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DISCONTINUOUS GALERKIN METHODS FOR FRIEDRICHS' SYSTEMS. PART III. MULTIFIELD THEORIES WITH PARTIAL COERCIVITY*

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Abstract. This paper is the third and last part of a work attempting to give a unified analysis of discontinuous Galerkin methods. The purpose of this paper is to extend the framework that has been developed in Part II for two-field Friedrichs' systems associated with second-order PDEs. We now consider two-field Friedrichs' systems with partial L^2-coercivity and three-field Friedrichs' systems with an even weaker L^2-coercivity hypothesis. In particular, this work generalizes the discontinuous Galerkin methods of Part II to compressible and incompressible linear continuum mechanics. We also show how the stabilizing parameters of the method must be set when the two-field Friedrichs' system is composed of terms that may be of different magnitude, thus accounting, for instance, for advection–diffusion equations at high Péclet numbers.

Key words. Friedrichs' systems, finite elements, partial differential equations, discontinuous Galerkin method

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1. Introduction.

The framework of Friedrichs' systems [21] is well adapted to the analysis of first-order PDEs for it accommodates many types of PDEs in a unified setting and systematically handles various types of boundary conditions. Friedrichs' systems were initially developed to deal with change of type in transonic flows, but the theory developed by Friedrichs turned out to give a "unified treatment" of PDEs that goes beyond the elliptic/hyperbolic/parabolic divide. The key ingredient of the theory is that the sum of the operator associated with the PDE and its formal adjoint is L^2-coercive (implying that the coefficients of the PDE satisfy symmetry properties).

As shown in Part I of this work [15], approximating Friedrichs' systems using discontinuous Galerkin (DG) methods is somewhat natural, and the setting of Friedrichs' systems lends itself nicely to analysis. Actually this observation is not new, and the analysis of DG approximations of Friedrichs' systems had been initiated in the 1970s by Lesaint and Raviart [25, 26] and further pursued by Johnson, Nävert, and Pitkäranta [24], but for some reason research remained focused for many years on hyperbolic equations. Meanwhile the development of DG methods for solving elliptic equations followed a different route and has been essentially based on the early work of Nitsche on boundary-penalty methods [27] and the use of interior penalties (IP) to weakly enforce continuity conditions imposed on the solution or its derivatives across the interfaces between adjoining elements; see Douglas and Dupont [12], Baker [4], Wheeler [29], Arnold [1], Bassi and Rebay [5], and Cockburn and Shu [11]. All these approaches have been recast into a single framework amenable to a unified error analysis in Arnold et al. [2] for the Poisson problem.

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In Part I [15] we reunited the two above diverging viewpoints by showing that the theory of Friedrichs' systems is a proper setting to analyze DG methods irrespective of the type of the PDE: The DG analysis of a transport equation and that of the Laplace equation are identical when all the components of the unknown are given equal attention. Sometimes, however, the unknown, say z, possesses a two-field structure, $z = (z^\sigma, z^u)$, and the σ-component can be eliminated to yield a system of second-order PDEs for the u-component. (We have in mind here the Maxwell equations, the mixed form of the Laplace equation, or the mixed form of the linear elasticity equations.) This situation, which gives more weight to one component of the unknown, say the u-component, has been analyzed in Part II [16]. Therein we developed a theory of two-field Friedrichs' systems and analyzed two-field DG methods for which z^σ can be locally eliminated on each mesh cell.

The goal of the third and last part of this work is to extend the analysis of Part II in three directions by weakening the assumption of L^2-coercivity on which the theory of the two-field Friedrichs' systems is based. First, the L^2-coercivity is assumed to hold only on the σ-component of the field $z = (z^\sigma, z^u)$. Examples include advection–diffusion equations and compressible linear continuum mechanics problems. Second, further weakening of the partial coercivity framework is done by introducing a three-field theory of Friedrichs' systems. This framework encompasses incompressible linear continuum mechanics, e.g., Stokes and Oseen flows. Third, the two-field DG method is revisited by performing a singular perturbation analysis. The goal of this third extension is to determine how the stabilizing parameters of the method must be set when the elliptic-like PDE associated with the two-field Friedrichs' system under scrutiny is composed of a second-order term and a first-order term that may be of different magnitude. The situation covered by this theory is that of advection–diffusion equations at high Péclet numbers. The recurrent theme of the present paper is to construct robust multifield DG methods that can efficiently handle type changes (elliptic/hyperbolic, compressible/incompressible, etc.) and are such that one subfield of the unknown, say z^σ, can be eliminated on each mesh element.

This paper is organized as follows. Section 2 sets the notation and briefly reviews the main results obtained in Parts I and II. This section can be skipped by readers who are familiar with the material. Section 3 treats two-field Friedrichs' systems for which L^2-coercivity holds only on the σ-component. The key difference with Part II is that a Poincaré-like inequality must be invoked to transfer the L^2-stability from z^σ to z^u. Section 4 deals with three-field Friedrichs' systems where the partial coercivity framework is further weakened. In both cases, the well-posedness of the Friedrichs' systems is established and the convergence of their DG approximation is analyzed under general design conditions. Finally, section 5 presents a singular perturbation analysis relevant to second-order PDEs where first- and second-order terms are not of the same magnitude. Sections 3, 4, and 5 are independent and can be read separately. The theme of section 4 is robustness with respect to the compressible/incompressible-type change and that of section 5 is robustness with respect to the elliptic/hyperbolic-type change.

2. DG approximation of Friedrichs' systems.

The objective of this section is to set the notation and briefly restate the main results of Parts I and II. The reader familiar with this material can jump to section 3.

2.1. One-field Friedrichs' systems.

Let Ω be a bounded, open, and connected Lipschitz domain in $\mathbb{R}^d$. Let m be a positive integer and set $L = [L^2(\Omega)]^m$ with inner product $(\cdot,\cdot)_L$. The two ingredients to build a Friedrichs' system are an operator

$K \in \mathcal{L}(L;L)$ and a family $\{\mathcal{A}^k\}_{1\le k\le d}$ of d functions on Ω with values in $\mathbb{R}^{m,m}$ such that

$$\text{(A1)} \qquad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \textstyle\sum_{k=1}^d \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m},$$

$$\text{(A2)} \qquad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k = (\mathcal{A}^k)^t \ \text{ a.e. in } \Omega,$$

$$\text{(A3)} \qquad \exists \mu_0 > 0, \quad \forall z \in L, \quad ((K + K^* - \nabla\cdot A)z, z)_L \ge 2\mu_0 \|z\|_L^2,$$

where K^* is the adjoint of K in $\mathcal{L}(L;L)$ and $\nabla\cdot A \in \mathcal{L}(L;L)$ is defined such that $\nabla\cdot A(z) = (\sum_{k=1}^d \partial_k \mathcal{A}^k)z$ for all $z \in L$.

