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# Vorticity–Velocity Formulations of the Stokes Problem in 3D

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This work studies the three-dimensional Stokes problem expressed in terms of vorticity and velocity variables. We make general assumptions on the regularity and the topological structure of the flow domain: the boundary is Lipschitz and possibly non-connected and the flow domain may be multiply connected. Upon introducing a new variational space for the vorticity, five weak formulations of the Stokes problem are obtained. All the formulations are shown to lead to well-posed problems and to be equivalent to the primitive variable formulation. The various formulations are discussed by interpreting the test functions for the vorticity (resp. velocity) equation as vector potentials for the velocity (resp. vorticity). Of the five sets of boundary conditions derived in the paper, three are already known, but only for domains with a trivial topological structure, while the remaining two are new. Copyright © 1999 John Wiley & Sons, Ltd.

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## 1. Introduction

Over the last decade, the vorticity–velocity formulation of the incompressible Navier–Stokes equations has emerged in the computational fluid dynamics community as an efficient approach to simulate fluid flows in two and three space dimensions. In this formulation, the dynamics are governed by the vorticity transport equation which is an extensively studied and well understood equation, while the kinematic aspects of the problem, embodied by the velocity, and controlled by an elliptic equation of vector type. From a numerical viewpoint, this formulation offers several advantages. First, vorticity and velocity appear to be appropriate variables for describing vortex dominated flows. Second, the treatment of boundary conditions for the vorticity–velocity formulation is easier than for the primitive variables or stream-function–vorticity formulations. Indeed, the vorticity–velocity formulation eliminates the numerical difficulties associated with pressure boundary conditions and numerically accurate vorticity boundary conditions are more easily implemented in terms of

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velocity than stream function. A third advantage of this formulation is its ability to easily handle non-inertial frames of reference. For a detailed review of the numerical solution of the incompressible Navier–Stokes equations using alternative formulations, we refer to [18] and to the references therein.

The vorticity–velocity formulation was first derived in [10] to investigate the stability of boundary layers in two dimensions and later extended to steady-state three-dimensional incompressible flows in [5]. Since then, several numerical techniques have been designed to enhance the efficiency and robustness of the method, including finite differences on staggered and non-staggered grids, finite element methods [14], influence matrix techniques [3] and spectral methods [2]. As a result, a large number of flow simulations using vorticity–velocity have been reported, including both internal and external flows. Moreover, the formulation has been recently extended to flows with variable density and viscosity in the low Mach number regime [7, 8] and has been used to investigate numerically laminar flame problems [9].

In spite of its numerical practicality, the vorticity–velocity formulation has been the subject of a rather limited number of mathematical studies. Well posedness results for the incompressible Stokes problem in two and three dimensions have been established in [19, 20]. These results have been generalized later to weakly compressible, two-dimensional flows in [8]. In these studies, the boundary of the flow domain was assumed to be of class $C^{1,1}$ or to be a convex polygon. A different approach has been followed in [22], but optimality regarding the regularity which can be assumed for the flow domain remains to be clarified. Another important point, which is almost always overlooked (with a few exceptions such as [4]), concerns the case of multiple connected domains with a non-connected boundary.

The goal of this paper is two-fold. Upon introducing appropriate functional spaces, we want to analyse in a systematic way the variational formulations that can be obtained for three-dimensional flow problems written in vorticity–velocity form. For the sake of simplicity, the analysis deals with the steady Stokes problem with homogeneous velocity boundary conditions. The extension of the present results to the Navier–Stokes equations, to unsteady problems or to non-homogeneous velocity boundary conditions is straightforward by using the same techniques as for the Navier–Stokes problem in the primitive variables formulation (see, for instance, [11]). Our second goal is to make general assumptions concerning both the regularity and the topological structure of the flow domain. Specifically, we assume that the boundary is only Lipschitz and we treat the case of a multiple connected domain with a non-connected boundary.

This paper is organized as follows. In section 2, we briefly restate some useful results, and upon introducing a new functional space for the vorticity, we obtain some preliminary results on the vorticity–velocity formulation for the Stokes problem. Weak formulations for the vorticity–velocity problem are discussed in sections 3–5. The formulations are classified by interpreting the test functions for the vorticity (resp. velocity) equation as vector potentials for the velocity (resp. vorticity) field. In all cases, we avoid using the divergence-free constraint in an essential way for both velocity and vorticity. We show that all the variational formulations lead to well posed systems and that they are equivalent to the primitive variable formulation. In addition, we derive five sets of boundary conditions in a strong formulation of the problem. Three of them are already well-known when both the flow domain and its boundary have a trivial topological structure, but the remaining two are new. In section 3 we treat the formulations based on a normal vector potential for the velocity, in section 4 those based on a tangential vector potential, and finally in section 5 the one based on a velocity vector potential vanishing on the boundary of the flow domain.

## 2. Preliminary results

In this section we present the assumptions and notation used in this work, we formulate the Stokes problem in velocity–pressure and vorticity–velocity variables and we introduce appropriate functional spaces for the vorticity and for the vector potentials of velocity and vorticity. For the proofs and more details on what follows, the reader is referred, for instance, to [1, 6, 11, 15, 21, 23].

### 2.1. *Assumptions and notation*

Let $\Omega$ be an open connected bounded domain of $\mathbb{R}^3$ with a Lipschitz boundary. The boundary of $\Omega$ is denoted by $\Gamma$ and its connected components are denoted by $\Gamma_i$, for $0 \leqslant i \leqslant I$, where $\Gamma_0$ is the exterior boundary. In addition, since $\Omega$ is not assumed to be simply connected, we introduce smooth cuts, namely, we assume that there exist surfaces $\Sigma_j$, $1 \leqslant j \leqslant J$, such that $\Sigma_j$ is an open part of a smooth manifold, the boundary of $\Sigma_j$ is contained in $\Gamma$, the intersection $\Sigma_j \cap \Sigma_{j'}$ is emply for $j \neq j'$, and the open set $\Omega^0 = \Omega \backslash \bigcup_{j=1}^{J} \Sigma_j$ is simply connected. The integer $J$ denotes the degree of multiple connectedness (the number of handles) of $\Omega$. The unit exterior normal vector to the boundary is defined almost everywhere on $\Gamma$ and is denoted by $\mathbf{n}$. Similarly, we denote by $\mathbf{n}_j$, $1 \leqslant j \leqslant J$, a unit normal vector to $\Sigma_j$.

We consider the classical spaces $\mathbf{H}(\mathbf{curl}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \nabla \times \mathbf{v} \in \mathbf{L}^2(\Omega)\}$ and $\mathbf{H}(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega);\ \nabla \cdot \mathbf{v} \in L^2(\Omega)\}$ and define

$$\mathbf{X}(\Omega) = \mathbf{H}(\mathrm{div}, \Omega) \cap \mathbf{H}(\mathbf{curl}, \Omega), \tag{2.1}$$

equipped with the norm $\|\mathbf{v}\|_X = \|\mathbf{v}\|_0 + \|\nabla \times \mathbf{v}\|_0 + \|\nabla \cdot \mathbf{v}\|_0$. Using the well-known trace properties of spaces $\mathbf{H}(\mathbf{curl}, \Omega)$ and $\mathbf{H}(\mathrm{div}, \Omega)$ [11], we define $\mathbf{H}_0(\mathrm{div}, \Omega) = \{\mathbf{v} \in \mathbf{H}(\mathrm{div}, \Omega);\ \mathbf{v} \cdot \mathbf{n}_{|\Gamma} = 0\}$, and we set

$$\mathbf{X}_{\mathrm{N}}(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega);\ \mathbf{v} \times \mathbf{n}_{|\Gamma} = 0\}, \tag{2.2}$$

$$\mathbf{X}_{\mathrm{T}}(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega);\ \mathbf{v} \cdot \mathbf{n}_{|\Gamma} = 0\}. \tag{2.3}$$

We have the following result.

