Glowinski–Pironneau method for the 3D ω-ψ equations

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Summary. Glowinski–Pironneau method for solving the 2D Stokes problem as two uncoupled scalar Poisson equations for the vorticity and the stream function is extended to the three-dimensional problem. The determination of the two tangential components of vorticity over the boundary is achieved by solving an auxiliary boundary problem characterized by a symmetric definite positive linear operator. In the discrete case, the explicit determination of the corresponding matrix and/or its solution involves the computation of solenoidal fields for the vorticity and the stream vector which are solution to Poisson equations supplemented by both essential and natural boundary conditions, the latter implying a coupling between the three Cartesian components of each vector unknown.

1. INTRODUCTION

A well known method for computing viscous incompressible flows in two dimensions relies upon formulating the Stokes problem in terms of the variables vorticity and stream function and using the uncoupling strategy proposed by Glowinski and Pironneau in their classical paper on the biharmonic equation [5]. Since its appearance, this work allowed substantial progresses in the numerical solution of the nonprimitive variable two-dimensional Navier–Stokes equations, by introducing the concept of *inflence matrix* to overcome the difficulty caused by the lack of boundary conditions in some mathematical formulations of the equations for incompressible viscous flows. As recent examples of the developments stemming from Glowinski–Pironneau method for the vorticity and stream function equations we can mention *e.g.* [3], [8], [1], [6]. The success of all these investigations being deeply rooted in Glowinski–Pironneau's

[1] in Proc. Comput. Sci. 21th Century, Bristeau et al. (Eds.), Wiley, (1997) 280-291

uncoupling strategy for the nonprimitive variables, it seems worthwhile to attempt to extend their idea to the three-dimensional equations for the vorticity and the velocity potential. In this case the two unknowns are however vector functions, which makes the problem somewhat different and more complicated, but not to the point of preventing a natural extension of the uncoupling strategy of the Stokes problem to the three-dimensional case, as our subsequent analysis will reveal.

Let Ω denote an open bounded domain of $\mathbb{R}^3$ which is assumed to be $C^{0,1}$, simply connected and such that its boundary, Γ, is connected. The last two assumptions are made to simplify the analysis presented in this work: they can be removed provided that suitable technical modifications to the theory are made, which however do not change its essential features.

We consider the following Stokes problem: Find $\boldsymbol{u}$ and p (up to a constant) so that

$$\left\{ \begin{aligned} -\Delta\boldsymbol{u} + \boldsymbol{\nabla} p &= \boldsymbol{f}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{u} &= 0, \\ \boldsymbol{u}_{|\Gamma} &= 0, \end{aligned} \right. \tag{1}$$

where $\boldsymbol{f}$ is some source term. It is clear that problem (1) has a unique solution: $\boldsymbol{u} \in \boldsymbol{H}_0^1$ and $p \in L_0^2$; this solution is stable with respect to that datum: $|\boldsymbol{u}|_1 + |p|_0 \le c|\boldsymbol{f}|_{-1}$.

To avoid unnecessary complicated technical details in the theory to be presented, we assume hereafter that $\boldsymbol{f}$ belongs to the dual (according to the distribution theory) of $\boldsymbol{H}_0(\mathrm{div})$: $\boldsymbol{f} \in (\boldsymbol{H}_0(\mathrm{div}))'$. Given the following series of continuous injections (with density):

$$\boldsymbol{H}_0^1 \subset \boldsymbol{H}_0(\mathrm{div}) \subset \boldsymbol{L}^2 \equiv (\boldsymbol{L}^2)' \subset (\boldsymbol{H}_0(\mathrm{div}))' \subset \boldsymbol{H}^{-1}$$

we have $|\boldsymbol{f}|_{-1} \le c|\boldsymbol{f}|_{(\boldsymbol{H}_0(\mathrm{div}))'}$.

The goal of this paper is to present two alternative formulations of this Stokes problem which are based on the vorticity and two possible definitions of vector potentials for the velocity. In particular, for the first formulation an uncoupled method of solution is derived in which the vorticity field and velocity vector potential (also called stream vector) are solved independently, much in the same manner as in the celebrated uncoupled formulation developed by Glowinski and Pironneau for the biharmonic scalar equation in two dimensions [5].

2. $\boldsymbol{\omega}$-$\boldsymbol{\psi}$ FORMULATION WITH $\boldsymbol{n} \times \boldsymbol{\psi}_{|\Gamma} = 0$

Let us consider the following Hilbert space

$$\boldsymbol{X}_N \stackrel{\mathrm{def}}{=} \boldsymbol{H}(\mathrm{div}) \cap \boldsymbol{H}_0(\mathrm{rot}).$$

We now give two Lemmas that will play a key role in the following (see *e.g.* Dautray–Lions [2] for a proof).

Lemma 1 *Provided Γ is connected and Ω is $C^{0,1}$, there is $c > 0$ so that*

$$\forall \boldsymbol{\varphi} \in \boldsymbol{X}_N, \qquad c|\boldsymbol{\varphi}|^2_{\boldsymbol{X}_N} \le |\boldsymbol{\nabla}\times\boldsymbol{\varphi}|_0^2 + |\boldsymbol{\nabla}\cdot\boldsymbol{\varphi}|_0^2.$$

Lemma 2 *Provided Γ is connected and Ω is simply connected and $C^{0,1}$, the following mapping is an isomorphism:*

$$\nabla\times \ : \ \boldsymbol{H}(\mathrm{div}=0)\cap\boldsymbol{H}_0(\mathrm{rot}) \longrightarrow \boldsymbol{H}_0(\mathrm{div}=0).$$