Let $\mathfrak{D}(\Omega)$ denote the space of $\mathfrak{C}^\infty$ functions that are compactly supported in Ω. A function z in L is said to have an A-weak derivative in L if the linear form $[\mathfrak{D}(\Omega)]^m \ni \phi \longmapsto -\int_\Omega \sum_{k=1}^d z^t \partial_k(\mathcal{A}^k\phi) \in \mathbb{R}$ is bounded on L. In this case, the function in L that can be associated with the above linear form by means of the Riesz representation theorem is denoted by Az. The so-called graph space $W = \{z \in L;\, Az \in L\}$ is endowed with a Hilbert structure when equipped with the scalar product $(z,y)_L + (Az, Ay)_L$. Define the operators $A \in \mathcal{L}(W;L)$ and $\tilde{A} \in \mathcal{L}(W;L)$ by

$$\text{(2.1)} \qquad Az = \sum_{k=1}^d \mathcal{A}^k \partial_k z, \qquad \tilde{A}z = -\sum_{k=1}^d \partial_k(\mathcal{A}^k z),$$

and set $T = K + A$, $\tilde{T} = K^* + \tilde{A}$. $\tilde{A}$ and $\tilde{T}$ are the formal adjoints of A and T, respectively. Assumption (A3), which implies that $T + \tilde{T}$ is L-coercive on L, is the full L^2-coercivity property alluded to in section 1.

Let $f \in L$ and consider the PDE system $Tz = f$. An important question we are facing now is to equip this problem with proper boundary conditions. The key idea underlying the theory of Friedrichs' systems is that boundary conditions can be enforced by making use of a boundary operator $M \in \mathcal{L}(W;W')$ such that

$$\text{(M1)} \qquad \forall z \in W,\ \langle Mz, z\rangle_{W',W} \ge 0,$$

$$\text{(M2)} \qquad W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M),$$

where $D \in \mathcal{L}(W;W')$ is defined by

$$\text{(2.2)} \qquad \forall (z,y) \in W \times W, \quad \langle Dz, y\rangle_{W',W} = (Az, y)_L - (z, \tilde{A}y)_L.$$

Observe that (2.2) is just an integration-by-parts formula and that D is self-adjoint by construction. It is shown in [17] that by setting $V = \mathrm{Ker}(D - M)$ and $V^* = \mathrm{Ker}(D + M^*)$, where M^* is the adjoint operator of M, the following problems are well-posed:

$$\text{(2.3)} \qquad \text{Seek } z \in V \text{ such that } Tz = f. \qquad \text{Seek } z^* \in V^* \text{ such that } \tilde{T}z^* = f.$$

The key idea sustaining the entire DG theory developed in Parts I, II, and hereafter is that it is possible to enforce boundary conditions weakly by introducing the following bilinear forms on $W \times W$,

$$\text{(2.4)} \qquad a(z,y) = (Tz, y)_L + \tfrac{1}{2}\langle (M - D)z, y\rangle_{W',W},$$

$$\text{(2.5)} \qquad a^*(z,y) = (\tilde{T}z, y)_L + \tfrac{1}{2}\langle (M^* + D)z, y\rangle_{W',W},$$

and by reformulating (2.3) as follows:

$$\text{Seek } z \in W \text{ such that } a(z, y) = (f, y)_L \quad \forall y \in W. \tag{2.6}$$

$$\text{Seek } z^* \in W \text{ such that } a^*(z^*, y) = (f, y)_L \quad \forall y \in W. \tag{2.7}$$

The key well-posedness result established in Part I is the following.

THEOREM 2.1. *Assume* (A1)–(A3) *and* (M1)–(M2). *Then there are unique solutions to* (2.6) *and* (2.7) *and these solutions solve* (2.3).

We finish this section by giving local representations of the operators D and M. Let $n = (n_1, \dots, n_d)^t$ be the unit outward normal to $\partial\Omega$. Whenever the fields $\{\mathcal{A}^k\}_{1 \le k \le d}$ are sufficiently smooth for the field $\mathcal{D} = \sum_{k=1}^d n_k \mathcal{A}^k : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ to be meaningful at the boundary, the following representation of D holds:

$$\langle Dz, y \rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{D} z \tag{2.8}$$

for every smooth function z and y. Likewise, we henceforth assume that there is a field $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$ such that following representation of M holds for every smooth function z and y:

$$\langle Mz, y \rangle_{W',W} = \int_{\partial\Omega} y^t \mathcal{M} z. \tag{2.9}$$

2.2. Two-field Friedrichs' systems. We now briefly recall the two-field theory developed in Part II. Elliptic-like PDEs in mixed form lead to Friedrichs' systems with the following 2×2 structure: There are two positive integers m_σ and m_u such that $m = m_\sigma + m_u$ and $L = L_\sigma \times L_u$, where $L_\sigma = [L^2(\Omega)]^{m_\sigma}$ and $L_u = [L^2(\Omega)]^{m_u}$, yielding the decomposition $v = (v^\sigma, v^u)$ for all $v \in L$. With obvious notation this leads to the following block decompositions

$$K = \left[\begin{array}{c:c} K^{\sigma\sigma} & K^{\sigma u} \\ \hdashline K^{u\sigma} & K^{uu} \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c:c} \mathcal{A}^{\sigma\sigma,k} & \mathcal{B}^k \\ \hdashline (\mathcal{B}^k)^t & \mathcal{C}^k \end{array}\right], \tag{2.10}$$

where for all $k \in \{1, \dots, d\}$, $\mathcal{B}^k$ is an $m_\sigma \times m_u$ matrix field and $\mathcal{C}^k$ an $m_u \times m_u$ matrix field. Assume now that the block $K^{\sigma\sigma}$ has a local representation, i.e., there is $\mathcal{K}^{\sigma\sigma} \in [L^\infty(\Omega)]^{m_\sigma, m_\sigma}$ such that $K^{\sigma\sigma} y^\sigma = \mathcal{K}^{\sigma\sigma} y^\sigma$ for all $y^\sigma \in L_\sigma$ (this localization hypothesis is needed to locally eliminate the σ-component in the two-field DG method described below). The two key hypotheses on which the two-field theory is based are

$$\forall k \in \{1, \dots, d\}, \quad \mathcal{A}^{\sigma\sigma,k} = 0, \tag{A4}$$

$$\exists k_0 > 0, \quad \mathcal{K}^{\sigma\sigma} \ge k_0 \mathcal{I}_{m_\sigma}, \tag{A5}$$

where $\mathcal{I}_{m_\sigma}$ is the identity matrix in $\mathbb{R}^{m_\sigma, m_\sigma}$. Assumptions (A4)–(A5) allow us to eliminate the σ-component of z in the PDE system $Tz = f$ leading to an elliptic-like PDE for the u-component. Furthermore, assumption (A4) yields

