16].

For sake of simplicity, we shall compare the solution of the projection scheme (6.2)–(6.3) to that of the following coupled scheme: set $w_h^0 = v_{o,h}$ and for $k \geq 0$ solve

$$\begin{cases} \dfrac{w_h^{k+1} - w_h^k}{\delta t} + A_h \tilde{w}_h^{k+1} + B_h^t q_h^{k+1} = f_h^{k+1}, \\ B_h w_h^{k+1} = 0. \end{cases} \tag{6.4}$$

We shall assume that for some k_0 large enough, the solution to this algorithm satisfies the following local in time convergence result:

$$\max_{k_0 \leq k \leq K} \left[|u(t^k) - w_h^k|_1 + |p(t^k) - q_h^k|_0 \right] \leq c(\delta t + h), \tag{6.5}$$

where c is a generic constant that does not depend on h but possibly depends on the regularity of the data of (2.1). We will restrict our analysis to $t^k \geq t^{k_0}$ in order to avoid a possible blow up of the error estimates at $t = 0$ due to a possible lack of smoothness of the solution to (2.1) (and/or incompatibility of the data at $t = 0$ [20]). We assume also that the extended gradient operator, C_h^t, is somewhat stable in $\mathrm{H}^1(\Omega)$ (a precise meaning of this stability is given in Guermond–Quartapelle [16]) so that we have

$$\max_{k_0 \leq k \leq K} |C_h^t (q_h^{k+1} - q_h^k)|_0 \leq c\delta t, \tag{6.6}$$

Of course (6.5) and (6.6) can be proved under reasonable hypotheses on X_h and M_h and on the data of (2.1); the reader is referred to Guermond–Quartapelle [16] for further details.

In order to initialize our fractional step technique, we assume that we have carried out k_0 steps of (6.4); that is to say, we set $u_h^{k_0} = w_h^{k_0}$ and $p_h^{k_0} = q_h^{k_0}$ and the projection algorithm (6.2)–(6.3) is implemented for $k \geq k_0$.

Let us denote by $e_h^k = w_h^k - u_h^k$, $\tilde{e}_h^k = w_h^k - \tilde{u}_h^k$ and $\epsilon_h^k = q_h^k - p_h^k$ the error functions. For conciseness we introduce the notation $\delta_t z^{k+1} = z^{k+1} - z^k$ for any function z. The ability of the solution to (6.2)–(6.3) to approximate that of (2.1) for $k_0 \leq k \leq K$ is stated in

THEOREM 6.1. *If δt is small enough, the solution to the projection scheme (6.2)–(6.3) satisfies:*

$$\max_{k_0 \leq k \leq K} |u(t^k) - u_h^k|_0 + \left[\delta t \sum_{k=k_0}^{K} |u(t^k) - \tilde{u}_h^k|_1^2 \right]^{1/2} \leq c(\delta t + h). \tag{6.7}$$

Proof. (a) By subtracting (6.2) from the first equation of (6.4), we derive the equation which controls the error $\tilde{e}_h^{k+1}$

$$\frac{\tilde{e}_h^{k+1} - i_h^t e_h^k}{\delta t} + A_h \tilde{e}_h^{k+1} + B_h^t \psi_h^k = 0 \tag{6.8}$$

where we have set

$$\psi_h^k = q_h^{k+1} - p_h^k = \delta_t q_h^{k+1} + \epsilon_h^k.$$

Note that we have used $i_h^t w_h^k = w_h^k$ since w_h^k is in X_h.

Furthermore, noticing that w_h^{k+1} is in X_h, $B_h w^{k+1} = 0$, and C_h is an extension of B_h, we obtain the system of equations which controls e_h^k and ϵ_h^k

$$\begin{cases} \dfrac{e_h^{k+1} - i_h \tilde{e}_h^{k+1}}{\delta t} + C_h^t(\epsilon_h^{k+1} - \psi_h^k) = 0, \\ C_h e_h^{k+1} = 0. \end{cases} \tag{6.9}$$

