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ON SENSITIVE VECTOR POISSON AND STOKES PROBLEMS

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Lions/Sanchez-Palencia's theory of sensitive boundary value problems is extended from the scalar biharmonic equation to the vector Poisson equation and the Stokes problem associated with the bilinear form $(\nabla\times u, \nabla\times v) + (\nabla\cdot u, \nabla\cdot v)$. For both problems the specification of completely natural conditions for the vector unknown on a part of the boundary leads to a variational formulation admitting a unique solution which is however sensitive to abitrarily small smooth perturbations of the data, as shown in the present paper.

Keywords: Sensitive boundary-value problem, Vector Poisson equation, Stokes problem.

1. Introduction

In the calculation of velocity fields for external incompressible flow problems it is common to impose nonstandard conditions at the outer boundary of the computational domain. At outflow boundaries these conditions assume typically the form of derivative boundary conditions for the velocity or can involve other unknown variables of the assumed mathematical model, the aim being to reduce as much as possible the perturbation on the solution caused by truncating the infinite domain at a large but finite distance from the internal boundary(ies). For instance, for incompressible viscous flows governed by the Stokes or Navier–Stokes equations in primitive variables, boundary conditions involving the tangential components of the curl of velocity (vorticity) or the value of pressure have been considered (see *eg.* Pironneau[9]).

In order to minimize the perturbations induced by the conditions imposed at the external boundaries, one could also be temped to consider completely natural boundary conditions for the velocity equation and thereby to build a variational formulation of the problem according to standard techniques of functional analysis. The aim of this paper is to investigate such a class of derivative boundary conditions for the vector Poisson problem as well as the Stokes problem and to show that the corresponding variational problems are sensitive in the sense of the theory of Lions/Sanchez-Palencia[7].

This paper is divided into three parts. Section 2 is devoted to some preliminaries which recall elementary results concerning well-posedness as it is classically understood in numerical analysis. The main results of this paper are presented in sections 3 and 4. In section 3 we study the vector Poisson problem associated with the bilinear form $(\nabla\times u, \nabla\times v)+(\nabla\cdot u, \nabla\cdot v)$. We show that, provided a Dirichlet-type boundary condition is enforced on the normal or the tangential component of the unknown, the problem is well-posed in some usual weak sense (*i.e.* in $\mathrm{H}^1(\Omega)$). We show also that, if neither the normal nor the tangential component of the unknown is controlled on a nonzero part of the boundary, the problem is necessarily sensitive in the sense defined by Lions/Sanchez-Palencia[7], *i.e.* no control of the solution in $\mathrm{H}^1(\Omega)$ is possible; actually, a unique solution possibly exists in a space which is not contained in the space of distributions. In section 4 we generalize our conclusions to the Stokes problem. We show that a sufficient and necessary condition for the Stokes problem associated with the bilinear form $(\nabla\times u, \nabla\times v)$ to be well-posed in the classical (variational) sense is that either the normal component or the tangential component of the velocity is controlled on the boundary of the domain. In other words, enforcing the tangential component of the vorticity together with the pressure yields a sensitive solution, though these boundary conditions may seem natural for the weak formulation.

2. Preliminaries

For the sake of completeness, in this section we recall some known results that will be used repeatedly hereafter. Since this paper is concerned with well-posedness of some linear problems, we recall a series of conditions that are sufficient and necessary for a linear continuous Banach operator to be bijective and its inverse to be continuous.

Let X and Y be two real Banach spaces and A be in $\mathcal{L}(X, Y)$. The analysis of the properties of A requires to consider the dual space of X and Y, say X' and Y', together with the transpose of A, say $A^{\mathrm{t}} : Y' \longrightarrow X'$ so that

$$\forall x \in X,\ \forall y' \in Y', \qquad \langle A^{\mathrm{t}} y', x\rangle = \langle y', Ax\rangle.$$

A first step towards the characterization of the null space $\ker(A)$ and the range $\mathrm{im}(A)$ of A is achieved by means of

Lemma 2.1 *For A in $\mathcal{L}(X, Y)$ we have*

(*i*) $\ker(A) = \mathrm{im}(A^{\mathrm{t}})^{\perp}$,
(*ii*) $\overline{\mathrm{im}(A)} = \ker(A^{\mathrm{t}})^{\perp}$.

Here we recall that for any subspace S of the dual Z' of any Banach space Z the notation $S^{\perp}$ represents the set $\{z \in Z;\quad \langle s, z\rangle = 0, \quad \forall s \in S\}$. These results are classical and may be found in Brezis[1], p. 28-30, or Yosida[11], p. 205–209.

Since surjectivity is an essential step towards the characterization of bijective operators, we recall an important corollary of Banach's closed range theorem, the proof of which may be found in *e.g.* Brezis[1], p. 29–30 or Yosida[11], p. 205.

Lemma 2.2 *For all A in $\mathcal{L}(X, Y)$, the following propositions are equivalent*

(*i*) *$A^{\mathrm{t}} : Y' \longrightarrow X'$ is surjective.*
(*ii*) *$A : X \longrightarrow Y$ is injective and the range of A is closed in Y.*
(*iii*) *There is a constant $c > 0$ so that*

$$\forall x \in X, \qquad \|Ax\|_Y \geq c\|x\|_X. \tag{2.1}$$

Remark 2.1 The first consequence of this result is that if $A \in \mathcal{L}(X, Y)$ is bijective, then its inverse is necessarily continuous. In fact, the bijectivity of A implies that A is injective and the range of A is closed in Y. From the lemma above we infer that there is a constant c so that $\|A^{-1}y\|_X \leq c\|A(A^{-1}y)\|_Y$. That is to say, A^{-1} is continuous $\square$.

For subsequent use we recall also the transpose counterpart of lemma 2.2.

Lemma 2.3 *For all A in $\mathcal{L}(X,Y)$, the following propositions are equivalent*

(i) *$A : X \longrightarrow Y$ is surjective.*
(ii) *$A^{\mathrm{t}} : Y' \longrightarrow X'$ is injective and the range of A^{t} is closed in X'.*
(iii) *There is a constant $c > 0$ so that*

$$\forall y' \in Y', \qquad \|A^{\mathrm{t}}y'\|_{X'} \geq c\|y'\|_{Y'}. \tag{2.2}$$

Remark 2.2 When A is associated with some bilinear form $a \in \mathcal{L}(X \times Z, \mathbb{R})$ by the definition $\langle Ax, z\rangle = a(x,z)$, then in the lemmas above Y is the dual of Z (*i.e.* $Y = Z'$). Under this condition the inequality (2.1) can be interpreted as an "inf-sup" condition in the sense of Brezzi[2]:

$$\inf_{x\in X} \sup_{z\in Z} \frac{a(x,z)}{\|x\|_X \, \|z\|_Z} \geq c > 0. \tag{2.3}$$

Furthermore, if Z is reflexive (*i.e.* $Z'' = Z$), the inequality (2.2) can also be understood as the following "inf-sup" condition:

$$\inf_{z\in Z} \sup_{x\in X} \frac{a(x,z)}{\|x\|_X \, \|z\|_Z} \geq c > 0 \; \square. \tag{2.4}$$

We are now a position of giving a first characterization of bijective Banach operators.

