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CONVERGENCE ANALYSIS OF A FINITE ELEMENT PROJECTION/LAGRANGE–GALERKIN METHOD FOR THE INCOMPRESSIBLE NAVIER–STOKES EQUATIONS*

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Abstract. This paper provides a convergence analysis of a fractional-step method to compute incompressible viscous flows by means of finite element approximations. In the proposed algorithm, the convection, the diffusion, and the incompressibility are treated in three different substeps. The convection is treated first by means of a Lagrange–Galerkin technique, whereas the diffusion and the incompressibility are treated separately in two subsequent substeps by means of a projection method. It is shown that provided the time step, δt, is of $\mathcal{O}(h^{d/4})$, where h is the meshsize and d is the space dimension ($2 \leq d \leq 3$), the proposed method yields for finite time T an error of $\mathcal{O}(h^{l+1} + \delta t)$ in the L^2 norm for the velocity and an error of $\mathcal{O}(h^l + \delta t)$ in the H^1 norm (or the L^2 norm for the pressure), where l is the polynomial degree of the approximate velocity.

Key words. incompressible Navier–Stokes equations, projection method, Lagrange–Galerkin method, fractional-step method, finite elements

AMS subject classifications. 35A40, 35Q30, 65M12, 65N30

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1. Introduction.

The Lagrange–Galerkin method is a numerical technique for solving convection-dominated–convection-diffusion problems. It consists of combining a Galerkin finite element procedure with a discretization of the Lagrangian material derivative along the characteristics. It combines the advantages of the methods that stabilize the convection (e.g., upwinding, Petrov–Galerkin, etc.) with the advantages of the methods that treat the convection in an explicit manner; that is to say, the linear systems to be solved at each time step involve only diffusion, are symmetric, and are time-invariant. For the Navier–Stokes equations, each time step of the algorithm is decomposed into two substeps: the first one accounts for the convection (i.e., transport along the characteristics), whereas the second one accounts for the incompressibility and diffusion effects. The second half step is a saddle point problem where the pressure is a Lagrange multiplier associated with the incompressibility constraint enforced on the velocity.

For convection-diffusion problems, the method has been analyzed by Bercovier & Pironneau [3], Russell [20, 21], and Douglas and Russell [9]. For the Navier–Stokes equations, the analysis has been done by Pironneau [17] and improved by Süli [23].

In recent years, renewed interest has developed in fractional-step projection methods for the incompressible Navier–Stokes equations in the primitive variables since the pioneering works of Chorin [7, 8], Temam [25, 26]. This method is based on a special time-discretization of the Navier–Stokes equations, in which the convection-diffusion and the incompressibility are dealt with in two different substeps. The velocity obtained in the convection-diffusion substep is projected in order to satisfy a weak incompressibility condition. A semidiscrete convergence analysis of Chorin's projection

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method can be found in Rannacher [19] and Shen [22]. A complete convergence analysis of an incremental version of the projection method for the Navier–Stokes equations is described in Guermond and Quartapelle [14]. A finite element implementation of the algorithm is reported in [15].

The aim of this paper is to combine the Lagrange–Galerkin method with the projection method. We propose and analyze an algorithm where the convection, the diffusion, and the incompressibility are treated in three different substeps. Even though it may a priori seem unwise to try to combine into a single algorithm a Lagrange–Galerkin approximation with a projection method—for the former relies heavily on the incompressibility of the flow for its stability, whereas the latter relaxes this constraint—the main result of this paper is that the two techniques can indeed be combined to yield a convergent method. It is shown that provided the time step δt is of $\mathcal{O}(h^{d/4})$, where h is the meshsize and d is the space dimension ($2 \leq d \leq 3$), the proposed method yields for finite time T an error of $\mathcal{O}(h^{l+1} + \delta t)$ in the L^2 norm for the velocity and an error of $\mathcal{O}(h^l + \delta t)$ in the H^1 norm (or the L^2 norm for the pressure), where l is the order of the approximation of the velocity.

This paper is divided into six parts. In section 2, we introduce some notations and hypotheses; we also introduce the finite element approximation and recall basic interpolation and stability results. The fractional step algorithm is presented in section 3. In section 4, we give preliminary results on the approximation on the material derivative. The error analysis is carried out in section 5 with mild regularity assumptions on the solution of the continuous problem. In section 6, we make stronger regularity assumptions, which yield a less restrictive stability condition on the time step and additional error estimates on the pressure.

2. The time-dependent Navier–Stokes problem.

2.1. Hypotheses and notations. Let Ω be an open connected bounded domain of $\mathbb{R}^d$ ($d \leq 3$) with a smooth boundary $\partial\Omega$. More specifically, the domain must be smooth enough so that the H^2 regularity of the Stokes operator holds; for instance we shall assume that $\partial\Omega$ is of class C^2 or Ω is a two-dimensional convex polygon (see Cattabriga [6]).

We consider the following time-dependent Navier–Stokes problem in which homogeneous Dirichlet condition has been assumed for simplicity. For a given body force f (possibly dependent on time) and a given divergence-free initial velocity field u_0, find a velocity field u and a pressure field p (with regularities yet to be clearly defined) such that $u = u_0$ at $t = 0$, and for $t > 0$,

$$\mathcal{P} \left\{ \begin{array}{l} \dfrac{\partial u}{\partial t} - \Delta u + (u \cdot \nabla) u + \nabla p = f \quad \text{in } \Omega \times (0,T), \\ \nabla \cdot u = 0 \quad \text{in } \Omega \times (0,T), \\ u = 0 \quad \text{on } \partial\Omega \times (0,T). \end{array} \right. \tag{2.1}$$

Of course, other types of boundary conditions are possible.

As usual, $W^{s,p}(\Omega)$ denotes the real Sobolev spaces, $0 \leq s < \infty$, $0 \leq p \leq \infty$, equipped with the norm $\|\cdot\|_{s,p}$ and seminorm $|\cdot|_{s,p}$. The space $W_0^{s,p}(\Omega)$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{s,p}$ norm. For $p = 2$, we denote the Hilbert spaces $W^{s,2}(\Omega)$ (resp., $W_0^{s,2}(\Omega)$) by $H^s(\Omega)$ (resp., $H_0^s(\Omega)$). The related norm is denoted by $\|\cdot\|_s$. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. For a fixed positive real number T and a Banach space X, we denote by $L^p(X)$, $H^s(X)$, and $C(X)$ the spaces $L^p(0,T;X)$, $H^s(0,T;X)$, and $C([0,T];X)$, respectively.

To formulate the Navier–Stokes problem in a variational form, we shall seek the velocity $u(t)$ in $\mathrm{H}^1_0(\Omega)^d$ and the pressure $p(t)$ in $\mathrm{L}^2_0(\Omega) = \{q \in \mathrm{L}^2(\Omega),\ \int_\Omega q = 0\}$. Furthermore, we set

$$(2.2)\quad V = \{v \in \mathrm{H}^1(\Omega)^d,\ \nabla\cdot v = 0\}, \quad H = \{v \in \mathrm{L}^2(\Omega)^d,\ \nabla\cdot v = 0,\ v\cdot n_{|\partial\Omega} = 0\}.$$

The importance of H lies in the following classical orthogonal decomposition of $\mathrm{L}^2(\Omega)^d$, whose discrete counterpart plays a key role in the projection technique that is described hereafter:

$$\mathrm{L}^2(\Omega)^d = H \oplus \nabla(\mathrm{H}^1(\Omega)). \tag{2.3}$$

For a given $f \in \mathrm{W}^{2,\infty}(\mathrm{L}^2(\Omega)^d)$ and a given initial velocity field $u_0 \in V \cap \mathrm{H}^2(\Omega)^d$, the variational formulation of problem $\mathcal{P}$ is as follows: find a pair (u, p) :

$$u \in \mathrm{L}^\infty(H) \cap \mathrm{L}^2(V),\ u_t \in \mathrm{L}^2(\mathrm{H}^{-1}(\Omega)^d),\ p \in \mathrm{L}^2(\mathrm{L}^2_0(\Omega)) \tag{2.4}$$

such that

$$(2.5)\ \mathcal{P}_{\mathrm{var}}\ \begin{cases} (u_t, v) + (\nabla u, \nabla v) + (u\cdot\nabla u, v) - (p, \nabla\cdot v) = (f, v) & \forall v \in \mathrm{H}^1_0(\Omega)^d, \\ (q, \nabla\cdot u) = 0 \qquad \forall q \in \mathrm{L}^2_0(\Omega), & \\ u(0) = u_0. & \end{cases}$$

It is known that there is some $T > 0$ for which $\mathcal{P}_{\mathrm{var}}$ has a solution. In the following, we shall assume that the solution to $\mathcal{P}_{\mathrm{var}}$ exists for all times and that it is as smooth as needed.

2.2. The spatial discretization. We introduce X_h and M_h, two continuous finite element approximations of $\mathrm{H}^1_0(\Omega)^d$ and $\mathrm{L}^2_0(\Omega)$ based on a regular, quasi-uniform triangulation $\mathcal{T}_h$ of Ω. The space X_h (resp., M_h) is composed of continuous piecewise polynomial functions of degree less than or equal to l with $l \geq 1$ (resp., of degree less than or equal to l', $\max(1, l-1) \leq l' \leq l$). It is assumed hereafter that the following properties hold (see, e.g., Bernardi and Raugel [4], Girault and Raviart [10], or Quarteroni and Valli [18] for other details).

(HA1) There exists $c > 0$ such that for $0 \leq r \leq l$,

$$\inf_{v_h\in X_h} [\|v - v_h\|_0 + h\|v - v_h\|_1] \leq ch^{r+1}\|v\|_{r+1} \qquad \forall v \in H^{r+1}(\Omega)^d \cap \mathrm{H}^1_0(\Omega)^d,$$
$$\inf_{v_h\in X_h} \|v - v_h\|_{1,p} \leq ch^r\|v\|_{r+1,p}, \qquad 2 \leq p \leq \infty \quad \forall v \in W^{r+1,p}(\Omega)^d \cap \mathrm{H}^1_0(\Omega)^d.$$

(HA2) There exists $c > 0$ such that $\forall$ q in $H^r(\Omega) \cap \mathrm{L}^2_0(\Omega)$, $0 \leq r \leq l'$,

$$\inf_{q_h\in M_h} \|q - q_h\|_0 \leq ch^r\|q\|_r.$$

(HA3) The Brezzi–Babuška inf-sup condition is verified [5], [2]; i.e., there exists $c > 0$ such that

$$\inf_{q_h\in M_h} \sup_{v_h\in X_h} \frac{(\nabla\cdot v_h, q_h)}{\|v_h\|_1\|q_h\|_0} \geq c.$$

(HA4) There exists $c > 0$ such that $\forall v_h$ in X_h, the following inverse inequalities hold:

$$\|v_h\|_{n,p} \le ch^{m-n+\frac{d}{p}-\frac{d}{q}}\|v_h\|_{m,q}, \qquad 0 \le m \le n \le 1, \qquad 0 \le q \le p \le \infty.$$

$$\|v_h\|_{0,\infty} \le c(1+|\log(h)|)^{1/2}\|v_h\|_{1,2} \text{ in two dimensions,}$$
$$\|v_h\|_{0,\infty} \le ch^{-\frac{1}{2}}\|v_h\|_{1,2} \text{ in three dimensions.}$$

We shall denote by $D(h)$ the quantity $c(1+|\log(h)|)^{1/2}$ in two dimensions (resp., $ch^{-\frac{1}{2}}$ in three dimensions) appearing in the inequality above. Numerous examples of pairs of finite element spaces satisfying these four assumptions can be found.

Example 2.1. Assume that Ω is a polyhedral domain in $\mathbb{R}^d$ $(2 \le d \le 3)$ and that $\mathcal{T}_h$ is a mesh of Ω consisting of triangles (resp., tetrahedron) in dimension 2 (resp., 3). One can choose for X_h the space of continuous piecewise P_2 vector functions with respect to $\mathcal{T}_h$ and for M_h the space of continuous piecewise linear scalar functions with respect to $\mathcal{T}_h$. This pair of spaces is usually referred to as the Taylor–Hood finite element spaces and satisfies the assumptions above with $l = 2$, $l' = 1$. □

Example 2.2. With the same hypothesis on the mesh as above and restricting ourselves to two dimensions in space, let $\mathcal{T}'_h$ be the mesh obtained by dividing each triangles of $\mathcal{T}_h$ into four smaller triangles by joining together the middle points of its sides. Then X_h (resp., M_h) can be chosen as the space of continuous piecewise linear vector functions with respect to $\mathcal{T}'_h$ (resp., continuous piecewise linear scalar functions with respect to $\mathcal{T}_h$). This pair of spaces is usually referred to as the P_1-iso-P_2/P_1 finite element spaces. It is frequently used because of the simplicity of its shape functions. The assumptions above hold with $l = 1$, $l' = 1$. □

Example 2.3. With the same hypothesis on the mesh as above, one can choose for X_h the space of continuous vector functions the restriction of which to a given element is the sum of linear functions and of a function vanishing on the boundary of the element (bubble function) and for M_h the space of continuous piecewise linear scalar functions with respect to $\mathcal{T}_h$. This pair of spaces is usually referred to as the MINI finite element spaces. The assumptions above hold with $l = 1$, $l' = 1$. □

We now introduce a discrete divergence operator $B_h : X_h \longrightarrow M_h$ and its transpose $B_h^{\mathrm{t}} : M_h \longrightarrow X_h'$ as follows: for every couple (v_h, q_h) in $X_h \times M_h$ we have $(B_h v_h, q_h) = -(\nabla \cdot v_h, q_h) = (v_h, B_h^{\mathrm{t}} q_h)$. Hypothesis (HA3) implies in particular that B_h is surjective.

3. The fractional-step scheme. Introduce a partition of the time interval $[0, T]$: $t^k = k\,\delta t$ for $0 \le k \le K$ where $\delta t = T/K$. This section is concerned with the time scheme for computing approximations to the velocity and pressure fields at each time step t^k.