**Proposition 2.1.** *If $\Omega$ is a Lipschitz domain, we have*

$$\mathbf{H}_0^1(\Omega) = \mathbf{X}_{\mathrm{N}}(\Omega) \cap \mathbf{X}_{\mathrm{T}}(\Omega). \tag{2.4}$$

 

For later use, we also introduce the space

$$\mathbf{H}(\mathrm{div}=0,\Omega)=\{\mathbf{v}\in\mathbf{H}(\mathrm{div},\Omega);\ \nabla\cdot\mathbf{v}=0\}, \tag{2.5}$$

with similar notation for $\mathbf{H}_0(\mathrm{div}\ =0,\Omega)$, $\mathbf{X}_\mathrm{N}(\mathrm{div}=0,\Omega)$ and $\mathbf{X}_\mathrm{T}(\mathrm{div}=0,\Omega)$.

We next introduce $\mathbf{H}_0(\mathrm{div},\Omega)'$, the dual space of $\mathbf{H}_0(\mathrm{div},\Omega)$, that we equip with the norm

$$\|\mathbf{f}\|_{,}=\sup_{\mathbf{v}\in\mathbf{H}_0(\mathrm{div},\Omega)}\frac{\langle\mathbf{f},\mathbf{v}\rangle}{\|\mathbf{v}\|_0+\|\nabla\cdot\mathbf{v}\|_0},\quad \mathbf{f}\in\mathbf{H}_0(\mathrm{div},\Omega)'. \tag{2.6}$$

We introduce the bilinear form defined on $\mathbf{X}(\Omega)\times\mathbf{X}(\Omega)$ as follows:

$$a(\mathbf{u},\mathbf{v})=(\nabla\times\mathbf{u},\nabla\times\mathbf{v})+(\nabla\cdot\mathbf{u},\nabla\cdot\mathbf{v}). \tag{2.7}$$

For all $\mathbf{u}$ in $\mathbf{X}(\Omega)$, we introduce the semi-norm $|\mathbf{u}|_X=a(\mathbf{u},\mathbf{u})^{1/2}$ and we set

$$\mathscr{K}_\mathrm{N}(\Omega)=\{\mathbf{v}\in\mathbf{X}_\mathrm{N}(\Omega);\ |\mathbf{v}|_X=0\}, \tag{2.8}$$

$$\mathscr{K}_\mathrm{T}(\Omega)=\{\mathbf{v}\in\mathbf{X}_\mathrm{T}(\Omega);\ |\mathbf{v}|_X=0\}, \tag{2.9}$$

It is well-known that $\mathscr{K}_\mathrm{N}(\Omega)\cap\mathscr{K}_\mathrm{T}(\Omega)=\{0\}$ and that $\mathscr{K}_\mathrm{N}(\Omega)$ and $\mathscr{K}_\mathrm{T}(\Omega)$ may be characterized as follows.

**Proposition 2.2.** $\dim(\mathscr{K}_\mathrm{N}(\Omega))=I$, *and there is a basis of* $\mathscr{K}_\mathrm{N}(\Omega)$: $\gamma_1,\gamma_2,\dots,\gamma_I$, *such that* $\int_{\Gamma_i}\gamma_k\cdot\mathbf{n}=\delta_{ik}$, *for* $1\leqslant i,k\leqslant I$, *where* $\delta_{ik}$ *is the Kronecker symbol and*

$$\forall\mathbf{v}\in\mathscr{K}_\mathrm{N}(\Omega),\quad \mathbf{v}=\sum_{i=1}^{I}\left(\int_{\Gamma_i}\mathbf{v}\cdot\mathbf{n}\right)\boldsymbol{\gamma}_i. \tag{2.10}$$

**Proposition 2.3.** $\dim(\mathscr{K}^\mathrm{T}(\Omega))=J$, *and there is a basis of* $\mathscr{K}_\mathrm{T}(\Omega)$: $\boldsymbol{\mu}_1,\boldsymbol{\mu}_2,\dots,\boldsymbol{\mu}_J$, *such that* $\int_{\Sigma_j}\boldsymbol{\mu}_k\cdot\mathbf{n}_j=\delta_{jk}$, *for* $1\leqslant j,k\leqslant J$, *and*

$$\forall\mathbf{v}\in\mathscr{K}_\mathrm{T}(\Omega),\quad \mathbf{v}=\sum_{j=1}^{J}\left(\int_{\Sigma_j}\mathbf{v}\cdot\mathbf{n}_j\right)\boldsymbol{\mu}_j. \tag{2.11}$$

For later use, we introduce the projection operators

$$\boldsymbol{\Pi}_\mathrm{N}(\mathbf{v})=\mathbf{v}-\sum_{i=1}^{I}\left(\int_{\Gamma_i}\mathbf{v}\cdot\mathbf{n}\right)\boldsymbol{\gamma}_i,\quad \mathbf{v}\in\mathbf{X}_\mathrm{N}(\Omega), \tag{2.12}$$

$$\boldsymbol{\Pi}_\mathrm{T}(\mathbf{v})=\mathbf{v}-\sum_{j=1}^{J}\left(\int_{\Sigma_j}\mathbf{v}\cdot\mathbf{n}_j\right)\boldsymbol{\mu}_j,\quad \mathbf{v}\in\mathbf{X}_\mathrm{T}(\Omega). \tag{2.13}$$

Finally, owing to the Peetre–Tartar lemma, the compacity of the injectors of $\mathbf{X}_\mathrm{N}(\Omega)$ and $\mathbf{X}_\mathrm{T}(\Omega)$ into $\mathbf{L}^2(\Omega)$ together with Propositions 2.2 and 2.3 implies the following coercivity property for the bilinear form $a(\cdot,\cdot)$.

 

**Proposition 2.4.** *There exists a constant* $c>0$ *such that*

$$\forall \mathbf{v} \in \mathbf{X}_{\mathrm{N}}(\Omega), \quad c\,\|\mathbf{v}\|_X \leqslant |\mathbf{v}|_X + \sum_{i=1}^{I} \left| \int_{\Gamma_i} \mathbf{v}\cdot\mathbf{n} \right|, \tag{2.14}$$

$$\forall \mathbf{v} \in \mathbf{X}_{\mathrm{T}}(\Omega), \quad c\,\|\mathbf{v}\|_X \leqslant |\mathbf{v}|_X + \sum_{j=1}^{J} \left| \int_{\Sigma_j} \mathbf{v}\cdot\mathbf{n}_j \right|. \tag{2.15}$$

Hereafter, $c$ will denote a generic positive constant.

### 2.2. *The Stokes problem*

The homogeneous Stokes problem formulated in the velocity–pressure formulation reads

$$\begin{aligned} &-\Delta\mathbf{u} + \nabla p = \mathbf{f},\\ &\nabla\cdot\mathbf{u} = 0,\\ &\mathbf{u}_{|\Gamma} = 0. \end{aligned} \tag{2.16}$$

If the body force $\mathbf{f}$ is in $\mathbf{H}^{-1}(\Omega)$, it is well-known that the Stokes problem (2.16) admits a unique solution $(\mathbf{u}, p)$ in $\mathbf{H}_0^1(\Omega)\times L_0^2(\Omega)$ [11]. One possible variational formulation for (2.16) consists in seeking $\mathbf{u}$ in $\mathbf{V} = \{\mathbf{v}\in\mathbf{H}_0^1(\Omega);\ \nabla\cdot\mathbf{v} = 0\}$ such that

$$\forall \mathbf{v}\in\mathbf{V}, \quad (\nabla\mathbf{u}, \nabla\mathbf{v}) = \langle \mathbf{f}, \mathbf{v}\rangle, \tag{2.17}$$

where $(\cdot,\cdot)$ denotes the duality pairing between $\mathbf{H}^{-1}(\Omega)$ and $\mathbf{H}_0^1(\Omega)$.