Lemma 2 implies that for each function $\boldsymbol{v}$ in $\boldsymbol{H}_0(\mathrm{div}=0)$, there is a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{H}(\mathrm{div}=0)\cap\boldsymbol{H}_0(\mathrm{rot})$ so that $\boldsymbol{v}=\nabla\times\boldsymbol{\psi}$. Hence we are led to consider the following Hilbert spaces

$$\boldsymbol{W} = \Big\{\boldsymbol{v}\in\boldsymbol{H}(\mathrm{div}) \ \Big| \ \nabla\times\boldsymbol{v}\in(\boldsymbol{H}_0(\mathrm{rot}))'\Big\},$$

$$\boldsymbol{\Psi} = \boldsymbol{X}_N \overset{\mathrm{def}}{=} \boldsymbol{H}(\mathrm{div})\cap\boldsymbol{H}_0(\mathrm{rot}).$$

Now consider the following problem: find $\boldsymbol{\omega}\in\boldsymbol{W}$ and $\boldsymbol{\psi}\in\boldsymbol{\Psi}$ so that

$$\begin{cases} \forall\boldsymbol{\varphi}\in\boldsymbol{\Psi} & \langle\nabla\times\boldsymbol{\varphi},\nabla\times\boldsymbol{\omega}\rangle+(\nabla\cdot\boldsymbol{\varphi},\nabla\cdot\boldsymbol{\omega}) \ = \ \langle\boldsymbol{f},\nabla\times\boldsymbol{\varphi}\rangle, \\ \forall\boldsymbol{v}\in\boldsymbol{W} & -(\boldsymbol{\omega},\boldsymbol{v})+\langle\nabla\times\boldsymbol{\psi},\nabla\times\boldsymbol{v}\rangle+(\nabla\cdot\boldsymbol{\psi},\nabla\cdot\boldsymbol{v}) \ = \ 0, \end{cases} \tag{2}$$

where $\langle\cdot,\cdot\rangle$ denotes the duality product between $\boldsymbol{H}_0(\mathrm{div})$ and $(\boldsymbol{H}_0(\mathrm{div}))'$. In the considered problem this duality pairing is meaningful since we clearly have $\nabla\times\boldsymbol{\Psi}\subset\boldsymbol{H}_0(\mathrm{div})$. Concerning problem (2) we have

Proposition 1 *Problem (2) has a unique solution and this solution is such that $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$ and $\nabla\times\boldsymbol{\psi}=\boldsymbol{u}$, $\boldsymbol{u}$ being the velocity field solution to the primitive variable Stokes problem (1).*

Proof. (a) Let us prove first uniqueness. Let $(\boldsymbol{\omega}_0,\boldsymbol{\psi}_0)$ be a solution to the homogeneous version of the problem. By taking $\boldsymbol{\psi}_0$ and $\boldsymbol{\omega}_0$ as test functions in (2) we infer

$$\begin{cases} \langle\nabla\times\boldsymbol{\psi}_0,\nabla\times\boldsymbol{\omega}_0\rangle+(\nabla\cdot\boldsymbol{\psi}_0,\nabla\cdot\boldsymbol{\omega}_0) \ = \ 0, \\ -|\boldsymbol{\omega}_0|^2+\langle\nabla\times\boldsymbol{\psi}_0,\nabla\times\boldsymbol{\omega}_0\rangle+(\nabla\cdot\boldsymbol{\psi}_0,\nabla\cdot\boldsymbol{\omega}_0) \ = \ 0. \end{cases}$$

Hence $|\boldsymbol{\omega}_0|^2=0$, that is $\boldsymbol{\omega}_0=0$. From this result we deduce that

$$\forall\boldsymbol{v}\in\boldsymbol{W} \qquad \langle\nabla\times\boldsymbol{\psi}_0,\nabla\times\boldsymbol{v}\rangle+(\nabla\cdot\boldsymbol{\psi}_0,\nabla\cdot\boldsymbol{v})=0.$$

Since $\boldsymbol{\Psi}$ is a subset of $\boldsymbol{W}$ ($\boldsymbol{\Psi}\subset\boldsymbol{W}$) we can take $\boldsymbol{\psi}_0$ as a test function, and we obtain

$$|\nabla\times\boldsymbol{\psi}_0|^2+|\nabla\cdot\boldsymbol{\psi}_0|^2=0.$$

From Lemma 1 we deduce that $\boldsymbol{\psi}_0$ is zero. Hence the solution to problem (2) is unique.

(b) We now prove existence. Let $(\boldsymbol{u},p)$ be the solution to the Stokes problem (1). Let us set $\boldsymbol{\omega}=\nabla\times\boldsymbol{u}$. It is clear that $\boldsymbol{\omega}\in\boldsymbol{H}(\mathrm{div})$. Furthermore, $-\Delta\boldsymbol{u}=\nabla\times\boldsymbol{\omega}=\boldsymbol{f}-\nabla p$; since $\boldsymbol{f}\in(\boldsymbol{H}_0(\mathrm{div}))'$ and $p\in L^2\Rightarrow\nabla p\in(\boldsymbol{H}_0(\mathrm{div}))'$, we infer that $\nabla\times\boldsymbol{\omega}$ is in $(\boldsymbol{H}_0(\mathrm{div}))'$. As a result $\boldsymbol{\omega}$ belongs to the space $\boldsymbol{W}$.