Theorem 2.1 *Let X and Y be two real Banach spaces and A an operator in $\mathcal{L}(X,Y)$. The following propositions are equivalent*

(i) *A is bijective.*
(ii) *$A^{\mathrm{t}} : Y' \longrightarrow X'$ is injective and there is a constant $c > 0$ so that*

$$\forall x \in X, \qquad \|Ax\|_Y \geq c\|x\|_X. \tag{2.5}$$

Proof. (i) $\Rightarrow$ (ii) Since A is surjective, $\ker(A^{\mathrm{t}}) = \mathrm{im}(A)^{\perp} = \{0\}$, that is, A^{t} is injective. Since $\mathrm{im}(A) = Y$ is closed and A is injective we deduce that there is $c > 0$ so that $\|Ax\|_Y \geq c\|x\|_X$.

(i) $\Leftarrow$ (ii) The injectivity of A^{t} implies $\overline{\mathrm{im}(A)} = (\ker(A^{\mathrm{t}}))^{\perp} = Y$, that is, the range of A is dense in Y. Let us prove now that the range of A is closed. Let x_n be a sequence in X so that Ax_n is a Cauchy sequence in Y. From the inequality $\|Ax_n\|_Y \geq c\|x_n\|_X$ we infer that x_n is a Cauchy sequence in X; let x be the limit of this sequence. The continuity of A implies that $Ax_n \to Ax$, *i.e.* $\mathrm{im}(A)$ is closed. The range of A being closed and dense, A is necessarily surjective. The injectivity of A is an easy consequence of the inequality (2.5) $\square$.

Remark 2.3 This theorem amounts to saying that a Banach operator is bijective if and only if it is injective, its range is closed, and its transpose is injective $\square$.

Corollary 2.1 *An operator $A \in \mathcal{L}(X, Y)$ is bijective if and only if there are two constants $c > 0$ and $c' > 0$ so that*

$$\forall x \in X, \quad \|Ax\|_Y \geq c\|x\|_X, \qquad \forall y' \in Y', \quad \|A^{\mathrm{t}} y'\|_{X'} \geq c'\|y'\|_{Y'}. \tag{2.6}$$

Proof. Saying that A is injective and $\mathrm{im}(A)$ is closed is equivalent to the first inequality (see lemma 2.2). The surjectivity of A is equivalent to the second inequality (see lemma 2.3) $\square$.

Remark 2.4 When A is associated with some bilinear form a in $\mathcal{L}(X \times Z, \mathbb{R})$, so that $\langle Ax, z\rangle = a(x, z)$, (*i.e.* $Y = Z'$), and if Z is reflexive then the two inequalities above are equivalent to two "inf-sup" conditions $\square$.

Let A be in $\mathcal{L}(X, X')$, we say that A is coercive if there is a constant $c > 0$ so that

$$\forall x \in X, \qquad \langle Ax, x\rangle \geq c\|x\|_X^2. \tag{2.7}$$

Corollary 2.2 *Given X a reflexive Banach space, coerciveness is a sufficient condition for an operator A in $\mathcal{L}(X, X')$ to be bijective.*

Proof. This is a direct consequence of corollary 2.1 $\square$.

Remark 2.5 Lax–Milgram theorem is a direct consequence of this result. Note that coerciveness is sufficient but not necessary for bijectivity. Actually, we recall below that coerciveness is also a necessary condition only in the special case of self-adjoint monotonous operators $\square$.

Let A be in $\mathcal{L}(X, X')$ (X being a Banach space), we say that A is monotonous if

$$\forall x \in X, \qquad \langle Ax, x\rangle \geq 0. \tag{2.8}$$

Assume that X is a reflexive Banach space, we say that $A \in \mathcal{L}(X, X')$ is self-adjoint if $A^{\mathrm{t}} = A$. We have the following characterization of bijective self-adjoint operators.

Corollary 2.3 *Assume that X is a reflexive Banach space and $A \in \mathcal{L}(X, X')$ is self-adjoint, then A is bijective if and only if there is a constant $c > 0$ so that*

$$\forall x \in X, \qquad \|Ax\|_{X'} \geq c\|x\|_X. \tag{2.9}$$

Proof. From theorem 2.1 we deduce that A is bijective if and only if A^{t} is injective and A satisfies the inequality (2.9). Conversely, the inequality (2.9) means that A is injective, that is to say A^{t} is injective, and theorem 2.1 yields the result $\square$.

We can further characterize bijective self-adjoint operators as follows

Corollary 2.4 *Let X be a reflexive Banach space and $A \in \mathcal{L}(X, X')$ be self-adjoint and monotonous; A is bijective if and only if it is coercive.*

Proof. Assume A is bijective; we will prove by contradiction that A is coercive. Assume that there is a sequence x_n in X so that $\|x_n\|_X = 1$ and $\langle Ax_n, x_n\rangle$ converges to zero as n tends to infinity. Since A is bijective, we have

$$c\|x_n\|_X \leq \|Ax_n\|_{X'} = \sup_{y\in X} \frac{\langle Ax_n, y\rangle}{\|y\|_X}.$$

Since A is monotonous we can prove the following inequality

$$\forall x, y \in X, \qquad \langle Ax, y\rangle \leq \langle Ax, x\rangle^{1/2}\langle Ay, y\rangle^{1/2}.$$

As a result we deduce

$$\|x_n\|_X \leq c \sup_{y\in X} \frac{\langle Ax_n, x_n\rangle^{1/2}\langle Ay, y\rangle^{1/2}}{\|y\|_X} \leq \|A\|^{1/2}\langle Ax_n, x_n\rangle^{1/2},$$

hence, $\|x_n\| \to 0$ when $n \to \infty$, which is a contradiction.

Conversely, assume A is coercive; we have

$$\|Ax\|_{X'} = \sup_{y\in X} \frac{\langle Ax, y\rangle}{\|y\|_X} \geq \frac{\langle Ax, x\rangle}{\|x\|_X} \geq c\|x\|_X.$$

From corollary 2.3 we infer that A is bijective $\square$.