Working directly with the space $\mathbf{V}$ is a usual way to eliminate the pressure. Alternative formulations of (2.17) can be built by replacing $\mathbf{V}$ by the curl of vector potentials. However, this way of proceeding is too restrictive since it yields formulations involving explicitly the biharmonic operator and vector potentials belonging to $\mathbf{H}^2(\Omega)$. An alternative approach consists in considering mixed formulations where the vorticity

$$\boldsymbol{\omega} = \nabla\times\mathbf{u}, \tag{2.18}$$

is introduced as an auxiliary variable. The momentum equation now reads

$$\nabla\times\boldsymbol{\omega} = \mathbf{f} - \nabla p. \tag{2.19}$$

To avoid unnecessary complications resulting from regularity requirements, we shall hereafter assume that the body force is in $\mathbf{H}_0(\mathrm{div},\Omega)'$. Provided this regularity assumption is made, the natural functional space for $\boldsymbol{\omega}$ is

$$\mathbf{W} = \{\mathbf{w}\in\mathbf{H}(\mathrm{div},\Omega);\ \nabla\times\mathbf{w}\in\mathbf{H}_0(\mathrm{div},\Omega)'\}, \tag{2.20}$$

equipped with the norm

$$\|\boldsymbol{\omega}\|_W = \|\boldsymbol{\omega}\|_0 + \|\nabla\cdot\boldsymbol{\omega}\|_0 + \|\nabla\times\boldsymbol{\omega}\|_{\prime}. \tag{2.21}$$

 

The weak form of the momentum equation reads

$$\forall \mathbf{v} \in \mathbf{H}_0(\operatorname{div} = 0, \Omega), \quad \langle \nabla \times \boldsymbol{\omega}, \mathbf{v} \rangle = \langle \mathbf{f}, \mathbf{v} \rangle. \tag{2.22}$$

The vorticity $\boldsymbol{\omega}$ will be sought in various subspaces of $\mathbf{W}$, depending on the way—essential or natural—the normal trace of the vorticity is enforced on the boundary. Specifically, since the velocity vanishes on the boundaries $\Gamma_i$, $0 \leqslant i \leqslant I$, we have $\mathbf{w} \cdot \mathbf{n}_{|\Gamma} = 0$ and a first subspace is

$$\mathbf{W}_\mathrm{T} = \{\mathbf{w} \in \mathbf{W};\ \mathbf{w} \cdot \mathbf{n}_{|\Gamma} = 0\}. \tag{2.23}$$

The condition on the normal trace of $\boldsymbol{\omega}$ can be weakened by seeking $\boldsymbol{\omega}$ in

$${}_I\mathbf{W} = \left\{\mathbf{w} \in \mathbf{W};\ \int_{\Gamma_i} \mathbf{w} \cdot \mathbf{n} = 0,\ 1 \leqslant i \leqslant I\right\}. \tag{2.24}$$

We shall also make use of a slightly stronger characterization of the vorticity by seeking it in

$${}_I\mathbf{W}_\star = \left\{\mathbf{w} \in {}_I\mathbf{W};\ \int_\Omega \nabla \cdot \mathbf{w} = 0\right\}. \tag{2.25}$$

2.3. *Functional spaces for vector potentials*

The two key ideas in deriving a weak formulation for the Stokes problem in vorticity–velocity form are the following:

(1) Replace in (2.22) the test functions $\mathbf{v}$, which are divergence free and with zero normal trace, by the curl of velocity vector potentials.
(2) Formulate the definition of the vorticity (2.18) in a weak form using test functions which are the curl of vorticity vector potentials.

The following two isomorphism results for the curl operator play a fundamental role to prove the existence of vector potentials for either velocity or vorticity. We first consider the case of vector potentials tangent to $\Gamma$.

**Theorem 2.1.** *Let $\Omega$ be a Lipschitz domain. Then the operator*

$$\nabla \times : \left\{\boldsymbol{\psi} \in \mathbf{X}_\mathrm{T}(\operatorname{div} = 0, \Omega);\ \int_{\Sigma_j} \boldsymbol{\psi} \cdot \mathbf{n}_j = 0,\ 1 \leqslant j \leqslant J\right\}$$

$$\to \left\{\mathbf{u} \in \mathbf{H}(\operatorname{div} = 0, \Omega);\ \int_{\Gamma_i} \mathbf{u} \cdot \mathbf{n} = 0,\ 1 \leqslant i \leqslant I\right\}$$

*is an isomorphism.*

In particular, $\nabla \times$ induces a one-to-one correspondence between a finite-dimensional subspace of $\{\boldsymbol{\psi} \in \mathbf{X}_\mathrm{T}(\operatorname{div} = 0, \Omega);\ \int_{\Sigma_j} \boldsymbol{\psi} \cdot \mathbf{n} = 0,\ 1 \leqslant j \leqslant J\}$ and $\mathscr{K}_\mathrm{T}(\Omega)$. For further reference we denote this space

$$\mathscr{J}_{\mathrm{T}} = (\nabla \times)^{-1}[\mathscr{K}_{\mathrm{T}}(\Omega)], \tag{2.26}$$

the reference to $\Omega$ being left implicit for brevity. We next consider the case of vector potentials normal to the boundary $\Gamma$.

**Theorem 2.2.** *Let $\Omega$ be a Lipschitz domain. Then the operator*

$$\nabla \times : \left\{ \boldsymbol{\psi} \in \mathbf{X}_{\mathrm{N}}(\mathrm{div} = 0, \Omega);\ \int_{\Gamma_i} \boldsymbol{\psi} \cdot \mathbf{n} = 0,\ 1 \leqslant i \leqslant I \right\} \oplus \mathscr{J}_{\mathrm{T}}$$

$$\rightarrow \mathbf{H}_0(\mathrm{div} = 0, \Omega)$$

*is an isomorphism.*

We now present the appropriate functional spaces for the vector potential of either velocity or vorticity. Upon introducing

$${}_I\boldsymbol{\Psi}_{\mathrm{N}} = \left\{ \boldsymbol{\psi} \in \mathbf{X}_{\mathrm{N}}(\Omega);\ \int_{\Gamma_i} \boldsymbol{\psi} \cdot \mathbf{n} = 0,\ 1 \leqslant i \leqslant I \right\}, \tag{2.27}$$

$${}_J\boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}} = \left\{ \boldsymbol{\psi} \in \mathbf{X}_{\mathrm{T}}(\Omega);\ \nabla \times \boldsymbol{\psi} \cdot \mathbf{n}_{|\Gamma} = 0;\ \int_{\Sigma_j} \boldsymbol{\psi} \cdot \mathbf{n}_j = 0,\ 1 \leqslant j \leqslant J \right\}, \tag{2.28}$$

$$\boldsymbol{\Psi}_0 = \mathbf{H}_0^1(\Omega), \tag{2.29}$$

the vector potentials for the velocity (test functions for the vorticity) will be selected in the following spaces:

$${}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}, \quad {}_J\boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}}, \quad \boldsymbol{\Psi}_0 \oplus \mathscr{J}_{\mathrm{T}}, \tag{2.30}$$

depending on whether the tangential, normal or all the components of $\boldsymbol{\psi}$ are set to zero at the boundary. We have the following proposition.