(b) In order to obtain a bound on $\tilde{e}_h^{k+1}$, we take the inner product of (6.8) by $2\delta t\, \tilde{e}_h^{k+1}$. Using the ellipticity of A_h (the ellipticity constant is denoted by α) together with the classical relation $2(a, a-b) = |a|^2 + |a-b|^2 - |b|^2$ we obtain:

$$|\tilde{e}_h^{k+1}|_0^2 + |\tilde{e}_h^{k+1} - i_h^t e_h^k|_0^2 + 2\alpha\delta t |\tilde{e}_{k+1}|_1^2 + 2\delta t(\tilde{e}_h^{k+1},\, B_h^t \psi_h^k) = |i_h^t e_h^k|_0^2.$$

The stability of i_h^t yields

$$|\tilde{e}_h^{k+1}|_0^2 + 2\alpha\delta t |\tilde{e}_{k+1}|_1^2 + 2\delta t(\tilde{e}_h^{k+1},\, B_h^t \psi_h^k) \le |e_h^k|_0^2. \tag{6.10}$$

(c) In order to obtain some control on $B_h^t \epsilon_h^{k+1}$, we take the inner product of the first equation of (6.9) by $2\delta t^2 C_h^t \psi_h^k$, and using the fact that C_h is an extension of B_h we obtain

$$-2\delta t(\tilde{e}_h^{k+1}, B_h^t \psi_h^k) + \delta t^2 |C_h^t \epsilon_h^{k+1}|_0^2 - |e_h^{k+1} - i_h \tilde{e}_h^{k+1}|_0^2 = \delta t^2 |C_h^t \psi_h^k|_0^2.$$

With the help of (6.6), the right hand side of this equation is bounded from above as follows.

$$\begin{aligned} \delta t^2 |C_h^t \psi_h^k|_0^2 &= \delta t^2 |\delta_t q_h^{k+1} + \epsilon_h^k|_0^2, \\ &\le \delta t^2 (1+\delta t) |C_h^t \epsilon_h^k|_0^2 + \delta t(1+\delta t)|C_h^t \delta_t q_h^{k+1}|_0^2, \\ &\le \delta t^2 (1+\delta t) |C_h^t \epsilon_h^k|_0^2 + c\delta t^3, \end{aligned}$$

that is to say

$$\begin{aligned} -2\delta t(\tilde{e}_h^{k+1}, B_h^t \psi_h^k) + \delta t^2 |C_h^t \epsilon_h^{k+1}|_0^2 &- |e_h^{k+1} - i_h \tilde{e}_h^{k+1}|_0^2 \\ &\le \delta t^2 (1+\delta t) |C_h^t \epsilon_h^k|_0^2 + c\delta t^3. \end{aligned} \tag{6.11}$$

(d) We obtain some control on e_h^{k+1} by taking the inner product of (6.9) by $2\delta t e_h^{k+1}$

$$|e_h^{k+1}|_0^2 + |e_h^{k+1} - i_h\tilde{e}_h^{k+1}|_0^2 - |\tilde{e}_h^{k+1}|_0^2 = 0. \tag{6.12}$$

(e) After summing up (6.10) + (6.11) + (6.12) we obtain

$$|e_h^{k+1}|_0^2 + \delta t^2|C_h^t\epsilon_h^{k+1}|_0^2 + 2\alpha\delta t|\tilde{e}_{k+1}|_1^2 \le |e_h^k|_0^2 + (1+\delta t)\delta t^2|C_h^t\epsilon_h^k|_0^2 + c\delta t^3.$$

By taking the sum from $k = k_0$ to some integer $n \le K$ we obtain

$$\begin{aligned} |e_h^{n+1}|_0^2 + \delta t^2|C_h^t\epsilon_h^{n+1}|_0^2 + 2\alpha\delta t \textstyle\sum_{k=k_0}^n |\tilde{e}_h^{k+1}|_1^2 \quad \le \quad & |e_h^{k_0}|_0^2 + \delta t^2|C_h^t\epsilon_h^{k_0}|_0^2 + c\delta t^2 \\ & +\delta t \sum_{k=k_0}^n \left[\delta t^2 |C_h^t\epsilon_h^k|_0^2\right] \end{aligned}$$

By our particular choice of the initial conditions we have

$$|e_h^{k_0}|_0 = 0, \text{ and } \qquad |C_h^t\epsilon_h^{k_0}|_0 = 0.$$

As a result, the discrete Gronwall lemma yields

$$|e_h^{n+1}|_0^2 + \delta t \sum_{k=k_0}^n |\tilde{e}_h^{k+1}|_1^2 \le c\delta t^2. \tag{6.13}$$

