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SIMULATION OF 2D EXTERNAL INCOMPRESSIBLE VISCOUS FLOWS BY MEANS OF A DOMAIN DECOMPOSITION METHOD

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1. INTRODUCTION – Visualization of incompressible viscous flows shows that vorticity concentrates in wakes and advection dominates viscous diffusion as the Reynolds number increases. These conditions are favourable for simulating such flows by means of particle methods. Furthermore since this class of methods is grid-free, it is suitable for tackling problems with moving boundaries. However, particle methods are inaccurate as viscous effects are of the same order as that of advection. Hence, in boundary layers, methods which are adapted to parabolic problems are needed (*eg.* finite differences, finite elements, *etc.*). The remarks above led us to develop a domain decomposition method that combines advantages of both approaches (*cf.* [3] [5]).

2. FORMULATION OF THE PROBLEM – Consider p moving solids $(S_i)_{i=1,\dots,p}$ in a Galilean frame of reference $(O, \boldsymbol{i}, \boldsymbol{j})$ of $\mathbb{R}^2$. Let $\boldsymbol{k} = \boldsymbol{i} \times \boldsymbol{j}$, define $(\boldsymbol{O}_i)_{i=1,\dots,p}$ origins of reference for each solid, and let $\boldsymbol{v}_i$ (resp. Ω_i) be the velocity of $\boldsymbol{O}_i$ (resp. the angular velocity of S_i). The solids are immersed in an incompressible Newtonian fluid which is at rest at infinity. The fluid domain, denoted by $\mathcal{D}$, is decomposed into $p+1$ open subdomains so that $\mathcal{D} = \mathcal{D}_0 \cup_{i=1,\dots,p} \overline{\mathcal{D}_i}$, where the subdomains $\mathcal{D}_i$ are homeomorphic to a ring. It is hereafter assumed that the domain decomposition has been done so that convective effects are dominant in $\mathcal{D}_0$. Let B_i (resp. Γ_i) be the interface between $\mathcal{D}_i$ and S_i (resp. $\mathcal{D}_0$), and $\boldsymbol{n}_i$ be the outward normal to the boundary of $\mathcal{D}_i$ for $i = 0, \dots, p$. In $\mathcal{D}_0$ the Navier–Stokes equations are formulated in terms of velocity and vorticity $(\boldsymbol{u}_0, \omega_0)$ and are approximated by means of a particle method, whereas in each subdomain $\mathcal{D}_i$ they are formulated in terms of stream function and vorticity (ψ_i, ω_i) and are approximated by means of finite differences. Let $T > 0$ and $N \in \mathbb{N}$, approximations of $(\boldsymbol{u}_0, \omega_0)$ and (ψ_i, ω_i) are sought in parallel in the time interval (t_k, t_{k+1}) where $\delta t = T/N$ and $t_k = k\delta t$ for $0 \le k \le N$

3. SOLUTION IN $\mathcal{D}_0$ – Let $\boldsymbol{v}_\infty(t)$ the mean velocity of the p solids. The fluid motion is studied in a frame of reference which moves with velocity $\boldsymbol{v}_\infty(t)$. In $\mathcal{D}_0$ the advection–diffusion equation of ω_0 is approximated by:

$$\partial \omega_0^{k+1}/\partial t + \nabla.(\omega_0^{k+1} \boldsymbol{u}_0^k) = \nu \nabla^2 \omega_0^{k+1}, \tag{3.1}$$

$$\boldsymbol{u}_0^k = -\boldsymbol{v}_\infty + \int_{\mathcal{D}} \omega^k \nabla G \times \boldsymbol{k} \mathrm{d}v + \sum_{j=1}^{p} \int_{B_j} [(\boldsymbol{n}_j \times \boldsymbol{v}_{ej}) \times \nabla G + (\boldsymbol{n}_j.\boldsymbol{v}_{ej}) \nabla G] \mathrm{d}l, \tag{3.2}$$

G is the Green function of the Laplace operator in $\mathbb{R}^2$, and $\boldsymbol{v}_{ej} = \boldsymbol{v}_j + \Omega_j \times (\boldsymbol{y} - \boldsymbol{O}_j)$.