**Proposition 2.5.** *Let $\Omega$ be a Lipschitz domain and let $\mathbf{u} \in \mathbf{H}_0(\mathrm{div} = 0, \Omega)$. Then there exist $\boldsymbol{\psi}_{\mathrm{N}} \in {}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}$, $\boldsymbol{\psi}_{\mathrm{T}c\mathrm{T}} \in {}_J\boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}}$, and $\boldsymbol{\psi}_0 \in \boldsymbol{\Psi}_0 \oplus \mathscr{J}_{\mathrm{T}}$ such that $\nabla \times \boldsymbol{\psi}_{\mathrm{N}} = \nabla \times \boldsymbol{\psi}_{\mathrm{T}c\mathrm{T}} = \nabla \times \boldsymbol{\psi}_0 = \mathbf{u}$.*

*Proof.* The existence of $\boldsymbol{\psi}_{\mathrm{N}}$ results from Theorem 2.2 while that of $\boldsymbol{\psi}_{\mathrm{T}c\mathrm{T}}$ results from Theorem 2.1. On the other hand, the existence of $\boldsymbol{\psi}_0$ is a classical consequence of Calderon's prolongation result [17]. $\square$

 

Similarly, we introduce the spaces

$$_I\mathbf{\Phi}_\mathrm{N} = \left\{\boldsymbol{\varphi} \in \mathbf{X}_\mathrm{N}(\Omega);\ \int_{\Gamma_i} \boldsymbol{\varphi}\cdot\mathbf{n} = 0,\ 1 \leqslant i \leqslant I\right\}, \tag{2.31}$$

$$_I\mathbf{\Phi}_{\mathrm{N}\star} = \left\{\boldsymbol{\varphi} \in {}_I\mathbf{\Phi}_\mathrm{N};\ \int_{\Omega} \nabla\cdot\boldsymbol{\varphi} = 0\right\}, \tag{2.32}$$

$$_J\mathbf{\Phi}_\mathrm{T} = \left\{\boldsymbol{\varphi} \in \mathbf{X}_\mathrm{T}(\Omega);\ \int_{\Sigma_j} \boldsymbol{\varphi}\cdot\mathbf{n}_j = 0,\ 1 \leqslant j \leqslant J\right\}. \tag{2.33}$$

The vector potentials for the vorticity (test functions for the velocity) will be selected in the following spaces:

$$_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{I}_\mathrm{T},\quad {}_I\mathbf{\Phi}_{\mathrm{N}\star} \oplus \mathscr{I}_\mathrm{T},\quad {}_J\mathbf{\Phi}_\mathrm{T}, \tag{2.34}$$

depending on whether the tangential or the normal components of $\boldsymbol{\varphi}$ are set to zero at the boundary. The choice between $_I\mathbf{\Phi}_\mathrm{N}$ and $_I\mathbf{\Phi}_{\mathrm{N}\star}$ will be motivated in section 6. We have the following proposition which is a direct consequence of Theorems 2.1 and 2.2.

**Proposition 2.6.** *Let $\Omega$ be a Lipschitz domain and $\boldsymbol{\omega} \in \mathbf{H}(\mathrm{div} = 0, \Omega)$ be such that $\int_{\Gamma_i} \boldsymbol{\omega}\cdot\mathbf{n} = 0$, for $1 \leqslant i \leqslant I$. Then there exists $\boldsymbol{\varphi}_\mathrm{T} \in {}_J\mathbf{\Phi}_\mathrm{T}$ such that $\nabla\times\boldsymbol{\varphi}_\mathrm{T} = \boldsymbol{\omega}$. Assume further that $\boldsymbol{\omega} \in \mathbf{H}_0(\mathrm{div} = 0, \Omega)$. Then there exists $\boldsymbol{\varphi}_\mathrm{N} \in {}_I\mathbf{\Phi}_{\mathrm{N}\star} \oplus \mathscr{I}_\mathrm{T}$, and a fortiori in $_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{I}_\mathrm{T}$, such that $\nabla\times\boldsymbol{\varphi}_\mathrm{N} = \boldsymbol{\omega}$.*

We finish this section by establishing a preliminary result to be used hereafter; its proof is straightforward and is omitted for brevity.

**Theorem 2.3.** *Let $\Omega$ be a Lipschitz domain and let $\mathbf{f}$ in $\mathbf{H}_0(\mathrm{div}, \Omega)'$. Let $(\mathbf{u}, p)$ in $\mathbf{H}^1_0(\Omega)\times L^2_0(\Omega)$ be the solution to the primitive variable Stokes problem (2.16) and let $\boldsymbol{\omega} = \nabla\times\mathbf{u}$. Then $\boldsymbol{\omega}$ belongs to space $\mathbf{W}_\mathrm{T}$, and thus also to $_I\mathbf{W}_\star$, $_I\mathbf{W}$, and $\mathbf{W}$. In addition, we have*

$$\langle \nabla\times\boldsymbol{\omega}, \nabla\times\boldsymbol{\psi}\rangle + (\nabla\cdot\boldsymbol{\omega}, \nabla\cdot\boldsymbol{\psi}) = \langle \mathbf{f}, \nabla\times\boldsymbol{\psi}\rangle, \tag{2.35}$$

*for all $\boldsymbol{\psi}$ belonging to $_I\mathbf{\Psi}_\mathrm{N} \oplus \mathscr{I}_\mathrm{T}$, $_J\mathbf{\Psi}_{\mathrm{TcT}}$, or $\mathbf{\Psi}_0 \oplus \mathscr{I}_\mathrm{T}$, and*

$$(\nabla\times\mathbf{u}, \nabla\times\boldsymbol{\varphi}) + (\nabla\cdot\mathbf{u}, \nabla\cdot\boldsymbol{\varphi}) = (\boldsymbol{\omega}, \nabla\times\boldsymbol{\varphi}), \tag{2.36}$$

*for all $\boldsymbol{\varphi}$ belonging to $_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{I}_\mathrm{T}$, $_I\mathbf{\Phi}_{\mathrm{N}\star} \oplus \mathscr{I}_\mathrm{T}$, or $_J\mathbf{\Phi}_\mathrm{T}$.*

*Remark* 2.1. Note that when $\boldsymbol{\psi}$ belongs to $_I\mathbf{\Psi}_\mathrm{N} \oplus \mathscr{I}_\mathrm{T}$, $_J\mathbf{\Psi}_{\mathrm{TcT}}$, or $\mathbf{\Psi}_0 \oplus \mathscr{I}_\mathrm{T}$ we have $\nabla\times\boldsymbol{\psi} \in \mathbf{H}_0(\mathrm{div}, \Omega)$ and the duality products are thus well defined.

*Remark* 2.2. It is readily seen that the constraints on the fluxes over $\Gamma_i$ and $\Sigma_j$ arising in the definition of the functional spaces for $\boldsymbol{\psi}$ and $\boldsymbol{\varphi}$ eliminate the test functions belonging to $\mathscr{K}_\mathrm{N}(\Omega)$ or $\mathscr{K}_\mathrm{T}(\Omega)$. Indeed, such functions would result in a trivial form for equations (2.35) and (2.36), namely $0 = 0$.

 

## 3. Formulations based on a normal velocity vector potential

In this section we review the vorticity–velocity formulations based on the existence of a normal vector potential for the velocity, i.e. a vector potential in ${}_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T}$. The only alternative left is the selection of the functional space for the test functions in the velocity equation. We shall see below that this choice amounts to specifying the normal trace of the vorticity as an essential boundary condition or a natural one.

### 3.1. *Normal test functions for the velocity equation*

We want first to enforce the normal trace of the vorticity as an essential boundary condition; hence, we shall seek $\boldsymbol{\omega}$ in $\mathbf{W}_\mathrm{T}$. Since we enforce four essential scalar conditions (three for the velocity and one for the vorticity), the spaces of the test functions must account for two natural scalar boundary conditions. Since ${}_I\mathbf{\Psi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T}$ allows for one natural scalar boundary condition, we are left with only one natural scalar boundary condition to test the velocity equation and a suitable space in then ${}_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T}$.

