FEM solution of the ψ-ω equations with explicit viscous diffusion[1]

J.-L. Guermond [2] and L. Quartapelle [3]

Abstract. This paper describes a variational formulation for solving the 2D time-dependent incompressible Navier–Stokes equations expressed in terms of the stream function and vorticity. The difference between the proposed approach and the standard one is that the vorticity equation is interpreted as a dynamical equation governing the evolution of (the weak Laplacian of) the stream function while the Poisson equation for the stream function is used as an expression to evaluate the distribution of the vorticity in the domain and on the boundary. A time discretization is adopted with the viscous diffusion made explicit, which leads to split the viscous effects from the incompressibility similarly to the fractional-step projection methods for the primitive variable equations. In some sense, the present method generalizes to the variational framework a well-known idea that is used in £nite differences approximations and which is based on a Taylor series expansion of the stream function on the boundary. Some error estimates and some numerical results are given.

1 INTRODUCTION

The prototype £nite element procedure for solving the 2D Stokes equations formulated in terms of the vorticity and stream function is the uncoupled solution method for the biharmonic problem introduced by Glowinski and Pironneau [4]. This approach can compute the solution of the Stokes problem by an uncoupled direct method. In the case of the time-dependent equations, this method assumes necessarily an implicit treatment of the viscous term and determines the unknown boundary value of the vorticity by means of an operator associated with that part of the boundary where no-slip velocity conditions are prescribed. While the techniques based on this method or on the related idea of an in¤uence matrix guarantee good stability properties, they are not so easy to implement; hence one may be tempted to make a trade-off between stability and simplicity. In the present paper we investigate one possible alternative technique for solving the evolutionary Navier–Stokes equations in two dimensions expressed in a weak variational form.

In the context of £nite differences, which are known for their simplicity, one classical approach is to assume an explicit treatment of the viscous term to derive vorticity boundary formulas. In this case, the Neumann condition for the stream function can be used as the last piece of the time-stepping algorithm. More precisely, the derivative boundary condition can be interpreted as a relationship specifying the boundary value of the new vorticity after the time advancement of the (internal distribution of) vorticity has been completed and after the new stream function has been determined, for details see *e.g.* Peyret and Taylor [7], E and Liu [2] or Napolitano *et al.* [6]. The aim of this paper is to develop the variational counterpart of this idea and to show that an interpretation alternative to the standard one can be given to the vorticity equation written in weak form which leads to a new and very simple algorithm for the numerical solution of the unsteady ψ-ω equations by means of £nite elements.

2 VARIATIONAL PROBLEM

The proposed numerical method stems from the following variational statement of the incompressible Navier–Stokes equation for 2D ¤ows in a simply connected,

[1] in Proc. Computational Fluid Dynamics'98, ECCOMAS96, (Septembre 1998, Athne, Grce), K.D. Papailiou, D. Tsahalis, J. Priaux, M. Pandol£ (Eds.), Wiley, Vol. **1** 2 (1998) 1258-1263.

[2] Laboratoire d'Informatique pour la Mécanique et les Sciences de l'Ingénieur, CNRS, BP 133, 91403, Orsay, France (`guermond@limsi.fr`).

[3] Dipartimento di Fisica, Politecnico di Milano, Piazza Leonardo da Vinci, 32, 20133 Milano, Italy.

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bounded domain Ω with a smooth boundary $\Gamma = \partial\Omega$. For the initial solenoidal velocity $\boldsymbol{u}_0$ belonging to $\boldsymbol{H}_0^1(\Omega)$ and the body force $\boldsymbol{f} \in L^2(0,T;\boldsymbol{L}^2(\Omega))$, we have the problem

$$\begin{cases} \text{Find } \psi \in L^2(0,T;H_0^2(\Omega)) \cap C(0,T;H_0^1(\Omega)) \text{ and} \\ \omega \in L^2(0,T;L^2(\Omega)) \cap C(0,T;H^{-1}(\Omega)), \text{ so that} \\ \forall \phi \in H_0^1(\Omega), \qquad ((\nabla\psi)_{|t=0}, \nabla\phi) = (\boldsymbol{u}_0, \boldsymbol{\nabla}\phi \times \widehat{\boldsymbol{z}}), \\ \text{and that, for all } t > 0, \\ \forall \psi' \in H_0^2(\Omega), \qquad \left(\frac{\partial \nabla\psi}{\partial t}, \nabla\psi'\right) + \nu \left\langle \nabla\omega, \nabla\psi' \right\rangle \\ \qquad\qquad + b(\omega,\psi,\psi') = (\boldsymbol{f}, \boldsymbol{\nabla}\psi' \times \widehat{\boldsymbol{z}}), \\ \forall v \in L^2(\Omega), \qquad (\omega, v) + (\nabla^2\psi, v) = 0. \end{cases} \tag{1}$$