3.1. The convective derivative and its approximation. Let $X(x, s; t)$ be the trajectory (or characteristics) under the action of the flow $u(\cdot, t)$ of a particle of fluid which is at point x at time s. The characteristics are solutions of the initial value problem

$$\left\{ \begin{array}{l} \dfrac{dX(x,s;t)}{dt} = u(X(x,s;t),t), \\ X(x,s;s) = x. \end{array} \right. \tag{3.1}$$

If $u \in C(C^{0,1}(\overline{\Omega})^d)$, this ODE has a unique solution thanks to the Cauchy–Lipschitz theorem. When no confusion may arise we set $X^k(x) = X(x, t^{k+1}; t^k)$. As shown in, e.g., Pironneau [17], Douglas and Russell [9], or Süli [23], the opportunity of introducing the characteristics is motivated by the following formal approximation property:

$$u(x,t^{k+1})\cdot\nabla u(x,t^{k+1})\approx\frac{u(x,t^k)-u(X(x,t^{k+1};t^k))}{\delta t}. \tag{3.2}$$

Likewise, if $\tilde{u}_h^k\in X_h$ is an approximation of $u(t^k),\forall\, x$ in Ω we define $X_h(x,t^{k+1};t)$ as the solution to the initial value problem

$$\begin{cases} \dfrac{d}{dt}X_h(x,t^{k+1};t)=\tilde{u}_h^k(X_h(x,t^{k+1};t)),\\ X_h(x,t^{k+1};t^{k+1})=x. \end{cases} \tag{3.3}$$

Note that this problem has a unique solution thanks to the Cauchy–Lipschitz theorem, for functions of X_h have Lipschitz regularity. When no confusion may arise, we set, for the sake of simplicity, $X_h^k(x)=X_h(x,t^{k+1};t^k)$. Hereafter, $(\tilde{u}_h^k(x)-\tilde{u}_h^k(X_h^k))/\delta t$ will be used as an approximation of $u(x,t^{k+1})\cdot\nabla u(x,t^{k+1})$.

The main interest of this approximation is twofold. On the one hand, it yields remarkable stability properties without relying on artificial diffusion (roughly speaking, unconditional stability of the material derivative; see [17]) and thus avoids both large spatial errors due to artificial diffusion and the need to cook up stabilizing parameters. On the other hand, since the treatment of the nonlinear convective term $u\cdot\nabla u$ is explicit, the linear system resulting from the implicit treatment of the diffusion and the incompressibility is symmetric and time-invariant; see, e.g., [1] for two dimensional and three dimension implementations of the method. In [1], different two dimensional computations around a cylinder at Reynolds 9500 using a Galerkin characteristics scheme and the (ψ,ω) formulation are also presented, assessing the good behavior of the method of characteristics.

3.2. An additional discrete setting. As in Guermond [11, 12], to relax the incompressibility constraint and to build a discrete version of the Helmholtz decomposition (2.3), we introduce an additional discrete setting. More precisely, we want to decompose each discrete vector field $\tilde{u}_h\in X_h$ into the sum of a discrete-divergence-free vector field u_h plus the discrete-gradient of a scalar field p_h in M_h. In practice, there are numerous ways of achieving this decomposition. For instance we could set $\tilde{u}_h=u_h+B_h^t p_h$ with $u_h\in X_h$ and $B_h u_h=0$. Another possibility could be to set $\tilde{u}_h=u_h+\nabla p_h$ where u_h is enforced to be orthogonal to ∇M_h. In this case we shall see that it is natural to choose u_h to be in $X_h+\nabla M_h$. Even though this alternative may seem weird, it turns out to be optimal and very easy to implement in practice (see (3.11), (3.13)). To unify the first approach, the second one, and all the intermediate ones, we introduce Y_h a finite-dimensional subspace of $\mathrm{L}^2(\Omega)^d$ in which we shall select u_h. For the sake of simplicity we assume that $X_h\subset Y_h$ and we denote by i_h the continuous injection of X_h into Y_h; the transpose of i_h is the L^2 projection of Y_h onto X_h. Furthermore, we assume that we can build an operator $C_h:Y_h\longrightarrow M_h$ such that we have the following.

(HA5) The operator C_h is an extension of B_h and $i_h^{\mathrm{t}}C_h^{\mathrm{t}}=B_h^{\mathrm{t}}$; i.e., the following commutative diagrams hold:

$$\begin{tikzcd} X_h \arrow[r, "B_h"] \arrow[d, "i_h"'] & M_h \\ Y_h \arrow[ur, "C_h"'] & \end{tikzcd} \qquad \begin{tikzcd} X_h' & M_h \arrow[l, "B_h^{\mathrm{t}}"'] \arrow[dl, "C_h^{\mathrm{t}}"] \\ Y_h \arrow[u, "i_h^{\mathrm{t}}"] & \end{tikzcd}$$

Example 3.1. The most trivial example consists in choosing $Y_h = X_h$ and $C_h = B_h$. Even though this choice seems natural, it is not the simplest one in terms of implementation.

Example 3.2. Since M_h is composed of continuous piecewise polynomial functions, we have $M_h \subset \mathrm{H}^1(\Omega)$; as a result the following space $Y_h = X_h + \nabla M_h$ is a subspace of $\mathrm{L}^2(\Omega)^d$. Furthermore, one may easily verify that C_h defined by

$$\forall (v_h, q_h) \in Y_h \times M_h, \qquad (C_h v_h, q_h) = (v_h, \nabla q_h) \tag{3.4}$$

is an extension of B_h, and C_h^t is the restriction of ∇ to M_h. □

From (HA3), we infer that C_h is also surjective, for C_h is an extension of B_h; as a consequence $\|C_h^{\mathrm{t}} q\|_0$ is a norm. The null space of C_h playing an important role in the sequel we set $H_h = \mathrm{Ker}\, C_h$. This definition enables us to build a discrete counterpart of the aforementioned orthogonal decomposition $\mathrm{L}^2(\Omega)^d = H \oplus \nabla(\mathrm{H}^1(\Omega))$.

Corollary 3.1. *We have the orthogonal decomposition*

$$Y_h = H_h \oplus C_h^{\mathrm{t}}(M_h). \tag{3.5}$$

We also assume that C_h^t satisfies the following hypothesis.

(HA6) There exists $c > 0$ such that $\forall\ q_h$ in M_h,

$$\|C_h^t q_h\|_0 \leq c \|q_h\|_1.$$

Remark 3.1. Note that the hypothesis (HA6) is automatically satisfied if we choose $Y_h = X_h + \nabla M_h$ and $C_h^t = \nabla$. In fact, (HA6) is assumed for the sake of simplicity and it could be somewhat weakened if a discontinuous approximation of the pressure was used (see Guermond [11] or Guermond and Quartapelle [14] for other details). □

3.3. Initial conditions. To avoid the technical difficulty of the blowing up of the error estimates at the initial time induced by the possible lack of regularity of the solution, we assume that the solution is as smooth as needed at $t = 0$.

Remark 3.2. Of course the smoothness hypothesis on u and p at $t = 0$ may be too optimistic in some cases (see [16]). We could relax it by assuming that from $k = 1$ to some k_0 $(1 < k_0 < K)$ such that $t_0 = k_0\, \delta t$ is some fixed time independent of K, the solution is approximated in time and space by means of some coupled, implicit Euler scheme of first order. Then, the approximate solution $\hat{u}_h^{k_0}$ and $\hat{p}_h^{k_0}$ that would be obtained from this preliminary step could serve as initial data for our fractional step algorithm at subsequent time steps, $k_0 \leq k \leq K$. □

Hereafter we denote by $\hat{u}_h^0 \in X_h$ and $\hat{p}_h^0 \in M_h$ an approximation to u_0 and $p(t = 0)$ such that

$$\|u_0 - \hat{u}_h^0\|_0 + h \left(\|u_0 - \hat{u}_h^0\|_1 + \|p(0) - \hat{p}_h^0\|_0 \right) \leq c h^{l+1}. \tag{3.6}$$

For instance, $\hat{u}_h^0$ can be obtained by solving

$$(\nabla \hat{u}_h^0, \nabla v_h) = (\nabla u_0, \nabla v_h) \qquad \forall v_h \in X_h, \tag{3.7}$$

and the initial pressure $\hat{p}_h^0$ can be obtained by calculating an approximation of $p(t = 0)$. For instance, one could (at least in principle) obtain it by solving

$$(\nabla \hat{p}_h^0, \nabla r_h) = (\nabla \hat{p}^0, \nabla r_h) = (f(0) + \Delta u(0) - u(0) \cdot \nabla u(0), \nabla r_h) \qquad \forall r_h \in M_h. \tag{3.8}$$

3.4. The fractional-step projection/Lagrange–Galerkin scheme. We are now interested in defining a projection/Lagrange–Galerkin scheme for $1 \leq k \leq K$. We define two sequences of approximate velocities $\{\tilde{u}_h^k \in X_h\}$ and $\{u_h^k \in Y_h\}$ and one sequence of approximate pressures $\{p_h^k \in M_h\}$ as follows:

- *Initialization.* The sequences $\{u_h^k\}$, $\{\tilde{u}_h^k\}$ are initialized by $u_h^0 = \tilde{u}_h^0 = \hat{u}_h^0$ and the sequence $\{p_h^k\}$ is initialized by $p_h^0 = \hat{p}_h^0$.
- *Time loop.* For $0 \leq k$, solve

$$\left\{ \begin{aligned} &\left(\frac{\tilde{u}_h^{k+1} - i_h^{\mathrm{t}} u_h^k}{\delta t}, v_h\right) + (\nabla \tilde{u}_h^{k+1}, \nabla v_h) \left(\frac{\tilde{u}_h^k - \tilde{u}_h^k(X_h^k)}{\delta t}, v_h\right) \\ &\qquad + (B_h^{\mathrm{t}} p_h^k, v_h) = (f(t^{k+1}), v_h) \qquad \forall v_h \in X_h \end{aligned} \right. \tag{3.9}$$

and

$$\left\{ \begin{aligned} &\frac{u_h^{k+1} - i_h \tilde{u}_h^{k+1}}{\delta t} + C_h^{\mathrm{t}}(p_h^{k+1} - p_h^k) = 0, \\ &C_h u_h^{k+1} = 0. \end{aligned} \right. \tag{3.10}$$

Remark 3.3. The problem (3.9) clearly has a unique solution. The problem (3.10) is also well posed thanks to Corollary 3.1: indeed, the pair $(u_h^{k+1}, \delta t\, C_h^{\mathrm{t}}(p_h^{k+1} - p_h^k))$ is the decomposition of $i_h \tilde{u}_h^{k+1}$ on $H_h \oplus C_h^{\mathrm{t}}(M_h)$; in other words, $u_h^{k+1} = P_{H_h}(i_h \tilde{u}_h^{k+1})$, where P_{H_h} is the orthogonal projector of Y_h onto H_h. ∎

Remark 3.4. In practice it is not convenient to solve the problem as presented here, for Y_h is possibly a very weird space. Actually, the projected velocity u_h^k may (must) be eliminated from the algorithm as follows (see Rannacher [19] or Guermond [11]). For $k \geq 1$, replace u_h^k in (3.9) by its definition which is given by (3.10) at the time step t^k; note that $i_h^{\mathrm{t}} C_h^{\mathrm{t}} = B_h^{\mathrm{t}}$, as already mentioned. In (3.10), u_h^{k+1} is eliminated by applying C_h to the first equation and by noting that C_h is an extension of B_h. Once u_h^{k+1} and u_h^k are eliminated, and by setting $p_h^{-1} = \hat{p}_h^0$, the algorithm that is implemented in practice read as follows for $k \geq 0$:

$$\begin{aligned} &\left(\frac{\tilde{u}_h^{k+1} - \tilde{u}_h^k(X_h^k)}{\delta t}, v_h\right) + (\nabla \tilde{u}_h^{k+1}, \nabla v_h) \\ &\qquad + (B_h^{\mathrm{t}}(2p_h^k - p_h^{k-1}), v_h) = (f^{k+1}, v_h) \qquad \forall v_h \in X_h \end{aligned} \tag{3.11}$$

and

$$C_h C_h^{\mathrm{t}}(p_h^{k+1} - p_h^k) = \frac{B_h \tilde{u}_h^{k+1}}{\delta t}. \qquad ∎ \tag{3.12}$$

Remark 3.5. If we choose $Y_h = X_h + \nabla M_h$, the projection step takes the following form: find p_h^{k+1} in M_h such that

$$\forall q_h \in M_h, \quad (\nabla(p_h^{k+1} - p_h^k), \nabla q_h) = -\frac{(\nabla \cdot \tilde{u}_h^{k+1}, q_h)}{\delta t}. \tag{3.13}$$

With this particular choice of Y_h, the projection step amounts to solving a discrete Poisson problem with homogeneous Neumann boundary condition. If needed, the velocity field u_h^{k+1} is given by

$$u_h^{k+1} = \tilde{u}_h^{k+1} - \delta t \nabla (p_h^{k+1} - p_h^k). \tag{3.14}$$

Note that in the particular case described here, the approximate velocity u_h^{k+1} is not an H^1-conforming approximation, for it is discontinuous. Indeed, it is shown below that u_h^{k+1} is an $\mathrm{L}^2(\Omega)^d$-approximation of $u(t)$, whereas $\tilde{u}_h^{k+1}$ is an $\mathrm{H}^1(\Omega)^d$-approximation; hence, $\tilde{u}_h^{k+1}$ is a better approximation than u_h^{k+1}, although its discrete divergence $B_h \tilde{u}_h^{k+1}$ is not zero. ◻

4. Preliminary results for the error analysis.

4.1. Preliminaries on the approximation of the material derivative. We recall in this section some results concerning the approximation of the material derivative by means of the Lagrange technique. Most of the results stated hereafter are largely inspired from Russell [20, 21], Douglas and Russell [9], and Süli [23]; they are recalled for the sake of completeness and most of the proofs are omitted.

Lemma 4.1. *Assume that $u \in C(C^{0,1}(\overline{\Omega})^d \cap V)$. If $|s-t|$ is sufficiently small, then $x \longrightarrow X(x, s; t)$ is a homeomorphism of Ω onto itself and its Jacobian equals* 1 *almost everywhere (a.e.) on Ω.*

Lemma 4.2. *Assume that for all k*

$$\delta t \|\tilde{u}_h^k\|_{1,\infty} \mathrm{e}^{\delta t \|\tilde{u}_h^k\|_{1,\infty}} \le 1/8;$$

then $\forall\ t$ in $[t^k, t^{k+1}]$, $x \longrightarrow X_h(x, t^{k+1}; t)$ is a homeomorphism of Ω onto itself and for a.e. x in Ω its Jacobian satisfies $1/2 \le J_h(x, t^{k+1}; t) \le 3/2$.

Lemma 4.3. *Assume that*

$$\delta t \|u\|_{L^\infty(W^{1,\infty}(\Omega)^d)} \le 1/6; \tag{HS1}$$

then the mapping $x \longrightarrow (1-\theta)x + \theta X(x, t^{k+1}; t^k)$ is a homeomorphism of Ω onto itself with a Jacobian $\ge 1/2\ \forall\ \theta$ in $[0,1]$.