Wellposedness of the problem requires that some transmission condition through Γ_i is imposed on ω_i. Such a condition is obtained by taking into account the fact

Figure 1: Streaklines about two cylinders, $g^* = 2.5$.

that in the vicinity of Γ_i viscous diffusion is dominated by advection. Hence, (3.1) can locally be considered as a hyperbolic equation whose right hand side, $\nu\nabla^2\omega_0^{k+1}$, can be explicited and considered as a source term. For this kind of problem, Dirichlet conditions are imposed on the subset of Γ_i where the flow enters $\mathcal{D}_0$ (*eg.* see [2], [4]):

$$j = 1, \ldots, p, \qquad \omega_0^{k+1}(\boldsymbol{x}) = \omega_j^k(\boldsymbol{x}), \text{ if } \boldsymbol{u}_0^k(\boldsymbol{x}).\boldsymbol{n}_0(\boldsymbol{x}) < 0 \tag{3.3}$$

Problem (3.1) (3.2) as presented above is approximated be means of a particle method that take into account Dirichlet data (3.3) (see [3] for details on this technique).

4. SOLUTION IN $\mathcal{D}_i$ – For each subdomain $\mathcal{D}_i$, the fluid motion is studied in a non-inertial frame of reference that is linked to S_i, and the Navier–Stokes equations are formulated in terms of stream function of the relative velocity and the vorticity of the absolute velocity. Hence, the PDE's to be solved are:

$$\partial\omega_i^{k+1}/\partial t + \nabla.(\omega_i^{k+1}\nabla \times (\psi_i^{k+1}\boldsymbol{k})) = \nu\nabla^2\omega_i^{k+1} \tag{4.1}$$

$$\nabla^2\psi_i^{k+1} = 2\Omega_i - \omega_i^{k+1} \tag{4.2}$$

The system above is complemented by the following boundary conditions on B_i:

$$\psi_i^{k+1} = \psi_{B_i}^{k+1}, \quad \frac{\partial\psi_i^{k+1}}{\partial n} = 0, \text{ and } \quad \int_{B_i}\frac{\partial\omega_i^{k+1}}{\partial n_i}\mathrm{d}l = \frac{\dot{\Omega}_i}{\nu}\int_{B_i}[\boldsymbol{k} \times (\boldsymbol{y} - \boldsymbol{O}_i)].\mathrm{d}\boldsymbol{l} \tag{4.3}$$

Furthermore, transmission conditions need to be enforced so that continuity of the physical variables along with their flux is ensured across Γ_i. The conditions in question depend on the local nature of the PDE's to which ψ_i and ω_i are solutions. Since ψ_i is solution to an elliptic problem (4.2) whatever the flow nature, a transmission condition on ψ_i must be enforced everywhere on Γ_i. Such a condition is provided by a Green identity based on (4.2) and (4.3):

$$\psi_i^{k+1} = -\psi_{ei} - \int_{\mathcal{D}}\omega^k G\mathrm{d}v + \sum_{j=1}^{p}\int_{B_j}[(\psi_{ej} + \psi_{B_j}^k)\frac{\partial G}{\partial n_j} - G\frac{\partial\psi_{ej}}{\partial n_j}]\mathrm{d}l, \tag{4.4}$$

$$\text{where,} \qquad \psi_{ej}(\boldsymbol{x}) = \boldsymbol{v}_j.[(\boldsymbol{x} - \boldsymbol{O}_j) \times \boldsymbol{k}] - \Omega_j|\boldsymbol{x} - \boldsymbol{O}_j|^2/2. \tag{4.5}$$

Note that (4.4) is global, *i.e.* it transmits the whole spectrum of information to each subdomain at once, whereas classical Dirichlet-Neumann coupling conditions (*eg.* see

Figure 2: Streaklines about two cylinders, $g^* = 1$.