Let us now show that $\boldsymbol{\omega}$ satisfies the first equation of problem (2). From the Stokes problem we have

$$\forall \boldsymbol{v} \in \boldsymbol{H}_0(\mathrm{div}=0) \qquad \begin{aligned} -\langle \Delta \boldsymbol{u}, \boldsymbol{v}\rangle + \langle \boldsymbol{\nabla} p, \boldsymbol{v}\rangle &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle, \\ \langle \boldsymbol{\nabla}\times\boldsymbol{\omega}, \boldsymbol{v}\rangle &= \langle \boldsymbol{f}, \boldsymbol{v}\rangle. \end{aligned}$$

Since the mapping $\boldsymbol{\nabla}\times : \boldsymbol{H}(\mathrm{div}=0) \cap \boldsymbol{H}_0(\mathrm{rot}) \longrightarrow \boldsymbol{H}_0(\mathrm{div}=0)$ is an isomorphism, according to Lemma 2, we deduce that the mapping $\boldsymbol{\nabla}\times : \boldsymbol{X}_N \stackrel{\mathrm{def}}{=} \boldsymbol{\Psi} \longrightarrow \boldsymbol{H}_0(\mathrm{div}=0)$ is surjective; as a result $\boldsymbol{\omega}$ satisfies the following equation:

$$\forall \boldsymbol{\varphi} \in \boldsymbol{\Psi} \qquad \langle \boldsymbol{\nabla}\times\boldsymbol{\omega}, \boldsymbol{\nabla}\times\boldsymbol{\varphi}\rangle + (\boldsymbol{\nabla}\cdot\boldsymbol{\omega}, \boldsymbol{\nabla}\cdot\boldsymbol{\varphi}) = \langle \boldsymbol{f}, \boldsymbol{\nabla}\times\boldsymbol{\varphi}\rangle,$$

where $\boldsymbol{\nabla}\cdot\boldsymbol{\omega} = 0$ by definition.

From Lemma 2 we infer that there is $\boldsymbol{\psi} \in \boldsymbol{H}(\mathrm{div}=0) \cap \boldsymbol{H}_0(\mathrm{rot})$ such that $\boldsymbol{\nabla}\times\boldsymbol{\psi} = \boldsymbol{u}$. This vector potential $\boldsymbol{\psi}$ satisfies

$$\forall \boldsymbol{v} \in \boldsymbol{W} \qquad \langle \boldsymbol{\nabla}\times\boldsymbol{\psi}, \boldsymbol{\nabla}\times\boldsymbol{v}\rangle + (\boldsymbol{\nabla}\cdot\boldsymbol{\psi}, \boldsymbol{\nabla}\cdot\boldsymbol{v}) = \langle \boldsymbol{u}, \boldsymbol{\nabla}\times\boldsymbol{v}\rangle,$$

from which we deduce that $\boldsymbol{\psi}$ is the solution to the second equation of problem (2), since $\langle \boldsymbol{u}, \boldsymbol{\nabla}\times\boldsymbol{v}\rangle = (\boldsymbol{\omega}, \boldsymbol{v})$. This concludes the proof of existence. □

Remark 1. Formally, problem (2) consists in solving the following PDE system:

$$\left\{ \begin{aligned} -\Delta\boldsymbol{\omega} &= \boldsymbol{\nabla}\times\boldsymbol{f}, \\ -\Delta\boldsymbol{\psi} &= \boldsymbol{\omega}, \\ \boldsymbol{\nabla}\cdot\boldsymbol{\omega}_{|\Gamma} &= 0, \\ \boldsymbol{\nabla}\cdot\boldsymbol{\psi}_{|\Gamma} &= 0, \\ \boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\psi}_{|\Gamma} &= 0, \\ \boldsymbol{n}\times\boldsymbol{\psi}_{|\Gamma} &= 0. \end{aligned} \right.$$

Note that the boundary conditions $\boldsymbol{\nabla}\cdot\boldsymbol{\omega}_{|\Gamma} = 0$, $\boldsymbol{\nabla}\cdot\boldsymbol{\psi}_{|\Gamma} = 0$ and $\boldsymbol{n}\times\boldsymbol{\nabla}\times\boldsymbol{\psi}_{|\Gamma} = 0$ are natural; *i.e*, they are naturally enforced by the variational formulation. The boundary condition $\boldsymbol{n}\cdot\boldsymbol{\omega}_{|\Gamma} = 0$ is not enforced, neither in a natural way nor in an essential one. The fact that $\boldsymbol{n}\cdot\boldsymbol{\omega}_{|\Gamma} = 0$ is a consequence of the present formulation.

3. APPROXIMATION OF THE ω-ψ VARIATIONAL PROBLEM

Let us assume that Ω is a polyhedron in $\mathbb{R}^3$, Ω is simply connected and Γ is connected. Let $\mathcal{T}_h$ be a (uniformly) regular triangulation of Ω. We denote by $\boldsymbol{W}_h$ and $\boldsymbol{\Psi}_h$ the discrete counterpart of $\boldsymbol{W}$ and $\boldsymbol{\Psi}$, respectively, so that

$$\begin{aligned} \boldsymbol{W}_h &= \Big\{ \boldsymbol{v}_h \in \boldsymbol{C}^0(\overline{\Omega}) \;\Big|\; \boldsymbol{v}_{h|T} \in P_k,\ \forall T \in \mathcal{T}_h \Big\}, \\ \boldsymbol{\Psi}_h &= \Big\{ \boldsymbol{\varphi}_h \in \boldsymbol{C}^0(\overline{\Omega}) \;\Big|\; \boldsymbol{\varphi}_{h|T} \in P_k,\ \forall T \in \mathcal{T}_h, \quad \boldsymbol{n}\times\boldsymbol{\varphi}_{h|\Gamma} = 0 \Big\}. \end{aligned}$$

The discrete version of the variational problem (2) reads: find $\boldsymbol{\omega}_h \in \boldsymbol{W}_h$ and $\boldsymbol{\psi}_h \in \boldsymbol{\Psi}_h$ so that

$$\begin{cases} \forall \boldsymbol{\varphi}_h \in \boldsymbol{\Psi}_h & \langle \nabla\times\boldsymbol{\varphi}_h, \nabla\times\boldsymbol{\omega}_h\rangle + (\nabla\cdot\boldsymbol{\varphi}_h, \nabla\cdot\boldsymbol{\omega}_h) \;=\; \langle \boldsymbol{f}, \nabla\times\boldsymbol{\varphi}_h\rangle, \\ \forall \boldsymbol{v}_h \in \boldsymbol{W}_h & -(\boldsymbol{\omega}_h, \boldsymbol{v}_h) + \langle \nabla\times\boldsymbol{\psi}_h, \nabla\times\boldsymbol{v}_h\rangle + (\nabla\cdot\boldsymbol{\psi}_h, \nabla\cdot\boldsymbol{v}_h) \;=\; 0. \end{cases} \tag{3}$$

We set hereafter $a(\boldsymbol{\varphi}, \boldsymbol{v}) = \langle \nabla\times\boldsymbol{\varphi}, \nabla\times\boldsymbol{v}\rangle + (\nabla\cdot\boldsymbol{\varphi}, \nabla\times\boldsymbol{v}),\ \forall\boldsymbol{\varphi} \in \boldsymbol{\Psi}, \forall\boldsymbol{v} \in \boldsymbol{W}$.