Lemma 4.4. *Assume* (HS1)*; then there exists ϵ such that if*

$$\delta t \|\tilde{u}_h^k\|_{1,\infty} \le \epsilon. \tag{4.1}$$

then the mapping $x \longrightarrow (1-\theta) X_h(x, t^{k+1}; t^k) + \theta X(x, t^{k+1}; t^k)$ is a homeomorphism of Ω onto itself with a Jacobian $\ge 1/2\ \forall\ \theta$ in $[0,1]$.

Denoting by ϵ the small parameter introduced in the lemma above, we shall assume hereafter that

$$\delta t \|\tilde{u}_h^k\|_{1,\infty} \le \epsilon. \tag{HS2}$$

Remark 4.1. Assumption (HS1) is a restriction on the time step, whereas (HS2) is a stability hypothesis on the approximate velocity that we shall prove by induction in the next subsection. ◻

Lemma 4.5. *Let $x \longrightarrow \psi_0(x)$ and $x \longrightarrow \psi_1(x)$ be two homeomorphisms of Ω onto itself such that for all $\theta \in [0,1]$, $x \longrightarrow (1-\theta)\psi_0(x) + \theta \psi_1(x)$ is a homeomorphism of*

Ω onto itself with Jacobian $\geq 1/2$. Then, $\forall$ p, q, $1 \leq q < \infty$, $1 \leq p \leq \infty$, and for $\eta \in W^{1,qp'}(\Omega)$, $(1/p + 1/p' = 1)$, we have

$$\|\eta \circ \psi_0 - \eta \circ \psi_1\|_{0,q} \leq 2\|\psi_0 - \psi_1\|_{0,pq}\|\nabla\eta\|_{0,qp'}. \tag{4.2}$$

From this lemma we infer a series of corollaries that will be used repeatedly hereafter.

COROLLARY 4.6. *Assume that $u \in C(C^{0,1}(\overline{\Omega})^d)$, $u_t \in L^\infty(\mathrm{L}^2(\Omega)^d)$, and that* (HS1) *and* (HS2) *hold, then there exists $c = 2\exp(\delta t\|u\|_{C(C^{0,1}(\overline{\Omega})^d)})$ such that*

$$\|X^k - X_h^k\|_{0,2} \leq c\delta t(\|u(t^k) - \tilde{u}_h^k\|_{0,2} + \delta t\,\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)}). \tag{4.3}$$

A direct consequence of this corollary together with Lemma 4.5 is the following corollary;

COROLLARY 4.7. *Assume that $u \in C(C^{0,1}(\overline{\Omega})^d)$, $u_t \in L^\infty(\mathrm{L}^2(\Omega)^d)$, and that* (HS1) *and* (HS2) *hold; then there exists $c = 4\exp(\delta t\|u\|_{C(C^{0,1}(\overline{\Omega})^d)})$ such that $\forall$ $\eta \in \mathrm{H}^1(\Omega)$*

$$\|\eta \circ X^k - \eta \circ X_h^k\|_{0,1} \leq c\delta t(\|u(t^k) - \tilde{u}_h^k\|_{0,2} + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)})\|\nabla\eta\|_{0,2}. \tag{4.4}$$

The following corollary is crucial for deriving L^2 optimal estimates; it is mainly due to Douglas and Russell [9].

COROLLARY 4.8. *Assume that $u \in C(C^{0,1}(\overline{\Omega})^d)$ and* (HS1) *holds; then $\forall$ $\eta \in \mathrm{L}^2(\Omega)$*

$$\|\eta - \eta \circ X^k\|_{-1} \leq 2\delta t\|u\|_{C(C^{0,1}(\overline{\Omega})^d)}\|\eta\|_0. \tag{4.5}$$

4.2. Definition of suitable interpolations. Before going through the details of the error analysis, we introduce some suitable interpolations of $u(t)$ and $p(t)$ which will preserve the high approximation order h^{l+1} on $u(t)$ in the $\mathrm{L}^2(\Omega)^d$-norm. For all t, we define $w_h(t) \in X_h$ and $q_h(t) \in M_h$ as the solutions of the following discrete Stokes problem:

$$\begin{cases} (\nabla w_h(t), \nabla v_h) + (B_h^{\mathrm{t}} q_h(t), v_h) = (\nabla u(t), \nabla v_h) - (p(t), \nabla\cdot v_h) & \forall v_h \in X_h, \\ (r_h, B_h w_h(t)) = -(r_h, \nabla\cdot u(t)) \qquad \forall r_h \in M_h. \end{cases} \tag{4.6}$$

Thanks to the H^2-regularity of the Stokes operator in regular domains together with the Aubin–Nitsche trick, these interpolations satisfy the following lemma.

LEMMA 4.9. *Provided hypotheses* (HA1), (HA2), *and* (HA3) *hold and for $1 \leq \beta \leq \infty$, $u \in L^\beta(H^{l+1}(\Omega)^d \cap V)$, $p \in L^\beta(H^l(\Omega) \cap M)$, there exists $c > 0$ such that*

$$\begin{aligned} &\|u - w_h\|_{L^\beta(\mathrm{L}^2(\Omega)^d)} + h(\|u - w_h\|_{L^\beta(\mathrm{H}^1(\Omega)^d)} + \|p - q_h\|_{L^\beta(\mathrm{L}^2(\Omega))}) \\ &\qquad \leq ch^{l+1}(\|u\|_{L^\beta(H^{l+1}(\Omega)^d)} + \|p\|_{L^\beta(H^l(\Omega))}). \end{aligned} \tag{4.7}$$

5. Error bounds on the velocity with mild regularity assumptions. In this section, we derive error bounds on the velocity. For the sake of simplicity, we assume the following regularity on the initial data.

(HR1) $u_0 \in H^2(\Omega)^d \cap W^{1,\infty}(\Omega)^d \cap V$.

Furthermore, denoting the material derivative $u_t + u\cdot\nabla u$ by $D_t u$, we assume that the solution (u, p) of the problem (2.1) satisfies

(HR2) $u \in L^\infty(V \cap H^{l+1}(\Omega)^d \cap W^{1,\infty}(\Omega)^d) \cap C(C^{0,1}(\overline{\Omega})^d)$,
$u_t \in L^2(V \cap H^{l+1}(\Omega)^d) \cap L^\infty(H)$,
$D_t^2 u \in L^2(\mathrm{L}^2(\Omega)^d)$, where $D_t^2 = D_t(D_t)$,
$p \in L^\infty(H^l(\Omega) \cap M)$ and $p_t \in L^2(H^l(\Omega))$.

For the sake of simplicity we assume that the initial values, $\hat{u}_h^0$ and $\hat{p}_h^0$, are chosen so that

(HI1) $\|w_h^0 - \hat{u}_h^0\|_0 \leq ch^{l+1}$, $\|w_h^0 - \hat{u}_h^0\|_1 \leq c\min(h^l, h^{l+1}/\delta t^{1/2})$, $\|q_h^0 - \hat{p}_h^0\|_1 \leq c$.

The main result of this section is summarized in the following theorem.

THEOREM 5.1. *Assume that the approximation hypotheses* (HA1)–(HA6)*, together with the regularity and initialization hypotheses* (HR1), (HR2)*, and* (HI1)*, hold. Then, there exist $c_s > 0$, $c_e > 0$, and $h_s > 0$ such that $\forall\ h \in]0, h_s]$ and $\forall\ \delta t \leq c_s h^{d/3}$*

$$\|u - u_h\|_{l^\infty(\mathrm{L}^2(\Omega)^d)} + \|u - \tilde{u}_h\|_{l^\infty(\mathrm{L}^2(\Omega)^d)} \leq c_e(h^{l+1} + \delta t), \tag{5.1}$$

$$\|u - \tilde{u}_h\|_{l^2(\mathrm{H}^1(\Omega)^d)} \leq c_e(h^l + \delta t). \tag{5.2}$$

Proof. Hereafter, we shall denote by c_i generic constants, independent of h, δt, and the time index k. These constants may depend on u, p, T, and Ω. The strategy that we adopt hereafter follows the ideas of Russell [20, 21], Douglas and Russell [9], Süli [23], and Guermond and Quartapelle [14]. We proceed by finite induction on k.

Our induction hypothesis is that there exists $c_s > 0$, $c_e > 0$, and $h_s > 0$ such that at time step t^m, $0 \leq m < K$, $\forall\ h \in]0, h_s]$ and $\forall\ \delta t \leq c_s h^{d/3}$

$$\|u - u_h\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega)^d)} + \|u - \tilde{u}_h\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega)^d)} \leq c_e(h^{l+1} + \delta t),$$

$$\|u - \tilde{u}_h\|_{l^2(0,t^m;\mathrm{H}^1(\Omega)^d)} \leq c_e(h^l + \delta t),$$

and that the stability hypothesis (HS2) holds for all time steps $0 \leq k \leq m$.

Initialization. We prove first that the induction hypothesis holds for $m = 0$. Given the initialization hypothesis (HI1) and the regularity of u_0, we have

$$\|u_0 - u_h^0\|_0 + \|u_0 - \tilde{u}_h^0\|_0 + h\|u_0 - \tilde{u}_h^0\|_1 \leq ch^{l+1}.$$

The error estimate in the induction hypothesis is satisfied if we denote by $c_{e,0}$ the constant in the inequality above. Denoting by $P_h u_0$ an interpolate of u_0 that satisfies the error estimates (HA1), we have

$$\begin{aligned}
\|\tilde{u}_h^0\|_{1,\infty} &\leq \|\tilde{u}_h^0 - P_h u_0\|_{1,\infty} + \|P_h u_0\|_{1,\infty} \\
&\leq c(h^{-\frac{d}{2}}\|\tilde{u}_h^0 - P_h u_0\|_{1,2} + \|u_0\|_{1,\infty}) \\
&\leq c(h^{1-\frac{d}{2}}\|u_0\|_{2,2} + \|u_0\|_{1,\infty}).
\end{aligned}$$

We can now choose a constant $c_{s,0}$ depending only on the regularity of u_0, such that for all $h \leq h_{s,0} = 1$ and for all $\delta t \leq c_{s,0}h^{d/3}$, (HS2) is verified for $\tilde{u}_h^0$.

Step 1. Assuming that the induction hypothesis holds for m such that $0 \leq m < K = T/\delta t$, we shall now prove that it also holds for $m+1$. Note that before verifying the induction hypothesis at step $m = 1$, we set $c_s = c_{s,0}$, $c_e = c_{e,0}$, $h_s = h_{s,0}$. The value of c_s and c_e will be modified once at the end of step $m = 1$. The value of h_s will be modified at the end of steps $m = 1$ and $m = 2$ since its value may depend on c_s and c_e. For shortness, we do not detail steps $m = 1$ and $m = 2$.

Step 2. First, we establish the equations that control the errors. For shortness, we denote by $e_h^k = w_h(t^k) - u_h^k$, $\tilde{e}_h^k = w_h(t^k) - \tilde{u}_h^k$ and $\epsilon_h^k = q_h(t^k) - p_h^k$ the error functions, $(u(t), p(t))$ being the solution of the continuous problem (2.5). The interpolation error

$u(t)-w_h(t)$ is denoted by $\eta(t)$. For any function, $\phi(t)$, which is continuous with respect to time, we set $\phi^k=\phi(t^k)$; furthermore, we introduce the notation $\delta_t\phi^{k+1}=\phi^{k+1}-\phi^k$.

Given the particular approximation (w_h,q_h) that we have chosen, the exact solution of the Navier–Stokes problem satisfies at time t^{k+1}

$$\left\{\begin{array}{l} \left(\dfrac{\delta_t w_h^{k+1}}{\delta t}, v_h\right)+(\nabla w_h^{k+1},\nabla v_h)+(u^{k+1}\cdot\nabla u^{k+1},v_h) \\ \qquad\qquad +(B_h^{\mathrm{t}}q_h^{k+1},v_h)=\left(f^{k+1}-u_t^{k+1}+\dfrac{\delta_t w_h^{k+1}}{\delta t},v_h\right) \quad \forall v_h\in X_h, \\ B_h w_h^{k+1}=0. \end{array}\right. \tag{5.3}$$

By subtracting (3.9) from (5.3), we derive the equation that controls the error $\tilde{e}_h^{k+1}$:

$$\begin{aligned} &\left(\frac{\tilde{e}_h^{k+1}-i_h^{\mathrm{t}}e_h^k}{\delta t},v_h\right)+(\nabla\tilde{e}_h^{k+1},\nabla v_h)+(B_h^{\mathrm{t}}\psi_h^k,v_h)=-\left(\frac{\delta_t\eta^{k+1}}{\delta t},v_h\right) \\ &+\left(\frac{u^{k+1}-u^k(X^k)}{\delta t}-D_t u^{k+1},v_h\right)-\left(\frac{u^k-\tilde{u}_h^k-(u^k-\tilde{u}_h^k)(X^k)}{\delta t},v_h\right) \\ &-\left(\frac{(u^k-\tilde{u}_h^k)(X^k)-(u^k-\tilde{u}_h^k)(X_h^k)}{\delta t},v_h\right)-\left(\frac{u^k(X_h^k)-u^k(X^k)}{\delta t},v_h\right), \end{aligned} \tag{5.4}$$

where we have set

$$\psi_h^k=q_h^{k+1}-p_h^k=\delta_t q_h^{k+1}+\epsilon_h^k.$$

We denote by $R_1(v_h),\dots,R_5(v_h)$ the five terms in the right-hand side of (5.4) .