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \mathcal{D}^{\sigma u} \\ \hdashline \mathcal{D}^{u\sigma} & \mathcal{D}^{uu} \end{array}\right], \tag{2.11}$$

with $\mathcal{D}^{\sigma u} = \sum_{k=1}^d n_k \mathcal{B}^k$, $\mathcal{D}^{u\sigma} = (\mathcal{D}^{\sigma u})^t$, and $\mathcal{D}^{uu} = \sum_{k=1}^d n_k \mathcal{C}^k$.

Henceforth, boundary conditions are enforced by taking

$$\mathcal{M} = \left[\begin{array}{c|c} 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hline \alpha\mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right], \tag{2.12}$$

where $\mathcal{M}^{uu} \in \mathbb{R}^{m_u,m_u}$ is positive and $\alpha \in \{-1,+1\}$. The choice $\alpha = +1$ leads to the Dirichlet boundary condition $z^u \in \mathrm{Ker}(\mathcal{D}^{\sigma u}) \cap \mathrm{Ker}(\mathcal{M}^{uu} - \mathcal{D}^{uu})$. The choice $\alpha = -1$ yields the Robin-type boundary condition $2\mathcal{D}^{u\sigma}z^\sigma + (\mathcal{D}^{uu} - \mathcal{M}^{uu})z^u = 0$; the boundary condition is of Neumann-type if $\mathcal{M}^{uu} = \mathcal{D}^{uu}$, provided $\mathcal{D}^{uu}$ is positive. In practice (see the examples in sections 3.3 and 3.4), $\mathrm{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$, so that the Dirichlet boundary condition amounts to $z^u = 0$, while the Robin-type boundary condition is enforced by taking $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$.

It will prove convenient in what follows to define the operators $B = \sum_{k=1}^d \mathcal{B}^k\partial_k$, $B^\dagger = \sum_{k=1}^d [\mathcal{B}^k]^t\partial_k$, and $C = \sum_{k=1}^d \mathcal{C}^k\partial_k$.

2.3. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of Ω. To simplify, we assume that the meshes are affine and that Ω is a polyhedron. For all $K \in \mathcal{T}_h$, $n_K = (n_{K,1}, \dots, n_{K,d})^t$ denotes the unit outward normal to K and h_K is the diameter of K. We set $h = \max_{K\in\mathcal{T}_h} h_K$ and we denote by $\mathfrak{h}$ the piecewise constant function such that for all $K \in \mathcal{T}_h$, $\mathfrak{h}|_K = h_K$. Henceforth, the notation $\xi \lesssim \zeta$ means that there is a positive c, *independent of* h, such that $\xi \le c\zeta$.

We denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of mesh interfaces, i.e., $F \in \mathcal{F}_h^{\mathrm{i}}$ if F is a $(d-1)$-manifold and there are $K_1(F)$ and $K_2(F) \in \mathcal{T}_h$ such that $F = K_1(F) \cap K_2(F)$. For $F \in \mathcal{F}_h^{\mathrm{i}}$, we set $\mathcal{T}(F) = K_1(F) \cup K_2(F)$. We denote by $\mathcal{F}_h^\partial$ the set of the faces that separate the mesh from the exterior of Ω, i.e., $F \in \mathcal{F}_h^\partial$ if F is a $(d-1)$-manifold and there is $K(F) \in \mathcal{T}_h$ such that $F = K(F) \cap \partial\Omega$. For $F \in \mathcal{F}_h^\partial$, we set $\mathcal{T}(F) = K(F)$. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, n_F is the unit normal vector on F pointing from $K_1(F)$ to $K_2(F)$, and for all $F \in \mathcal{F}_h^\partial$, n_F is the unit normal vector on F pointing outside Ω. Finally, we set $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^\partial$ and for all $F \in \mathcal{F}_h$, h_F denotes the diameter of F. The sole assumption we make on the matching of element faces is that for all $F \in \mathcal{F}_h$, $\max_{K\in\mathcal{T}(F)} h_K \lesssim h_F$. This assumption implies, in particular, that the mesh family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular.

For any measurable subset E of Ω, $(\cdot,\cdot)_{L,E}$ denotes the usual scalar product in $[L^2(E)]^m$. For ease of notation, we define the operators B_h, $B_h^\dagger$, and C_h as the elementwise versions of B, $B^\dagger$, and C, respectively; for instance, for v smooth enough, $(B_h v)|_K = \sum_{k=1}^d \mathcal{B}^k\partial_k(v|_K)$ for all $K \in \mathcal{T}_h$.

Let p be a *nonnegative* integer and consider the DG finite element space

$$W_h = [P_{h,p}]^m, \qquad P_{h,p} = \{v_h \in L^2(\Omega);\, \forall K \in \mathcal{T}_h,\, v_h|_K \in \mathbb{P}_p\}, \tag{2.13}$$

$\mathbb{P}_p$ denoting the vector space of polynomials with real coefficients and with total degree less than or equal to p. Observe that the usual inverse and trace inverse inequalities hold in W_h. For a function v that admits a possibly two-valued trace on mesh interfaces, we define the jump and mean-value of v on $F \in \mathcal{F}_h^{\mathrm{i}}$ as

$$[\![v]\!] = v^1 - v^2, \qquad \{v\} = \tfrac{1}{2}(v^1 + v^2), \tag{2.14}$$

where $v^\gamma(x) = \lim_{y\to x} v(y)|_{K_\gamma(F)}$, $\gamma \in \{1,2\}$. The field $\mathcal{D}$ is extended to $\mathcal{F}_h$ by setting for all $K \in \mathcal{T}_h$, $\mathcal{D} = \sum_{k=1}^d n_{K,k}\mathcal{A}^k$ a.e. on ∂K. Observe that $\mathcal{D}$ is two-valued on $\mathcal{F}_h^{\mathrm{i}}$ with $\{\mathcal{D}\} = 0$ on $\mathcal{F}_h^{\mathrm{i}}$ and that $|\mathcal{D}|$ is well-defined and single-valued since $\mathcal{D}$ is symmetric. We also define $\mathcal{D}_F = \sum_{k=1}^d n_{F,k}\mathcal{A}^k$.