Proposition 2 *The discrete variational problem (3) has a unique solution and this solution is stable in the sense that*

$$|\boldsymbol{\omega}_h|_0 + |\boldsymbol{\psi}_h|_{\boldsymbol{X}_N} \le c|\boldsymbol{f}|_{(\boldsymbol{H}_0(\mathrm{div}))'}.$$

Proof. By taking $\boldsymbol{\psi}_h$ as test function in the first equation of (3) and $\boldsymbol{\omega}_h$ in the second equation, we obtain, after subtracting one equation from the other,

$$\begin{aligned} |\boldsymbol{\omega}_h|_0^2 &= \langle \boldsymbol{f}, \nabla\times\boldsymbol{\psi}_h\rangle \\ &\le |\boldsymbol{f}|_{(\boldsymbol{H}_0(\mathrm{div}))'}\, |\boldsymbol{\psi}_h|_{\boldsymbol{X}_N}. \end{aligned}$$

Since $\boldsymbol{\Psi}_h \subset \boldsymbol{W}_h$, we take $\boldsymbol{\psi}_h$ as test function in the second equation of (3)

$$\begin{aligned} |\nabla\times\boldsymbol{\psi}_h|_0^2 + |\nabla\cdot\boldsymbol{\psi}_h|_0^2 &= (\boldsymbol{\omega}_h, \boldsymbol{\psi}_h) \\ &\le |\boldsymbol{\omega}_h|_0\, |\boldsymbol{\psi}_h|_0 \le |\boldsymbol{\omega}_h|_0\, |\boldsymbol{\psi}_h|_{\boldsymbol{X}_N}. \end{aligned}$$

Thanks to Lemma 1 we deduce $c|\boldsymbol{\psi}_h|_{\boldsymbol{X}_N}^2 \le |\boldsymbol{\omega}_h|_0\, |\boldsymbol{\psi}_h|_{\boldsymbol{X}_N}$. Hence $|\boldsymbol{\psi}_h|_{\boldsymbol{X}_N} \le |\boldsymbol{\omega}_h|_0$; as a result, we obtain $|\boldsymbol{\omega}_h|_0 + |\boldsymbol{\psi}_h|_{\boldsymbol{X}_N} \le c|\boldsymbol{f}|_{(\boldsymbol{H}_0(\mathrm{div}))'}$. This completes the proof. □

We now perform a crude error analysis to give an idea of the convergence properties of the approximate solution $(\boldsymbol{\omega}_h, \boldsymbol{\psi}_h)$. A first convergence result is given by

Lemma 3 *The solution to the discrete variational problem (3) satisfies the following unequality*

$$|\boldsymbol{\omega}_h - \boldsymbol{\omega}|_0 + |\boldsymbol{\psi}_h - \boldsymbol{\psi}|_{\boldsymbol{X}_N} \le c\left(\inf_{\boldsymbol{\alpha}_h\in\boldsymbol{W}_h} |\boldsymbol{\omega}_h - \boldsymbol{\alpha}_h|_1 + h^{-1}\inf_{\boldsymbol{\beta}_h\in\boldsymbol{\Psi}_h} |\boldsymbol{\psi}_h - \boldsymbol{\beta}_h|_1\right).$$

As a consequence we infer

Proposition 3 *If $\boldsymbol{\omega} \in \boldsymbol{H}^\ell$ and $\boldsymbol{\psi} \in \boldsymbol{H}^{\ell+1}$, we have*

$$|\nabla\times\boldsymbol{\psi}_h - \boldsymbol{u}|_0 + |\boldsymbol{\omega}_h - \boldsymbol{\omega}|_0 + |\boldsymbol{\psi}_h - \boldsymbol{\psi}|_{\boldsymbol{X}_N} \le ch^{\ell-1}\left(|\boldsymbol{\omega}|_{\boldsymbol{H}^\ell} + |\boldsymbol{\psi}|_{\boldsymbol{H}^{\ell+1}}\right),$$

where $\boldsymbol{u}$ is the velocity field solution to the Stokes problem (1).

4. GLOWINSKI–PIRONNEAU METHOD IN 3D

This section is devoted to the presentation of a method aiming at reducing the discrete problem (3) to a set of simpler ones which involve only the solution of vector Poisson problems plus a small problem for the tangential trace of the vorticity. The method to be described is templated from the one proposed by Glowinski and Pironneau for the uncoupled solution of the two-dimensional ω-ψ equations [5]. The strategy for uncoupling the vector equations governing three-dimensional flows was outlined in [7].