On the other hand, since $w_h^{k+1}\in X_h$, $B_h w_h^{k+1}=0$, and C_h is an extension of B_h, we obtain the system of equations that controls e_h^{k+1} and ϵ_h^{k+1}:

$$\left\{\begin{array}{l} \dfrac{e_h^{k+1}-i_h\tilde{e}_h^{k+1}}{\delta t}+C_h^{\mathrm{t}}(\epsilon_h^{k+1}-\psi_h^k)=0, \\ C_h e_h^{k+1}=0. \end{array}\right. \tag{5.5}$$

Step 3. To obtain a bound on $\tilde{e}_h^{k+1}$, we take the inner product of (5.4) by $2\delta t\,\tilde{e}_h^{k+1}$. Using the algebraic relation $2(a,a-b)=|a|^2+|a-b|^2-|b|^2$ and the Poincaré–Friedrichs inequality (α is the constant such that $\|v\|_1^2\le\alpha\|\nabla v\|_0^2\ \forall v\in H_0^1(\Omega)$), we have

$$\begin{aligned} \|\tilde{e}_h^{k+1}\|_0^2+\|\tilde{e}_h^{k+1}-i_h^{\mathrm{t}}e_h^k\|_0^2-\|i_h^{\mathrm{t}}e_h^k\|_0^2+2\alpha\delta t\|\tilde{e}_h^{k+1}\|_1^2 \\ +2\delta t(\tilde{e}_h^{k+1},\,B_h^{\mathrm{t}}\psi_h^k)\le 2\delta t\sum_{i=1}^5|R_i(\tilde{e}_h^{k+1})|. \end{aligned}$$

The terms in the right-hand side of the inequality above may be bounded from above as follows:

$$2\delta t|R_1(\tilde{e}_h^{k+1})|\le\delta t\|\tilde{e}_h^{k+1}\|_0^2+\|\eta_t\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)}.$$

Exactly as in [23], and from assumption (HR2),

$$2\delta t|R_2(\tilde{e}_h^{k+1})|\le\delta t\|\tilde{e}_h^{k+1}\|_0^2+\delta t^2\|D_t^2 u\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)}.$$

Corollary 4.8 implies

$$
\begin{aligned}
2\delta t|R_3(\tilde{e}_h^{k+1})| &\le 2\|\tilde{e}_h^{k+1}\|_1\|u^k - \tilde{u}_h^k - (u^k - \tilde{u}_h^k)(X^k)\|_{-1} \\
&\le \frac{\alpha}{5}\delta t\|\tilde{e}_h^{k+1}\|_1^2 + c\delta t\|u^k - \tilde{u}_h^k\|_0^2.
\end{aligned}
$$

By using Corollary 4.7, we have

$$
\begin{aligned}
2\delta t|R_4(\tilde{e}_h^{k+1})| &\le 2\|\tilde{e}_h^{k+1}\|_{0,\infty}\|(u^k - \tilde{u}_h^k)(X^k) - (u^k - \tilde{u}_h^k)(X_h^k)\|_{0,1} \\
&\le c\delta t D(h)\|\tilde{e}_h^{k+1}\|_1\left(\|u^k - \tilde{u}_h^k\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)}\right)\|\nabla(u^k - \tilde{u}_h^k)\|_0 \\
&\le c\delta t D(h)\|\tilde{e}_h^{k+1}\|_1\left(\|u^k - \tilde{u}_h^k\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)}\right)(\|\eta^k\|_1 + \|\tilde{e}_h^k\|_1).
\end{aligned}
$$

The induction hypothesis implies that

$$
\|u^k - \tilde{u}_h^k\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)} \le c(c_e)(h^{l+1} + \delta t).
$$

There exists $h_{s,1} > 0$ which does not depend on m but possibly depends on c_s and c_e such that $\forall\ h \le h_{s,1}$ and $\delta t \le c_s h^{d/3}$,

$$
cD(h)\left(\|u^k - \tilde{u}_h^k\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)}\right) \le \frac{\alpha}{5}.
$$

Furthermore, given the regularity of u (i.e., hypothesis (HR2)), there exists $h_{s,2} > 0$ which does not depend on m such that $\forall\ h \le h_{s,2}$,

$$
cD(h)\|\eta^k\|_1 \le 1.
$$

As a result, there are positive constants c_1 and c_2 independent of c_e, c_s, h and δt so that $\forall\ h \le \min(h_{s,1}, h_{s,2})$ and $\delta t \le c_s h^{d/3}$,

$$
\begin{aligned}
2\delta t|R_4(\tilde{e}_h^{k+1})| &\le \delta t\|\tilde{e}_h^{k+1}\|_1\left(\frac{\alpha}{5}\|\tilde{e}_h^k\|_1 + \|u^k - \tilde{u}_h^k\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)}\right) \\
&\le \delta t\frac{3\alpha}{5}\|\tilde{e}_h^{k+1}\|_1^2 + \delta t\frac{\alpha}{5}\|\tilde{e}_h^k\|_1^2 + c_1\delta t\|u^k - \tilde{u}_h^k\|_0^2 + c_2\delta t^3.
\end{aligned}
$$

The last residual $R_5(\tilde{e}_h^{k+1})$ is bounded from above as follows:

$$
\begin{aligned}
2\delta t|R_5(\tilde{e}_h^{k+1})| &\le 2\|\tilde{e}_h^{k+1}\|_0\|u^k(X_h^k) - u^k(X^k)\|_0 \\
&\le c\|\tilde{e}_h^{k+1}\|_0\|X^k - X_h^k\|_0\|u\|_{L^\infty(t^k,t^{k+1};W^{1,\infty}(\Omega)^d)} \\
&\le c\delta t\|\tilde{e}_h^{k+1}\|_0(\|u^k - \tilde{u}_h^k\|_{0,2} + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)})\|u\|_{L^\infty(W^{1,\infty}(\Omega)^d)} \\
&\le \delta t\|\tilde{e}_h^{k+1}\|_0^2 + c_1\delta t\|u^k - \tilde{u}_h^k\|_0^2 + c_2\delta t^3,
\end{aligned}
$$

After collecting all the bounds we obtain

$$
\begin{aligned}
&\|\tilde{e}_h^{k+1}\|_0^2 + \|\tilde{e}_h^{k+1} - i_h^t e_h^k\|_0^2 + 2\delta t(\tilde{e}_h^{k+1}, B_h^{\mathrm{t}}\psi_h^k) + 2\alpha\delta t\|\tilde{e}_h^{k+1}\|_1^2 \\
&\quad\le \|e_h^k\|_0^2 + 3\delta t\|\tilde{e}_h^{k+1}\|_0^2 + \frac{4\alpha}{5}\delta t\|\tilde{e}_h^{k+1}\|_1^2 + \frac{\alpha}{5}\delta t\|\tilde{e}_h^k\|_1^2 + c_1\delta t\|u^k - \tilde{u}_h^k\|_0^2 \\
&\quad + c_2\delta t^3 + \|\eta_t\|_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)}^2 + \delta t^2\|D_t^2 u\|_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)}^2.
\end{aligned}
$$

Now we use

$$
\|\tilde{e}_h^{k+1}\|_0^2 \le 2\|\tilde{e}_h^{k+1} - i_h^t e_h^k\|_0^2 + 2\|e_h^k\|_0^2,
$$

and using the convention $e_h^{-1} = 0$, we have

$$\|u^k - \tilde{u}_h^k\|_0^2 \leq 2\|\eta^k\|_0^2 + 4\|\tilde{e}_h^k - i_h^t e_h^{k-1}\|_0^2 + 4\|e_h^{k-1}\|_0^2.$$

The final bound is

$$\begin{aligned} \|\tilde{e}_h^{k+1}\|_0^2 + (1-6\delta t)\|\tilde{e}_h^{k+1} - i_h^t e_h^k\|_0^2 + 2\delta t(\tilde{e}_h^{k+1}, B_h^{\mathrm{t}}\psi_h^k) + \frac{6\alpha}{5}\delta t\|\tilde{e}_h^{k+1}\|_1^2 \\ \leq \|e_h^k\|_0^2 + 6\delta t\|e_h^k\|_0^2 + \frac{\alpha}{5}\delta t|\tilde{e}_h^k|_1^2 \\ + c_1\delta t\|e_h^{k-1}\|_0^2 + c_2\delta t\|\tilde{e}_h^k - i_h^t e_h^{k-1}\|_0^2 + c_3\delta t\|\eta^k\|_0^2 \\ + c_4\delta t^3 + \|\eta_t\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} + \delta t^2\|D_t^2 u\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)}. \end{aligned} \tag{5.6}$$

We can pick $h_{s,3} > 0$ (independent of m, but dependent of c_s) such that for all $h \leq h_{s,3}$ and $\delta t \leq c_s h^{d/3}$, the following inequality holds: $c_2\delta t \leq 1/4 < 3/4 \leq 1 - 6\delta t$, where c_2 is the constant in bound (5.6) above. Finally the current values of h_s are replaced by

$$\min(h_s, h_{s,1}, h_{s,2}, h_{s,3}).$$

Step 4. To obtain some control on $\delta t(\tilde{e}_h^{k+1}, B_h^{\mathrm{t}}\psi_h^k)$, we take the inner product of the first equation of (5.5) by $2\delta t^2 C_h^{\mathrm{t}}\psi_h^k$ and we obtain

$$\begin{aligned} -2\delta t(\tilde{e}_h^{k+1}, B_h^{\mathrm{t}}\psi_h^k) + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^{k+1}\|_0^2 - \|e_h^{k+1} - i_h\tilde{e}_h^{k+1}\|_0^2 = \delta t^2\|C_h^{\mathrm{t}}\psi_h^k\|_0^2 \\ = \delta t^2\|C_h^{\mathrm{t}}(\delta_t q_h^{k+1} + \epsilon_h^k)\|_{0,2}^2; \end{aligned}$$

that is to say, given the stability property (HA6),

$$\begin{aligned} -2\delta t(\tilde{e}_h^{k+1}, B_h^{\mathrm{t}}\psi_h^k) + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^{k+1}\|_0^2 - \|e_h^{k+1} - i_h\tilde{e}_h^{k+1}\|_0^2 \\ \leq \delta t^2(1+\delta t)\|C_h^{\mathrm{t}}\epsilon_h^k\|_0^2 + c\delta t^2\|q_{ht}\|^2_{L^2(t^k,t^{k+1};\mathrm{H}^1(\Omega))}. \end{aligned} \tag{5.7}$$

Step 5. We obtain some control on e_h^{k+1} by taking the inner product of (5.5) by $2\delta t e_h^{k+1}$:

$$\|e_h^{k+1}\|_0^2 + \|e_h^{k+1} - i_h\tilde{e}_h^{k+1}\|_0^2 - \|\tilde{e}_h^{k+1}\|_0^2 = 0. \tag{5.8}$$

Step 6. After summing up (5.6) + (5.7) + (5.8), we obtain

$$\begin{aligned} \|e_h^{k+1}\|_0^2 + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^{k+1}\|_0^2 + \frac{3}{4}\|\tilde{e}_h^{k+1} - i_h^t e_h^k\|_0^2 + \frac{6\alpha}{5}\delta t\|\tilde{e}_h^{k+1}\|_1^2 \\ \leq \|e_h^k\|_0^2 + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^k\|_0^2 + 6\delta t\|e_h^k\|_0^2 + \delta t^3\|C_h^{\mathrm{t}}\epsilon_h^k\|_0^2 + \frac{\alpha}{5}\delta t\|\tilde{e}_h^k\|_1^2 \\ + c_1\delta t\|e_h^{k-1}\|_{0,2}^2 + \frac{1}{4}\|\tilde{e}_h^k - i_h^t e_h^{k-1}\|_{0,2}^2 + c_2\delta t\|\eta^k\|_{0,2}^2 + c_3\delta t^3 \\ + \|\eta_t\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} + \delta t^2\|D_t^2 u\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} + c_4\delta t^2\|q_{ht}\|^2_{L^2(t^k,t^{k+1};\mathrm{H}^1(\Omega))}. \end{aligned}$$

By taking the sum from $k = 0$ to m, we obtain

$$\begin{aligned} \|e_h^{m+1}\|_0^2 + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^{m+1}\|_0^2 + \frac{1}{2}\sum_{k=0}^m \|\tilde{e}_h^{k+1} - i_h^t e_h\|_0^2 + \alpha\delta t\sum_{k=0}^m \|\tilde{e}_h^{k+1}\|_1^2 \\ \leq c_1\delta t\sum_{k=0}^m \left[\|e_h^k\|_0^2 + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^k\|_0^2\right] \\ + c_2\delta t\sum_{k=0}^m \|\eta^k\|_0^2 + c_3\delta t^2 + \|e_h^0\|_0^2 + \delta t^2\|\epsilon_h^0\|_1^2 + \frac{\alpha}{5}\delta t\|\tilde{e}_h^0\|_1^2 \\ + \|\eta_t\|^2_{L^2(\mathrm{L}^2(\Omega)^d)} + \delta t^2\|D_t^2 u\|^2_{L^2(\mathrm{L}^2(\Omega)^d)} + c_4\delta t^2\|q_{ht}\|^2_{L^2(\mathrm{H}^1(\Omega))}. \end{aligned}$$

From the initialization hypothesis, we infer that the term $\|e_h^0\|_0^2 + \delta t^2\|\epsilon_h^0\|_1^2 + \alpha/5\delta t\|\tilde{e}_h^0\|_1^2$ is bounded from above by $c(\delta t + h^{l+1})$. From the definition of $(w_h(t), q_h(t))$ (see (4.6)) we infer that $\|q_{ht}\|_{L^2(\mathrm{H}^1(\Omega))}$ is bounded from above by $C(\|u_t\|_{L^2(\mathrm{H}^2(\Omega)^d)} + \|p_t\|_{L^2(\mathrm{H}^1(\Omega))})$, which is well defined from assumption (HR2); we may write $\|q_{ht}\|_{L^2(\mathrm{H}^1(\Omega))} \leq c$. Likewise, the approximation error η satisfies

$$\delta t \sum_{k=0}^{m} \|\eta^k\|_0^2 \leq ch^{2(l+1)}\delta t \sum_{k=0}^{m} (\|u^k\|_{l+1}^2 + \|p^k\|_l^2) \leq ch^{2(l+1)},$$

and from assumption (HR2)

$$\|\eta_t\|_{L^2(\mathrm{L}^2(\Omega)^d)} \leq ch^{l+1}(\|u_t\|_{L^2(H^{l+1}(\Omega)^d)} + \|p_t\|_{L^2(H^l(\Omega))}) \leq ch^{l+1}.$$

As a result, we can apply the discrete Gronwall lemma, which leads to

$$\begin{aligned}\|e_h^{m+1}\|_0^2 + \delta t^2\|C_h^{\mathrm{t}}\epsilon_h^{m+1}\|_0^2 + \frac{1}{2}\sum_{k=0}^{m} \|\tilde{e}_h^{k+1} - i_h^t e_h^k\|_0^2 + \alpha\delta t \sum_{k=0}^{m+1} \|\tilde{e}_h^k\|_1^2 \\ \leq c_1 \exp\left(c_2 \delta t^m\right)\left(\delta t + h^{l+1}\right)^2.\end{aligned}$$

From this bound we deduce

$$\begin{aligned}\|u^{m+1} - u_h^{m+1}\|_0 &\leq \|e_h^{m+1}\|_0 + \|\eta^{m+1}\|_0 \\ &\leq c_1 \exp(c_2 T)\left(\delta t + h^{l+1}\right).\end{aligned}$$

Now we set $c_{e,1} = c_1 \exp(c_2 T)$, where c_1 and c_2 are the constants used in the bound above. Now, using

$$\begin{aligned}\|\tilde{e}_h^{m+1}\|_0 &\leq \|\tilde{e}_h^{m+1} - i_n^t e_h^m\|_0 + \|e_h^m\|_0 \\ &\leq c_1 \exp\left(c_2 T\right)\left(\delta t + h^{l+1}\right),\end{aligned}$$

we infer that

$$\begin{aligned}\|u^{m+1} - \tilde{u}_h^{m+1}\|_0 &\leq \|\eta^{m+1}\|_0 + \|\tilde{e}_h^{m+1}\|_0 \\ &\leq c_1 \exp\left(c_2 T\right)\left(\delta t + h^{l+1}\right).\end{aligned}$$

Now we set $c_{e,2} = c_1 \exp(c_2 T)$, where c_1 and c_2 are the constants used in the bound above. The error bound in the H^1 norm is obtained as follows:

$$\begin{aligned}\|u - \tilde{u}_h\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)}^2 &\leq 2\|\eta\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)}^2 + 2\|\tilde{e}_h\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)}^2 \\ &\leq c_1 \exp\left(c_2 T\right)\left(\delta t + h^l\right).\end{aligned}$$

Note that although the approximation error $\|\tilde{e}_h^{m+1}\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)}$ is of order $\mathcal{O}(\delta t + h^{l+1})$, the total error is spoiled by the interpolation error in space which is of order $\mathcal{O}(h^l)$. Now we set $c_{e,3} = c_1 \exp(c_2 T)$, where c_1 and c_2 are the constants used in the bound above, and we redefine c_e as being

$$\max(c_{e,0}, c_{e,1}, c_{e,2}, c_{e,3}).$$

It is important to note that c_e is redefined once at the end of time step $m = 1$; as a result, it is definitely independent of m, h, and δt.