To write a DG method starting from (2.6), we introduce three families of boundary and interface operators: $\{M_F\}_{\mathcal{F}_h^\partial}$, $\{S_F\}_{\mathcal{F}_h^{\mathrm{i}}}$, and $\{R_F\}_{\mathcal{F}_h}$. For all $F \in \mathcal{F}_h^\partial$, the role of the operator $M_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$ is to weakly enforce the boundary conditions on F. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, the role of the operator $S_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$ is to penalize the jump of the discrete unknowns across F. For all $F \in \mathcal{F}_h$, the operator $R_F \in \mathcal{L}([L^2(\mathcal{F}_h)]^m; [L^2(F)]^m)$ is user-defined so as to facilitate the implementation of the method. The default option is to take $R_F \equiv 0$ in general for the one-field approach, but a nonzero choice must be made if, when using the multifield approach, one insists on obtaining an IP-like method; cf. Part II. The reader can take $R_F \equiv 0$ in a first reading. The design of the above operators depends on whether the one-field, two-field, or three-field approach is used.

Let $W(h) = W_h + [H^1(\Omega)]^m$ and define on $W(h) \times W(h)$ the DG bilinear form

(2.15)
$$a_h(z,y) = \sum_{K\in\mathcal{T}_h} (Tz,y)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F(z) - \mathcal{D}z, y)_{L,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}z\}, \{y\})_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![z]\!]), [\![y]\!])_{L,F} + \sum_{F\in\mathcal{F}_h} (R_F([\![z]\!]), [\![y]\!])_{L,F}.$$

The first and second terms on the right-hand side are the discrete counterparts of (2.4). The third term is a consistency term; it is zero whenever z is smooth and it is meant to guarantee the L-coercivity of a_h (recall that (A3) and (M1) imply that a is L-coercive). The discrete counterpart of (2.6) is formulated as follows:

(2.16) $$\text{Seek } z_h \in W_h \text{ such that } a_h(z_h, y_h) = (f, y_h)_L \quad \forall y_h \in W_h.$$

Problem (2.16) can be equivalently reformulated in local form by introducing the notion of flux: Seek $z_h \in W_h$ such that for all $K \in \mathcal{T}_h$ and for all $y \in [\mathbb{P}_p(K)]^m$,

(2.17) $$(Kz_h, y)_{L,K} + (z_h, \tilde{A}y)_{L,K} + (\phi_{\partial K}(z_h), y)_{L,\partial K} = (f, y)_{L,K},$$

where the element fluxes are defined on a face $F \subset \partial K$ by

(2.18) $$\phi_{\partial K}(z)|_F = \begin{cases} \frac{1}{2}\mathcal{D}_F z + \frac{1}{2}M_F(z) + R_F(z) & \text{if } F \in \mathcal{F}_h^\partial, \\ n_F{\cdot}n_K(\mathcal{D}_F\{z\} + S_F([\![z]\!]) + R_F([\![z]\!])) & \text{if } F \in \mathcal{F}_h^{\mathrm{i}}. \end{cases}$$

2.4. One-field DG approximation. For the one-field DG method, the operators $\{R_F\}_{\mathcal{F}_h}$ are generally set to zero and the operators $\{M_F\}_{\mathcal{F}_h^\partial}$ and $\{S_F\}_{\mathcal{F}_h^{\mathrm{i}}}$ are designed as follows: For all $v, w \in [L^2(F)]^m$,

(DG1A) $$\mathrm{Ker}(\mathcal{M} - \mathcal{D}) \subset \mathrm{Ker}(M_F - \mathcal{D}),$$

(DG1B) $$(M_F(v), v)_{L,F} \geq 0,$$

(DG1C) $$|(M_F(v) - \mathcal{D}v, w)_{L,F}| \lesssim |v|_{M,F} \|w\|_{L,F},$$

(DG1D) $$|(M_F(v) + \mathcal{D}v, w)_{L,F}| \lesssim \|v\|_{L,F} |w|_{M,F},$$

(DG1E) $$S_F = (S_F)^* \text{ and } |\mathcal{D}| \lesssim S_F \lesssim \mathcal{I}_m,$$

where $|v|^2_{M,F} = (M_F(v), v)_{L,F}$, $\mathcal{I}_m$ is the identity matrix in $\mathbb{R}^{m,m}$, and $(S_F)^*$ is the adjoint operator of S_F. Assumption (DG1A) is a consistency assumption meaning that for all $F \in \mathcal{F}_h^\partial$ and for all $v \in [L^2(F)]^m$, $\mathcal{M}v = \mathcal{D}v$ implies $M_F(v) = \mathcal{D}v$. Design conditions for $\{S_F\}_{\mathcal{F}_h^{\mathrm{i}}}$ slightly more general than (DG1E) are stated in Part I.

To formulate the convergence result, we equip $W(h)$ with the following norms:

$$\|y\|_{h,A}^2 = \|y\|_L^2 + |y|_J^2 + |y|_M^2 + \sum_{K\in\mathcal{T}_h} h_K \|Ay\|_{L,K}^2, \tag{2.19}$$

$$\|y\|_{h,\frac{1}{2}}^2 = \|y\|_{h,A}^2 + \sum_{K\in\mathcal{T}_h} [h_K^{-1}\|y\|_{L,K}^2 + \|y\|_{L,\partial K}^2], \tag{2.20}$$

with $|y|_J^2 = \sum_{F\in\mathcal{F}_h^{\rm i}} |[\![y]\!]|_{J,F}^2$, $|y|_{J,F}^2 = (S_F([\![y]\!]),[\![y]\!])_{L,F}$, and $|y|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |y|_{M,F}^2$.

The main convergence result derived in Part I is the following.