[2] [4]) poorly transmit low frequencies. By using the same arguments as that of §3, transmission of information on ω_i is achieved by:

$$\omega_i^{k+1}(\boldsymbol{x}) = \omega_0^k(\boldsymbol{x}), \text{ if } \boldsymbol{u}_i^k(\boldsymbol{x}).\boldsymbol{n}_i(\boldsymbol{x}) < 0. \tag{4.6}$$

As far as information transfert is concerned, this condition is sufficient. Nevertheless, since (4.1) is approximated by means of a centered finite differences scheme, a boundary condition for ω_i^{k+1} on Γ_i is required. Since the flow regime is almost hyperbolic, the piece of information that is missing on the subset of Γ_i where the flow goes out is obtained by doing an approximate Lagrangian integration of (4.1):

$$\omega_i^{k+1}(\boldsymbol{x}) = \omega_i^k(\boldsymbol{x} - \boldsymbol{u}_i^k\delta t) + \nu\delta t\nabla^2\omega_i^k(\boldsymbol{x} - \boldsymbol{u}_i^k\delta t), \text{ if } \boldsymbol{u}_i^k(\boldsymbol{x}).\boldsymbol{n}_i(\boldsymbol{x}) \geq 0. \tag{4.7}$$

The (ψ_i, ω_i) problem as formulated above is linearized and solved by means of a finite differences method that has been developed in [1].

5. NUMERICAL EXAMPLES – The present method has been coded and tested; comparisons with experimental data have shown reasonable agreement (see [3] for details on tests). In the three examples shown below we try to emphasize the versatility of the present approach and give some flavor of its possibilities.

The first example concerns the flow about two interacting cylinders. In figures 1 and 2 are shown the streaklines about two impulsively started cylinders at times $t = 80$ (fig. 1) and $t = 70$ (fig. 2). The Reynolds number based on the diameter of the cylinders is equal to 110. In each case the finite differences domains are composed of rings the width of which are set to one cylinder radius. The dimensionless gap g^* between the cylinders (i.e. gap/diameter) is equal to 2.5 and 1 in case 1 and case 2 respectively. It is clearly shown here that the stable flow regime consists of two out of phase Karman streets. The phase between the two vortex sheddings depends on the dimensionless gap g^*. A stability analysis of the wake interactions by means of the present method is under way. Since the finite differences subdomains are disconnected, it is possible, at the same numerical cost, to let the cylinders oscillate.

In figures 3, 4, and 5 we present numerical simulations of the flow about tandem airfoils. The leading airfoil oscillates in pitch and the rear one is fixed. This configuration may be viewed as a model for the rotor/stator interaction in turbines and rotating machines. The fluid domain is decomposed into three subdomains as shown

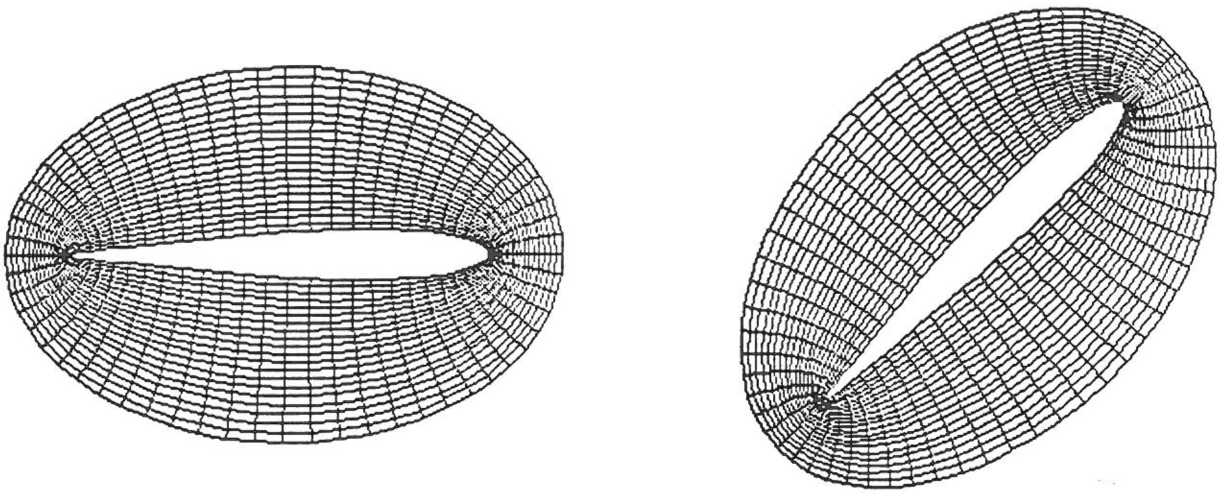

Figure 3: Definition of the finite differences subdomains for the tandem airfoils.