Let us introduce $\boldsymbol{T}_h$ the space of the tangential trace of the vector fields of $\boldsymbol{W}_h$:

$$\boldsymbol{T}_h = \Big\{ \boldsymbol{v}_h \in \boldsymbol{W}_h \;\Big|\; \boldsymbol{v}_h(a_h) = 0,\ \forall a_h,\ a_h \text{ interior node},\ \boldsymbol{n}\cdot\boldsymbol{v}_{h|\Gamma} = 0 \Big\}.$$

By construction we have

Proposition 4 *The space $\boldsymbol{W}_h$ can be decomposed as follows: $\boldsymbol{W}_h = \boldsymbol{\Psi}_h \oplus \boldsymbol{T}_h$.*

We now introduce a "lifting" of the source term $\boldsymbol{f}$ as follows

$$\begin{cases} \boldsymbol{\omega}_{0,h} \in \boldsymbol{\Psi}_h \text{ is solution to:} & a(\boldsymbol{\varphi}_h, \boldsymbol{\omega}_{0,h}) = (\boldsymbol{f}, \boldsymbol{\nabla}\times\boldsymbol{\varphi}_h), \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\Psi}_h; \\ \boldsymbol{\psi}_{0,h} \in \boldsymbol{\Psi}_h \text{ is solution to:} & a(\boldsymbol{\psi}_{0,h}, \boldsymbol{\varphi}_h) = (\boldsymbol{\omega}_{0,h}, \boldsymbol{\varphi}_h), \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\Psi}_h. \end{cases}$$

Note that the two problems above are well posed, since, thanks to Lemma 1, the bilinear form $a\ :\ \boldsymbol{\Psi}\times\boldsymbol{\Psi} \longrightarrow \mathbb{R}$ is coercive. Now we shall introduce a bilinear form on $\boldsymbol{T}_h \times \boldsymbol{T}_h$ as follows. Let $\boldsymbol{t}_h$ be in $\boldsymbol{T}_h$, we define $\boldsymbol{\omega}_h(\boldsymbol{t}_h) \in \boldsymbol{W}_h$ and $\boldsymbol{\psi}_h(\boldsymbol{t}_h) \in \boldsymbol{\Psi}_h$ so that

$$\begin{cases} \boldsymbol{\omega}_h(\boldsymbol{t}_h) - \boldsymbol{t}_h \in \boldsymbol{\Psi}_h, \\ a(\boldsymbol{\varphi}_h, \boldsymbol{\omega}_h(\boldsymbol{t}_h)) = 0, \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\Psi}_h; \end{cases}$$

$$\begin{cases} \boldsymbol{\psi}_h(\boldsymbol{t}_h) \in \boldsymbol{\Psi}_h, \\ a(\boldsymbol{\psi}_h(\boldsymbol{t}_h), \boldsymbol{\varphi}_h) = (\boldsymbol{\omega}_h(\boldsymbol{t}_h), \boldsymbol{\varphi}_h), \qquad \forall \boldsymbol{\varphi}_h \in \boldsymbol{\Psi}_h. \end{cases}$$

The vorticity fields $\boldsymbol{\omega}_h(\boldsymbol{t}_h)$ allows us to introduce the bilinear form $i_h\ :\ \boldsymbol{T}_h\times\boldsymbol{T}_h \longrightarrow \mathbb{R}$ defined by

$$i_h(\boldsymbol{t}_h, \boldsymbol{t}'_h) = (\boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{\omega}_h(\boldsymbol{t}'_h)).$$

Then we have the following properties

Proposition 5 *The bilinear form i_h is symmetric definite positive.*

Proof. The symmetry and positiveness being evident, let us show that i_h is definite. Let $\boldsymbol{t}_h$ be in $\boldsymbol{T}_h$ so that $i_h(\boldsymbol{t}_h, \boldsymbol{t}_h) = 0$. From the definition of i_h we infer that $\boldsymbol{\omega}_h(\boldsymbol{t}_h)$ is zero. From the definition $\boldsymbol{\omega}_h(\boldsymbol{t}_h) - \boldsymbol{t}_h \in \boldsymbol{\Psi}_h$ we infer $\boldsymbol{t}_h \in \boldsymbol{\Psi}_h \cap \boldsymbol{T}_h$, but the sum $\boldsymbol{\psi}_h \oplus \boldsymbol{T}_h$ being direct, we obtain $\boldsymbol{t}_h = 0$. □

We have the following decomposition of the discrete problem (3).

Proposition 6 *The solution of the discrete variational problem (3) is given by:*

$$\boldsymbol{\omega}_h = \boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h) \qquad \textit{and} \qquad \boldsymbol{\psi}_h = \boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h), \tag{4}$$

where $\boldsymbol{t}_h \in \boldsymbol{T}_h$ is solution to the problem

$$\forall \boldsymbol{t}'_h \in \boldsymbol{T}_h, \qquad i_h(\boldsymbol{t}_h, \boldsymbol{t}'_h) = a(\boldsymbol{\psi}_{0,h}, \boldsymbol{t}'_h) - (\boldsymbol{\omega}_{0,h}, \boldsymbol{t}'_h). \tag{5}$$

Proof. Thanks to Proposition 5, problem (5) has a unique solution. Let us now prove that $\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h)$ and $\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h)$ are indeed the solutions to problem (3).

(a) Note first that by construction we have

$$a(\boldsymbol{\varphi}'_h,\, \boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h)) = 0, \qquad \forall \boldsymbol{\varphi}'_h \in \boldsymbol{\Psi}_h,$$

hence $\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h)$ satisfies the first equation of problem (3).