THEOREM 2.2. *Assume* (DG1A)–(DG1E) *and* $\mathcal{A}^k \in [\mathfrak{C}^{0,\frac{1}{2}}(K)]^{m,m}$ *for all* $K \in \mathcal{T}_h$ *and all* $1 \le k \le d$. *Let* $z \in [H^1(\Omega)]^m \cap V$ *solve* (2.6) *and let* z_h *solve* (2.16). *Then*

$$\|z - z_h\|_{h,A} \lesssim \inf_{y_h\in W_h} \|z - y_h\|_{h,\frac{1}{2}}. \tag{2.21}$$

Theorem 2.2 yields $(p+\frac{1}{2})$-order convergence in the L-norm and p-order convergence in the broken graph norm if the mesh family is quasi-uniform and the exact solution is in the broken Sobolev space $[H^{p+1}(\mathcal{T}_h)]^m$.

2.5. Two-field DG approximation. Let $p_u > 0$ be a *positive* integer and take $p_\sigma \in \mathbb{N}$ such that $p_u - 1 \le p_\sigma$. Define the finite element spaces

$$\Sigma_h = [P_{h,p_\sigma}]^{m_\sigma}, \qquad U_h = [P_{h,p_u}]^{m_u}, \qquad W_h = \Sigma_h \times U_h. \tag{2.22}$$

The bilinear form a_h is still defined by (2.15) and the discrete problem is still (2.16).

The design of the operators $\{M_F\}_{F\in\mathcal{F}_h^\partial}$, $\{S_F\}_{F\in\mathcal{F}_h^{\rm i}}$, and $\{R_F\}_{F\in\mathcal{F}_h^\partial}$ for the two-field DG approximation hinges on the requirement that we be able to locally eliminate the discrete component z_h^σ. To this purpose, these operators are designed such that

(DG2A) $M_F = \left[\begin{array}{c:c} 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hdashline \alpha\mathcal{D}^{u\sigma} & M_F^{uu}\end{array}\right]$, $\alpha \in \{-1,+1\}$, $\quad S_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & S_F^{uu}\end{array}\right]$, $\quad R_F = \left[\begin{array}{c:c} 0 & 0 \\ \hdashline 0 & R_F^{uu}\end{array}\right]$;

(DG2B) if $\alpha = +1$, $\begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \operatorname{Ker}(\mathcal{D}^{\sigma u}) \subset \operatorname{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}; \end{cases}$

(DG2C) if $\alpha = -1$, $M_F^{uu}(v) = \mathcal{M}^{uu}v$ and $|\mathcal{D}^{uu}| \lesssim \mathcal{M}^{uu} \lesssim \mathcal{I}_{m_u}$;

(DG2D) $S_F^{uu} = (S_F^{uu})^*$ and $h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}$.

Furthermore, letting $|y^u|_J^2 = \sum_{F\in\mathcal{F}_h^{\rm i}} |y^u|_{J,F}^2$ and $|y^u|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |y^u|_{M,F}^2$, where

$$|y^u|_{J,F}^2 = (S_F^{uu}([\![y^u]\!]),[\![y^u]\!])_{L_u,F}, \qquad |y^u|_{M,F}^2 = (M_F^{uu}(y^u),y^u)_{L_u,F}, \tag{2.23}$$

the user-dependent operator R_F^{uu} must be designed such that for all $z_h^u \in U_h$ and all $(z^u, y_h^u) \in U(h)\times U_h$,

(DG2E) $$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z_h^u]\!]),[\![z_h^u]\!])_{L_u,F} \ge -\frac{1}{4}(|z_h^u|_J^2 + |z_h^u|_M^2),$$

(DG2F) $$\sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z^u]\!]),[\![y_h^u]\!])_{L_u,F} \lesssim (|z^u|_J + |z^u|_M)(|y_h^u|_J + |y_h^u|_M).$$

Owing to (DG2A), the σ-component of the element fluxes defined by (2.18) does not depend on z_h^σ, thus allowing for the local elimination of z_h^σ. Design conditions slightly more general than (DG2A)–(DG2D) are stated in Part II. Furthermore, assumptions (DG2A)–(DG2C) imply the consistency condition $\mathrm{Ker}(\mathcal{M}-\mathcal{D}) \subset \mathrm{Ker}(M_F-\mathcal{D})$ and the adjoint-consistency condition $\mathrm{Ker}(\mathcal{M}^t+\mathcal{D}) \subset \mathrm{Ker}(M_F^*+\mathcal{D})$. Indeed, both conditions are evident if $\alpha = -1$ since (DG2A) and (DG2C) imply $M_F = \mathcal{M}$. If $\alpha = +1$, the consistency condition directly results from (DG2B), while the adjoint-consistency condition results from the fact that if $z \in \mathrm{Ker}(\mathcal{M}^t+\mathcal{D})$, $\mathcal{D}^{\sigma u} z^u = 0$ and using (DG2B) yields $(\mathcal{M}^{uu})^t z^u = -\mathcal{D}^{uu} z^u = -M_F^{uu}(z^u) = -(M_F^{uu})^*(z^u)$. Finally, we observe that the second part of assumption (DG2C) imposes a condition on the way the Robin–Neumann boundary condition is enforced rather than on the DG setting; in practice, $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ (see section 3.3.2) so that (DG2C) holds.

Equip $W(h)$ with the following norms:

$$\|y\|_{h,B}^2 = \|y\|_L^2 + |y^u|_J^2 + |y^u|_M^2 + \|B_h y^u\|_{L_\sigma}^2, \tag{2.24}$$

$$\|y\|_{h,1}^2 = \|y\|_{h,B}^2 + \sum_{K\in\mathcal{T}_h} [h_K^{-2}\|z^u\|_{L_u,K}^2 + h_K^{-1}\|z^u\|_{L_u,\partial K}^2 + h_K\|z^\sigma\|_{L_\sigma,\partial K}^2], \tag{2.25}$$

$$\|y\|_{h,1^+}^2 = \|y\|_{h,1}^2 + \sum_{K\in\mathcal{T}_h} [h_K^2\|y^\sigma\|_{[H^1(K)]^{m_\sigma}}^2 + h_K\|y^\sigma\|_{L_\sigma,\partial K}^2]. \tag{2.26}$$

The main convergence result proved in Part II is the following.