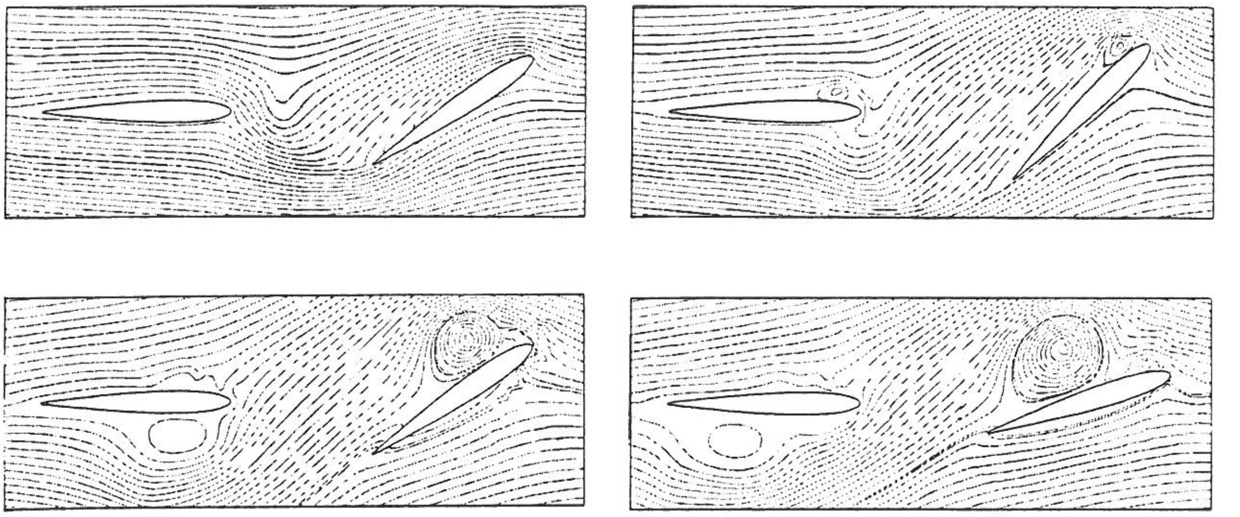

Figure 4: Streamline patterns about impulsively started tandem airfoils.