Remark 2.6 This result emphasizes that coerciveness is a necessary and sufficient condition for monotonous self-adjoint operators to be bijective $\square$.

We finish this section by recalling a condition which is sufficient to prove that the range of an injective operator is closed.

Lemma 2.4 *(Petree–Tartar) Let X, Y, Z be three Banach spaces. Let $A \in \mathcal{L}(X, Y)$ be an injective operator and $T \in \mathcal{L}(X, Z)$ be a compact operator. Assume that there is a constant $c > 0$ so that $c\|x\|_X \leq \|Ax\|_Y + \|Tx\|_Z$. Then the range of A is closed, or equivalently, there is a constant $c > 0$ so that*

$$\forall x \in X, \qquad c\|x\|_X \leq \|Ax\|_Y. \tag{2.10}$$

Proof. (By contradiction) Assume that there is a sequence x_n in X so that $\|x_n\|_X = 1$ and $\|Ax_n\|_Y$ converges to zero when n tends to infinity. Since T is compact and the sequence x_n is bounded, we deduce that there is a subsequence x_{n_k} so that Tx_{n_k} is a Cauchy sequence in Z. From the inequality

$$c\|x_{n_k} - x_{m_k}\|_X \leq \|Ax_{n_k} - Ax_{m_k}\|_Y + \|Tx_{n_k} - Tx_{m_k}\|_Z,$$

we infer that x_{n_k} is a Cauchy sequence in X. Let x be the limit of this subsequence in X. By continuity we deduce $Ax_{n_k} \to Ax$ and by unicity of the limit we obtain $Ax = 0$, for $Ax_{n_k} \to 0$. The injectivity of A implies that x is zero, which is in contradiction with the assumption $\|x_{n_k}\|_X = 1$ $\square$.

Remark 2.7 As an illustration of this result, note that Poincaré and Poincaré–Wirtinger inequalities are direct consequences of it (use $\|u\|_1 \leq \|\nabla u\|_1 + \|u\|_0$ together with the compactness of the injection of $\mathrm{H}^1(\Omega)$ into $\mathrm{L}^2(\Omega)$) $\square$.

3. The vector Poisson problem

3.1. *Hypotheses and notations*

Let Ω be an open connected bounded domain of $\mathbb{R}^d$ ($d = 2$ or 3) with a smooth boundary Γ. For the sake of simplicity we assume that Γ is of class C^2. Of course, some of our arguments can be generalized to rougher boundaries, but these considerations are out of the scope of the present paper, since our goal is to show that even with very smooth boundaries the solution can be very rough.

In the sequel, the set of real functions infinitely differentiable with compact support in Ω is denoted by $\mathrm{D}(\Omega)$. The set of distributions on Ω is denoted by $\mathrm{D}'(\Omega)$. As usual, $\mathrm{L}^2(\Omega)$ denotes the space of real-valued functions, the squares of which are summable in Ω. We denote the inner product in $\mathrm{L}^2(\Omega)$ by $(\,\cdot\,,\,\cdot\,)$ and its norm by $\|\cdot\|_0$. $\mathrm{H}^m(\Omega)$, $m \geq 0$, is the set of distributions the successive derivatives of which, up to order m, can be identified with square summable functions. The space $\mathrm{H}^m(\Omega)$, equipped with the norm $\|u\|_m^2 = (\sum_{|\alpha|=0}^m \|D^\alpha u\|_0^2)^{1/2}$, expressed in the multi-index notation, is a Hilbert space. Recall that a Hilbert space is also a reflexive Banach space.

In this section we are mainly concerned with the solution of vector Poisson problems in Ω. Recall that in terms of distributions we have the following relation

$$\nabla^2 u = -\nabla\times\nabla\times u + \nabla(\nabla\cdot u). \tag{3.1}$$

This yields to consider the following bilinear form

$$a(u,v) = (\nabla\times u, \nabla\times v) + (\nabla\cdot u, \nabla\cdot v), \tag{3.2}$$

where u and v are smooth enough so that this expression makes sense. In the following we analyze variational problems based on this form and we show that fully natural boundary conditions associated with it yield sensitive solutions.

3.2. *A vector Poisson problem with partially natural BCs*

We assume in this section that Γ, the boundary of Ω, is partitioned into three smooth pieces, Γ_1, Γ_2, and Γ_3, so that

$$\overline{\Gamma} = \bigcup_{k=1}^{3} \overline{\Gamma}_k \qquad \text{and} \qquad \bigcap_{k=1}^{3} \Gamma_k = \emptyset. \tag{3.3}$$

We define

$$X = \{v \in \mathrm{H}^1(\Omega)^d; \quad v_{|\Gamma_1} = 0,\ v\cdot n_{|\Gamma_2} = 0,\ v\times n_{|\Gamma_3} = 0\}. \tag{3.4}$$

X is a Hilbert space when equipped with the scalar product of $\mathrm{H}^1(\Omega)^d$. Note that for v in X, the traces $v_{|\Gamma}$, $v\cdot n_{|\Gamma}$, and $v\times n_{|\Gamma}$ are meaningful in $\mathrm{H}^{1/2}(\Gamma)$. Note also that the bilinear form $a : X\times X \longrightarrow \mathbb{R}$, defined above, is continuous. We associate with a the continuous linear operator $A : X \longrightarrow X'$ so that $a(u,v) = \langle Au, v\rangle$; A is clearly self-adjoint and monotonous. Furthermore, provided $\mathrm{meas}(\Gamma_1) > 0$, we have

Lemma 3.5 *A is bijective.*

Proof. The demonstration of this result is not very well-known when boundary conditions are mixed, so we give a proof (without claiming originality).

(i) First we prove that A is injective. If Ω is not simply connected, *i.e.* Ω is p-connected, we define p cuts $\Sigma_1,\ldots,\Sigma_p$ so that the cuts in question are smooth manifolds of dimension $d-1$, $\Sigma_i \cap \Sigma_j = \emptyset$ if $i \neq j$, and $\dot{\Omega} = \Omega \setminus \bigcup_{i=1}^{p} \Sigma_i$ is simply connected and smooth. Let u be in X and assume that $Au = 0$, then $\nabla \cdot u = 0$ and $\nabla \times u = 0$ in $\dot{\Omega}$. Given the simple connectedness of $\dot{\Omega}$, this means that u is the gradient of a harmonic scalar function ϕ. The boundary condition $u_{|\Gamma_1} = 0$ means that $(\partial\phi/\partial n)_{|\Gamma_1} = 0$ and $\phi_{|\Gamma_1} =$ constant. Given the hypothesis $\mathrm{meas}(\Gamma_1) > 0$ and the extension theorem of harmonic functions (*cf. e.g.* Dautray and Lions[3], Chap. II, p 308) we infer that ϕ is a constant in $\dot{\Omega}$; as a result, ϕ is constant almost everywhere in Ω. That is to say u is zero almost everywhere in Ω; hence, the class representative of u in X (in the sense of the Lebesgue measure) is zero.