Homogeneous velocity boundary conditions for the sake of simplicity. The trilinear form $b(\omega,\psi,\psi')$ associated with the advection term is written in rotational form, namely,

$$b(\omega,\psi,\psi') = (\omega\, \boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi' \times \widehat{\boldsymbol{z}}), \tag{2}$$

to guarantee conservation of the (kinetic) energy, when $\nu = 0$, also in the spatially discrete case of H^1-conformal approximations.

Note the unusual form of the evolutionary term which involves mixed time and space derivatives of ψ. Actually, this weak form of the dynamical equation is the most natural one within the present variational setting since it stems from the momentum equation where

- the velocity is replaced by $\boldsymbol{\nabla}\psi \times \widehat{\boldsymbol{z}}$,
- the viscous term is written as the curl of ω, and
- the velocity test functions belonging to

$$\boldsymbol{V} = \{\boldsymbol{v} \in \boldsymbol{H}_0^1(\Omega),\ \boldsymbol{\nabla}\cdot\boldsymbol{v} = 0\}$$

  are expressed as $\boldsymbol{\nabla}\psi' \times \widehat{\boldsymbol{z}}$ by virtue of the the well-known isomorphim (see Girault and Raviart [3])

$$\boldsymbol{\nabla}\ldots \times \widehat{\boldsymbol{z}} \ :\ H_0^2(\Omega) \longrightarrow \boldsymbol{V}.$$

At variance with more usual ways of writing the vorticity transport equation, no integration by parts is to be performed; in other words, the curl of the momentum equation has not been taken in a strong form. In this way the momentum equation becomes an evolutionary equation for the stream function (actually for its Laplacian in weak form) whereas the Poisson equation is used as the de£nition of vorticity.

3 SPATIAL DISCRETIZATION

Let W_h and $\Psi_{0,h}$ be two £nite dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively. We assume that $\Psi_{0,h} \subset W_h$, and W_h, $\Psi_{0,h}$ satisfy standard interpolation and inverse properties (described in detail in [5]).

4 FIRST ORDER SCHEME

In this section we approximate the time derivative by means of the £rst order Euler scheme. Let $[0,T]$ be a £nite time interval and N be an integer. We denote $\delta t = T/N$ and $t_n = n\delta t$ for $0 \le n \le N$. For any function of time, $\varphi(t)$, we denote $\varphi^n = \varphi(t_n)$.

The fully discrete problem is formulated as follows. The initialization step reads:

$$\begin{cases} \text{Find } \psi_h^0 \in \Psi_{0,h} \text{ so that, } \forall \phi_h \in \Psi_{0,h}, \\ (\nabla\psi_h^0, \nabla\phi_h) = (\boldsymbol{u}_0, \boldsymbol{\nabla}\phi_h \times \widehat{\boldsymbol{z}}), \end{cases} \tag{3}$$

$$\begin{cases} \text{Find } \omega_h^0 \in W_h \text{ so that, } \forall v_h \in W_h, \\ (\omega_h^0, v_h) = (\nabla\psi_h^0, \nabla v_h). \end{cases} \tag{4}$$

Then for each $n \ge 0$, carry out the following two uncoupled steps:

$$\begin{cases} \text{Find } \psi_h^{n+1} \in \Psi_{0,h} \text{ so that, } \forall \phi_h \in \Psi_{0,h}, \\ \dfrac{(\nabla(\psi_h^{n+1} - \psi_h^n), \nabla\phi_h)}{\delta t} + b(\omega_h^n, \psi_h^{n+1}, \phi_h) \\ \quad = -\nu\,(\nabla\omega_h^n, \nabla\phi_h) + (\boldsymbol{f}^{n+1}, \boldsymbol{\nabla}\phi_h \times \widehat{\boldsymbol{z}}), \end{cases} \tag{5}$$

and

$$\begin{cases} \text{Find } \omega_h^{n+1} \in W_h \text{ so that, } \forall v_h \in W_h, \\ (\omega_h^{n+1}, v_h) = (\nabla\psi_h^{n+1}, \nabla v_h). \end{cases} \tag{6}$$

The nonlinear term is accounted for in a semi-implicit form for the sake of simplicity. All that is said afterwards holds with minor modi£cations if this term is made explicit. From the theoretical point of view, the modi£cations in question essentially amount to deriving slightly sharper bounds for the nonlinear residuals.