The final result is a consequence of

$$u(t^k) - u_h^k = e_h^k + u(t^k) - w_h^k, \qquad u(t^k) - \tilde{u}_h^k = \tilde{e}_h^k + u(t^k) - w_h^k,$$

together with the convergence hypothesis (6.5). □

The ability of $\delta_t u_h^{k+1}/\delta t$ to approximate $\delta_t w_h^{k+1}/dt$ is explicited by

Proposition 6.1. *If δt is small enough, the solution to the projection scheme (6.2)–(6.3) satisfies:*

$$\max_{k_0\le k\le K} |\delta_t e_h^k|_0 + \left[\delta t \sum_{k=k_0}^K |\delta_t\tilde{e}_h^k|_1^2\right]^{1/2} \le c\delta t^2. \tag{6.14}$$

Proof. (a) In a first step we control $\delta_t\tilde{e}_h^{k_0+1}$, $\delta_t e_h^{k_0+1}$, and $\delta_t\epsilon^{k_0+1}$. We note first that

$$\begin{aligned} |\tilde{e}_h^{k_0+1}|_0^2 \quad & \le \quad |i_h^t e_h^{k_0}|_0^2 - 2\delta t(\tilde{e}_h^{k_0+1}, B_h^t\psi_h^{k_0}) \\ & \le \quad -2\delta t(\tilde{e}_h^{k_0+1}, B_h^t[\delta_t q_h^{k_0+1} + \epsilon_h^{k_0}]) \\ & \le \quad \frac{1}{2}|\tilde{e}_h^{k_0+1}|_0^2 + c\delta t^4. \end{aligned}$$

We obtain the bound

$$|\tilde{e}_h^{k_0+1}|_0 \le c\delta t^2,$$

which yields

$$|\delta_t \tilde{e}_h^{k_0+1}|_0 \le c\delta t^2. \tag{6.15}$$

Furthermore, from the projection step (6.9) we obtain

$$\begin{cases} |e_h^{k_0+1}|_0 & \le & |\tilde{e}_h^{k_0+1}|_0, \\ |C_h^t(\epsilon_h^{k_0+1} - \psi_h^{k_0})|_0 & \le & |\tilde{e}_h^{k_0+1}|_0/\delta t. \end{cases}$$

The first bound easily yields

$$|\delta_t e_h^{k_0+1}|_0 \le c\delta t^2. \tag{6.16}$$

The other bound yields

$$\begin{aligned} |C_h^t(\epsilon_h^{k_0+1} - \epsilon_h^{k_0})|_0 & \le |\tilde{e}_h^{k_0+1}|_0/\delta t + |C_h^t \delta_t q_h^{k_0+1}|_0 \\ & \le c\delta t. \end{aligned}$$

In other words, we have

$$|C_h^t \delta_t \epsilon_h^{k_0+1}|_0 \le c\delta t. \tag{6.17}$$

(b) Now we proceed as in the proof of theorem 6.1. For $k \ge k_0 + 1$, the equation which controls the error $\delta_t \tilde{e}_h^{k+1}$ is

$$\frac{\delta_t \tilde{e}_h^{k+1} - i_h^t \delta_t e_h^k}{\delta t} + A_h \delta_t \tilde{e}_h^{k+1} + B_h^t \delta_t \psi_h^k = 0, \tag{6.18}$$

and the system of equations which controls $\delta_t e_h^k$ and $\delta_t \epsilon_h^k$ is found to be

$$\begin{cases} \dfrac{\delta_t e_h^{k+1} - i_h \delta_t \tilde{e}_h^{k+1}}{\delta t} + C_h^t[\delta_t \epsilon_h^{k+1} - \delta_t \psi_h^k] = 0, \\ C_h \delta_t e_h^{k+1} = 0. \end{cases} \tag{6.19}$$

(c) By reasoning as in the proof of theorem 6.1, we see that the final bound is a consequence of (a) and (b). □

We are now in measure of establishing a convergence result on the pressure.