(b) We prove now that the second equation of (3) is also satisfied. For $\boldsymbol{t}'_h$ and $\boldsymbol{\varphi}'_h$ arbitrarily chosen in $\boldsymbol{T}_h$ and $\boldsymbol{\Psi}_h$ we have:

$$\begin{aligned}
(\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) &= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h) + (\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) - a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&\quad + (\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) - (\boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{\omega}_h(\boldsymbol{t}'_h)) \\
&\quad - a(\boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) + (\boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) + (\boldsymbol{\omega}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h - \boldsymbol{\omega}_h(\boldsymbol{t}'_h)) \\
&\quad - a(\boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) + a(\boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h - \boldsymbol{\omega}_h(\boldsymbol{t}'_h)) \\
&\quad - a(\boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{t}'_h) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h) - a(\boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\omega}_h(\boldsymbol{t}'_h)) \\
&= a(\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h),\, \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h).
\end{aligned}$$

Since every $\boldsymbol{v}'_h \in \boldsymbol{W}_h$ can be decomposed into the sum $\boldsymbol{v}'_h = \boldsymbol{\varphi}'_h + \boldsymbol{t}'_h$ (thanks to Proposition 5), we infer that the second equation of problem (3) is satisfied. Hence $\boldsymbol{\omega}_{0,h} + \boldsymbol{\omega}_h(\boldsymbol{t}_h) = \boldsymbol{\omega}_h$ and $\boldsymbol{\psi}_{0,h} + \boldsymbol{\psi}_h(\boldsymbol{t}_h) = \boldsymbol{\psi}_h$. □

To interpret the meaning of the bilinear form i_h in the spatial continuum, we follow a path very similar to that of Glowinski and Pironneau [5] and build a bilinear form on $\boldsymbol{H}^{1/2}(\Gamma) \times \boldsymbol{H}^{1/2}(\Gamma)$.

Consider $\boldsymbol{t}$ in $\boldsymbol{H}^{1/2}(\Gamma)$ so that $\boldsymbol{n}\cdot\boldsymbol{t}_{|\Gamma} = 0$, and introduce $\widetilde{\boldsymbol{\omega}} \in \boldsymbol{H}^1(\Omega)$, a lifting of $\boldsymbol{t}$, $\widetilde{\boldsymbol{\omega}}_{|\Gamma} = \boldsymbol{t}$. We now consider the following problem: find $\boldsymbol{\omega}(\boldsymbol{t}) - \widetilde{\boldsymbol{\omega}} \in \boldsymbol{\Psi}$ so that $a(\boldsymbol{\omega}(\boldsymbol{t}) - \widetilde{\boldsymbol{\omega}},\, \boldsymbol{\varphi}) = -a(\widetilde{\boldsymbol{\omega}}, \boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in \boldsymbol{\Psi}$. This problem is well posed thanks to Lax–Milgram's theorem; $\boldsymbol{\omega}(\boldsymbol{t})$ is in $\boldsymbol{H}(\mathrm{rot}) \cap \boldsymbol{H}(\mathrm{div})$ and satisfies $a(\boldsymbol{\omega}(\boldsymbol{t}), \boldsymbol{\varphi})) = 0$. Formally $\boldsymbol{\omega}(\boldsymbol{t})$ is solution to

$$\begin{cases}
\Delta\boldsymbol{\omega}(\boldsymbol{t}) = 0, \\
\boldsymbol{\nabla}\cdot\boldsymbol{\omega}(\boldsymbol{t})_{|\Gamma} = 0, \\
\boldsymbol{n}\times\boldsymbol{\omega}(\boldsymbol{t})_{|\Gamma} = \boldsymbol{n}\times\boldsymbol{t}.
\end{cases}$$

By simple arguments, we infer that $\boldsymbol{\nabla}\cdot\boldsymbol{\omega}(\boldsymbol{t}) = 0$. Let us now introduce $\boldsymbol{\psi}(\boldsymbol{t}) \in \boldsymbol{\Psi}$ so that $a(\boldsymbol{\psi}(\boldsymbol{t}),\boldsymbol{\varphi}) = (\boldsymbol{\omega}(\boldsymbol{t}),\boldsymbol{\varphi})$ for all $\boldsymbol{\varphi} \in \boldsymbol{\Psi}$. Formally $\boldsymbol{\psi}(\boldsymbol{t})$ is solution to

$$\left\{ \begin{array}{rcl} -\Delta\boldsymbol{\psi}(\boldsymbol{t}) &=& \boldsymbol{\omega}(\boldsymbol{t}), \\ \nabla\cdot\boldsymbol{\psi}(\boldsymbol{t})_{|\Gamma} &=& 0, \\ \boldsymbol{n}\times\boldsymbol{\psi}(\boldsymbol{t})_{|\Gamma} &=& 0. \end{array} \right.$$

Clearly $\nabla\cdot\boldsymbol{\psi}(\boldsymbol{t}) = 0$ in Ω, as a result $\nabla\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t}) = \boldsymbol{\omega}(\boldsymbol{t}) \in \boldsymbol{L}^2(\Omega)$, hence by classical trace theorem one infers that $[\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t})]_{|\Gamma} \in \boldsymbol{H}^{-1/2}(\Gamma)$, with no other hypotheses than that of Ω being $C^{0,1}$ (and Ω is a polyhedron, for the sake of simplicity of the approximation theory).

Let us define $\boldsymbol{\mathcal{I}} : \boldsymbol{H}^{1/2}(\Gamma) \longrightarrow \boldsymbol{H}^{-1/2}(\Gamma)$ so that $\boldsymbol{\mathcal{I}}(\boldsymbol{t}) = [\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t})]_{|\Gamma}$, where $\boldsymbol{\psi}(\boldsymbol{t})$ is defined as above. This operator is clearly bounded, *i.e.*, $\boldsymbol{\mathcal{I}} \in \mathcal{L}(\boldsymbol{H}^{1/2}(\Gamma), \boldsymbol{H}^{-1/2}(\Gamma))$, since

$$\begin{array}{rcl} |\boldsymbol{\mathcal{I}}(\boldsymbol{t})|_{-1/2} &=& \big|[\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t})]_{|\Gamma}\big|_{\boldsymbol{H}^{-1/2}(\Gamma)} \\ &\leq& c|\nabla\times\boldsymbol{\psi}(\boldsymbol{t})|_{\boldsymbol{H}(\mathrm{rot})} \\ &\leq& c|\boldsymbol{\omega}(\boldsymbol{t})|_0 \\ &\leq& c|\widetilde{\boldsymbol{\omega}}|_1 \\ &\leq& c|\boldsymbol{t}|_{1/2}. \end{array}$$

Let $i : \boldsymbol{H}^{1/2}(\Gamma)\times\boldsymbol{H}^{1/2}(\Gamma) \longrightarrow \mathbb{R}$ be the bilinear form defined by $i(\boldsymbol{t},\boldsymbol{t}') = \langle\boldsymbol{\mathcal{I}}(\boldsymbol{t}),\boldsymbol{t}'\rangle_\Gamma$.