Step 7. Now we turn our attention to the stability hypothesis (HS2). Let us denote by $P_h u^{m+1}$ some interpolate of u^{m+1} that satisfies the error estimates of (HA1). We have

$$\begin{aligned}\|\tilde{u}_h^{m+1}\|_{1,\infty} &\le \|\tilde{u}_h^{m+1} - P_h u^{m+1}\|_{1,\infty} + \|P_h u^{m+1}\|_{1,\infty}\\ &\le c_1 h^{-\frac{d}{2}}(\|\tilde{u}_h^{m+1} - u^{m+1}\|_{1,2} + \|u^{m+1} - P_h u^{m+1}\|_{1,2}) + c_2\|u_{m+1}\|_{1,\infty}.\end{aligned}$$

Given the error bound we have obtained at time step $m+1$ (after using a classical inverse inequality in time), we clearly have

$$\|\tilde{u}_h^{m+1} - u^{m+1}\|_{1,2} \le c_e(\delta t^{1/2} + h^l \delta t^{-1/2}).$$

As a result, setting $\delta t \le c_s h^{d/3}$ we obtain

$$\begin{aligned}\delta t\|\tilde{u}_h^{m+1}\|_{1,\infty} &\le c_1 h^{-\frac{d}{2}}(c_e(\delta t^{\frac{3}{2}} + h^l \delta t^{\frac{1}{2}}) + c_2 h^l \delta t) + c_3 \delta t \|u_{m+1}\|_{1,\infty}\\ &\le c_1(c_e(c_s^{\frac{3}{2}} + c_s^{\frac{1}{2}}) + c_2 c_s) + c_3 c_s.\end{aligned}$$

From this bound we can define $c_{s,1}$ such that (HS2) holds. Then we redefine a new value of c_s as being equal to

$$\min(c_s, c_{s,1}).$$

Note that c_e and c_s are fixed after the first time step $m = 1$. As a result, h_s is fixed at the end of time step $m = 2$.

In conclusion the induction hypothesis is satisfied at step $m+1$. ∎

Remark 5.1. We obtained a stability condition of the type $\delta t = \mathcal{O}(h^{d/3})$ with mild regularity assumptions on the solution of the continuous problem, whereas Süli [23] obtained $\delta t = \mathcal{O}(h^{d/4})$. The main reason for this more demanding condition on the time step is the fact that the projection algorithm does not easily yield an estimate in the $l^\infty(0, t^{m+1}; \mathrm{H}^1(\Omega)^d)$ norm. Since we are not able to obtain it with these regularity assumptions, we have derived it from the $l^2(0, t^{m+1}; \mathrm{H}^1(\Omega)^d)$ norm by using an inverse inequality in time. By doing so we lose a $\delta t^{1/2}$ factor, which in turn yields the stability condition $\delta t = \mathcal{O}(h^{d/3})$. However, if such a $l^\infty(0, t^{m+1}; \mathrm{H}^1(\Omega)^d)$ estimate was at hand then Süli's stability condition would imply the stability condition (HS2). In the next section, we shall make stronger regularity assumptions on the solution of the continuous problem as well as additional hypothesis on the initialization of the scheme; then, assuming $\delta t = \mathcal{O}(h^{d/4})$, we shall obtain the result of Theorem 5.1 as well as error estimates on the pressure. ∎

6. Error bounds on velocity and pressure with stronger regularity assumptions. In this section, we derive error bounds on the velocity and the pressure with the weaker condition $\delta t = \mathcal{O}(h^{d/4})$. For that, we reproduce exactly the proof of 5.1 up to Step 6, but we replace Step 7 of the above mentioned proof by a much more involved argument, which yields an error estimate for the pressure and for the velocity in the $l^\infty(0, t^{m+1}; \mathrm{H}^1(\Omega)^d)$ norm.

To this end, we reconstruct a momentum equation on the error by summing (5.4) and (5.5). Then, the inf-sup condition yields a bound on ϵ_h^{k+1} if we can control the approximate time derivative $(e_h^{k+1} - e_h^k)/\delta t$. The classical way for obtaining such a control consists of using $\delta_t e_h^{k+1}$ as a test function in the momentum equation. In the present case, this procedure does not seem to be feasible since $\delta_t e_h^{k+1}$ does not belong to the right function space. To avoid this difficulty, we shall construct the equations that control the incremental errors $\delta_t\tilde{e}_h^{k+1}$, $\delta_t e_h^{k+1}$, and $\delta_t\epsilon_h^{k+1}$; then we shall proceed as in the proof of Theorem 5.1.

Throughout this section we shall make the following assumptions.

(HI2) The algorithm is initialized so that

$$\|e_h^0\|_0 \le \min(h^{l+1}, \delta t h^l), \quad \|\tilde{e}_h^0\|_1 \le c\delta t^{1/2} h^l, \quad \|\epsilon_h^0\|_1 \le ch^l.$$

(HR3) In addition to (HR1) and (HR2) the solution (u,p) is assumed to be such that

$$\begin{aligned}
&u \in W^{1,\infty}(V\cap H^{l+1}(\Omega)^d \cap W^{1,\infty}(\Omega)^d) \cap = L^\infty(W^{2,\infty}(\Omega)^d),\\
&u_{tt} \in L^2(V\cap H^{l+1}(\Omega)^d) \cap L^\infty(H),\\
&\partial_t D_t^2 u \in L^2(\mathrm{L}^2(\Omega)^d),\\
&D_t^3 u \in L^2(\mathrm{L}^2(\Omega)^d),\\
&p_t \in L^\infty(H^l(\Omega)\cap M) \text{ and } p_{tt} \in L^2(H^l(\Omega)).
\end{aligned}$$

6.1. Preliminary results. We need first to establish a series of preliminary lemmas: the next lemma can be proved as in [23].

LEMMA 6.1. *Setting* $X_\theta^k = \theta X^k + (1-\theta)X^{k-1}$ *and* $X_{h,\theta}^k = \theta X_h^k + (1-\theta)X_h^{k-1}$, *and* $X_{h,\zeta,\theta}^k = \zeta X_\theta^k + (1-\zeta)X_{h,\theta}^k$, *there exists* ϵ *such that if*

$$\delta t(\|u\|_{L^\infty(W^{1,\infty}(\Omega)^d)} + \|\tilde{u}_h^k\|_{1,\infty} + \|\tilde{u}_h^{k-1}\|_{1,\infty}) \le \epsilon, \tag{6.1}$$

then the mappings $x \longrightarrow (1-\theta)X_h^k(x) + \theta X^k(x)$, X_θ^k ,$X_{h,\theta}^k$, $X_{h,\zeta,\theta}^k$ *are homeomorphisms of* Ω *onto itself with a Jacobian* $\ge 1/2$ *for all* θ *in* $[0,1]$.

Denoting by ϵ the small parameter introduced in the lemma above, we shall make the assumption

(HS3) $\forall k \in \{1,\dots,K\}, \quad \delta t(\|u\|_{L^\infty(W^{1,\infty}(\Omega)^d)} + \|\tilde{u}_h^k\|_{1,\infty} + \|\tilde{u}_h^{k-1}\|_{1,\infty}) \le \epsilon.$

LEMMA 6.2. *Assume* (HR2) *and* (HR3)*; moreover, assume that there is* $c_e > 0$ *so that* $\max(\|\tilde{e}_h^k\|_1, \|\tilde{e}_h^{k-1}\|_1) \le c_e(\delta t + h^l)$. *Then, there exist* $h_s(c_e) > 0$ *and* c_1, c_2 *independent of* c_e *but possibly depending on* u *so that* $\forall\ h \in]0, h_s(c_e)]$ *and* $\forall\ \delta t \le h^{d/4}$ *we have*

$$\frac{1}{\delta t}\|\delta_t X^k - \delta_t X_h^k\|_0 \le c_1\delta t\left[\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1)\right] + c_2\|\delta_t\tilde{e}_h^k\|_0. \tag{6.2}$$

Proof. For shortness, we shall adopt the following notations: $X^l(s) = X(x,t^l;s)$ and $X_h^l(s) = X(x,t^l;s)$; note that $X^{k+1}(t^k)$ is by definition equal to X^k. For any function $\phi^k(s)$, the quantity $\phi^k(s) - \phi^{k-1}(s-\delta t)$ is denoted by $\delta_t\phi^k(s)$. The quantity $\delta_t X^{k+1}(s) - \delta_t X_h^{k+1}(s)$ satisfies

$$\begin{aligned}
\frac{d}{ds}(\delta_t X^{k+1}(s) - \delta_t X_h^{k+1}(s)) &= u(X^{k+1}(s),s) - u(X^k(s-\delta t), s-\delta t)\\
&\quad -\tilde{u}_h^k(X_h^{k+1}(s)) + \tilde{u}_h^{k-1}(X_h^k(s-\delta t))\\
&= \int_{t^k}^s [u_\tau(X^{k+1}(s),\tau) - u_\tau(X^k(s-\delta t), \tau-\delta t)]d\tau\\
&\quad +u^k(X^{k+1}(s)) - u^k(X_h^{k+1}(s)) - u^{k-1}(X^k(s-\delta t)) + u^{k-1}(X_h^k(s-\delta t))\\
&\quad +\eta^k(X_h^{k+1}(s)) - \eta^{k-1}(X_h^{k+1}(s)) + \eta^{k-1}(X_h^{k+1}(s)) - \eta^{k-1}(X_h^k(s-\delta t))\\
&\quad +\tilde{e}_h^k(X_h^{k+1}(s)) - \tilde{e}_h^{k-1}(X_h^{k+1}(s)) + \tilde{e}_h^{k-1}(X_h^{k+1}(s)) - \tilde{e}_h^{k-1}(X_h^k(s-\delta t)).
\end{aligned}$$

From (HR3), we infer that

$$\left\| \int_{t^k}^{s} [u_\tau(X^{k+1}(s), \tau) - u_\tau(X^k(s-\delta t), \tau - \delta t)] d\tau \right\|_0 \le c\delta t^2.$$

For the second term, we use Lemma 4.5 together with Corollary 4.6:

$$\begin{aligned} \|u^k(X^{k+1}(s)) - u^k(X_h^{k+1}(s))\|_0 &\le c\|X^{k+1}(s)) - X_h^{k+1}(s)\|_0 \\ &\le c\delta t(\delta t + h^{l+1} + \|\tilde{e}_h^k\|_0). \end{aligned}$$

In the same manner, for the third term we have

$$\|u^{k-1}(X^k(s-\delta t)) - u^{k-1}(X_h^k(s-\delta t))\|_0 \le c\delta t(\delta t + h^{l+1} + \|\tilde{e}_h^{k-1}\|_0).$$

For the fourth term, we use Lemma 4.2 together with (HR3):

$$\begin{aligned} \|\eta^k(X_h^{k+1}(s)) - \eta^{k-1}(X_h^{k+1}(s))\|_0 &\le c\|\eta^k - \eta^{k-1}\|_0 \\ &\le c\delta t h^{l+1}. \end{aligned}$$

Similarly, for the sixth term, we have

$$\|\tilde{e}_h^k(X_h^{k+1}(s)) - \tilde{e}_h^{k-1}(X_h^{k+1}(s))\|_0 \le c\|\delta_t \tilde{e}_h^k\|_0.$$

Finally, for bounding the two remaining terms, we shall need an estimate on $\|X_h^{k+1}(s))-X_h^k(s-\delta t))\|_{0,\infty}$. Using the triangular inequality together with the Gronwall lemma we obtain

$$\begin{aligned} \|X_h^{k+1}(s) - X_h^k(s-\delta t)\|_{0,\infty} &\le \|X^{k+1}(s) - X_h^{k+1}(s)\|_{0,\infty} + \|X^{k+1}(s) - X^k(s-\delta t)\|_{0,\infty} \\ &\quad + \|X^k(s-\delta t) - X_h^k(s-\delta t)\|_{0,\infty} \\ &\le c\delta t(\delta t + D(h)h^l + \|\tilde{e}_h^k\|_{0,\infty} + \|\tilde{e}_h^{k-1}\|_{0,\infty}) \\ &\le cD(h)\delta t(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1), \end{aligned}$$

where we have used the fact that $\|\eta_h(t)\|_{0,\infty} \le cD(h)h^l\|u(t)\|_{l+1}$ and assumption (HR2). By hypothesis, the right-hand side is bounded by $c(c_e)\delta t(h^{d/4} + h^l)D(h)$. We can choose $h_s(c_e)$ so that $\forall\ h \in]0, h_s(c_e)]$, $c(c_e)(h^{d/4} + h^l)D(h) \le 1$. As a result we obtain

$$\|X_h^{k+1}(s) - X_h^k(s-\delta t)\|_{0,\infty} \le \delta t.$$

Therefore, the last two terms are bounded from above as follows:

$$\begin{aligned} \|\tilde{e}_h^{k-1}(X_h^{k+1}(s)) - \tilde{e}_h^{k-1}(X_h^k(s-\delta t))\|_0 &\le c\|\tilde{e}_h^{k-1}\|_{1,2}\|X_h^{k+1}(s) - X_h^k(s-\delta t))\|_{0,\infty} \\ &\le c\delta t\|\tilde{e}_h^{k-1}\|_1, \end{aligned}$$

$$\begin{aligned} \|\eta^{k-1}(X_h^{k+1}(s)) - \eta^{k-1}(X_h^k(s-\delta t))\|_0 &\le c\|\eta^{k-1}\|_{1,2}\|\tilde{X}_h^{k+1}(s) - X_h^k(s-\delta t))\|_{0,\infty} \\ &\le c\delta t h^l. \end{aligned}$$

By combining the bounds obtained above we infer

$$\frac{1}{\delta t}\|\delta_t X^k - \delta_t X_h^k\|_0 \le c_1\delta t(\delta t + h^l + \|\tilde{e}_h^{k-1}\|_1 + \|\tilde{e}_h^k\|_1) + c_2\|\delta_t \tilde{e}_h^k\|_0.$$

The proof of the lemma is complete. $\square$

We shall also need the next lemma.