(ii) Let u and v be some smooth functions X. Irrespective of any boundary condition assumed by u and v, an integration by parts yields

$$(\nabla u, \nabla v) = a(u,v) + \int_\Gamma \left[(\nabla \times u) \cdot v \times n - (\nabla \cdot u)\, v \cdot n + \frac{\partial u}{\partial n} \cdot v \right].$$

Denote by $I_\Gamma(u,v)$ the boundary integral in the right-hand side. Provided Γ is smooth enough and given the boundary conditions enforced on the functions of X (namely $w_{|\Gamma_1} = 0$, $w \cdot n_{|\Gamma_2} = 0$, $w \times n_{|\Gamma_3} = 0$), the surface integral can be bounded (after some calculus) as follows

$$I_\Gamma(u,v) \leq c \int_\Gamma |u \cdot v|.$$

For other details on the way of obtaining this inequality, the reader is referred to Dautray and Lions[3], Chap. IX, p. 246. Given the Poincaré inequality, the inequality above and the continuous injection of $\mathrm{H}^{1/2}(\Gamma)$ into $\mathrm{L}^2(\Gamma)$ we infer

$$\begin{aligned} c\|u\|_1 &\leq \frac{\|\nabla u\|_0^2}{\|u\|_1} \\ &\leq \sup_{v \in X} \frac{(\nabla u, \nabla v)}{\|v\|_1} \\ &\leq \|Au\|_{X'} + c' \Big(\int_\Gamma |u|^2 \Big)^{1/2}, \end{aligned}$$

that is

$$c\|u\|_1 \leq \|Au\|_{X'} + c' \|\gamma(u)\|_{\mathrm{L}^2(\Gamma)^d}, \tag{3.5}$$

where $\gamma : X \longrightarrow \mathrm{L}^2(\Gamma)^d$ is the trace operator.

(iii) We are now in measure of applying Petree–Tartar's lemma 2.4: $A \in \mathcal{L}(X, X')$ is injective, the trace operator $\gamma : X \longrightarrow \mathrm{L}^2(\Gamma)^d$ is compact, and we have the inequality (3.5). As a result, we have

$$c\|u\|_1 \leq \|Au\|_{X'}.$$

Since A is self-adjoint, we infer from corollary 2.3 that A is bijective $\square$.

Remark 3.1 As a by-product of the lemma above and the fact that A is self-adjoint and monotonous, we deduce the following equivalence of norms (*cf.* corollary 2.4):

$$\forall v \in X, \qquad c_1\|v\|_1 \leq (\|\nabla \times v\|_0^2 + \|\nabla \cdot v\|_0^2)^{1/2} \leq c_2 \|v\|_1 \,\square. \tag{3.6}$$

Remark 3.2 The hypothesis $\mathrm{meas}(\Gamma_1) > 0$ plays a key role in the proof of the injectivity of A. For other details on this matter the reader is referred to Girault and Raviart[5], p. 51–56 $\square$.

For $f \in \mathrm{L}^2(\Omega)^d$ we consider the following vector Poisson problem: find u in X so that

$$\forall v \in X, \qquad a(u,v) = (f,v). \tag{3.7}$$

Clearly, this problem amounts to looking for a solution to the problem $Au = f$ in X' (where we have continuously embedded $\mathrm{L}^2(\Omega)^d$ in X'). As a consequence of lemma 3.5 we deduce

Proposition 3.1 *Problem (3.7) has a unique solution and this solution is stable with respect to the datum:* $\|u\|_1 \leq c\|f\|_0$.

Remark 3.3 Note that if u and v are smooth enough we have the following integration by parts formula

$$(-\nabla^2 u, v) = a(u,v) + \int_\Gamma (\nabla\times u)\cdot v \times n - \int_\Gamma (\nabla\cdot u)\, v\cdot n. \tag{3.8}$$

As a result, we deduce that u is solution to the following problem

$$\left\{\begin{array}{rclrcl} & & -\nabla^2 u = f, & & & \\ u_{|\Gamma_1} & = & 0, & & & \\ u\cdot n_{|\Gamma_2} & = & 0, & (\nabla\times u)\times n_{|\Gamma_2} & = & 0, \\ u\times n_{|\Gamma_3} & = & 0, & \nabla\cdot u_{|\Gamma_3} & = & 0, \end{array}\right. \tag{3.9}$$

if u is smooth enough, say u is in $\mathrm{H}^2(\Omega)^d$ $\square$.

Remark 3.4 Note that the boundary conditions $(\nabla\times u)\times n_{|\Gamma_2} = 0$ and $\nabla\cdot u_{|\Gamma_3} = 0$ are natural; that is, they are naturally enforced by the bilinear form a. These boundary conditions are to be understood in some weak sense the exact meaning of which is out of the scope of the present paper, *cf.* Lions and Magenes[6] $\square$.

3.3. *A sensitive Poisson problem*

We show in this section that if there is a nonzero portion of the boundary where neither the normal trace nor the tangential trace of the unknown is controlled, the Poisson problem associated with the bilinear form a is ill-posed in the usual sense (*i.e.* in $\mathrm{H}^1(\Omega)^d$ or $\mathrm{L}^2(\Omega)^d$).

For sake of simplicity, we assume that Γ is partitioned into two smooth pieces Γ_1 and Γ_2 so that

$$\overline{\Gamma} = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \qquad \text{and} \qquad \Gamma_1 \cap \Gamma_2 = \emptyset. \tag{3.10}$$

Accordingly, we define

$$X = \{v \in \mathrm{H}^1(\Omega)^d; \quad v_{|\Gamma_1} = 0\}. \tag{3.11}$$

X is a Hilbert space when equipped with the scalar product of $\mathrm{H}^1(\Omega)^d$. Note that the bilinear form $a : X \times X \longrightarrow \mathrm{I\!R}$, defined above, is continuous and the operator $A : X \longrightarrow X'$ is self-adjoint and monotonous.