Observe that, when compared to the classical ω-ψ formulation, the present method interchanges the role of the variables ψ and ω. Here, the dynamical equation for the transport of ω has turned into an equation governing the evolution of (the weak form of) $-\nabla^2\psi_h$, whereas the Poisson equation for ψ_h has become an expression controlling ω_h explicitly.

4.1 Vorticity integral conditions

Note that the explicit evaluation of the new vorticity £eld ω_h^{n+1} through the solution of the mass matrix problem (6) *does* enforce the integral conditions for the vorticity which underlay the uncoupled method due to Glowinski and Pironneau [4]. In fact, considering more general, *i.e.*, nonhomogeneous, boundary conditions

$$\psi_{|\Gamma}^{n+1} = a^{n+1} \quad \text{and} \quad \frac{\partial \psi^{n+1}}{\partial n}_{|\Gamma} = b^{n+1},$$

the vorticity problem would read

$$\begin{cases} \text{Find } \omega_h^{n+1} \in W_h \text{ so that, } \forall v_h \in W_h, \\ (\omega_h^{n+1}, v_h) = (\nabla \psi_h^{n+1}, \nabla v_h) - \displaystyle\int_\Gamma b^{n+1} v_h. \end{cases}$$

Selecting the functions v_h in the subspace of the discrete harmonic functions, namely, $v_h = \eta_h \in W_h$ so that $(\nabla \eta_h, \nabla v_h) = 0$, $\forall v_h \in W_h$, the weak equation above gives

$$(\omega_h^{n+1}, \eta_h) \approx \int_\Gamma \Big(a^{n+1} \frac{\partial \eta_h}{\partial n} - b^{n+1} \eta_h \Big),$$

since it can be shown that $\int_\Gamma a^{n+1} \partial \eta_h / \partial n \approx (\nabla \psi_h^{n+1}, \nabla \eta_h)$. One recovers the vorticity integral conditions for the transient problem at time t_{n+1}. Thus, the proposed method, with the viscous diffusion made explicit, allows the vorticity integral conditions to be ful£lled *a posteriori*, as already pointed out in [6].

In the present formulation the vorticity boundary value is determined in a way that is very similar to the classical procedure used in the context of £nite differences. In fact the vorticity boundary formula used in second-order accurate central differences is obtained by means of a Taylor series expansion as follows

$$\begin{aligned} \psi_h(\Delta x) &= \psi(0) + \Delta x \frac{\partial \psi(0)}{\partial x} \\ &- \frac{(\Delta x)^2}{2} \left[\omega_h(0) + \frac{\partial^2 \psi(0)}{\partial y^2} \right] + \mathcal{O}((\Delta x)^3). \end{aligned}$$

This argument uses the Poisson equation for ψ on the boundary together with the Dirichlet and Neumann boundary data for ψ. In some sense, the Taylor expansion above mimics the weak vorticity equation for a weighting function v_h such that $v_{h|\Gamma} \neq 0$.

5 STABILITY ANALYSIS

The following convergence result is established in [5]:

Theorem 1 *Under convenient regularity hypothesis on the solution (ψ, ω) of the continuous problem (1), there is $c_s(\Omega) > 0$ and $c_e(T, \nu, \psi, \Omega) > 0$ so that if $\delta t \leq c_s h^2/\nu$, then*

$$\|\psi - \psi_h\|_{l^\infty(H^1(\Omega))} + \|\omega - \omega_h\|_{l^2(L^2(\Omega))} \leq c_e(\delta t + h^\ell).$$

6 SECOND ORDER BDF SCHEME

The present technique is not restricted to £rst order; it can be modi£ed to obtain high order accuracy in time. This can be done simply by approximating the time derivative by a high order £nite differencing (Crank–Nicolson, three-level backward differencing, etc...) and by extrapolating the terms that involve ω, accordingly. To illustrate this possibility we present in the following a second order scheme based on the three-level backward differencing of the time derivative and using a semi-implicit evaluation of the nonlinear term by means of linear extrapolation in time of the the vorticity.