**Theorem 3.1.** *Let $\Omega$ be a Lipschitz domain and let* $\mathbf{f}$ *in* $\mathbf{H}_0(\mathrm{div}, \Omega)'$. *Consider the following problem: seek* $\boldsymbol{\omega} \in \mathbf{W}_\mathrm{T}$ *and* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{\psi} \in {}_I\mathbf{\Psi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T},\ \langle \nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\psi} \rangle + (\nabla \cdot \boldsymbol{\omega}, \nabla \cdot \boldsymbol{\psi}) = \langle \mathbf{f}, \nabla \times \boldsymbol{\psi} \rangle,\\
&\forall \boldsymbol{\varphi} \in {}_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T},\ (\nabla \times \mathbf{u}, \nabla \times \boldsymbol{\varphi}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi}).
\end{aligned} \tag{3.1}$$

*Then, this problem has a unique solution and we have the a priori estimate*

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\omega}\|_W \leqslant c\|\mathbf{f}\|_{\prime}. \tag{3.2}$$

*Furthermore,* $\mathbf{u}$ *is the solution to the Stokes problem* (2.16) *and* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

*Proof.* The existence of a solution directly results from Theorem 2.3. Hence, we only need to prove uniqueness and the a priori estimate (3.2). To this purpose, we show first that $\mathbf{u}$ and $\boldsymbol{\omega}$ are solenoidal. We consider an arbitrary function $h$ in $L^2(\Omega)$ and define $q$ such that $\Delta q = h$ with homogeneous Dirichlet boundary conditions. Then, $\boldsymbol{\psi} = \mathbf{\Pi}_\mathrm{N}(\nabla q)$ is in ${}_I\mathbf{\Psi}_\mathrm{N}$ and we have $\nabla \times \boldsymbol{\psi} = 0$ and $\nabla \cdot \boldsymbol{\psi} = h$. Upon using $\boldsymbol{\psi}$ as a test function in the vorticity equation, we get $(\nabla \cdot \boldsymbol{\omega}, h) = 0$ which implies $\nabla \cdot \boldsymbol{\omega} = 0$. By using the same argument, we prove that $\nabla \cdot \mathbf{u}$ is zero.

Since $\boldsymbol{\omega}$ is in $\mathbf{H}_0(\mathrm{div} = 0, \Omega)$, we deduce from Proposition 2.6 that there is a vector potential $\boldsymbol{\varphi}(\boldsymbol{\omega})$ in ${}_I\mathbf{\Phi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T}$ such that $\boldsymbol{\omega} = \nabla \times \boldsymbol{\varphi}(\boldsymbol{\omega})$. Similarly, from Proposition 2.5, there exists a vector potential $\boldsymbol{\psi}(\mathbf{u})$ in ${}_I\mathbf{\Psi}_\mathrm{N} \oplus \mathscr{J}_\mathrm{T}$ such that $\mathbf{u} = \nabla \times \boldsymbol{\psi}(\mathbf{u})$. By using $\boldsymbol{\varphi}(\boldsymbol{\omega})$ as a test function in the velocity equation, we obtain

$$(\nabla \times \mathbf{u}, \boldsymbol{\omega}) = \|\boldsymbol{\omega}\|_0^2. \tag{3.3}$$

Furthermore, by using $\boldsymbol{\psi}(\mathbf{u})$ as a test function in the vorticity equation, we obtain

$$\langle \nabla \times \boldsymbol{\omega}, \mathbf{u} \rangle = (\mathbf{f}, \mathbf{u}). \tag{3.4}$$

Finally, by using $\mathbf{u}$ as test function in the velocity equation and by Proposition 2.4, we derive the bound

$$\|\mathbf{u}\|_1 \leqslant c\|\boldsymbol{\omega}\|_0. \tag{3.5}$$

By combining the three estimates above, we obtain the a priori estimate

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\omega}\|_0 \leqslant c\|\mathbf{f}\|_\prime, \tag{3.6}$$

from which we easily infer uniqueness.

To obtain estimate (3.2), we need only to derive an upper bound for $\|\nabla \times \boldsymbol{\omega}\|_\prime$, keeping in mind that $\boldsymbol{\omega}$ is solenoidal. Since we know that $\boldsymbol{\omega} = \nabla \times \mathbf{u}$ where $\mathbf{u}$ is the solution of (2.16), we have $\nabla \times \boldsymbol{\omega} = \mathbf{f} - \nabla p$ and the pressure $p$ satisfies $\|p\|_0 \leqslant c\|\mathbf{f}\|_\prime$. Since $\|\nabla p\|_\prime \leqslant \|p\|_0$, we easily obtain estimate (3.2). □

*Remark* 3.1. Formally, the solution to (3.1) satisfies the following system:

$$
\begin{aligned}
&-\Delta\boldsymbol{\omega} = \nabla \times \mathbf{f},\ -\Delta\mathbf{u} = \nabla \times \boldsymbol{\omega},\\
&\mathbf{u}_{|\Gamma} = 0,\ \nabla \cdot \mathbf{u}_{|\Gamma} = 0,\ \boldsymbol{\omega} \cdot \mathbf{n}_{|\Gamma} = 0,\ \nabla \cdot \boldsymbol{\omega}_{|\Gamma} = 0,\\
&(\nabla \times \boldsymbol{\omega} - \mathbf{f}) \perp \mathscr{K}_{\mathrm{T}}(\Omega),\ (\nabla \times \mathbf{u} - \boldsymbol{\omega}) \perp \mathscr{K}_{\mathrm{T}}(\Omega).
\end{aligned}
$$

The conditions $(\nabla \times \boldsymbol{\omega} - \mathbf{f}) \perp \mathscr{K}_{\mathrm{T}}(\Omega)$ and $(\nabla \times \mathbf{u} - \boldsymbol{\omega}) \perp \mathscr{K}_{\mathrm{T}}(\Omega)$ account for the multiple connectedness of the domain and enforce $(\nabla \times \boldsymbol{\omega} - \mathbf{f})$ and $(\nabla \times \mathbf{u} - \boldsymbol{\omega})$ to be gradients. Note also that the last condition can be equivalently replaced by $\int_{\Sigma_j} \boldsymbol{\omega} \cdot \mathbf{n}_j = 0$, for $1 \leqslant j \leqslant J$. This formulation has been considered in [4] for the time-dependent Navier–Stokes equations.

### 3.2. *Tangential test functions for the velocity equation*

As in section 3.1, we still use normal vector potentials to test the vorticity equation. However, we now choose to enforce the normal trace of the vorticity in a natural way. Since there essential scalar boundary conditions are enforced (three for the velocity and none for the vorticity), the test functions must globally account for three scalar natural boundary conditions. Since ${}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}$ allows for one natural scalar boundary condition, we are left with two natural scalar boundary conditions to test the velocity equation. Then, a suitable space of vector potentials is ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$. Since $\boldsymbol{\omega}$ is sought in $\nabla \times {}_J\boldsymbol{\Phi}_{\mathrm{T}}$, the fluxes $\int_{\Gamma_i} \boldsymbol{\omega} \cdot \mathbf{n}$ must be zero for $1 \leqslant i \leqslant I$; as a result, we shall seek $\boldsymbol{\omega}$ in ${}_I\mathbf{W}$.