THEOREM 2.3. *Assume* (DG2A)–(DG2F) *and* $\mathcal{B}^k \in [\mathfrak{C}^{0,1}(K)]^{m_\sigma,m_u}$ *for all* $K \in \mathcal{T}_h$ *and* $1 \le k \le d$. *Let* $z \in [H^1(\Omega)]^m \cap V$ *solve* (2.6) *and let* z_h *solve* (2.16). *Then*

$$\|z - z_h\|_{h,B} \lesssim \inf_{y_h\in W_h} \|z - y_h\|_{h,1}. \tag{2.27}$$

Moreover, if for every $y^u \in L_u$, *the solution* $\psi \in V^*$ *to the dual problem* $\tilde{T}\psi = (0, y^u)$ *is such that* $\|\psi^u\|_{[H^2(\Omega)]^{m_u}} + \|\psi^\sigma\|_{[H^1(\Omega)]^{m_\sigma}} \lesssim \|y^u\|_{L_u}$, *then*

$$\|z^u - z_h^u\|_{L_u} \lesssim h \inf_{y_h\in W_h} \|z - y_h\|_{h,1^+}. \tag{2.28}$$

If the exact solution is in $[H^{p_u}(\mathcal{T}_h)]^{m_\sigma}\times[H^{p_u+1}(\mathcal{T}_h)]^{m_u}$, Theorem 2.3 yields p_u-order convergence in the L_σ-norm for the σ-component, (p_u+1)-order convergence in the L_u-norm for the u-component, and p_u-order convergence in the broken graph norm for the u-component.

3. Two-field theory with L_σ-coercivity only. The goal of this section is to weaken assumption (A3), so as to be able to account for two-field Friedrichs' systems with no L_u-coercivity on the u-component. The model problems we have in mind are advection–diffusion equations with no zero-order term, i.e., no reaction (see section 3.3), and compressible linear continuum mechanics (see section 3.4).

3.1. Two-field Friedrichs' systems with L_σ-coercivity only. We assume

$$\forall z \in L, \quad ((K + K^* - \nabla\cdot A)z, z)_L \gtrsim \|z^\sigma\|_{L_\sigma}^2, \tag{A3A}$$

$$K^{\sigma u} = (K^{u\sigma})^* = 0 \text{ and the fields } \mathcal{B}^k \text{ are constant over } \Omega, \tag{A3B$'$}$$

$$\forall z \in V \cup V^*, \quad \|z^u\|_{L_u} \lesssim a(z,z)^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma}. \tag{A3B$''$}$$

Observe that (A3B″) is meaningful since a is L_σ-coercive on W. Indeed, the definition of D and assumptions (M1) and (A3A) yield, for all $z \in W$,

$$a(z,z) = \tfrac{1}{2}[(Tz,z)_L + (z,\tilde{T}z)_L] + \tfrac{1}{2}\langle Mz,z\rangle_{W',W} \gtrsim \|z^\sigma\|^2_{L_\sigma}. \tag{3.1}$$

Moreover, T (resp., T^*) is L_σ-coercive on V (resp., V^*).

THEOREM 3.1. *The conclusions of Theorem* 2.1 *still hold if assumption* (A3) *is replaced by assumptions* (A3A)–(A3B′)–(A3B″).

Proof. (1) Let us first prove that $T : V \to L$ is an isomorphism by using the so-called Banach–Nečas–Babuška (BNB) theorem, which states that the bijectivity of $T \in \mathcal{L}(V;L)$ is equivalent to the following conditions [14, p. 85]:

$$\forall z \in V, \quad \sup_{y \in L\setminus\{0\}} \frac{(Tz,y)_L}{\|y\|_L} = \|Tz\|_L \gtrsim \|z\|_W, \tag{3.2}$$

$$\forall y \in L, \quad ((Tz,y)_L = 0\ \forall z \in V) \implies (y = 0). \tag{3.3}$$

Recall that the graph norm is $\|z\|_W = \|z\|_L + \|Az\|_L$ with $\|z\|_L = \|z^\sigma\|_{L_\sigma} + \|z^u\|_{L_u}$ and $\|Az\|_L = \|Bz^u\|_{L_\sigma} + \|B^\dagger z^\sigma + Cz^u\|_{L_u}$.

(1a) Proof of (3.2). Let $z \in V$. Owing to (A3B′), $Bz^u = (Tz)^\sigma - \mathcal{K}^{\sigma\sigma}z^\sigma$. Hence, $\|Bz^u\|_{L_\sigma} \lesssim \|z^\sigma\|_{L_\sigma} + \|Tz\|_L$. Then, since $a(z,z) = (Tz,z)_L$, (A3B″) implies

$$\|z^u\|_{L_u} \lesssim (Tz,z)_L^{\frac{1}{2}} + c(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L) \lesssim \gamma\|z^u\|_{L_u} + (\|z^\sigma\|_{L_\sigma} + \|Tz\|_L),$$

where $\gamma > 0$ can be chosen as small as needed. Hence, $\|z^u\|_{L_u} \lesssim \|z^\sigma\|_{L_\sigma} + \|Tz\|_L$. Combining this result with the L_σ-coercivity of T on V yields

$$\|z^\sigma\|^2_{L_\sigma} \lesssim \frac{(Tz,z)_L}{\|z\|_L}(\|z^\sigma\|_{L_\sigma} + \|z^u\|_{L_u}) \lesssim \|Tz\|_L(\|z^\sigma\|_{L_\sigma} + \|Tz\|_L),$$

whence $\|z^\sigma\|_{L_\sigma} \lesssim \|Tz\|_L$. Collecting the above bounds leads to $\|z\|_L \lesssim \|Tz\|_L$, and hence $\|z\|_W = \|z\|_L + \|Az\|_L \lesssim \|z\|_L + \|Tz\|_L \lesssim \|Tz\|_L$.

(1b) Proof of (3.3). Assume that $y \in L$ is such that $(Tz,y)_L = 0$ for all $z \in V$. Following the same arguments as in the proof of [15, Theorem 2.5] in Part I or Corollary 5.8 in [14], we infer that $y \in V^*$ and $\tilde{T}y = 0$. The L_σ-coercivity of $\tilde{T}$ on V^* yields $y^\sigma = 0$. Proceeding as above using (A3B′) leads to $\|z^u\|_{L_u} \lesssim \|z^\sigma\|_{L_\sigma} + \|\tilde{T}z\|_L$ for all $z \in V^*$. Applying this estimate to y yields $y^u = 0$. Hence, $y = 0$.

(2) Since $T : V \to L$ is an isomorphism and $V = \mathrm{Ker}(D - M)$, a solution to (2.6) is readily constructed by setting $z = T^{-1}f$. To prove uniqueness, let us prove that the only solution to (2.6) with $f = 0$ is $z = 0$. Since a is L_σ-coercive on W, $z^\sigma = 0$. In addition, taking $y \in [\mathfrak{D}(\Omega)]^m$ in (2.6) yields $Tz = 0$ in L and $z \in V$. Hence, the bound $\|Bz^u\|_{L_\sigma} \lesssim \|z^\sigma\|_{L_\sigma} + \|Tz\|_L$ implies $Bz^u = 0$ and owing to (A3B″), $z^u = 0$.