Initialize the scheme by evaluating (ψ_h^0, ω_h^0) and (ψ_h^1, ω_h^1). ψ_h^0 and ω_h^0 are evaluated from the initial data through (3) and (4). ψ_h^1 can be obtained by many means; for instance, it can be calculated by using a second order Runge–Kutta technique; from ψ_h^1 one evaluates ω_h^1 easily. Then, for each $n \geq 1$, carry out the following two steps:

$$\begin{cases} \text{Find } \psi_h^{n+1} \in \Psi_{0,h} \text{ so that, } \forall \phi_h \in \Psi_{0,h}, \\ \dfrac{(\nabla(3\psi_h^{n+1} - 4\psi_h^n + \psi_h^{n-1}), \nabla \phi_h)}{2\delta t} \\ \qquad\qquad + b(2\omega_h^n - \omega_h^{n-1}, \psi_h^{n+1}, \phi_h) \\ = -\nu\,(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla \phi_h) + (\boldsymbol{f}^{n+1}, \boldsymbol{\nabla} \phi_h \times \widehat{\boldsymbol{z}}), \end{cases} \tag{7}$$

and

$$\begin{cases} \text{Find } \omega_h^{n+1} \in W_h \text{ so that, } \forall v_h \in W_h, \\ (\omega_h^{n+1}, v_h) = (\nabla \psi_h^{n+1}, \nabla v_h). \end{cases} \tag{8}$$

This scheme is second order accurate in time. Its error analysis follows the same ideas as those that have been used to analyze the £rst order scheme and a uniform stability result can be found in [5].

7 NUMERICAT TESTS

We have implemented the three-level BDF scheme above in two different manners: £rst by evaluating the nonlinear term in a fully explicit manner by time extrapolation

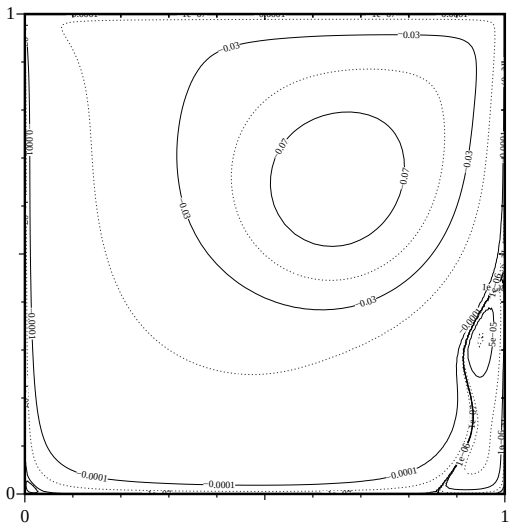

Figure 1. Streamlines of FEM solution of the driven cavity problem for $\mathrm{Re} = 1000$ at $t = 6.25$: fully explicit scheme with direct solution of the symmetric linear systems.

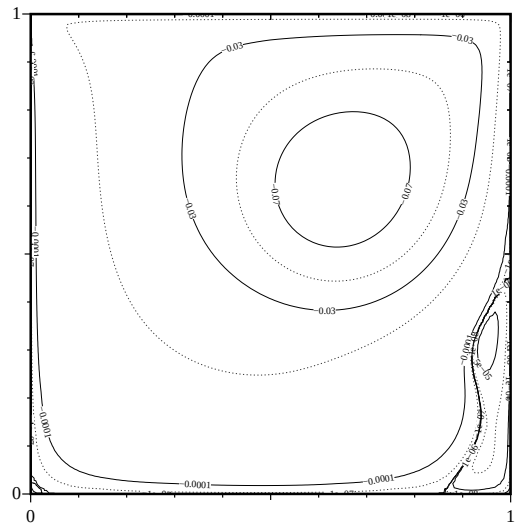

Figure 2. Streamlines of FEM solution of the driven cavity problem for $\mathrm{Re} = 1000$ at $t = 6.25$: semi-implicit scheme with iterative solution of the linear systems.