LEMMA 6.3. *With the same assumptions as in Lemma* 6.2 *and assuming* (HS3), *there is $h_s(c_e) > 0$ so that $\forall\ h \in]0, h_s(c_e)], \forall\ \delta t \leq h^{d/4}, \forall\ 2 \leq q \leq \infty$, $\xi \in W^{1,q}(\Omega)$, $v_h \in X_h$, we have*

$$\frac{1}{\delta t}\left|\int_\Omega \left[\delta_t \xi(X^k) - \delta_t \xi(X_h^k)\right] v_h dx\right|$$
$$\leq c(|\xi|_{1,q}\|v_h\|_{0,q'} + |\xi|_{1,2}\|v_h\|_{1,2})\left[\delta t(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^k\|_1) + \|\delta_t \tilde{e}_h^k\|_0\right],$$

where we have set $\delta_t \xi(\phi^k) = \xi(\phi^k) - \xi(\phi^{k-1})$ and $1/q + 1/q' = 1/2$.

Proof. Setting $X_\theta^k = \theta X^k + (1-\theta)X^{k-1}$ and $X_{h,\theta}^k = \theta X_h^k + (1-\theta)X_h^{k-1}$, for an arbitrary function v_h in X_h we have

$$\begin{aligned}\int_\Omega \left[\delta_t \xi(X^k) - \delta_t \xi(X_h^k)\right] v_h dx &= \int_\Omega \int_{\theta=0}^1 \left[\nabla\xi(X_\theta^k)\cdot \delta_t X^k - \nabla\xi(X_{h,\theta}^k)\cdot\delta_t X_h^k\right] v_h d\theta dx\\ &= \int_{\theta=0}^1 \int_\Omega \left[\nabla\xi(X_\theta^k) - \nabla\xi(X_{h,\theta}^k)\right]\cdot \delta_t X^k v_h dx d\theta\\ &+ \int_{\theta=0}^1\int_\Omega \nabla\xi(X_{h,\theta}^k)\cdot(\delta_t X^k - \delta_t X_h^k) v_h dx d\theta.\end{aligned}$$

As a result

$$\begin{aligned}\int_\Omega \left[\delta_t \xi(X^k) - \delta_t \xi(X_h^k)\right] v_h dx \leq& \int_{\theta=0}^1 \int_\Omega \left[\nabla\xi(X_\theta^k) - \nabla\xi(X_{h,\theta}^k\right]\cdot \delta_t X^k v_h dx d\theta\\ &+ c\|\nabla\xi\|_{0,q}\|\delta_t X^k - \delta_t X_h^k\|_{0,2}\|v_h\|_{0,q'}.\end{aligned}$$

Here, we have used the hypothesis (HS3)(to be verified by induction) to bound $\|\nabla\xi(X_{h,\theta}^k)\|_{0,q}$ by $C\|\nabla\xi\|_{0,q}$. Let us denote by I_0 the second term in the right-hand side. This term is easily bounded from above by using Lemma 6.2:

$$|I_0| \leq c|\xi|_{1,q}\|v_h\|_{0,q'}\left[\delta t^2(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1) + \delta t\|\delta_t \tilde{e}_h^k\|_0\right].$$

For the other term we proceed as follows:

$$\begin{aligned}\int_\Omega \left[\nabla\xi(X_\theta^k) - \nabla\xi(X_{h,\theta}^k)\right]\cdot\delta_t X^k v_h dx &= \int_\Omega \nabla\xi(X_\theta^k)\cdot\delta_t X^k v_h\left[1 - \det(\nabla X_\theta^k)\right]dx\\ &- \int_\Omega \nabla\xi(X_{h,\theta}^k)\cdot\delta_t X^k v_h\left[1-\det(\nabla X_{h,\theta}^k)\right]dx\\ &+ \int_\Omega\left[\det(\nabla X_\theta^k)\nabla\xi(X_\theta^k) - \det(\nabla X_{h,\theta}^k)\nabla\xi(X_{h,\theta}^k)\right]\cdot\delta_t X^k v_h dx.\end{aligned}$$

Let us denote, respectively, by I_1, I_2, and I_3 the three integrals in the right-hand side above. For I_1 we have

$$|I_1| \leq |\xi(X_\theta^k)|_{1,q}\|\delta_t X^k\|_{0,\infty}\|1 - \det(\nabla X_\theta^k)\|_{0,2}\|v_h\|_{0,q'}.$$

From Lemma 6.1 we infer

$$|I_1| \leq c|\xi|_{1,q}\|\delta_t X^k\|_{0,\infty}\|1 - \det(\nabla X_\theta^k)\|_{0,2}\|v_h\|_{0,q'}.$$

Provided X^{k+1} and X^k satisfy the hypothesis of Lemma 4.5 (i.e., δt small enough) and $u \in L^\infty(W^{1,\infty}(\Omega)^d)$, $u_t \in L^\infty(L^\infty(\Omega)^d)$, we infer

$$\|\delta_t X^k\|_{0,\infty} \le c\delta t^2.$$

For the remaing term involving $1 - \det(\nabla X_\theta^k)$ we use the equality

$$\|1 - \det(\nabla X_\theta^k)\|_{0,2} = \| \det \left[\theta \nabla(X^k) - (1-\theta)\nabla(X^k)\right]) - \det(\nabla X_\theta^k)\|_{0,2},$$

together with the inequality

$$\begin{aligned} \| \det(\nabla\psi_{1,\theta}^k) - \det(\nabla\psi_{2,\theta}^k)\|_{0,2} &\le c \max(\|\psi_1^k\|_{1,\infty}, \|\psi_1^{k-1}\|_{1,\infty}, \|\psi_2^k\|_{1,\infty}, \|\psi_2^{k-1}\|_{1,\infty}) \\ &\qquad \left[\|\psi_1^k - \psi_2^k\|_{1,2} + \|\psi_1^{k-1} - \psi_2^{k-1}\|_{1,2}\right], \end{aligned} \tag{6.3}$$

where ψ_1^k, ψ_1^{k-1}, ψ_2^k, and ψ_2^{k-1} are four arbitrary mappings of Ω onto Ω. By setting $\psi_1^k = \psi_1^{k-1} = X^k$, $\psi_2^k = X^k$, and $\psi_2^{k-1} = X^{k-1}$ we infer from (6.3)

$$\|1 - \det(\nabla X_\theta^k)\|_{0,2} \le c\delta t^2.$$

As a result, we obtain

$$|I_1| \le c\delta t^4 |\xi|_{1,q} \|v_h\|_{0,q'}.$$

For I_2 we proceed similarly:

$$\begin{aligned} |I_2| &\le |\xi|_{1,q} \|\delta_t X^k\|_{0,\infty} \|1 - \det(\nabla X_{h,\theta}^k)\|_{0,2} \|v_h\|_{0,q'}, \\ &\le c\delta t^2 |\xi|_{1,q} \|1 - \det(\nabla X_{h,\theta}^k)\|_{0,2} \|v_h\|_{0,q'}. \end{aligned}$$

Given the assumed regularity $u \in L^\infty(W^{2,\infty}(\Omega)^d) \cap W^{1,\infty}(L^2(\Omega)^d)$, we infer

$$\|X^k - X_h^k\|_1 + \|X^{k-1} - X_h^{k-1}\|_1 \le c\delta t \left[\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1\right].$$

Furthermore, given the hypothesis (HS3) (to be verified by induction), we infer

$$\|X_h^k\|_{1,\infty} + \|X_h^{k-1}\|_{1,\infty} \le c.$$

By using formula (6.3), we obtain

$$|I_2| \le c\delta t^3 \left[\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1\right] |\xi|_{1,q} \|v_h\|_{0,q'}.$$

For I_3 we have

$$|I_3| = \int_\Omega \nabla\xi \cdot \left[(v_h \delta_t X)(X_\theta^{-k}) - (v_h \delta_t X)(X_{h,\theta}^{-k})\right] dx,$$

where X_θ^{-k} and $X_{h,\theta}^{-k}$ denote the inverse mappings of X_θ^k and $X_{h,\theta}^k$, respectively. By using Lemmas 4.5 and 6.1, we infer

$$\begin{aligned} |I_3| &\le |\xi|_{1,2} |v_h \delta_t X|_{0,2} \|X_\theta^{-k} - X_{h,\theta}^{-k}\|_{0,\infty} \\ &\le |\xi|_{1,2} \|v_h\|_{1,2} \|\delta_t X\|_{1,\infty} \|X_\theta^{-k} - X_{h,\theta}^{-k}\|_{0,\infty}. \end{aligned}$$

It is possible to prove

$$\|X_\theta^{-k} - X_{h,\theta}^{-k}\|_{0,\infty} \le cD(h)\delta t(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1),$$

which yields $\|X_\theta^{-k} - X_{h,\theta}^{-k}\|_{0,\infty} \le \delta t$ if $\delta t \le h^{d/4}$ and h is small enough. As a result, for I_3 we obtain

$$|I_3| \le c\delta t^3 |\xi|_{1,2} \|v_h\|_{1,2}.$$

Note that the bound on $|I_3|$ is rather coarse, but we do not need a finer one here since the bound on I_0 is already more restrictive. The final bound is obtained by collecting the bounds on I_0, I_1, I_2, and I_3. $\square$

6.2. Error bounds. We now state the main theorem of section 6.

THEOREM 6.4. *Under the approximations hypotheses* (HA1)–(HA6)*, the regularity assumptions* (HR1)–(HR3)*, and the initialization hypothesis* (HI2)*, there exist* c_e*,* c_s*, and* h_s *such that* $\forall\ h$ *in* $]0, h_s]$ *and for* $\delta t \leq c_s h^{\frac{d}{4}}$*, the solution of the projection scheme* (3.12)–(3.13) *satisfies*

$$\|u - u_h\|_{l^\infty(\mathrm{L}^2(\Omega)^d)} + \|u - \tilde{u}_h\|_{l^\infty(\mathrm{L}^2(\Omega)^d)} \leq c_e(h^{l+1} + \delta t), \tag{6.4}$$

$$\|u - \tilde{u}_h\|_{l^\infty(\mathrm{H}^1(\Omega)^d)} \leq c_e(h^l + \delta t), \tag{6.5}$$

$$\|p - \tilde{p}_h\|_{l^\infty(\mathrm{L}^2(\Omega)^d)} \leq c_e(h^l + \delta t). \tag{6.6}$$

Proof. The proof is done by induction: the induction hypothesis is that there exist $c_s > 0$, $c_e > 0$, $c'_e > 0$, and $h_s > 0$ such that at time step t^m, $0 \leq m < K$, $\forall\ h \in]0, h_s]$ and $\forall\ \delta t \leq c_s h^{d/4}$

$$\left\{ \begin{array}{l} \|e_h\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega)^d)} + \|\tilde{e}_h\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega)^d)} \leq c_e(h^{l+1} + \delta t), \\ \|\tilde{e}_h\|_{l^2(0,t^m;\mathrm{H}^1(\Omega)^d)} \leq c_e(h^l + \delta t), \end{array} \right. \tag{6.7}$$

$$\left\| \frac{\delta_t e_h}{\delta t} \right\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega)^d)} + \left\| \frac{\delta_t \tilde{e}_h}{\delta t} \right\|_{l^2(0,t^m;\mathrm{H}^1(\Omega)^d)} \leq c'_e(\delta t + h^l), \tag{6.8}$$

$$\|\epsilon_h\|_{l^\infty(0,t^m;\mathrm{L}^2(\Omega))} \leq c'_e(\delta t + h^l), \tag{6.9}$$

$$\|\tilde{e}_h\|_{l^\infty(0,t^m;\mathrm{H}^1(\Omega)^d)} \leq c'_e(\delta t + h^l), \tag{6.10}$$

and that the stability hypothesis (HS3) holds for all time steps $0 \leq k \leq m$.

Initialization. It is possible to find a constant $c_{s,0}$ such that $\forall\ h \leq h_{s,0} = 1$ and $\delta t \leq c_{s,0} h^{\frac{d}{4}}$, (HS3) is verified. We also need to find bounds on the error on the first time increment of velocity and pressure, namely, $\delta_t \tilde{e}^1$ and $\delta_t \tilde{\epsilon}^1$. We first control $\tilde{e}^1$; it is clear that

$$\begin{aligned} \|\tilde{e}_h^1\|_0^2 &\leq \|e_h^0\|_0^2 + 2\delta t |(\tilde{e}^1, B_h^t \psi_h^0)| + 2\delta t \sum_{i=1}^5 |R_i(\tilde{e}_h^1)| \\ &\leq \|e_h^0\|_0^2 + \gamma \|\tilde{e}_h^1\|_0^2 + c\delta t^2 \|\psi_h^0\|_1^2 + 2\delta t \sum_{i=1}^5 |R_i(\tilde{e}_h^1)|, \end{aligned}$$

with the same notations as in the proof of Theorem 5.1, and γ is a positive real number that can be chosen as small as needed.