For $f \in \mathrm{L}^2(\Omega)^d$ (or possibly smoother) we consider the vector Poisson problem: find u in X so that

$$\forall v \in X, \qquad a(u,v) = (f,v). \tag{3.12}$$

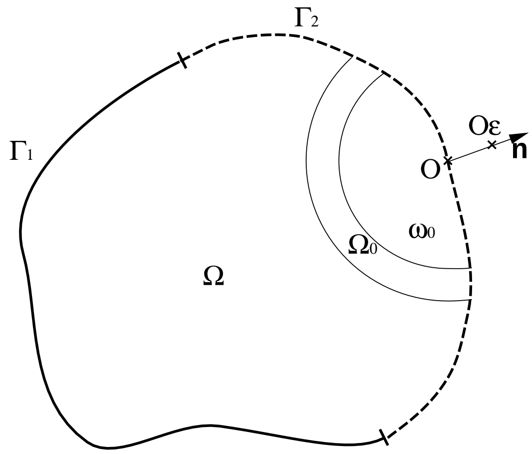

Fig. 1. Schematic representation of domain Ω.

If this problem has a solution and if this solution is smooth enough, say u is in $\mathrm{H}^2(\Omega)^d$, we deduce from the distributions theory that u is solution to the following problem

$$\left\{\begin{array}{rcl} & -\nabla^2 u = f, & \\ u_{|\Gamma_1} & = & 0, \\ (\nabla\times u)\times n_{|\Gamma_2} & = & 0, \qquad \nabla\cdot u_{|\Gamma_2} \ = \ 0. \end{array}\right. \tag{3.13}$$

Note here that only natural boundary conditions are enforced on Γ_2, in contrast with problem (3.7) where a piece of Dirichlet condition was enforced together with the natural condition on Γ_2 and Γ_3.

Problem (3.12) is equivalent to looking for a solution to the problem $Au = f$ in X' (recall that $\mathrm{L}^2(\Omega)^d$ is the pivot space). This problem is well-posed provided A is bijective. Since A is self-adjoint and monotonous, A is bijective if and only if it is coercive (*cf.* corollary 2.4). The striking characteristic of operator A is that:

Proposition 3.2 *The operator A is not coercive.*

Proof. We shall build a counter-example, that is to say, we shall build a sequence of functions w_ϵ of X, the L^2 norm of which is equal to 1 for all values of $\epsilon > 0$, whereas $a(w_\epsilon, w_\epsilon)$ converges to 0 when ϵ tends to zero.

Let O be a point of Γ_2 in the vicinity of which Γ is C^2. Let n be the outward normal to Γ_2 at O. For $\epsilon > 0$ we set $O_\epsilon = O + \epsilon n$. Let ϕ be a function which is harmonic in $\mathbb{R}^d \setminus \{0\}$ (*i.e.* $\mathbb{R}^d$ without zero); recall that such a function is in $C^\infty(\mathbb{R}^d \setminus \{0\})$. Furthermore, we assume that for every cone C with vertex 0, we have

$$\int_{C\cap B(0,1)} \left|\frac{\partial\phi(x)}{\partial r}\right|^2 \,\mathrm{d}x = \infty,$$

where we have set $r = |x|$ and $B(0,1)$ is the ball of center 0 and radius 1. Define $\phi_\epsilon(x) = \phi(x - O_\epsilon)$; it is clear that in $\mathbb{R}^d \setminus \{O_\epsilon\}$ we have $\nabla\cdot\nabla\phi_\epsilon = 0$ and $\nabla\times\nabla\phi_\epsilon = 0$.

Let D_1 be the distance from O to Γ_1, *i.e.* $D_1 = \inf_{x\in\Gamma_1} \mathrm{dist}(O,x)$; we set $D = \min(D_1, 4)$, $\omega_0 = \Omega \cap B(O, D/4)$, and $\Omega_0 = \Omega \cap B(O, D/2)$ (see figure 1). Furthermore, define θ_0 in $C^\infty(\mathbb{R}^d)$ so that $\theta_0 = 1$ in $B(O, D/4)$ and $\theta_0 = 0$ in the complement of $B(O, D/2)$. We denote by $v_\epsilon(x)$ the restriction to Ω of $\theta_0(x)\nabla\phi_\epsilon(x)$. It is clear that v_ϵ belongs to $[C^\infty(\mathbb{R}^d)]^d \cap X$. Furthermore, we have

$$\begin{aligned} a(v_\epsilon, v_\epsilon) &= \int_\Omega |\nabla\times v_\epsilon|^2 + |\nabla\cdot v_\epsilon|^2 \\ &= \int_{\Omega_0\setminus\omega_0} |\nabla\times(\theta_0\nabla\phi_\epsilon)|^2 + |\nabla\cdot(\theta_0\nabla\phi_\epsilon)|^2 \\ &= \int_{\Omega_0\setminus\omega_0} |\nabla\theta_0\times\nabla\phi_\epsilon|^2 + |\nabla\theta_0\cdot\nabla\phi_\epsilon|^2 \\ &\leq c\int_{\Omega_0\setminus\omega_0} |\nabla\phi_\epsilon|^2. \end{aligned}$$

Assume that $|O - O_\epsilon| = \epsilon \leq D/8$, then $D/8 \leq \mathrm{dist}(O_\epsilon, \Omega_0\setminus\omega_0) \leq 5D/8$. As a result

$$a(v_\epsilon, v_\epsilon) \leq c\ \mathrm{meas}(\Omega_0\setminus\omega_0) \max_{x\in B(0,\frac{5D}{8})\setminus B(0,\frac{D}{8})} |\nabla\phi(x)| \leq c'.$$

However, for the L^2 norm of v_ϵ we have

$$\begin{aligned} \|v_\epsilon\|_0^2 &= \int_\Omega |\theta_0\nabla\phi_\epsilon|^2 \\ &\geq \int_{\omega_0} |\nabla\phi_\epsilon|^2 \\ &\geq \int_{\omega_0} \left|\frac{\partial\phi_\epsilon}{\partial r}\right|^2. \end{aligned}$$

Let C_0 be a cone with vertex O so that $C_0\cap B(O, D/4) = C_0\cap\omega_0$; we set $C_\epsilon = C_0 + \epsilon n$ (C_ϵ is a cone with vertex O_ϵ). For all subset S of $\mathbb{R}^d$, we denote by χ_S the characteristic function of S.

$$\|v_\epsilon\|_0^2 \geq \int_{C_\epsilon\cap\omega_0} \left|\frac{\partial\phi_\epsilon}{\partial r}\right|^2 = \int_{C_\epsilon} \chi_{\omega_0}(x)\left|\frac{\partial\phi(x-O_\epsilon)}{\partial|x-O_\epsilon|}\right|^2.$$