(3) Proceed similarly to prove that problem (2.7) is well-posed. ◻

3.2. Two-field DG approximation with L_σ-coercivity only. Consider the two-field DG method introduced in section 2.5 and assume that conditions (DG2A)–(DG2F) are fulfilled. The objective of this section is to analyze the convergence of the two-field DG approximation in the framework of the partial coercivity assumptions (A3A), (A3B′), and (A3B″). The discrete counterpart of assumption (A3B″) is

$$\forall z_h \in W_h, \quad \|z_h^u\|_{L_u} \lesssim a_h(z_h,z_h)^{\frac{1}{2}} + \|B_h z_h^u\|_{L_\sigma}. \tag{3.4}$$

Recall that the norm $\|\cdot\|_{h,B}$ is defined by (2.24).

LEMMA 3.2. *Assume* (A3A)–(A3B′), (3.4), *and* (DG2A)–(DG2F). *Then*

$$\forall z_h \in W_h, \qquad \|z_h\|_{h,B} \lesssim \sup_{y_h \in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,B}}. \tag{3.5}$$

Proof. Let $z_h \in W_h$. Owing to the definition of a_h, (DG2A), (DG2E), and (A3A),

$$\|z_h^\sigma\|_{L_\sigma}^2 + |z_h^u|_J^2 + |z_h^u|_M^2 \lesssim a_h(z_h, z_h). \tag{3.6}$$

Set $\varpi_h = (B_h z_h^u, 0)$ and observe that $\varpi_h \in W_h$ since the fields $\mathcal{B}^k$ are constant over Ω and $p_u - 1 \le p_\sigma$. Moreover,

$$\|B_h z_h^u\|_{L_\sigma}^2 = a_h(z_h, \varpi_h) - (\mathcal{K}^{\sigma\sigma} z_h^\sigma, B_h z_h^u)_{L_\sigma} - \sum_{F\in\mathcal{F}_h^\partial} \tfrac{\alpha+1}{2}(\mathcal{D}^{\sigma u} z_h^u, B_h z_h^u)_{L_\sigma,F}$$
$$+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{B_h z_h^u\})_{L_\sigma,F} := a_h(z_h, \varpi_h) + R_1 + R_2 + R_3.$$

Clearly, $|R_1| \lesssim \|z_h^\sigma\|_{L_\sigma}^2 + \gamma\|B_h z_h^u\|_{L_\sigma}^2$, where $\gamma > 0$ can be chosen as small as needed. If $\alpha = +1$, use (DG2B) and a trace inverse inequality to infer

$$|R_2| \lesssim \sum_{F\in\mathcal{F}_h^\partial} h_F^{\frac{1}{2}} |z_h^u|_{M,F} h_F^{-\frac{1}{2}} \|B_h z_h^u\|_{L_\sigma,\mathcal{T}(F)} \lesssim |z_h^u|_M^2 + \gamma\|B_h z_h^u\|_{L_\sigma}^2,$$

while if $\alpha = -1$, $R_2 = 0$. Finally, using $\{\mathcal{D}^{\sigma u}\} = 0$ and (DG2D) leads to $|R_3| \lesssim |z_h^u|_J^2 + \gamma\|B_h z_h^u\|_{L_\sigma}^2$. Collecting the above bounds yields

$$\|B_h z_h^u\|_{L_\sigma}^2 \lesssim a_h(z_h, \varpi_h) + a_h(z_h, z_h),$$

and owing to (3.4) and (3.6), it is inferred that $\|z_h\|_{h,B}^2 \lesssim a_h(z_h, \varpi_h) + a_h(z_h, z_h)$. Conclude using the fact that $\|\varpi_h\|_{h,B} = \|B_h z_h^u\|_{L_\sigma} \lesssim \|z_h\|_{h,B}$. □

It is now straightforward to verify the following convergence result.

THEOREM 3.3. *The statement of Theorem* 2.3 *remains valid under the assumptions of Lemma* 3.2.

3.3. Example 1: Advection–diffusion. Let $f \in L^2(\Omega)$ and let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$. Let $\mu \in L^\infty(\Omega)$ and let $\kappa = (\kappa_{kl})_{1\le k,l\le d}$ be a symmetric positive definite tensor-valued field defined on Ω whose lowest eigenvalue is uniformly bounded away from zero. Consider the PDE $-\nabla\cdot(\kappa\nabla u) + \beta\cdot\nabla u + \mu u = f$ in mixed form

$$\begin{cases} \kappa^{-1}\sigma + \nabla u = 0, \\ \mu u + \nabla\cdot\sigma + \beta\cdot\nabla u = f. \end{cases} \tag{3.7}$$

Letting $m = d+1$, $m_\sigma = d$, and $m_u = 1$, the mixed formulation (3.7) fits the two-field framework by setting, for all $z \in L$ and for all $k \in \{1, \dots, d\}$,

$$K(z) = \left[\begin{array}{c:c} \kappa^{-1} & 0 \\ \hdashline 0 & \mu \end{array}\right] z, \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & \beta^k \end{array}\right], \tag{3.8}$$

where e^k is the kth vector in the canonical basis of $\mathbb{R}^d$, and β^k is the kth component of β. Clearly, hypotheses (A1), (A2), (A4), and (A5) hold. We further assume that

$$\inf_\Omega \operatorname{ess}(\mu - \tfrac{1}{2}\nabla\cdot\beta) \ge 0 \tag{3.9}$$

so that (A3) does not hold, but (A3A) holds instead with μ_0 equal to the reciprocal of the largest eigenvalue of κ. This situation covers, in particular, the Laplace/Poisson equation where $\mu = 0$ and $\beta = 0$.