**Theorem 3.2.** *Let* $\Omega$ *be a Lipschitz domain and* $\mathbf{f}$ *be in* $\mathbf{H}_0(\mathrm{div}, \Omega)'$. *Consider the following problem*: *seek* $\boldsymbol{\omega} \in {}_I\mathbf{W}$ and $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$
\begin{aligned}
&\forall \boldsymbol{\psi} \in {}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}},\ \langle \nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\psi} \rangle + (\nabla \cdot \boldsymbol{\omega}, \nabla \cdot \boldsymbol{\psi}) = \langle \mathbf{f}, \nabla \times \boldsymbol{\psi} \rangle,\\
&\forall \boldsymbol{\varphi} \in {}_J\boldsymbol{\Phi}_{\mathrm{T}},\ (\nabla \times \mathbf{u}, \nabla \times \boldsymbol{\varphi}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi}).
\end{aligned} \tag{3.7}
$$

*Then, this problem has a unique solution and we have the a priori estimate*

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\omega}\|_W \leqslant c\|\mathbf{f}\|_\prime. \tag{3.8}$$

*Furthermore*, $\mathbf{u}$ *is the solution to the Stokes problem* (2.16) *and* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

 

*Proof.* The proof follows the same ideas of that of Theorem 3.1 and we only outline the differences. We prove that $\boldsymbol{\omega}$ is solenoidal as before. Regarding the velocity, we proceed as follows. Let $h$ be an arbitrary function in $L^2_0(\Omega)$, and consider $q$ in $H^1(\Omega)$ such that $\Delta q = h$ with a homogeneous Neumann boundary condition. Then, $\boldsymbol{\varphi} = \boldsymbol{\Pi}_{\mathrm{T}}(\boldsymbol{\nabla} q)$ is in ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$ and we have $\boldsymbol{\nabla} \times \boldsymbol{\varphi} = 0$ and $\boldsymbol{\nabla} \cdot \boldsymbol{\varphi} = h$. Upon using $\boldsymbol{\varphi}$ as a test function in the velocity equation, we obtain $(\boldsymbol{\nabla} \cdot \mathbf{u}, h) = 0$. Hence, $\boldsymbol{\nabla} \cdot \mathbf{u}$ is constant and since the normal trace of $\mathbf{u}$ is zero, we deduce that $\mathbf{u}$ is actually solenoidal.

Since $\boldsymbol{\omega}$ is now in $\mathbf{H}(\mathrm{div} = 0, \Omega)$ and since $\int_{\Gamma_i} \boldsymbol{\omega} \cdot \mathbf{n} = 0$, for $1 \leqslant i \leqslant I$, by definition of ${}_I\mathbf{W}$, we deduce from Proposition 2.6 the existence of $\boldsymbol{\varphi}(\boldsymbol{\omega})$ in ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$ such that $\boldsymbol{\omega} = \boldsymbol{\nabla} \times \boldsymbol{\varphi}(\boldsymbol{\omega})$. On the other hand, as in the previous proof, there exists $\boldsymbol{\psi}(\mathbf{u})$ in ${}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}$ such that $\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\psi}(\mathbf{u})$. Using $\boldsymbol{\psi}(\mathbf{u})$, $\boldsymbol{\varphi}(\boldsymbol{\omega})$, and $\boldsymbol{\Pi}_{\mathrm{T}}(\mathbf{u})$ as test functions, uniqueness and the a priori estimate (3.8) are proved as before. □

*Remark* 3.2. Formally, the solution to (3.7) satisfies the following system:

$$-\Delta\boldsymbol{\omega} = \boldsymbol{\nabla} \times \mathbf{f}, \; -\Delta\mathbf{u} = \boldsymbol{\nabla} \times \boldsymbol{\omega},$$

$$\mathbf{u}_{|\Gamma} = \mathbf{0}, \; \boldsymbol{\nabla} \cdot \boldsymbol{\omega}_{|\Gamma} = 0, \; (\boldsymbol{\omega} - \boldsymbol{\nabla} \times \mathbf{u}) \times \mathbf{n}_{|\Gamma} = 0,$$

$$(\boldsymbol{\nabla} \times \boldsymbol{\omega} - \mathbf{f}) \perp \mathscr{K}_{\mathrm{T}}(\Omega), \; \int_{\Gamma_i} \boldsymbol{\omega} \cdot \mathbf{n} = 0, \; 1 \leqslant i \leqslant I.$$

The condition $\int_{\Gamma_i} \boldsymbol{\omega} \cdot \mathbf{n} = 0$ for $1 \leqslant i \leqslant I$, results from the fact that $\Gamma$ is not connected, whereas the requirement $(\boldsymbol{\nabla} \times \boldsymbol{\omega} - \mathbf{f}) \perp \mathscr{K}_{\mathrm{T}}(\Omega)$ accounts for the multiple connectedness of the domain.

## 4. Formulations based on a tangential velocity vector potential

In this section we review the vorticity–velocity formulations based on the existence of a velocity vector potential which is tangential to $\Gamma$. As in section 3, two cases are considered depending on whether the test functions for the velocity equation are normal or tangential to the boundary. This choice amounts to specifying the normal trace of the vorticity as an essential boundary condition or not.

### 4.1. *Normal test functions for the velocity equation*

Let us enforce the normal trace on the vorticity as an essential boundary condition. Since four essential scalar conditions are enforced on the velocity and the vorticity (three for $\mathbf{u}$ and one for $\boldsymbol{\omega}$), the spaces spanned by the test functions must account for two natural scalar boundary conditions. Since ${}_J\boldsymbol{\psi}_{\mathrm{TcT}}$ allows for only one natural scalar condition, we must require the vector potentials used to test the velocity equation to allow for another one. A space of vector potentials satisfying this requirement is ${}_I\boldsymbol{\Phi}_{\mathrm{N}\star} \oplus \mathscr{J}_{\mathrm{T}}$.

 

**Theorem 4.1.** *Let* $\Omega$ *be a Lipschitz domain and* $\mathbf{f}$ *be in* $\mathbf{H}_0(\mathrm{div}, \Omega)'$. *Consider the following problem: seek* $\boldsymbol{\omega} \in \mathbf{W}_{\mathrm{T}}$ *and* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$\begin{aligned} &\forall \boldsymbol{\psi} \in {}_J\boldsymbol{\Psi}_{\mathrm{TcT}}, \quad \langle \nabla \times \boldsymbol{\omega}, \nabla \times \boldsymbol{\psi} \rangle + (\nabla \cdot \boldsymbol{\omega}, \nabla \cdot \boldsymbol{\psi}) = \langle \mathbf{f}, \nabla \times \boldsymbol{\psi} \rangle, \\ &\forall \boldsymbol{\varphi} \in {}_I\boldsymbol{\Phi}_{\mathrm{N}\star} \oplus \mathscr{J}_{\mathrm{T}}, \quad (\nabla \times \mathbf{u}, \nabla \times \boldsymbol{\varphi}) + (\nabla \cdot \mathbf{u}, \nabla \cdot \boldsymbol{\varphi}) = (\boldsymbol{\omega}, \nabla \times \boldsymbol{\varphi}). \end{aligned} \tag{4.1}$$

*Then, this problem has a unique solution and we have the a priori estimate*

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\omega}\|_W \leqslant c\|\mathbf{f}\|_{\prime}. \tag{4.2}$$

*Furthermore,* $\mathbf{u}$ *is the solution to the Stokes problem* (2.16) *and* $\boldsymbol{\omega} = \nabla \times \mathbf{u}$.

*Proof.* Let $h$ be an arbitrary function in $L_0^2(\Omega)$ and consider $q$ in $H^1(\Omega)$ such that $\Delta q = h$ with a homogeneous Neumann boundary condition. Then it is readily seen that $\boldsymbol{\psi} = \boldsymbol{\Pi}_{\mathrm{T}}(\nabla q)$ is in ${}_J\boldsymbol{\Psi}_{\mathrm{TcT}}$ and that $\nabla \times \boldsymbol{\psi} = 0$ and $\nabla \cdot \boldsymbol{\psi} = h$. By using $\boldsymbol{\psi}$ as a test function in the vorticity equation, we obtain $(\nabla \cdot \boldsymbol{\omega}, h) = 0$. Hence $\nabla \cdot \boldsymbol{\omega}$ is constant in $\Omega$ and since the normal trace of $\boldsymbol{\omega}$ is zero, we deduce that $\boldsymbol{\omega}$ is solenoidal. Let $h$ be again an arbitrary function in $L_0^2(\Omega)$ and consider now $\eta$ in $H^1(\Omega)$ such that $\Delta \eta = h$ with homogeneous Dirichlet boundary condition. Then it is readily seen that $\boldsymbol{\varphi} = \boldsymbol{\Pi}_{\mathrm{N}}(\nabla \eta)$ is an admissible test function for the velocity equation, yielding $(\nabla \cdot \mathbf{u}, h) = 0$. As for $\boldsymbol{\omega}$, this condition readily implies that $\nabla \cdot \mathbf{u}$ is zero.