Theorem 6.2. *The approximate pressure given by the projection scheme (6.2)–(6.3) satisfies:*

$$\left[\delta t \sum_{k=k_0}^{K} |p(t^k) - p_h^k|_0^2 \right]^{1/2} \le c(\delta t + h). \tag{6.20}$$

Proof. By summing (6.8) and i_h^t(6.9), and using the relation $i_h^t C_h^t = B_h^t$, we obtain

$$B_h^t \epsilon_h^{k+1} = -\frac{i_h^t \delta_t e_h^{k+1}}{\delta t} - A_h \tilde{e}_h^{k+1}. \tag{6.21}$$

The inf-sup condition on $B_h^t : M_h \longrightarrow X_h'$ implies that

$$c_1 |\epsilon_h^k|_0 \leq \frac{|\delta_t e_h^{k+1}|_0}{\delta t} + c_2 |\tilde{e}_h^{k+1}|_1.$$

The final bound is a consequence of this inequality and (6.13), (6.14) together with the convergence hypothesis (6.5) and the identity

$$p(t^k) - p_h^k = \epsilon_h^k + p(t^k) - q_h^k.$$

□

Note that in order to obtain an error estimate on the pressure we have reconstructed an approximation of the momentum equation (6.21), by combining the viscous step (6.8) and the projection step (6.9). The discrete momentum equation (6.21) clearly shows that the discrete gradient operator which comes into play is not $C_h^t : M_h \longrightarrow Y_h$ but $B_h^t : M_h \longrightarrow X_h'$. Hence, although the fractional steps (6.2)–(6.3) do not seem to require B_h^t to be onto, convergence in time is ensured only if X_h and M_h satisfy a inf-sup condition.

7. Relationship with some Uzawa preconditioning. We show in this section how the three projection algorithms presented above can be interpreted as particular preconditioning techniques.

Our starting point is still the time-dependent Stokes problem (2.11). Assume we wish to approximate it by means of the one step backward Euler scheme (6.4). The formulation (6.4) is classical; it couples the pressure and the velocity by means of the kinematical constraint $B_h w_h^{k+1} = 0$. One way of looking at this problem consists in eliminating the velocity by means of a Gauss elimination, for the first pivot in the system (6.4), $I_d/\delta t + A_h$, is invertible. As a result the problem (6.4) consists in finding the pressure field q_h^{k+1} so that

$$B_h (I_d - \delta t A_h)^{-1} B_h^t (q_h^{k+1}) = \frac{1}{\delta t} B_h (I_d - \delta t A_h)^{-1} (\delta t f_h^{k+1} + w_h^k). \tag{7.1}$$

In the literature, the operator $B_h(I_d - \delta t A_h)^{-1} B_h^t$ is frequently referred to as the Uzawa operator; it is hereafter denoted by $U_{\delta t}$. Note that this operator has a condition number of $\mathcal{O}(1)$ if δt is of $\mathcal{O}(1)$. However, δt is bound to tend to zero. As a result, the condition number of $U_{\delta t}$ is very large in practice, and iterative solutions of (7.1) are possible only if $U_{\delta t}$ is preconditioned. If a preconditioner were at hand, one crude iterative technique would consist of Picard iterations. In the following we show how such an algorithm can be implemented.

Assume that q_h^k is the first guess of the pressure, then the pressure increment, $q_h^{k+1} - q_h^k$, is solution to

$$U_{\delta t}(q_h^{k+1} - q_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}, \tag{7.2}$$

where we have set $\tilde{u}_h^{k+1} = B_h(I_d - \delta t A_h)^{-1}(\delta t f_h^{k+1} + w_h^k - B_h^t q_h^k)$; in other words, $\tilde{u}_h^{k+1} \in X$ is the solution to

$$\frac{\tilde{u}_h^{k+1} - w_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} + B_h^t q_h^k = f_h^{k+1}. \tag{7.3}$$

Note that this problem coincides exactly with the provisional step of the three projection algorithms studied in the previous sections.