of both ψ_h^{n+1} and ω_h^{n+1}, leading to the weak equation, $\forall \phi_h \in \Psi_{0,h}$,

$$\begin{aligned} & (3\nabla\psi_h^{n+1}/2\delta t, \nabla\phi_h) \\ = \; & -b(2\omega_h^n - \omega_h^{n-1}, 2\psi_h^n - \psi_h^{n-1}, \phi_h) + g^n(\phi_h), \end{aligned}$$

second, by evaluating the nonlinear term by means of a semi-implicit approximation which gives a nonsymmetric contribution to the elliptic problem for ψ_h^{n+1}, as follows,

$$\begin{aligned} (3\nabla\psi_h^{n+1}/2\delta t, \nabla\phi_h) \; + \; & b(2\omega_h^n - \omega_h^{n-1}, \psi_h^{n+1}, \phi_h) \\ = \; & g^n(\phi_h). \end{aligned}$$

In both cases the source term $g^n(\phi_h)$ is de£ned by

$$\begin{aligned} g^n(\phi_h) \; = \; & (\nabla(2\psi_h^n - \psi_h^{n-1}/2)/\delta t \\ & - \nu\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\phi_h). \end{aligned}$$

The £rst scheme allows using direct algorithms for solving the two symmetric and time-independent linear systems (stiffness and mass matrix) for ψ_h^{n+1} and ω_h^{n+1}, respectively. Beside the mass matrix problem, the second scheme has a nonsymmetric system of linear equations to be solved at each time step. As a consequence, we used the GMRES technique with preconditioning based on incomplete factorization of the (constant) stiffness matrix.

The two algorithms have been tested and compared by solving the unsteady driven cavity problem using nonuniform meshes of $\approx \; 2 \times 80^2$ linear triangles, £rst for $\mathrm{Re} = 1000$.

In £gures 1 and 2 we report the streamlines at time $t = 6.25$ calculated by the direct and the iterative FEM scheme using $\delta t = 0.001$ and $\delta t = 0.0005$, respectively. Figures 3 and 4 contains the vorticity distribution at the same time, to be compared with the spectral solution calculated by a Galerkin spectral method of Glowinski–Pironneau type [1] with $\delta t = 0.001$ and 150 Legendre polynomials in each direction, shown in £gure 5. The two FE methods are equally accurate, the slight differ-

ences between them being caused by the use of different meshes.

The second example is the calculation of the same problem for a moderately high Reynolds number $\mathrm{Re} = 10^4$. The streamlines at time $t = 5.0$ provided by the explicit direct FEM scheme with $\delta t = 0.001$ are shown in £gure 6. This solution is in fair agreement with a £nite difference solution obtained on a uniform 501^2 grid and using a centered h^4-accurate approximation of the fully explicit Jacobian reported in £gure 7.

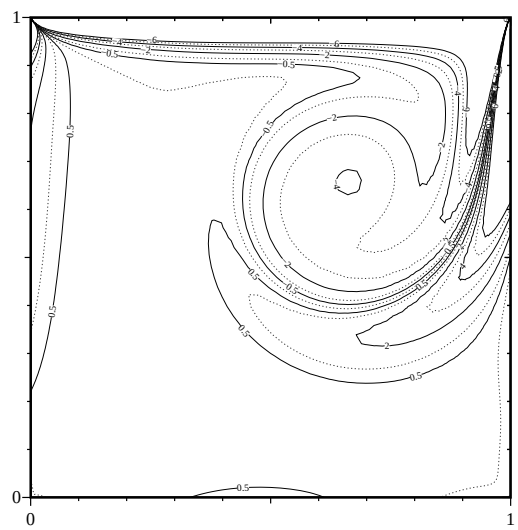

Figure 3. Vorticity of FEM solution of the driven cavity problem for $\mathrm{Re} = 1000$ at $t = 6.25$: fully explicit scheme with direct solution of the symmetric linear systems.

8 CONCLUSIONS

In this paper we have presented a £nite element scheme for solving the time-dependent Navier–Stokes equations formulated in terms of the stream function and the vorticity. The calculation of the stream function and the vorticity are uncoupled owing to an explicit treatment of the viscous diffusion together with a non-standard writing of the evolutionary term in the weak form of the momentum equation. The explicit treatment of viscous diffusion

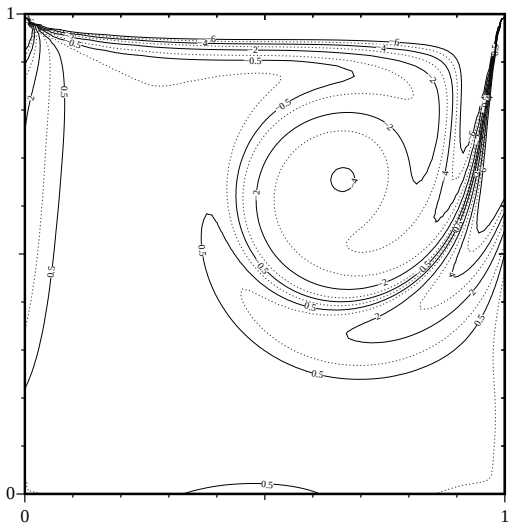

Figure 4. Vorticity of FEM solution of the driven cavity problem for $\mathrm{Re} = 1000$ at $t = 6.25$: semi-implicit scheme with iterative solution of the linear systems.