In order to control $\|e_h^1\|_0$, the terms $|R_i(\tilde{e}_h^1)|$, $1 \leq i \leq 5$ will be treated differently from what we did in the proof of Theorem 5.1:

$$2\delta t |R_1(\tilde{e}_h^1)| \leq \gamma \|\tilde{e}_h^1\|_0^2 + c(\delta t h^{l+1})^2,$$

$$2\delta t |R_2(\tilde{e}_h^1)| \leq \gamma \|\tilde{e}_h^1\|_0^2 + c(\delta t)^4,$$

$$\begin{aligned} 2\delta t|R_3(\tilde{e}_h^1)| &\le 2\|\tilde{e}_h^1\|_0 \|u^0 - \tilde{u}_h^0 - (u^0 - \tilde{u}_h^0)(X^0)\|_0 \\ &\le c\|\tilde{e}_h^1\|_{0,2}\|u^0 - \tilde{u}_h^0\|_{0,2} \\ &\le \gamma\|\tilde{e}_h^1\|_0^2 + c(\delta t h^l)^2. \end{aligned}$$

Similarly, for $|R_4(\tilde{e}_h^1)|$ one obtains

$$\begin{aligned} 2\delta t|R_4(\tilde{e}_h^1)| &\le 2\|\tilde{e}_h^1\|_0 \|(u^0 - \tilde{u}_h^0)(X^0) - (u^0 - \tilde{u}_h^0)(X_h^0)\|_0 \\ &\le c\|\tilde{e}_h^1\|_{0,2}\|u^0 - \tilde{u}_h^0\|_0 \\ &\le \gamma\|\tilde{e}_h^1\|_0^2 + c(\delta t h^l)^2. \end{aligned}$$

Finally,

$$\begin{aligned} 2\delta t|R_5(\tilde{e}_h^1)| &\le 2\|\tilde{e}_h^1\|_0 \|u^0(X_h^0) - u^0(X^0)\|_0 \\ &\le c\|\tilde{e}_h^1\|_0 \|X^0 - X_h^0\|_0 \|u^0\|_{L^\infty(t^0,t^1;W^{1,\infty}(\Omega))} \\ &\le c\delta t\|\tilde{e}_h^1\|_0 (\|u^0 - \tilde{u}_h^0\|_0 + \delta t\|u_t\|_{L^\infty(\mathrm{L}^2(\Omega)^d)})\|u\|_{L^\infty(W^{1,\infty}(\Omega))} \\ &\le \gamma\|\tilde{e}_h^1\|_0^2 + c\delta t^2(\delta t + h^{l+1})^2. \end{aligned}$$

From these bounds and from the assumptions on e_h^0 and ϵ_h^0 (recalling that $\psi_h^0 = (q_h^1 - q_h^0) + \epsilon_h^0$ and that $\|q_h^1 - q_h^0\|_1 \le c\delta t$), we deduce that

$$\|\tilde{e}_h^1\|_0 \le c\delta t(\delta t + h^l).$$

Thence,

$$\|\delta_t \tilde{e}_h^1\|_0 \le c\delta t(\delta t + h^l).$$

Furthermore, from the projection step (5.5) we obtain

$$\left\{ \begin{array}{rcl} \|e_h^1\|_0 &\le& \|\tilde{e}_h^1\|_0, \\ \|C_h^{\mathrm{t}}(\epsilon_h^1 - \psi_h^0)\|_0 &\le& \|\tilde{e}_h^1\|_0/\delta t. \end{array} \right.$$

The first bound yields easily

$$\|\delta_t e_h^1\|_0 \le c\delta t(\delta t + h^l). \tag{6.11}$$

The other bound yields

$$\|C_h^{\mathrm{t}}\delta_t \epsilon_h^1\|_0 \le c(\delta t + h^l). \tag{6.12}$$

Thus the induction hypothesis is verified at step $m = 0$ with $c'_e = c'_{e,0}$. To verify it for $m+1 > 0$, the strategy is the following: by reproducing Steps 1 to 6 of the proof of Theorem 5.1, we obtain that (6.7) holds at time step $m+1$. Then we derive the equations that controls the incremental errors $\delta_t e_h/\delta t$, $\delta_t \tilde{e}_h/\delta t$, and $\delta_t \epsilon_h/\delta t$. Assuming that (HS3) holds, we obtain bound (6.8) by proceeding as in the proof of Theorem 5.1. By summing (5.4) and $i_h^{\mathrm{t}}(5.5)$ we obtain the momentum equation and the inf-sup inequality yields the bound (6.9). Then using $\delta_t \tilde{e}_h^{m+1}/\delta t$ as a test function in the same momentum equation yields (6.10). The last steps consist of verifying (HS3) at step $m+1$ provided $\delta t \le c_s h^{d/4}$. Thus the induction hypothesis is verified in five steps.

Step 1. We reproduce the arguments of Theorem 5.1 up to Step 6 and prove that (6.7) holds at time step t^{m+1}.

Step 2. We derive an error estimate on the time increments of the errors $\delta_t\tilde{e}^{k+1}$. Let us consider the equations that control the time increments of the errors $\delta_t\tilde{e}^{k+1}$, $\delta_t\tilde{e}^{k+1}$, and $\delta_t\tilde{\psi}^{k+1}$:

$$\left(\frac{\delta_t\tilde{e}_h^{k+1} - i_h^{\mathrm{t}}\delta_t e_h^k}{\delta t}, v_h\right) + (\nabla\delta_t\tilde{e}_h^{k+1}, \nabla v_h) + (B_h^{\mathrm{t}}\delta_t\psi_h^k, v_h) = (\delta_t R^{k+1}, v_h) \tag{6.13}$$

and

$$\begin{cases} \dfrac{\delta_t e_h^{k+1} - i_h\delta_t\tilde{e}_h^{k+1}}{\delta t} + C_h^{\mathrm{t}}(\delta_t\epsilon_h^{k+1} - \delta_t\psi_h^k) = 0, \\ C_h\delta_t e_h^{k+1} = 0, \end{cases} \tag{6.14}$$

where $\delta_t R^{k+1}$ is given by

$$\begin{aligned} \delta_t R^{k+1} = &-\frac{1}{\delta t}(\delta_t{}^2\eta^{k+1}, v_h) + \left(\frac{u^{k+1}-u^k}{\delta t} - D_t u^{k+1}, v_h\right) - \frac{1}{\delta t}(\tilde{u}_h^k(X_h^k) - \tilde{u}_h^k, v_h), \\ &-\left(\frac{u^k - u^{k-1}}{\delta t} - D_t u^k, v_h\right) + \frac{1}{\delta t}(\tilde{u}_h^{k-1}(X_h^{k-1}) - \tilde{u}_h^{k-1}, v_h). \end{aligned}$$

The term $\delta_t R^{k+1}$ is in turn decomposed into the sum of nine terms:

$$\delta_t R^{k+1} = \sum_{i=0}^{8} \delta_t R_i^{k+1},$$

where

$$\begin{aligned}
\delta_t R_0^{k+1}(v_h) &= -\frac{1}{\delta t}(\delta_{tt}\eta^{k+1}, v_h), \\
\delta_t R_1^{k+1}(v_h) &= \left(\frac{u^{k+1} - u^k(X^k)}{\delta t} - D_t u^{k+1}, v_h\right) - \left(\frac{u^k - u^{k-1}(X^{k-1})}{\delta t} - D_t u^k, v_h\right), \\
\delta_t R_2^{k+1}(v_h) &= \frac{1}{\delta t}\left(\eta^k(X^k) - \eta^k - \eta^{k-1}(X^{k-1}) + \eta^{k-1}, v_h\right), \\
\delta_t R_3^{k+1}(v_h) &= \frac{1}{\delta t}\left(\delta_t\tilde{e}_h^k(X^k) - \delta_t\tilde{e}_h^k, v_h\right), \\
\delta_t R_4^{k+1}(v_h) &= \frac{1}{\delta t}\left(\tilde{e}_h^{k-1}(X^k) - \tilde{e}_h^{k-1}(X^{k-1}), v_h\right), \\
\delta_t R_5^{k+1}(v_h) &= \frac{1}{\delta t}\left(\delta_t\tilde{e}_h^k(X_h^k) - \delta_t\tilde{e}_h^k(X^k), v_h\right), \\
\delta_t R_6^{k+1}(v_h) &= \frac{1}{\delta t}\left(\tilde{e}_h^{k-1}(X_h^k) - \tilde{e}_h^{k-1}(X^k) - \tilde{e}_h^{k-1}(X_h^{k-1}) + \tilde{e}_h^{k-1}(X^{k-1}), v_h\right), \\
\delta_t R_7^{k+1}(v_h) &= \frac{1}{\delta t}\left(w_h^k(X^k) - w_h^k(X_h^k) - w_h^k(X^{k-1}) + w_h^k(X_h^{k-1}), v_h\right), \\
\delta_t R_8^{k+1}(v_h) &= \frac{1}{\delta t}\left(\delta_t w_h^k(X^{k-1}) - \delta_t w_h^k(X_h^{k-1}), v_h\right).
\end{aligned}$$

Recall that $w_h(t)$ is the approximate of $u(t)$ that has been defined in (4.6). From the regularity assumption (HR3) on u, it is clear that

$$|\delta_t R_0^{k+1}(v_h)| \le \delta t\|v_h\|_0^2 + c\|\eta_{tt}\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)},$$

and from (HR3)

$$
\begin{aligned}
&|\delta_t R_1^{k+1}(v_h)| \\
&\leq \frac{1}{\delta t}\left\| \int_{t^k}^{t^{k+1}} (s-t^k) D_t^2 u(X^{k+1}(s),s) - \int_{t^{k-1}}^{t^k} (s-t^{k-1}) D_t^2 u(X^k(s),s) \right\|_0 \|v_h\|_0 \\
&\leq c\delta t^{\frac{3}{2}} \left(\left\| \frac{\partial}{\partial t} D_t^2 u \right\|_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} + \|D_t^3 u\|_{L^2(t^{k-1},t^{k+1};\mathrm{L}^2(\Omega)^d)} \right) \|v_h\|_0 \\
&\leq c\delta t \|v_h\|_0^2 + \delta t^2 \left(\left\| \frac{\partial}{\partial t} D_t^2 u \right\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} + \|D_t^3 u\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)} \right).
\end{aligned}
$$

Using Lemma 4.5 and Corollary 4.8, the term $\delta_t R_2^{k+1}(v_h)$ is bounded as follows:

$$
\begin{aligned}
|\delta_t R_2^{k+1}(v_h)| &= \frac{1}{\delta t}(\eta^k(X^k) - \eta^k(X^{k-1}) + \delta_t \eta^k(X^{k-1}) - \delta_t \eta^k, v_h) \\
&\leq \frac{1}{\delta t}\|\eta^k(X^k) - \eta^k(X^{k-1})\|_0 \|v_h\|_0 + \frac{1}{\delta t}\|\delta_t \eta^k(X^{k-1}) - \delta_t \eta^k\|_{-1} \|v_h\|_1 \\
&\leq \frac{c_1}{\delta t}\|\nabla \eta^k\|_{0,2} \|X^k - X^{k-1}\|_{0,\infty} \|v_h\|_{0,2} + c_2 \|\delta_t \eta^k\|_0 \|v_h\|_1 \\
&\leq \delta t \|v_h\|_0^2 + \gamma \delta t \|v_h\|_1^2 + c_1 \delta t \|\eta^k\|_1^2 + c_2 \|\eta_t\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)},
\end{aligned}
$$

where γ is an arbitrary constant. In the same manner,

$$
\begin{aligned}
|\delta_t R_3^{k+1}(v_h)| &\leq \frac{1}{\delta t}\|\delta_t \tilde{e}_h^k(X^k) - \delta_t \tilde{e}_h^k\|_{-1} \|v_h\|_1 \\
&\leq c\|\delta_t \tilde{e}_h^k\|_0 \|v_h\|_1 \\
&\leq \delta t \gamma \|v_h\|_1^2 + c\frac{\|\delta_t \tilde{e}_h^k\|_0^2}{\delta t}.
\end{aligned}
$$

Given the regularity of u and from Lemma 4.5, the fourth term is bounded from above as follows:

$$
\begin{aligned}
|\delta_t R_4^{k+1}(v_h)| &\leq \frac{1}{\delta t}\|\tilde{e}_h^{k-1}(X^k) - \tilde{e}_h^{k-1}(X^{k-1})\|_0 \|v_h\|_0 \\
&\leq c\delta t \|\nabla \tilde{e}_h^{k-1}\|_0 \|v_h\|_0 \\
&\leq \delta t \gamma \|v_h\|_1^2 + c\delta t \|\tilde{e}_h^{k-1}\|_1^2.
\end{aligned}
$$

For the fifth term, we use Corollary 4.7, and if h is small enough we have

$$
\begin{aligned}
|\delta_t R_5^{k+1}(v_h)| &\leq \frac{1}{\delta t}\|\delta_t \tilde{e}_h^k(X_h^k) - \delta_t \tilde{e}_h^k(X^k)\|_{0,1} \|v_h\|_{0,\infty} \\
&\leq cD(h)\|v_h\|_1 \|\delta_t \tilde{e}_h^k\|_1 (h^{l+1} + \delta t) \\
&\leq \delta t \gamma \|v_h\|_1^2 + \gamma \frac{\|\delta_t \tilde{e}_h^k\|_1^2}{\delta t}.
\end{aligned}
$$

Given the induction hypothesis, we can apply Lemma 6.3 with $q = 2$; hence, the sixth term is bounded by

$$
|\delta_t R_6^{k+1}(v_h)| \leq cD(h)\|v_h\|_1 \|\tilde{e}_h^{k-1}\|_1 \left[\delta t(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1) + \|\delta_t \tilde{e}_h^k\|_0 \right].
$$

If h is small enough, $cD(h)(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1)$ is bounded by an arbitrary constant, say γ; as a result, we have

$$|\delta_t R_6^{k+1}(v_h)| \leq \gamma \delta t \|v_h\|_1^2 + \gamma \frac{\|\delta_t \tilde{e}_h^k\|_0^2}{\delta t} + \gamma \delta t \|\tilde{e}_h^{k-1}\|_1^2.$$

The seventh term is treated similarly: by using Lemma 6.3 with $q = 2$ and $q = \infty$, we obtain

$$\begin{aligned}
|\delta_t R_7^{k+1}(v_h)| &= \frac{1}{\delta t} \int_\Omega \left[\eta_h^k(X^k) - \eta_h^k(X_h^k) - \eta_h^k(X^{k-1}) + \eta_h^k(X_h^{k-1})\right] v_h dx \\
&\quad + \int_\Omega \left[u^k(X^k) - u^k(X_h^k) - u^k(X^{k-1}) + u^k(X_h^{k-1})\right] v_h dx \\
&\leq c\big(D(h)\|v_h\|_1 \|\eta_h^k\|_1 + \|v_h\|_{1,2} \|u^k\|_{1,\infty}\big)\Big(\|\delta_t \tilde{e}_h^k\|_0 \\
&\qquad\qquad + \delta t\big(\delta t + h^l + \|\tilde{e}_h^k\|_1 + \|\tilde{e}_h^{k-1}\|_1\big)\Big).
\end{aligned}$$

Since it can be shown that $\|\eta_h^k\|_1 \leq ch^l$, and since $h^l D(h)$ is smaller than any fixed constant γ if h is small enough, we infer

$$|\delta_t R_7^{k+1}(v_h)| \leq \gamma \delta t \|v_h\|_1^2 + c_1 \frac{\|\delta_t \tilde{e}_h^k\|_0^2}{\delta t} + c_2 \delta t \big((\delta t + h^l)^2 + \|\tilde{e}_h^k\|_1^2 + \|\tilde{e}_h^{k-1}\|_1^2\big).$$