The difficult task, now, consists in solving (7.2) approximately. One possibility consists in assuming that if δt is small, $U_{\delta t}$ should not be very different from $U_0 = B_h B_h^t$. As a result, if δt is small enough, it is legitimate to solve approximately the pressure problem in the following form (see *eg.* [13])

$$B_h B_h^t (p_h^{k+1} - q_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \tag{7.4}$$

Note that (7.4) is equivalent to the projection step of the first algorithm. Likewise, by defining C_h as the discrete divergence operator introduced in the second projection algorithm (4.5), an other preconditioning technique consists in solving

$$C_h C_h^t (p_h^{k+1} - q_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \tag{7.5}$$

This approximate solution corresponds to the projection step of the second algorithm. A third alternative that is equivalent to the projection step of the third algorithm consists in solving

$$D_h (p_h^{k+1} - q_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \tag{7.6}$$

In conclusion, one may interpret the projection step of the three projection algorithms presented above as a one step preconditioned Picard iteration on $U_{\delta t}$, the three preconditioners being either $B_h B_h^t$, $C_h C_h^t$, or D_h. Note also that if instead of q_h^k, the first guess of the pressure is zero, the preconditioning techniques in question reduce to the non incremental version of the three projection algorithms (see Cahouet–Chabard [6], Guermond [14], or Lababie–Lasbleiz [17] for other details on these preconditioners). Incidentally, it is shown in [6] and confirmed in [14] that $B_h B_h^t$ is really a preconditioner of $U_{\delta t}$ in purely algebraic terms only if δt is very small; for finite elements this condition reads $\delta t \leq ch^3$. The fact that such a condition did not show up in the analysis of the stability of projection algorithms may possibly mean that $B_h B_h^t$ acts as a preconditioner of $U_{\delta t}$ if the spatial spectral content of the right hand side is dominated by low frequencies.

The next and final step consists is correcting the provisional velocity $\tilde{u}_h^{k+1}$. In a pure fixed point strategy, the end-of-step velocity should be set to $(I_d - \delta t A_h)^{-1}(\delta t f_h^{k+1} + w_h^k - B_h^t p_h^{k+1})$, and depending on the satisfaction of some convergence criterion, new iterations could be performed as advocated in [2]. It is at this very point that classical iterative techniques differ from the projection techniques. Actually, if the velocity is corrected as suggested above, the stability on the discrete divergence of this velocity cannot be ensured in only one step. As a consequence, in practice this technique requires more than one iteration (see Cahouet–Chabard [6] or Lababie–Lasbleiz [17] for other details on this technique). In contrast, the distinctive feature of projection algorithms is that the provisional velocity $\tilde{u}_h^{k+1}$ is corrected in a way so that the new velocity is discretely divergence free (see second equation in (3.3) and (4.5), and (5.7) in proposition 5.1), the consequence being that the scheme is stable and converges (*ie.* reachs the consistency level) in only one iteration per time step.

Finally, note that when it comes to implementing the three projection algorithms described above, it is not necessary to calculate explicitly the end-of-step velocities $(u_h^k)_{k=0,\ldots,K}$ since they can be eliminated. Indeed, by replacing the velocity u_h^k in the prediction steps (3.2), (4.4), or (5.4) by its value calculated at the previous projection step (3.3), (4.5), or (5.6) one obtains

$$\frac{\tilde{u}_h^{k+1} - \tilde{u}_h^k}{\delta t} + A_h \tilde{u}_h^{k+1} = f_h^{k+1} - B_h^t(2p_h^k - p_h^{k-1}). \tag{7.7}$$

$$C_h C_h^t (p_h^{k+1} - p_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}, \tag{7.8}$$

where C_h denotes any extension of B_h that has already been defined.

Proof. This is a consequence of the fact that $i_h^t \tilde{u}_h^k = \tilde{u}_h^k$ and $i_h^t C_h^t = B_h^t$. □

This remark means that in practice the possibly weird space Y_h is never used.

Once again, this algorithm could be interpreted as a one step preconditioned Picard algorithm if the initial guess q_h^k in (7.8) was replaced by $2p_h^k - p_h^{k-1}$, but in this case the algorithm would not be stable. Global stability is ensured by taking $q_h^k = 2p_h^k - p_h^{k-1}$ in the first step (7.7) and $q_h^k = p_h^k$ in the second step (7.8).

Acknowledgment. The author is grateful to P. Le Quéré and L. Quartapelle for helpful discussions and remarks that improved the content of this paper.

REFERENCES

[1] M. Azaiez, C. Bernardi and M. Grundmann, Méthodes spectrales pour les équations du milieu poreux, R 93029, Laboratoire d'Analyse Numérique, Paris VI, 1993.

[2] J. B. Bell, P. Colella and H. M. Glaz, A second order projection method for the incompressible Navier–Stokes equations, *J. of Comput. Phys.*, **85**, 1989, 257–283.