Proposition 7 *The bilinear form i has the following alternative expression*

$$i(\boldsymbol{t},\boldsymbol{t}') = (\boldsymbol{\omega}(\boldsymbol{t}),\boldsymbol{\omega}(\boldsymbol{t}')).$$

Proof.

$$\begin{array}{rcl} i(\boldsymbol{t},\boldsymbol{t}') &=& \langle[\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t})]_{|\Gamma},\, \boldsymbol{t}'\rangle_\Gamma \\ &=& (\nabla\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t}),\, \boldsymbol{\omega}(\boldsymbol{t}')) - (\nabla\times\boldsymbol{\psi}(\boldsymbol{t}), \nabla\times\boldsymbol{\omega}(\boldsymbol{t}')) \\ &=& (\boldsymbol{\omega}(\boldsymbol{t}),\, \boldsymbol{\omega}(\boldsymbol{t}')) - a(\boldsymbol{\psi}(\boldsymbol{t}),\, \boldsymbol{\omega}(\boldsymbol{t}')), \qquad \text{since } \nabla\cdot\boldsymbol{\omega}(\boldsymbol{t}') = 0 \\ &=& (\boldsymbol{\omega}(\boldsymbol{t}),\, \boldsymbol{\omega}(\boldsymbol{t}')), \qquad \text{since } a(\boldsymbol{\varphi},\boldsymbol{\omega}(\boldsymbol{t}')) = 0,\ \forall\boldsymbol{\varphi} \in \boldsymbol{\Psi}.\ \Box \end{array}$$

Proposition 7 shows that i_h is the discrete counterpart of i. By definition i is associated to the linear operator $\boldsymbol{\mathcal{I}}$ which associates $[\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}(\boldsymbol{t})]_{|\Gamma}$ to the tangential trace of the vorticity field $\boldsymbol{\omega}(\boldsymbol{t})$. As a result, problem (5) can be interpreted in the following way: it is the discrete counterpart of the problem consisting in finding the *right* tangential trace of the vorticity field which makes the tangential trace of $\nabla\times\boldsymbol{\psi}$ to vanish, ($\boldsymbol{n}\times\nabla\times\boldsymbol{\psi}_{|\Gamma} = 0$), that is which makes the tangential velocity field to vanish on the boundary.

This formulation of the discrete problem (3) parallels completely the approach of Glowinski and Pironneau for the ω-ψ problem in 2D [5]. Recall that in 2D the Glowinski–Pironneau method consists roughly in looking for the right trace of the (scalar) vorticity on the boundary so that $(\partial\psi/\partial n)_{|\Gamma}$ is zero, *i.e.*, the tangential trace of velocity is zero.

We now give a bound on the condition number of the linear system which is generated by problem (5). To this purpose, the two following hypotheses on Ω are introduced:

Hypothesis (H$_1$): For $\boldsymbol{l} \in \boldsymbol{L}^2(\Omega)$, the solution to problem

$$\begin{cases} \text{Find } \boldsymbol{e} \in \boldsymbol{\Psi} \text{ so that} \\ a(\boldsymbol{e}, \boldsymbol{\varphi}) = (\boldsymbol{l}, \boldsymbol{\varphi}) \end{cases}$$

is in $\boldsymbol{H}^2(\Omega)$ and $|\boldsymbol{e}|_2 \leq c|\boldsymbol{l}|_0$.

Hypothesis (H$_2$): The operator $\boldsymbol{\mathcal{I}}$ defined above can be extended to $\boldsymbol{L}^2(\Gamma)$ with range in $\boldsymbol{L}^2(\Gamma)$, still denoted by $\boldsymbol{\mathcal{I}} \ : \ \boldsymbol{L}^2(\Gamma) \longrightarrow \boldsymbol{L}^2(\Gamma)$, and this operator is such that $|\boldsymbol{\omega}(\boldsymbol{t})|_{\boldsymbol{L}^2(\Omega)} \leq c|\boldsymbol{t}|_{\boldsymbol{L}^2(\Gamma)}$.

Remark 2. Note that the vector function $\boldsymbol{e}$ defined in (H$_1$) is solution to

$$\begin{cases} -\Delta \boldsymbol{e} \ = \ \boldsymbol{l}, \\ \boldsymbol{\nabla} \cdot \boldsymbol{e}_{|\Gamma} \ = \ 0, \\ \boldsymbol{n} \times \boldsymbol{e}_{|\Gamma} \ = \ 0. \end{cases}$$

If Ω is $C^{1,1}$, (H$_1$) is satisfied automatically.

Remark 3. It can be shown that (H$_2$) holds if Ω is such that there is some $\epsilon > 0$ so that $\boldsymbol{X}_T \overset{\text{def}}{=} \boldsymbol{H}_0(\text{div}) \cap \boldsymbol{H}(\text{rot})$ is continuously embedded in $\boldsymbol{H}^{1/2+\epsilon}(\Omega)$. In particular, this is true with $\epsilon = 1/2$ if Ω is convex or Ω is $C^{1,1}$.

Proposition 8 *If Ω is such that the hypotheses (H$_1$) and (H$_2$) are satisfied, there are $c_1 > 0$ and $c_2 > 0$ so that*

$$c_1 h |\gamma_0 \boldsymbol{t}_h|_{\boldsymbol{L}^2(\Gamma)} \leq i_h(\boldsymbol{t}_h, \boldsymbol{t}_h) \leq c_2 |\gamma_0 \boldsymbol{t}_h|_{\boldsymbol{L}^2(\Gamma)}$$

as $h \rightarrow 0$, where γ_0 denotes the trace operator.