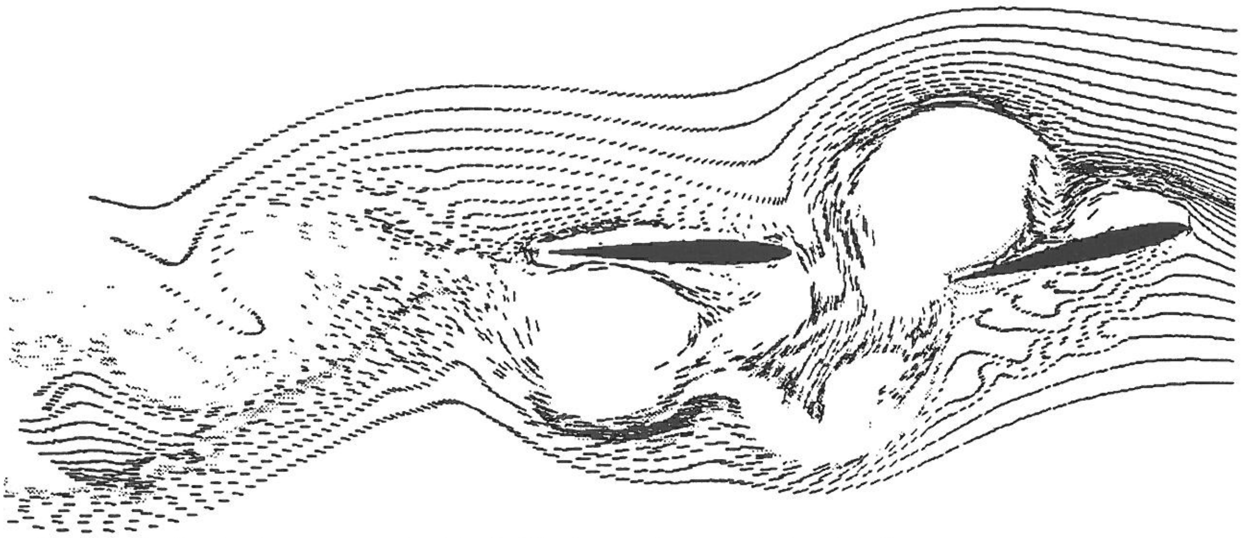

Figure 5: Instantaneous velocity field about the tamdem airfoil at time $t = 10$. Notice the large eddies shed by the leading airfoil.

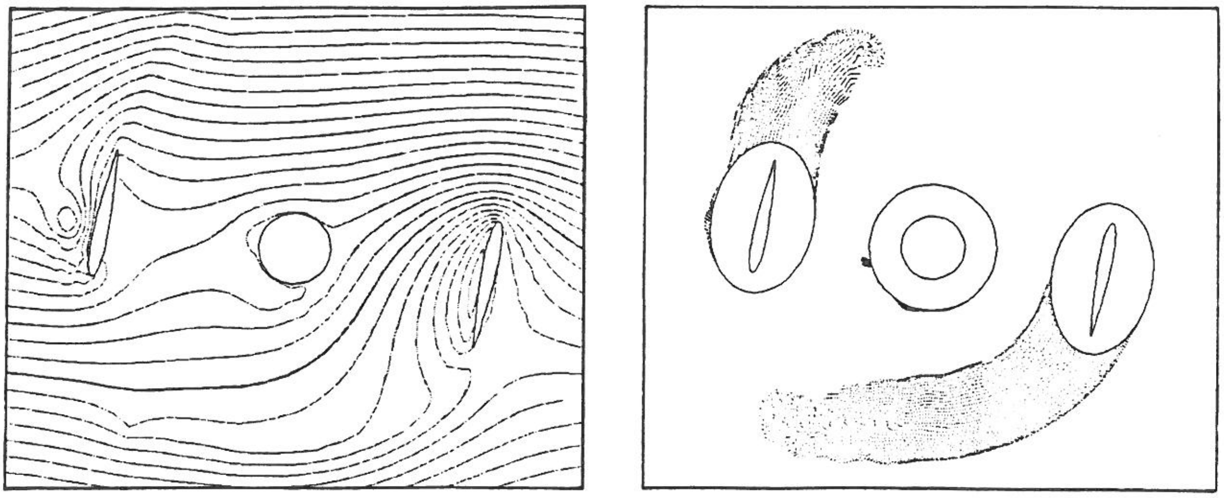

Figure 6: Shed particles and streamline patterns about a Darrieus wind turbine at $t = 2$.

in figure 3. Each airfoil is embedded in a small finite differences subdomain. Since each subdomain moves with the airfoil it embeds, no regridding is required as time evolves.

In figure 4 we present streamline patterns about the impulsively started tandem airfoils at times $t = 1.5$, 2.5, 3.5 and 4.5. Shown here is the interaction between the rear airfoil and the starting vortex that has been shed by the leading airfoil. The Reynolds number $v_\infty C/2\nu$ is set to 3000, the reduced frequency of the oscillating airfoil $fC/2v_\infty$ is equal to 0.2 and $\alpha_{\max} = 45^o$.

In figure 5 is shown the instantaneous velocity field at time $t = 10$. One may verify in figures 4 and 5 that the velocity field is smooth across the interfaces of the subdomains. Note that the tandem airfoil problem or other problems of this kind would be difficult to treat by means of classical global approaches, since for these class of methods the flow domain would have to be either regridded or deformed at each time step.

The third examples concerns the simulation of a Darrieus wind turbine. In figure 6 is shown shed particles and streamline patterns about a Darrieus-like wind turbine at $t = 2.2$ after an impulsive start. There are four subdomains. The wind turbine rotates in the anti-clockwise direction and the fluid moves from right to left with velocity v_∞. denote by R be the windmill radius, $\Omega R/v_\infty = 2.16$ and $v_\infty 2r/\nu = 3000$.

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