[3] M. Bercovier, O. Pironneau, Error estimates for finite element solution of the Stokes problem in the primitive variables, *Numer. Math.*, **33**, 1979, 211–224.

[4] C. Bernardi, Y. Maday, *Approximations spectrales des problèmes aux limites elliptiques*, Mathématiques et Applications, **10**, Springer-Verlag, 1992.

[5] F. Brezzi, On the existence uniqueness and approximation of saddle-point problems arising from Lagrangian multipliers, *R.A.I.R.O.*, **R.2**, 1974, 129–151.

[6] J. Cahouet, J.-P. Chabard, Some fast 3-D finite element solvers for generalized Stokes problem, *Int. J. Num. Meth. in Fluids*, **8**, 1988, 269–295.

[7] C. Canuto, A. Quarteroni, Approximation results for orthogonal polynomials in Sobolev spaces, *Math. Comp.*, **38**, 1982, 67–86.

[8] A. Chorin, Numerical simulation of the Navier–Stokes equations, *Math. Comp.*, **22**, 1968, 745–762.

[9] J. Donea, S. Giuliani, H. Laval, L. Quartapelle, Finite element solution of the unsteady Navier–Stokes equations by a fractional step method, *Comput. Meths. Appl. Mech. Engrg.*, **30**, 1982, 53–73.

[10] V. Girault and P.-A. Raviart, *Finite Element Methods for Navier–Stokes Equations*, Springer Series in Computational Mathematics, **5**, Springer–Verlag, 1986.

[11] K. Goda, A multistep technique with implicit difference schemes for calculating two- or three-dimensional cavity flows, *J. of Comput. Phys.*, **30**, 1979, 76–95.

[12] P. M. Gresho and S. T. Chan, On the theory of semi-implicit projection methods for viscous incompressible flow and its implementation via finite element method that also introduces a nearly consistent mass matrix. Part I and Part II, *Int. J. Numer. Methods Fluids*, **11**, 1990, 587–620.

[13] J.-L. Guermond, Sur l'approximation des équations de Navier–Stokes instationnaires par une méthode de projection, *C. R. Acad. Sc. Paris*, Série I, **319**, 1994, 887–892.

[14] J.-L. Guermond, Perturbations singulières des problèmes de point selle et préconditionnement du problème de Stokes, *Modél. Math. Anal. Numér.*, **28**, 3, 1994, 357–374.

[15] J.-L. Guermond, Remarques sur les méthodes de projection pour l'approximation des équations de Navier–Stokes, *Numer. Math.*, **67**, 1994, 465–473.

[16] J.-L. Guermond and L. Quartapelle, On the approximation of the unsteady Navier–Stokes equations by finite element projection methods, submitted to *Numer. Math.*, 1995, and LIMSI reports 95-06, 95-14.

[17] G. Labadie, P. Lasbleiz, *Quelques méthodes de résolution du problème de Stokes en éléments finis*, E.D.F rapport HE41/83.01, 1983.

[18] J.-L. Lions and E. Magenes, *Problèmes aux limites non homogènes et applications*, Dunod, Paris, 1968.

[19] L. Quartapelle, *Numerical solution of the incompressible Navier–Stokes equations*, ISNM **113**, Birkhäuser, Basel, 1993.

[20] R. Rannacher, On Chorin's projection method for the incompressible Navier–Stokes equations, *Lectures Notes in Mathematics*, **1530**, Springer, Berlin, 1992, 167–183.

[21] J. Shen, On error estimates of projection methods for Navier–Stokes equations: first-order schemes, *SIAM J. Numer. Anal.*, **29**, 1, 1992, 57–77.

[22] R. Temam, *Navier–Stokes Equations*, Studies in Mathematics and its Applications, **2**, North-Holland, 1977.

[23] R. Temam, Une méthode d'approximation de la solution des équations de Navier–Stokes, *Bull. Soc. Math. France*, **98**, 1968, 115–152.

[24] R. Temam, Remark on the pressure boundary condition for the projection method, *Theoret. Comput. Fluid Dynamics*, **3**, 1991, 181–184.

[25] J. Van Kan, A second-order accurate pressure-correction scheme for viscous incompressible flow, *SIAM J. Sci. Stat. Comput.*, **7**, 3, 1986, 870–891.