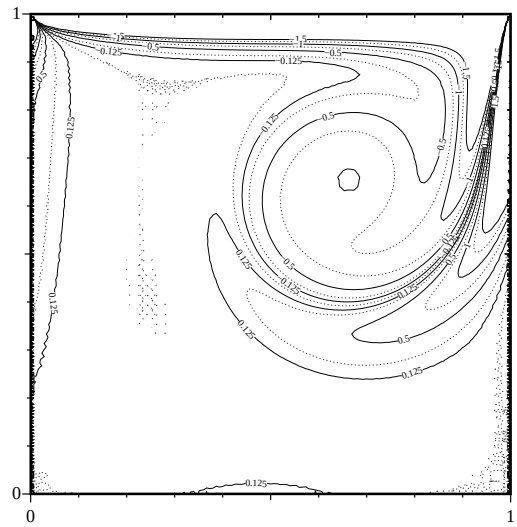

Figure 5. Vorticity of Galerkin–Legendre spectral solution of the driven cavity problem for $\mathrm{Re} = 1000$ at $t = 6.25$: Glowinski–Pironneau uncoupled method.

implies a stability condition of the type: $\nu \delta t / h^2 \leq c$. This stability constraint is the price to be paid for the extreme algorithmic simplicity of the proposed uncoupled scheme, especially when compared to Glowinski–Pironneau method and related techniques. While the stability restriction may be severe for creeping ¤ows, the matter improves for convection dominated ¤ows since the combination of the cell Reynolds number condition for adequate spatial resolution with the stability condition gives a condition $\delta t \leq ch$. The uncoupling strategy proposed in the paper is not limited to time discretizations of low order; a second order accurate scheme based on the three-level backward difference formula has been presented.

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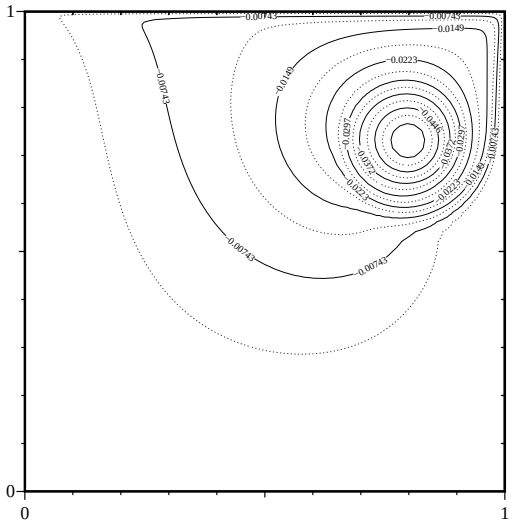

Figure 6. Streamlines of FEM solution of the driven cavity problem for $\mathrm{Re} = 10^4$ at $t = 5.0$.

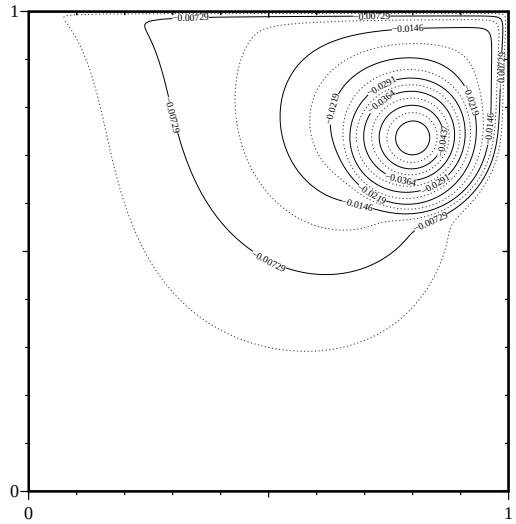

Figure 7. Streamlines of FDM solution of the driven cavity problem for $\mathrm{Re} = 10^4$ at $t = 5.0$.