Finally, using the fact that $\|\delta_t u^k\|_{1,\infty} \leq c\delta t$ provided u_t belongs to $L^\infty(0, T, W^{1,\infty}(\Omega)^d)$ according to (HR3), we have

$$\begin{aligned}
|\delta_t R_8^{k+1}(v_h)| &\leq \frac{1}{\delta t} \|\delta_t w_h^k(X^{k-1}) - \delta_t w_h^k(X_h^{k-1})\|_{0,2} \|v_h\|_{0,2} \\
&\leq \frac{1}{\delta t} c\|X^{k-1} - X_h^{k-1}\|_{0,2} \left[\|\delta_t u^k\|_{1,\infty} \|v_h\|_{0,2} + D(h)\|\delta_t \eta_h^k\|_1 \|v_h\|_1\right] \\
&\leq \gamma \delta t \|v_h\|_1^2 + c\delta t (h^{l+1} + \delta t)^2.
\end{aligned}$$

With these estimates, it is now possible to apply the same arguments as in the proof of Theorem 5.1 to prove that at time step t^{m+1} the following bound applies:

$$\|\frac{\delta_t e_h}{\delta t}\|_{l^\infty(0,t^{m+1};\mathrm{L}^2(\Omega)^d)} + \|\frac{\delta_t \tilde{e}_h}{\delta t}\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)} \leq c'_{e,1}(\delta t + h^l). \tag{6.15}$$

Step 3. We derive the bound on $\|\epsilon_h\|_{l^2(0,t^{m+1};\mathrm{L}^2(\Omega))}$. By summing (5.4) and $i_h^{\mathrm{t}}(5.5)$ we obtain

$$(B_h^{\mathrm{t}} \epsilon_h^{k+1}, v_h) = -\left(\frac{i_h^{\mathrm{t}} \delta_t e_h^{k+1}}{\delta t}, v_h\right) - (\nabla \tilde{e}_h^{k+1}, \nabla v_h) + \sum_{i=1}^5 R_i^{k+1}(v_h). \tag{6.16}$$

The inf-sup condition yields

$$c_1 \|\epsilon_h^{k+1}\|_0 \leq \frac{\|\delta_t e_h^{k+1}\|_0}{\delta t} + c_2 \|\tilde{e}_h^{k+1}\|_1 + \sup_{|v_h|_1 = 1} \left|\sum_{i=1}^5 R_i^{k+1}(v_h)\right|,$$

which implies

$$c_1 \|\epsilon_h^{k+1}\|_0 \leq \frac{\|\delta_t e_h^{k+1}\|_0}{\delta t} + c_2 \|\tilde{e}_h^{k+1}\|_1 + c_3 \|\tilde{e}_h^k\|_1 + c_4(\delta t + h^{l+1}).$$

The desired bound can be derived directly from this inequality and the bounds on $\|\tilde{e}_h\|_{l^2(0,t^{m+1};\mathrm{H}^1(\Omega)^d)}$ and on $\|\delta_t e_h^{k+1}\|_{l^\infty(0,t^{m+1};\mathrm{L}^2(\Omega)^d)}$ that have been obtained above:

$$\|\epsilon_h\|_{l^2(0,t^{m+1};\mathrm{L}^2(\Omega)^d)} \le C(c'_{e,1})(\delta t + h^l) \le c'_{e,2}(\delta t + h^l),$$

where $c'_{e,2}$ depends of $c'_{e,1}$.

Step 4. The bound on $\|\tilde{e}_h\|_{l^\infty(\mathrm{H}^1(\Omega)^d)}$ is derived as follows by taking $2\delta_t\tilde{e}_h^{m+1}$ as a test function in (6.16): we obtain for $0 \le k \le m$ (given the algebraic identity $2(a, a-b) = |a|^2 + |a-b|^2 - |b|^2$),

$$\begin{aligned}\|\nabla\tilde{e}_h^{k+1}\|_0^2 + \|\nabla\delta_t\tilde{e}_h^{k+1}\|_0^2 + \frac{2}{\delta t}(\delta_t e_h^{k+1}, \delta_t\tilde{e}_h^{k+1}) \le \|\nabla\tilde{e}_h^k\|_0^2 + \delta t\|\epsilon_h^{k+1}\|_0^2 + \frac{1}{\delta t}\|\nabla\delta_t\tilde{e}_h^{k+1}\|_0^2 \\ + \sum_{i=1}^{5} R_i^{k+1}(2\delta_t\tilde{e}_h^{k+1}).\end{aligned} \tag{6.17}$$

But

$$(\delta_t e_h^{k+1}, \delta_t\tilde{e}_h^{k+1}) = (\delta_t e_h^{k+1}, \delta_t e_h^{k+1} + \delta t C_h^t(\delta_t\epsilon_h^{k+1} - \delta_t\psi_h^{k+1})) = \|\delta_t e_h^{k+1}\|_0^2$$

is nonnegative, and

$$|R_1^{k+1}(2\delta_t\tilde{e}_h^{k+1})| \le \frac{1}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0^2 + ch^{2(l+1)}||u_t||^2_{L^2(t_k,t_{k+1},H^{l+1}(\Omega)^d)},$$

$$|R_2^{k+1}(2\delta_t\tilde{e}_h^{k+1})| \le \frac{1}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0^2 + c\delta t^2\|D_t^2 u\|^2_{L^2(t^k,t^{k+1};\mathrm{L}^2(\Omega)^d)},$$

$$|R_3^{k+1}(2\delta_t\tilde{e}_h^{k+1})| \le \frac{1}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0^2 + c_1\delta t\|\tilde{e}_h^k\|_1^2 + c_2\delta t h^{2l},$$

$$\begin{aligned}|R_4^{k+1}(2\delta_t\tilde{e}_h^{k+1})| &\le \frac{2}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0\|(u^k - \tilde{u}_h^k)(X^k) - (u^k - \tilde{u}_h^k)(X_h^k)\|_0 \\ &\le \frac{1}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0^2 + c_2\delta t(\|\tilde{e}_h^k\|_1^2 + h^{2l}),\end{aligned}$$

$$|R_5^{k+1}(2\delta_t\tilde{e}_h^{k+1})| \le \frac{1}{\delta t}\|\delta_t\tilde{e}_h^{k+1}\|_0^2 + c\delta t(\delta t + h^{l+1})^2.$$

Therefore, by summing (6.17) from $k = 0$ to $k = m$ and by making use of the bounds above on the residuals, we obtain that

$$\|\nabla\tilde{e}_h^{m+1}\|_0 \le C(c_e, c'_{e,1}, c'_{e,2})(\delta t + h^l) \le c'_{e,3}(\delta t + h^l), \tag{6.18}$$

which yields (6.10) by taking $c'_e = \max(c'_{e,0}, c'_{e,1}, c'_{e,2}, c'_{e,3})$.

Step 5. From (6.16), (6.15), and (6.10), we deduce

$$\|\epsilon_h^{k+1}\|_0 \le c(\delta + h^l).$$

Step 6. There remains to check the stability hypothesis (HS3). Let us denote by $P_h u^{m+1}$ some interpolate of u^{m+1} that satisfies the error estimates of (HA1). We have

$$\begin{aligned}\|\tilde{u}_h^{m+1}\|_{1,\infty} &\le \|\tilde{u}_h^{m+1} - P_h u^{m+1}\|_{1,\infty} + \|P_h u^{m+1}\|_{1,\infty} \\ &\le c_1 h^{-\frac{d}{2}}(\|\tilde{u}_h^{m+1} - w_h^{m+1}\|_{1,2} + \|w_h^{m+1} - P_h u^{m+1}\|_{1,2}) + c_2\|u_{m+1}\|_{1,\infty}.\end{aligned}$$

Given the error bound we have obtained at time step $m+1$, and setting $\delta t \le c_s h^{d/4}$ we obtain

$$\delta t \|\tilde{u}_h^{m+1}\|_{1,\infty} \le \delta t \left[C(c'_e) h^{-\frac{d}{2}} (\delta t + h^l) + c \right].$$

From this bound we can define c_s such that (HS3) holds.

Step 7. The bounds (6.4), (6.5), and (6.6) are easy consequences of (6.7), (6.8), (6.9), and (6.10) since

$$\begin{aligned} u^{m+1} - u_h^{m+1} &= \eta^{m+1} + e_h^{m+1}, \\ u^{m+1} - \tilde{u}_h^{m+1} &= \eta^{m+1} + \tilde{e}_h^{m+1}, \\ p^{m+1} - p_h^{m+1} &= p^{m+1} - q_h^{m+1} + \epsilon_h^{m+1}. \end{aligned}$$

This completes the proof of Theorem 6.4. ∎

7. Conclusions. We have carried out the convergence analysis of a fractional step method to compute incompressible viscous flows. The algorithm is composed of three substeps: a Lagrange–Galerkin step (advection), a Helmholtz problem (diffusion), and a Poisson problem (incompressibility). Provided the solution is smooth enough on a finite time interval $[0, T]$ and the time step is smaller than $c_s h^{d/4}$, the method is shown to yield an error of $\mathcal{O}(h^{l+1} + \delta t)$ in the L^2 norm for the velocity and an error of $\mathcal{O}(h^l + \delta t)$ in the H^1 norm (or the L^2 norm for the pressure). In practice the Lagrange–Galerkin step cannot be performed exactly: the velocity $\tilde{u}_h^k(X_h(\cdot, t^{k+1}; t^k))$ is evaluated at some Gauss-points (x_l) and $X_h(x_l, t^{k+1}; t^k)$ is obtained by solving approximately the ODE (3.3). For a detailed analysis of the effects of nonexact integration on the Lagrange–Galerkin technique the reader is referred to Süli [24].

The method proposed above has been implemented in a three dimensional Navier–Stokes code; see [1], for simulating flows in cavities and around cylinders. The observed behavior of the scheme seems to be in agreement with the estimates reported above, but no systematic numerical study has been carried out yet.

We finish this paper by giving an improvement of the algorithm to obtain second order accuracy in time. First we build a second order approximation of the advection derivative $u^{k+1} \cdot \nabla u^{k+1}$. Denote by $u_h^{\star,k+1}$ the second order extrapolation of the velocity: $2\tilde{u}_h^k - \tilde{u}_h^{k-1}$. For all x in Ω we define $X_h(x, t^{k+1}; t)$ as the solution to the initial value problem

$$\text{(7.1)} \qquad \left\{ \begin{array}{l} \dfrac{dX_h(x, t^{k+1}; t)}{dt} = u_h^{\star,k+1}(X_h(x, t^{k+1}; t)), \qquad t^{k-1} \le t < t^{k+1}, \\ X_h(x, t^{k+1}; t^{k+1}) = x. \end{array} \right.$$

Let us set $X_h^{k+1,k} = X_h(x, t^{k+1}; t^k)$ and $X_h^{k+1,k-1} = X_h(x, t^{k+1}; t^{k-1})$; then we may use $2\delta t^{-1}(\tilde{u}_h^k - \tilde{u}_h^k(X_h^{k+1,k})) - 0.5\delta t^{-1}(\tilde{u}_h^{k-1} - \tilde{u}_h^k(X_h^{k+1,k-1}))$ as an approximation to $u^{k+1} \cdot \nabla u^{k+1}$. Now we assume that $\hat{u}_h^0$, $\hat{u}_h^1$, and $\hat{p}_h^1$ are some known approximations of $u(0)$, $u(\delta t)$, and $p(\delta t)$. For $k = 0, 1$ we set $u_h^k = \tilde{u}_h^k = \hat{u}_h^k$ and $p_h^1 = \hat{p}_h^1$. For $1 \le k \le K-1$, we define $\tilde{u}_h^{k+1} \in X_h$ as being the solution to the following problem:

$$\text{(7.2)} \qquad \left\{ \begin{array}{l} \left(\dfrac{3\tilde{u}_h^{k+1} - 4 i_h^t u_h^k + i_h^t u_h^{k-1}}{2\delta t}, v_h \right) + (\nabla u_h^{k+1}, \nabla v_h) \\ \quad + \left(\dfrac{4(\tilde{u}_h^k - \tilde{u}_h^k(X_h^{k+1,k})) - (\tilde{u}_h^{k-1} - \tilde{u}_h^k(X_h^{k+1,k-1}))}{2\delta t}, v_h \right) \\ \quad + (B_h^t p_h^k, v_h) = (f^{k+1}, v_h) \qquad \forall v_h \in X_h; \end{array} \right.$$

then the projection step reads as follows: find u_h^{k+1} in Y_h and p_h^{k+1} in M_h so that

$$\begin{cases} \dfrac{3u_h^{k+1} - 3i_h\tilde{u}_h^{k+1}}{2\delta t} + C_h^{\mathrm{t}}(p_h^{k+1} - p_h^k) = 0, \\ C_h u_h^{k+1} = 0. \end{cases} \tag{7.3}$$

In practice, the projected velocities u_h^k are eliminated: for $k \geq 3$, the algorithm takes the form

$$\begin{aligned} &\left(\frac{3\tilde{u}_h^{k+1} - 4\tilde{u}_h^k(X_h^{k+1,k}) + \tilde{u}_h^{k-1}(X_h^{k+1,k-1})}{2\delta t}, v_h\right) + (\nabla\tilde{u}_h^{k+1}, \nabla v_h) \\ &\qquad + \frac{1}{3}(B_h^{\mathrm{t}}(7p_h^k - 5p_h^{k-1} + p_h^{k-2}), v_h) = (f^{k+1}, v_h) \qquad \forall v_h \in X_h \end{aligned} \tag{7.4}$$

and

$$C_h C_h^{\mathrm{t}}(p_h^{k+1} - p_h^k) = \frac{3B_h\tilde{u}_h^{k+1}}{2\delta t}. \tag{7.5}$$

For $k = 3$, $\tilde{u}_h^3$, is computed by eliminating u_h^2, whereas the computation of $\tilde{u}_h^2$ does not require any elimination since u_h^1 and u_h^0 are known. Note that in the diffusion step the pressure term $(7p_h^k - 5p_h^{k-1} + p_h^{k-2})/3$ can be written in the form $2p_h^k - p_h^{k-1} + (p_h^k - 2p_h^{k-1} + p_h^{k-2})/3$; hence, it can be seen as a second order extrapolation. This scheme is being investigated numerically and numerical results will be reported in a forthcoming work. The convergence analysis of a scheme similar to the one proposed here where the advection term is made semi-implicit is given in Guermond [13]; it is shown in this reference that the error on the velocity in the L^2 norm is of $\mathcal{O}(\delta t^2 + h^{l+1})$.

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