By the change of variable $x - O_\epsilon = y - O$, we obtain

$$\|v_\epsilon\|_0^2 \geq \int_{C_0} \chi_{\omega_0}(y+\epsilon n)\left|\frac{\partial\phi(y-O)}{\partial|y-O|}\right|^2.$$

Now, without loss of generality, we assume that D is small enough (we can choose it this way) so that for all y in ω_0, $y+\epsilon n$ crosses Γ only once as ϵ goes from D to 0. As a result, for all $y\in\omega_0$, $\chi_{\omega_0}(y+\epsilon n)$ is a monotonously increasing sequence as ϵ goes from D to 0. This means that $\chi_{\omega_0}(y+\epsilon n)(\partial\phi(y-O)/\partial|y-O|)^2$ is an increasing monotonous sequence of functions of $L^1(C_0)$, converging to $\chi_{\omega_0}(y)(\partial\phi(y-O)/\partial|y-O|)^2$ for all $y\in C_0\setminus\{O\}$. Since, by hypothesis, $C_0\cap\omega_0 = C_0\cap B(O, D/4)$ and $|\partial\phi/\partial r|$ is not in $L^1(C_0\cap B(O,1))$, we infer from Beppo-Levi's monotonous convergence theorem that

$$\int_{C_0} \chi_{\omega_0}(y+\epsilon n)\left|\frac{\partial\phi(y-O)}{\partial|y-O|}\right|^2 \to \infty \quad \text{when} \quad \epsilon\to 0.$$

Now we set $w_\epsilon = v_\epsilon/\|v_\epsilon\|_0$; the sequence w_ϵ is such that $\|w_\epsilon\|_0 = 1$ and $a(w_\epsilon, w_\epsilon) \to 0$ when $\epsilon \to 0$ $\square$.

Remark 3.5 There are many choices for the singular harmonic function that we used in the proof above. By using polar coordinates and assuming separation of variable one can built such functions in $\mathbb{R}^d$; by setting $r = |x|$ and $\sigma = x/r$, these functions have the general form $\phi(x) = f(\sigma)/r^k$, where k is any positive integer. For instance, in two dimension we have (see Dautray–Lions[3], Chap. II, p. 662):

$$\phi(x) = \frac{\cos(\varphi - \varphi_0)}{r^k}, \qquad k \geq 1. \tag{3.14}$$

In $\mathbb{R}^3$ we have

$$\phi(x) = \frac{P_k^{|m|}(\cos\theta)\cos(\varphi - \varphi_0)}{r^k}, \qquad -k \leq m \leq k, \qquad k \geq 1, \tag{3.15}$$

where (r, θ, φ) are the spherical coordinates in $\mathbb{R}^3$, $P_k^{|m|}(z)$ is a polynomial the exact form of which may be found in Dautray–Lions[3], Chap. II, p. 624 $\square$.

Remark 3.6 The lack of coerciveness of A is equivalent to the fact that the range of A is not closed in X'. Indeed, since A is injective, the statement: $\mathrm{im}(A)$ is closed, is equivalent to A^{t} is surjective (lemma 2.1); A being self-adjoint, this statement is equivalent to A is injective and surjective, *i.e.* A is bijective, or in other words A is coercive (corollary 2.4) $\square$.

To recover some well-posedness we can follow Lions and Sanchez-Palencia's strategy[7]. Since A is injective, self-adjoint, and monotonous, $a(u, v)$ induces a scalar product on X. Introduce X^a the completion of X with respect to the metric induced by the norm $a(u, u)^{1/2}$. It is clear that X^a is a Hilbert space when equipped with the scalar product $a(u, v)$. Assume that $v \mapsto \int_\Omega fv$ can be extended by continuity as a continuous linear form on X^a, then Riesz's representation theorem implies that problem (3.12) has unique (stable) solution in X^a (recall that a is the norm that induces the Hilbertian structure of X^a). This conclusion could be of practical purpose if the elements of X^a could be characterized as "good" functions or at least "not too bad" distributions. Unfortunately (*cf.* Lions and Sanchez-Palencia[7]), we have the following negative result

Proposition 3.3 *X^a is not contained in $D(\Omega)'$ (i.e. the space of distributions).*

Proof. (from Sanchez-Palencia[10]) We take a subspace of X and we show that the completion of this subspace with respect to the norm $a^{1/2}$ is not contained in $\mathrm{D}(\Omega)'$.

Define $\Phi = \{\phi \in \mathrm{H}^2(\Omega); \quad \phi_{|\Gamma_1} = 0,\ (\partial\phi/\partial n)_{|\Gamma_1} = 0\}$. Define $Y = \nabla\Phi$; it is clear that Y is a subset of X. For $y = \nabla\phi$, we have $a(y, y) = (\nabla^2\phi, \nabla^2\phi)$; as a result, the completion of Y for the norm $a^{1/2}$ is the gradient of the completion of Φ for the norm induced by $b(\phi, \phi) = (\nabla^2\phi, \nabla^2\phi)$. Lions and Sanchez-Palencia[7] have shown that Φ^b, the completion of Φ for the norm $b^{1/2}$, is not a contained in $\mathrm{D}(\Omega)'$ $\square$.

Remark 3.7 As a consequence of this proposition we infer that the dual of X^a does not contain $\mathrm{D}(\Omega)$ (although X clearly contains $D(\Omega)$), that is to say, even if f is a C^∞ function with compact support, problem (3.12) has not necessarily a solution. Furthermore, let f be an admissible source term (*i.e.* f is in the dual of X^a) and

δf be an arbitrarily small C^∞ function with a compact support, then $f + \delta f$ is not necessarily admissible; in other words, the solution of (3.12) is not continuous with respect to C^∞ perturbations with compact support of the datum. In this sense problem (3.12) is sensitive, according to Lions and Sanchez-Palencia's terminology $\square$.