Since $\boldsymbol{\omega}$ is in $\mathbf{H}_0(\mathrm{div} = 0, \Omega)$, we deduce from Proposition 2.6 that there is a vector potential $\boldsymbol{\varphi}(\boldsymbol{\omega})$ in ${}_I\boldsymbol{\Phi}_{\mathrm{N}\star} \oplus \mathscr{J}_{\mathrm{T}}$ such that $\boldsymbol{\omega} = \nabla \times \boldsymbol{\varphi}(\boldsymbol{\omega})$. On the other hand, we infer from Proposition 2.5 that there is $\boldsymbol{\Psi}(\mathbf{u})$ in ${}_J\boldsymbol{\Psi}_{\mathrm{TcT}}$ such that $\mathbf{u} = \nabla \times \boldsymbol{\psi}(\mathbf{u})$. By using $\boldsymbol{\varphi}(\boldsymbol{\omega})$, $\boldsymbol{\psi}(\mathbf{u})$, and $\mathbf{u}$ as test functions in the vorticity and velocity equations, we complete the proof as before. □

*Remark* 4.1. Formally, the solution to (4.1) satisfies the following system:

$$\begin{aligned} &-\Delta \boldsymbol{\omega} = \nabla \times \mathbf{f}, \quad -\Delta \mathbf{u} = \nabla \times \boldsymbol{\omega}, \\ &\mathbf{u}_{|\Gamma} = \mathbf{0}, \quad \boldsymbol{\omega} \cdot \mathbf{n}_{|\Gamma} = 0, \quad \nabla \cdot \mathbf{u}_{|\Gamma} = 0, \\ &\textstyle\int_\Gamma (\nabla \times \boldsymbol{\omega} - \mathbf{f}) \cdot \mathbf{n} \times \boldsymbol{\psi}_\Gamma = 0, \quad \forall \boldsymbol{\psi}_\Gamma \in \boldsymbol{\Psi}_{\mathrm{TcT}}(\Gamma). \end{aligned}$$

Here, $\boldsymbol{\Psi}_{\mathrm{TcT}}(\Gamma)$ denotes the space obtained by taking the trace on $\Gamma$ of the functions in ${}_J\boldsymbol{\Psi}_{\mathrm{TcT}}(\Omega)$, i.e. $\boldsymbol{\Psi}_{\mathrm{TcT}}(\Gamma) = [{}_J\boldsymbol{\Psi}_{\mathrm{TcT}}(\Omega)]_{|\Gamma}$. The surface integral condition on the vorticity curl can be interpreted as a condition enforcing that the tangential trace of $(\nabla \times \boldsymbol{\omega} - \mathbf{f})$ be the gradient of a scalar function defined on the boundary. The same condition has also been considered in connection with a formulation of the Stokes problem in terms of vorticity and vector potential variables [13].

### 4.2. *Tangential test functions for the velocity equation*

While still using tangential vector potentials in the vorticity equation, we now seek the normal trace of the vorticity as a natural boundary condition. By using arguments similar to those of section 3.2, we are led to use ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$ to test the velocity equation while $\boldsymbol{\omega}$ will be sought in ${}_I\mathbf{W}_\star$.

 

**Theorem 4.2.** *Let* $\Omega$ *be a Lipschitz domain and let* $\mathbf{f}$ *in* $\mathbf{H}_0(\mathrm{div},\Omega)'$. *Consider the following problem: seek* $\boldsymbol{\omega} \in {}_I\mathbf{W}_\star$ *and* $\mathbf{u} \in \mathbf{H}_0^1(\Omega)$ *such that*

$$\begin{aligned}
&\forall \boldsymbol{\psi} \in {}_J\boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}}, \quad \langle \nabla\times\boldsymbol{\omega}, \nabla\times\boldsymbol{\psi}\rangle + (\nabla\cdot\boldsymbol{\omega}, \nabla\cdot\boldsymbol{\psi}) = \langle \mathbf{f}, \nabla\times\boldsymbol{\psi}\rangle,\\
&\forall \boldsymbol{\varphi} \in {}_J\boldsymbol{\Phi}_{\mathrm{T}}, \quad (\nabla\times\mathbf{u}, \nabla\times\boldsymbol{\varphi}) + (\nabla\cdot\mathbf{u}, \nabla\cdot\boldsymbol{\varphi}) = (\boldsymbol{\omega}, \nabla\times\boldsymbol{\varphi}).
\end{aligned} \tag{4.3}$$

*Then, this problem has a unique solution and we have the a priori estimate*

$$\|\mathbf{u}\|_1 + \|\boldsymbol{\omega}\|_W \leqslant c\|\mathbf{f}\|_{,}. \tag{4.4}$$

*Furthermore,* $\mathbf{u}$ *is the solution to the Stokes problem* (2.16) *and* $\boldsymbol{\omega} = \nabla\times\mathbf{u}$.

*Proof.* Upon considering an appropriate Neumann problem as in the proof of Theorem 4.1, we prove that $\nabla\cdot\boldsymbol{\omega}$ is constant in $\Omega$. Since $\int_\Omega \nabla\cdot\boldsymbol{\omega} = 0$ by definition of ${}_I\mathbf{W}_\star$, we deduce that $\boldsymbol{\omega}$ is indeed solenoidal. On the other hand, we prove that $\mathbf{u}$ is solenoidal as in the proof of Theorem 3.2.

Since $\boldsymbol{\omega}$ is solenoidal and is such that $\int_{\Gamma_i} \boldsymbol{\omega}\cdot\mathbf{n} = 0$ for $1 \leqslant i \leqslant I$, by definition of ${}_I\mathbf{W}_\star$, we deduce from Proposition 2.6 the existence of $\boldsymbol{\varphi}(\boldsymbol{\omega})$ in ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$ such that $\boldsymbol{\omega} = \nabla\times\boldsymbol{\varphi}(\boldsymbol{\omega})$. On the other hand, by Proposition 2.5, there exists $\boldsymbol{\psi}(\mathbf{u})$ in ${}_J\boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}}$ such that $\mathbf{u} = \nabla\times\boldsymbol{\psi}(\mathbf{u})$. Finally, it is easily seen that $\boldsymbol{\Pi}_{\mathrm{T}}(\mathbf{u})$ is in ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$. By using the test functions $\boldsymbol{\psi}(\mathbf{u})$, $\boldsymbol{\varphi}(\boldsymbol{\omega})$, and $\boldsymbol{\Pi}_{\mathrm{T}}(\mathbf{u})$, the proof is then completed as before. □

*Remark* 4.2. Note that the constraint on $\nabla\cdot\boldsymbol{\omega}$ appearing in the definition of space ${}_I\mathbf{W}_\star$ can be replaced by $\int_{\Gamma_0} \boldsymbol{\omega}\cdot\mathbf{n} = 0$. We also point out that this constraint is needed in order to establish that $\boldsymbol{\omega}$ is solenoidal.

*Remark* 4.2. Formally, the solution to (4.3) satisfies the following system:

$$\begin{aligned}
&-\Delta\boldsymbol{\omega} = \nabla\times\mathbf{f}, \quad -\Delta\mathbf{u} = \nabla\times\boldsymbol{\omega},\\
&\mathbf{u}_{|\Gamma} = 0, \quad (\boldsymbol{\omega} - \nabla\times\mathbf{u})\times\mathbf{n}_{|\Gamma} = 0,\\
&\int_\Gamma (\nabla\times\boldsymbol{\omega} - \mathbf{f})\cdot\mathbf{n}\times\boldsymbol{\psi}_\Gamma = 0, \quad \forall\boldsymbol{\psi}_\Gamma \in \boldsymbol{\Psi}_{\mathrm{T}c\mathrm{T}}(\Gamma),\\
&\int_{\Gamma_i} \boldsymbol{\omega}\cdot\mathbf{n} = 0, \quad 0 \leqslant i \leqslant I.
\end{aligned}$$

## 5. Formulation based on a velocity vector potential vanishing on the boundary

In this section we are interested in the vorticity–velocity formulation based on the existence of a velocity vector potential which vanishes on $\Gamma$. Accordingly, we shall require the velocity vector potential to span the space $\boldsymbol{\Psi}_0 \oplus \mathscr{J}_{\mathrm{T}}$.