5. $\boldsymbol{\omega}$-$\boldsymbol{\phi}$ FORMULATION WITH $\boldsymbol{n} \cdot \boldsymbol{\phi}_{|\Gamma} = 0$ AND $\boldsymbol{n} \cdot \nabla \times \boldsymbol{\phi}_{|\Gamma} = 0$

We terminate this paper by introducing another possible vorticity/stream vector formulation of the Stokes problem (1) which is based on the existence of tangential vector potentials, as expressed by the following lemma (see [2] for a proof).

Lemma 4 *Provided Γ is connected, Ω is simply connected and $C^{1,1}$, then the following mapping is an isomorphism:*

$$\boldsymbol{\nabla} \times \ : \ \boldsymbol{H}_0(\text{div} = 0) \cap \boldsymbol{H}(\text{rot}) \longrightarrow \boldsymbol{H}(\text{div} = 0).$$

This Lemma implies that for every velocity field, $\boldsymbol{v}$ in $\boldsymbol{H}(\mathrm{div}=0)$, there is a vector potential $\boldsymbol{\psi}$ in $\boldsymbol{H}_0(\mathrm{div}=0) \cap \boldsymbol{H}(\mathrm{rot})$ so that $\boldsymbol{v} = \nabla\times\boldsymbol{\psi}$. If we restrict $\boldsymbol{v}$ to $\boldsymbol{H}_0(\mathrm{div}=0)$, the space of the test functions for the Stokes problem, we are led to introduce:

$$\boldsymbol{\Phi} = \Big\{ \boldsymbol{\phi} \in \boldsymbol{H}(\mathrm{div}) \cap \boldsymbol{H}(\mathrm{rot}) \;\Big|\; \boldsymbol{n}\cdot\boldsymbol{\phi}_{|\Gamma} = 0,\ \boldsymbol{n}\cdot\nabla\times\boldsymbol{\phi}_{|\Gamma} = 0 \Big\}.$$

The new vector potential for the velocity $\boldsymbol{u}$ will be sought in $\boldsymbol{\Phi}$, while the vorticity field will be sought in the same space $\boldsymbol{W}$, as before. Hence, we consider the following problem: For $\boldsymbol{f} \in (\boldsymbol{H}_0(\mathrm{div}))'$, find $\boldsymbol{\omega} \in \boldsymbol{W}$ and $\boldsymbol{\phi} \in \boldsymbol{\Phi}$ so that

$$\begin{cases} \forall \boldsymbol{\varphi} \in \boldsymbol{\Phi} & \langle \nabla\times\boldsymbol{\varphi}, \nabla\times\boldsymbol{\omega}\rangle + (\nabla\cdot\boldsymbol{\varphi}, \nabla\cdot\boldsymbol{\omega}) \;=\; \langle \boldsymbol{f}, \nabla\times\boldsymbol{\varphi}\rangle, \\ \forall \boldsymbol{v} \in \boldsymbol{W} & -(\boldsymbol{\omega}, \boldsymbol{v}) + \langle \nabla\times\boldsymbol{\phi}, \nabla\times\boldsymbol{v}\rangle + (\nabla\cdot\boldsymbol{\phi}, \nabla\cdot\boldsymbol{v}) \;=\; 0. \end{cases} \tag{6}$$

Proposition 9 *The variational problem (6) has a unique solution and this solution is such that $\boldsymbol{\omega} = \nabla\times\boldsymbol{u}$ and $\nabla\times\boldsymbol{\phi} = \boldsymbol{u}$, $\boldsymbol{u}$ being the velocity field solution to the Stokes problem (1).*

This problem consists formally in solving the following PDE system:

$$\begin{cases} -\Delta\boldsymbol{\omega} \;=\; \nabla\times\boldsymbol{f}, \\ -\Delta\boldsymbol{\phi} \;=\; \boldsymbol{\omega}, \\ \oint_\Gamma (\nabla\times\boldsymbol{\omega})\cdot\boldsymbol{n}\times\boldsymbol{\varphi}_\Gamma \;=\; \oint_\Gamma \boldsymbol{f}\cdot\boldsymbol{n}\times\boldsymbol{\varphi}_\Gamma, \quad \forall\boldsymbol{\varphi}_\Gamma \in \boldsymbol{\Phi}_{|\Gamma}, \\ \boldsymbol{n}\cdot\boldsymbol{\phi}_{|\Gamma} \;=\; 0, \\ \boldsymbol{n}\cdot\nabla\times\boldsymbol{\phi}_{|\Gamma} \;=\; 0, \\ \nabla\cdot\boldsymbol{\phi}_{|\Gamma} \;=\; 0, \\ \boldsymbol{n}\times\nabla\times\boldsymbol{\phi}_{|\Gamma} \;=\; 0. \end{cases}$$

Note the quite peculiar character of the (surfacial) integral condition on the vorticity which involves only the trace of $\boldsymbol{\Phi}$: such a condition can be interpreted as a condition enforcing that the tangential trace $[\boldsymbol{n}\times(\nabla\times\boldsymbol{\omega} - \boldsymbol{f})]_{|\Gamma}$ is the (tangential) gradient of a scalar function defined on the boundary.

To conclude with this formulation, we emphasize that, although admissible, it is difficult to envisage its practical implementation since it has the condition $\boldsymbol{n}\cdot\nabla\times\boldsymbol{\phi}_{|\Gamma} = 0$ as an essential boundary condition. Relaxing such a condition would be equivalent to re-introducng the trace of the pressure field on Γ.

Acknowledgments

This work has been partly supported by the French/Italian *Galileo Program*.

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