Remark 3.8 One can have some insight on the structure of some pathological functions of X^a by looking back at the proof of proposition 3.2. Let O be a point of Γ_2 in the vicinity of which Γ is C^2 and set $O_\epsilon = O + \epsilon n$. Let ϕ be any function which is harmonic in $\mathbb{R}^d \setminus \{0\}$; recall that such a function is in $C^\infty(\mathbb{R}^d \setminus \{0\})$. Define $\phi_\epsilon(x) = \phi(x - O_\epsilon)$ and $v_\epsilon(x) = \theta_0(x)\nabla\phi_\epsilon(x)$, where θ_0 is defined as in the proof of proposition 3.2; we have

$$a(v_\epsilon - v_{\epsilon'}, v_\epsilon - v_{\epsilon'}) \leq c \max_{x\in\Omega_0\setminus\omega_0} |\nabla(\phi(x - O_\epsilon) - \phi(x - O_{\epsilon'}))|^2.$$

If ϵ is small enough (*i.e.* $\epsilon < D/8$), $|\nabla\nabla\phi_\epsilon|$ is bounded in $\Omega_0 \setminus \omega_0$ independently of ϵ; as a result, $\nabla\phi_\epsilon$ is Lipschitzian; that is to say

$$a(v_\epsilon - v_{\epsilon'}, v_\epsilon - v_{\epsilon'})^{1/2} \leq c|\epsilon - \epsilon'|.$$

This means that v_ϵ is a Cauchy sequence in X^a; as a result, the limit function v_0 is an element of X^a. The function v_0 is in $[C^\infty(\mathbb{R}^d \setminus \{O\})]^d$ but is possibly singular in O. Actually, as shown in remark 3.5, by using spherical harmonic functions one can build for any integer k a function v_0 of X^a which behaves like $1/|x - O|^k$ in the vicinity of O $\square$.

Now let us look at the Poisson problem we can build on the sequence of functions v_ϵ. Let us set $f_\epsilon(x) = -\nabla^2 v_\epsilon(x)$, $d_\epsilon(x_{\Gamma_2}) = \nabla\cdot v_\epsilon(x)_{|\Gamma_2}$, and $r_\epsilon(x_{\Gamma_2}) = (\nabla\times v_\epsilon(x)) \times n_{|\Gamma_2}$; these functions have the following "good" properties:

(i) Functions f_ϵ are smooth and all their derivatives of any order are bounded uniformly with respect to ϵ.
(ii) For all ϵ, f_ϵ is zero in ω_0.
(iii) For all ϵ, d_ϵ and r_ϵ are zero on $\Gamma \setminus \partial(\Omega_0 \setminus \omega_0)$ and are as smooth as the normal, n, on $\Gamma \cap \partial(\Omega_0 \setminus \omega_0)$, and up to the maximum degree of smoothness of n their derivatives are bounded uniformly.

As a result, one could say that f_ϵ, d_ϵ, and r_ϵ are "very nice" source term and boundary data. It is clear that v_ϵ is solution to the following problem

$$\left\{\begin{array}{rclrcl} & -\nabla^2 v_\epsilon = f_\epsilon, & & & & \\ v_{\epsilon|\Gamma_1} & = & 0, & & & \\ (\nabla\times v_\epsilon) \times n_{|\Gamma_2\setminus\partial(\Omega_0\setminus\omega_0)} & = & 0, & \nabla\cdot v_{\epsilon|\Gamma_2\setminus\partial(\Omega_0\setminus\omega_0)} & = & 0, \\ (\nabla\times v_\epsilon) \times n_{|\Gamma_2\cap\partial(\Omega_0\setminus\omega_0)} & = & r_\epsilon, & \nabla\cdot v_{\epsilon|\Gamma_2\cap\partial(\Omega_0\setminus\omega_0)} & = & d_\epsilon. \end{array}\right. \tag{3.16}$$

This example concretely illustrates the fact that, from a uniformly smooth sequence of source terms f_ϵ, d_ϵ, and r_ϵ, one can obtain a sequence of solutions to the Poisson problem (3.16) which converges to a function v_0 which is arbitrarily rough in O (recall that the degree k of the singularity is arbitrary!). Note however that v_0 is in $C^\infty(\Omega)$ (*i.e.* v_0 is a distribution). We have not yet been able to exhibit a sequence of functions of X^a the limit of which is not a distribution. For such an exemple in another context, the reader is referred to Lions/Sanchez-Palencia[8].

4. A Stokes problem

In this section we briefly extend to the Stokes problem the results obtained for the vector Poisson problem.

4.1. *A well posed Stokes problem*

We show in this section that provided the normal component or the tangential component of the velocity is controlled on the boundary, natural boundary conditions can be taken into account within the classical variational framework. For sake of simplicity let us assume that Γ is partitioned into three smooth pieces, Γ_1, Γ_2, and Γ_3 so that

$$\overline{\Gamma} = \bigcup_{k=1}^{3} \overline{\Gamma}_k \qquad \text{and} \qquad \bigcap_{k=1}^{3} \Gamma_k = \emptyset. \tag{4.1}$$

We introduce

$$V = \{v \in \mathrm{H}^1(\Omega)^d; \quad \nabla\cdot v = 0,\ v_{|\Gamma_1} = 0,\ v \cdot n_{|\Gamma_2} = 0,\ v \times n_{|\Gamma_3} = 0\}. \tag{4.2}$$

Let us denote by a_V the restriction to $V \times V$ of the bilinear form a; for all u, v in V we have

$$a_V(u,v) = (\nabla\times u, \nabla\times v). \tag{4.3}$$

We associate with a_V the continuous linear operator $A_V : V \longrightarrow V'$ so that $a_V(u,v) = \langle A_V u, v\rangle$; A_V is self-adjoint and monotonous, and provided $\mathrm{meas}(\Gamma_1) > 0$ we have

Lemma 4.6 *A_V is bijective.*

Proof. From the definition of A_V and remark 3.1 we infer

$$\begin{aligned} \langle A_V u, u\rangle &= a_V(u,u) \\ &= a(u,u) \\ &\geq c\|u\|_1^2. \end{aligned}$$

Hence A_V is coercive, which is a necessary and sufficient condition for A_V to be bijective (see corollary 2.4) $\square$.

For f in $\mathrm{L}^2(\Omega)^d$ we consider the following Stokes problem in variational form: find u in V so that

$$\forall v \in V, \qquad a_V(u,v) = (f,v). \tag{4.4}$$

Equivalently, this problem amounts to looking for a solution to the problem $A_V u = f$ in V' (where the duality product has been identified as an extension of the L^2 scalar product). From lemma 4.6 we deduce

Proposition 4.4 *Problem (4.4) has a unique solution and this solution is stable with respect to the datum: $\|u\|_1 \leq c\|f\|_0$.*

A formal interpretation of problem (4.4) is given by

Proposition 4.5 *If u, the solution to the variational problem (4.4), is in $\mathrm{H}^2(\Omega)^d$, there is a function p in $\mathrm{H}^1(\Omega)$ so that (u,p) is solution to the following Stokes problem*

$$\left\{\begin{array}{rlrl} & -\nabla^2 u + \nabla p = f, & & \\ & \nabla\cdot u = 0, & & \\ u_{|\Gamma_1} & = 0, & & \\ u\cdot n_{|\Gamma_2} & = 0, & (\nabla\times u)\times n_{|\Gamma_2} & = 0, \\ u\times n_{|\Gamma_3} & = 0, & p_{|\Gamma_3} & = 0. \end{array}\right. \tag{4.5}$$

Proof. This is an easy consequence of De Rham's theorem and the formula of integration by parts (3.8) $\Box$.