At variance with the two previous sections, an essential boundary condition must be enforced here on the normal trace of the vorticity. More precisely, if we had chosen not to enforce essential boundary conditions on $\boldsymbol{\omega}$, we would have tested the velocity equation by means of vector potentials with no boundary conditions. However, as shown in [12], enforcing completely natural boundary conditions on the bilinear form $(\nabla\times\mathbf{u}, \nabla\times\mathbf{v}) + (\nabla\cdot\mathbf{u}, \nabla\cdot\mathbf{v})$ yields a sensitive problem in the sense of Lions–Palençia [16]. As a result, the vorticity will be sought in space $\mathbf{W}_{\mathrm{T}}$ and the velocity equation will be tested with functions in ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$.

 

**Theorem 5.1.** *Let $\Omega$ be a Lipschitz domain and let* $\mathbf{f}$ *in* $\mathbf{H}_0(\mathrm{div},\Omega)'$. *Consider the following problem*: *seek* $\boldsymbol{\omega}\in\mathbf{W}_\mathrm{T}$ *and* $\mathbf{u}\in\mathbf{H}_0^1(\Omega)$ *such that*

$$\begin{aligned}&\forall\boldsymbol{\psi}\in\boldsymbol{\Psi}_0\oplus\mathscr{J}_\mathrm{T},\quad \langle\nabla\times\boldsymbol{\omega},\nabla\times\boldsymbol{\psi}\rangle+(\nabla\cdot\boldsymbol{\omega},\nabla\cdot\boldsymbol{\psi})=\langle\mathbf{f},\nabla\times\boldsymbol{\psi}\rangle,\\ &\forall\boldsymbol{\varphi}\in{}_J\boldsymbol{\Phi}_\mathrm{T},\quad (\nabla\times\mathbf{u},\nabla\times\boldsymbol{\varphi})+(\nabla\cdot\mathbf{u},\nabla\cdot\boldsymbol{\varphi})=(\boldsymbol{\omega},\nabla\times\boldsymbol{\varphi}).\end{aligned}\tag{5.1}$$

*Then this problem has a unique solution and we have the a priori estimate*

$$\|\mathbf{u}\|_1+\|\boldsymbol{\omega}\|_W\leqslant c\|\mathbf{f}\|_{\prime}.\tag{5.2}$$

*Furthermore*, $\mathbf{u}$ *is the solution to the Stokes problem* (2.16) *and* $\boldsymbol{\omega}=\nabla\times\mathbf{u}$.

*Proof.* By considering an appropriate Neumann problem as in the proof of Theorem 3.2, one first proves that $\mathbf{u}$ is divergence free. Consider now $\mathbf{v}=\boldsymbol{\omega}-\nabla\times\mathbf{u}$. We first note that $\nabla\cdot\mathbf{v}$ is in $L^2(\Omega)$ and $\mathbf{v}\cdot\mathbf{n}_{|\Gamma}=0$ by definition. In addition, the velocity equation implies that $\nabla\times\mathbf{v}=0$ in the distribution sense and hence in $\mathbf{L}^2(\Omega)$ and also that $\mathbf{v}\times\mathbf{n}_{|\Gamma}=0$. As a result, $\mathbf{v}$ is in $\mathbf{X}_\mathrm{N}(\Omega)\cap\mathbf{X}_\mathrm{T}(\Omega)$ and, by Proposition 2.1, $\mathbf{v}$ is thus an admissible test function for the vorticity equation. This readily yields that $\mathbf{v}=0$ and thus $\boldsymbol{\omega}=\nabla\times\mathbf{u}$. The proof is then completed as before. □

*Remark* 5.1. Formally, the solution to (5.1) satisfies the following system:

$$\begin{aligned}&-\Delta\boldsymbol{\omega}=\nabla\times\mathbf{f},\ \ -\Delta\mathbf{u}=\nabla\times\boldsymbol{\omega},\\ &\mathbf{u}_{|\Gamma}=0,\ \boldsymbol{\omega}\cdot\mathbf{n}_{|\Gamma}=0,\ (\nabla\times\mathbf{u}-\boldsymbol{\omega})\times\mathbf{n}_{|\Gamma}=0,\\ &(\nabla\times\boldsymbol{\omega}-\mathbf{f})\perp\mathscr{K}_\mathrm{T}(\Omega).\end{aligned}$$

This formulation has been considered in [20] in the case of a simply connected domain with a connected boundary.

## 6. Concluding remarks

In this paper we have considered five weak formulations for the incompressible Stokes problem in vorticity–velocity form. We have classified the weak formulations in terms of the boundary conditions imposed on the associated velocity and vorticity vector potentials. For the five formulations, we have proved existence, uniqueness and stability of the solution and we have also recovered the equivalence with the Stokes problem written in velocity–pressure variables.

With an eye towards the numerical approximation of the formulations presented above, consider a discretization of the flow domain $\Omega$ and its boundary $\Gamma$ with $N_\Omega$ mesh nodes in $\Omega$ and $N_\Gamma$ mesh nodes on $\Gamma$. Assume that all the degrees of freedom for both the unknowns and the test functions are located at the mesh nodes. Then it is easily verified that the functional spaces for the vorticity $\mathbf{W}_\mathrm{T}$, ${}_I\mathbf{W}_\star$, ${}_I\mathbf{W}$, and $\mathbf{W}$ yield, respectively, $2N_\Gamma$, $3N_\Gamma-I-1$, $3N_\Gamma-I$ and $3N_\Gamma$ degrees of freedom on the boundary.

On the other hand, for the test functions $\boldsymbol{\psi}$, the functional spaces ${}_I\boldsymbol{\Psi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}$, ${}_J\boldsymbol{\Psi}_{TcT}$, and $\boldsymbol{\Psi}_0 \oplus \mathscr{J}_{\mathrm{T}}$ yield, respectively, $N_\Gamma - I + J$, $N_\Gamma + I - J + 1$ and $J$ degrees of freedom on the boundary. Finally, for the test functions $\boldsymbol{\varphi}$, the functional spaces ${}_I\boldsymbol{\Phi}_{\mathrm{N}} \oplus \mathscr{J}_{\mathrm{T}}$, ${}_I\boldsymbol{\Phi}_{\mathrm{N}_\star} \oplus \mathscr{J}_{\mathrm{T}}$, and ${}_J\boldsymbol{\Phi}_{\mathrm{T}}$ yield, respectively, $N_\Gamma - I + J$, $N_\Gamma - I + J - 1$ and $2N_\Gamma - J$ degrees of freedom on the boundary. It is then interesting to compare the number of discrete equations and unknowns obtained from the various formulations. It is readily seen that (3.7), (4.1) and (5.1) yield as many equations as unknowns (this result motivates the choice of space ${}_I\boldsymbol{\Phi}_{\mathrm{N}_\star}$ for (4.1)). On the other hand, (3.1) yields $2(J - I)$ more equations than unknows and (4.3) yields $2(I + 1 - J)$ more equations than unknowns. As a result of this mismatch, we expect the use of spatial discretizations of unstaggered type to be impossible for the two latter formulations.

The natural extension of this work is now to consider finite element approximations for the various weak formulations and investigate the compatibility conditions between discrete subspaces of the ones considered here for the velocity and vorticity vector potentials. This study is quite lengthy and will be reported in a forthcoming work.

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