Remark 4.1 Note that in problem (4.4) a natural boundary condition is enforced on the pressure on Γ_3 though this variable is not explicitly involved in the variational formulation. The boundary conditions $(\nabla\times u)\times n_{|\Gamma_2} = 0$ and $p_{|\Gamma_3} = 0$ are natural in the sense that they are naturally enforced by the bilinear form a_V $\Box$.

4.2. *A sensitive Stokes problem*

As for the vector Poisson problem we show now that if there is a nonzero portion where neither the normal trace nor the tangential trace of the unknown is controlled, the Stokes problem associated with the bilinear form a_V is sensitive.

For sake of simplicity, we partition Γ into two smooth pieces Γ_1 and Γ_2 so that

$$\overline{\Gamma} = \overline{\Gamma}_1 \cup \overline{\Gamma}_2 \qquad \text{and} \qquad \Gamma_1 \cap \Gamma_2 = \emptyset. \tag{4.6}$$

Accordingly, we introduce

$$V = \{v \in \mathrm{H}^1(\Omega)^d; \quad \nabla\cdot v = 0,\ v_{|\Gamma_1} = 0\}. \tag{4.7}$$

V is a Hilbert space when equipped with the scalar product of $\mathrm{H}^1(\Omega)^d$. The definition of bilinear form $a_V : V \times V \longrightarrow \mathbb{R}$ together its associated operator $A_V : V \longrightarrow V'$ remains unchanged.

For a source term f in $\mathrm{L}^2(\Omega)^d$ (or possibly smoother) we consider the Stokes problem: find u in V so that

$$\forall v \in V, \qquad a_V(u,v) = (f,v). \tag{4.8}$$

Let us assume that this problem has a solution and that the solution is smooth, say u is in $\mathrm{H}^2(\Omega)^d$; we infer from classical arguments that u is solution to the following formal problem

$$\left\{\begin{array}{rlrl} & -\nabla^2 u + \nabla p = f, & & \\ & \nabla\cdot u = 0, & & \\ u_{|\Gamma_1} & = 0, & & \\ (\nabla\times u)\times n_{|\Gamma_2} & = 0, & p_{|\Gamma_2} & = 0. \end{array}\right. \tag{4.9}$$

Note here that on Γ_2 only natural boundary conditions are enforced. Problem (4.8) is well-posed provided A_V is bijective. Since A_V is self-adjoint and monotonous, A_V is bijective if and only if it is coercive (*cf.* corollary 2.4). Concerning the possible coerciveness of A_V we have the following negative result

Proposition 4.6 *The operator A_V is not coercive.*

Proof. We proceed as in the proof of proposition 3.2. We built a sequence of functions w_ϵ the L^2 norm of which is equal to one and so that $a_V(w_\epsilon, w_\epsilon)$ converges to zero as ϵ tends to zero.

Using the same notation as in the proof of proposition 3.2, we set $v_\epsilon(x) = \nabla\phi_\epsilon(x)$ for all x in Ω_0. Furthermore, in $\Omega \setminus \Omega_0$, we define v_ϵ as being a divergence free lifting of the trace of $\nabla\phi_\epsilon$ on $\partial\Omega_0$ so that v_ϵ is zero on Γ_1 (such a lifting exists *cf.* Girault–Raviart[5], p. 24). It is easily shown that thus defined v_ϵ is in V and the sequence $w_\epsilon = v_\epsilon/\|v_\epsilon\|_0$ is the counter-example we look for $\square$.

Remark 4.2 Since A_V is injective, self-adjoint, and monotonous, $a_V(u, v)$ induces a scalar product on V; hence, we can recover some well-posedness by considering V^a the completion of V with respect to the metric induced by the norm $a_V(u, u)^{1/2}$; V^a is a Hilbert space when equipped with the scalar product $a_V(u, v)$. If $v \mapsto \int_\Omega fv$ can be extended by continuity as a continuous linear form on V^a, then Riesz's representation theorem implies that problem (4.8) has a unique (stable) solution in V^a. However, concerning the characterization of V^a we could show that this space is not contained in the space of distributions $\mathrm{D}(\Omega)'$. We only sketch a possible proof of this fact. Define $\Psi = \{\psi \in \mathrm{H}^2(\Omega)^d; \quad \nabla\cdot\psi = 0,\ \psi \times n_{|\Gamma_1} = 0,\ (\nabla\times\psi) \times n_{|\Gamma_1} = 0\}$. Define $Y = \nabla\times\Psi$; it is clear that Y is a subset of V. For $y = \nabla\times\psi$, we have $a_V(y, y) = (\nabla^2\psi, \nabla^2\psi)$; as a result, the completion of Y for the norm $a_V^{1/2}$ is the rotational of the completion of Ψ for the norm induced by $b(\psi, \psi) = (\nabla^2\psi, \nabla^2\psi)$. By using arguments similar to that of Lions and Sanchez-Palencia[7] we could show that Ψ^b, the completion of Ψ for the norm $b^{1/2}$, is not a contained in $\mathrm{D}(\Omega)'$. From classical results on distributions, this means that Y^a is not contained in $\mathrm{D}(\Omega)'$. As a result, Y being a subspace of V, we conclude that V^a is not contained in $\mathrm{D}(\Omega)'$ $\square$.

5. Conclusions

This note has investigated the possibility of building up well-posed boundary value problems of vector type in the presence of purely natural boundary conditions on a part of the boundary. By generalizing Lions/Sanchez-Palencia theory of sensitive boundary value problems for the scalar biharmonic problem in two dimensions, we have demonstrated that the vector Poisson equation and the Stokes system in two and three dimensions under the aformentioned natural boundary conditions give rise to a variational formulation of the problem which admits uniqueness of the solution but which is nevertheless possibly sensitive to arbitrarily small smooth perturbations of the data. This result establishes a limitation on the kind of natural boundary conditions which can be enforced in vector elliptic and incompressible viscous flows problems. In particular, the tangential components of vorticity cannot be prescribed together with the value of pressure on any part of the boundary, as one could instead think advisable at outflow boundaries to minimize the perturbation on the solution due to lack of knowledge of the exact velocity field in the downstream region.

In conclusion, the proposed analysis confirms that the well-posedness of the considered vector problems (*i.e.* using the bilinear form (3.2)), as classically understood in numerical analysis, is guaranteed only provided that at least one of either the normal or the tangential component of the vector unknown is specified on every part of the entire boundary.

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