The graph space is $W = H(\text{div};\Omega) \times H^1(\Omega)$ and the boundary operator D is such that for all $z, y \in W$,

$$\langle Dz, y\rangle_{W',W} = \langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta{\cdot}n) z^u y^u, \tag{3.10}$$

where $\langle , \rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $H^{-\frac{1}{2}}(\partial\Omega)$ and $H^{\frac{1}{2}}(\partial\Omega)$. Dirichlet boundary conditions can be enforced by setting

$$\langle Mz, y\rangle_{W',W} = \langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{3.11}$$

yielding $V = H(\text{div};\Omega) \times H^1_0(\Omega)$. Furthermore, mixed Robin–Neumann boundary conditions can be enforced by setting

$$\langle Mz, y\rangle_{W',W} = -\langle z^\sigma{\cdot}n, y^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle y^\sigma{\cdot}n, z^u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho + \beta{\cdot}n) z^u y^u, \tag{3.12}$$

where $\varrho \in L^\infty(\partial\Omega)$ is such that $2\varrho + \beta{\cdot}n \geq 0$ a.e. on $\partial\Omega$. Then $V = \{z \in W;\, z^\sigma{\cdot}n - \varrho z^u|_{\partial\Omega} = 0\}$ and $V^* = \{z \in W;\, z^\sigma{\cdot}n + (\varrho + \beta{\cdot}n) z^u|_{\partial\Omega} = 0\}$. In terms of boundary fields, (3.10), (3.11), (3.12), respectively, yield

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & n \\ \hdashline n^t & \beta{\cdot}n \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & 0 \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c:c} 0 & n \\ \hdashline -n^t & 2\varrho + \beta{\cdot}n \end{array}\right]. \tag{3.13}$$

Observe that $\text{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$. Furthermore, a possible choice for the mixed Robin–Neumann boundary condition is $\varrho = -\min(\beta{\cdot}n, 0)$ (or, equivalently, $\mathcal{M}^{uu} = |\beta{\cdot}n| = |\mathcal{D}^{uu}|$), yielding the usual Robin (inflow) condition $(z^\sigma + \beta z^u){\cdot}n = 0$ on $\partial\Omega^- = \{x \in \partial\Omega;\, \beta(x){\cdot}n(x) < 0\}$ and the usual Neumann (outflow) condition $z^\sigma{\cdot}n = 0$ on $\partial\Omega \backslash \partial\Omega^-$.

3.3.1. Well-posedness. Let us verify that the advection–diffusion equation equipped with the above boundary conditions fits the theoretical framework analyzed in section 3.1.

PROPOSITION 3.4. *Assumptions* (A3A)*,* (A3B$'$)*, and* (A3B$''$) *hold for Dirichlet boundary conditions and for mixed Robin–Neumann boundary conditions, provided either $\mu - \frac{1}{2}\nabla{\cdot}\beta \neq 0$ or $\varrho + \frac{1}{2}\beta{\cdot}n \neq 0$ (this means that either $\mu - \frac{1}{2}\nabla{\cdot}\beta$ or $\varrho + \frac{1}{2}\beta{\cdot}n$ is uniformly bounded away from zero on a measurable subset of Ω or of $\partial\Omega$, respectively, of nonzero measure).*

Proof. Assumptions (A3A) and (A3B$'$) are evident. Observe also that $Bz^u = \nabla z^u$.

(1) For Dirichlet boundary conditions, (A3B$''$) directly results from the Poincaré inequality since $z \in V = V^*$ implies $z^u \in H^1_0(\Omega)$ so that $\|z^u\|_{L_u} \lesssim \|\nabla z^u\|_{L_\sigma}$.

(2) For mixed Robin–Neumann boundary conditions, observe that for all $z \in W$,

$$a(z, z) \geq \int_\Omega (\mu - \tfrac{1}{2}\nabla{\cdot}\beta)(z^u)^2 + \int_{\partial\Omega} (\varrho + \tfrac{1}{2}\beta{\cdot}n)(z^u)^2. \tag{3.14}$$

We apply Lemma 3.5 below; it is a simple variant of the Petree–Tartar lemma (the proof is omitted for brevity). Take $X = H^1(\Omega)$, $Y = [L^2(\Omega)]^d$, $Z = L^2(\Omega)$, $Fx = \nabla x$, $Gx = x$, and $\Phi(x) = \|(\mu - \frac{1}{2}\nabla{\cdot}\beta)^{\frac{1}{2}} x\|_{L^2(\Omega)} + \|(\varrho + \frac{1}{2}\beta{\cdot}n)^{\frac{1}{2}} x\|_{L^2(\partial\Omega)}$. Properties (i) and (iii) are evident. Property (ii) results from the fact that if $\|Fx\|_Y + \Phi(x) = 0$, then x is constant (since Ω is connected) and $\Phi(x) = 0$ implies that $x = 0$ since $\mu - \frac{1}{2}\nabla\cdot\beta \neq 0$ or $\varrho + \frac{1}{2}\beta\cdot n \neq 0$. Moreover, Φ is continuous since for all $x, y \in X$, $|\Phi(x) - \Phi(y)| \leq \Phi(x-y) \lesssim \|x-y\|_X$. Hence, Lemma 3.5 yields $\|z^u\|_{L_u} \lesssim \Phi(z^u) + \|\nabla z^u\|_{L_\sigma}$, whence (A3B$''$) follows owing to (3.14). ∎

LEMMA 3.5. *Let X, Y, Z be Banach spaces, let $F \in \mathcal{L}(X;Y)$, and let $G \in \mathcal{L}(X;Z)$. Let $\Phi : X \to \mathbb{R}_+$ be a continuous functional. Assume that*

(i) *G is compact;*
(ii) *for all $x \in X$, $(\|Fx\|_Y + \Phi(x) = 0) \Rightarrow (x = 0)$;*
(iii) *there is $\gamma_1 > 0$ such that for all $x \in X$, $\gamma_1\|x\|_X \leq \|Fx\|_Y + \|Gx\|_Z$.*

Then there is $\gamma_2 > 0$ such that for all $x \in X$, $\gamma_2\|x\|_X \leq \|Fx\|_Y + \Phi(x)$.

Remark 3.1. When mixed Robin–Neumann boundary conditions are enforced and $\mu - \frac{1}{2}\nabla\cdot\beta = 0$ and $2\varrho + \beta\cdot n = 0$, (A3B$''$) no longer holds and the analysis proceeds as follows. If $\mu \neq 0$ or $\varrho \neq 0$, it is easily verified that T is injective on V and that $\tilde{T}$ is injective on V^*. The injectivity of T on V combined with Lemma 3.5 easily yields (3.2), while the injectivity of $\tilde{T}$ on V^* yields (3.3); thus well-posedness holds. If $\mu = \nabla\cdot\beta = 0$ and $\varrho = \beta\cdot n = 0$, then T is no longer injective on V, the compatibility condition $\langle f\rangle_\Omega = 0$ must be imposed on the data, and the solution u is subjected to the constraint $\langle u\rangle_\Omega = 0$ (here, for a function $\phi \in L^2(\Omega)$, $\langle\phi\rangle_\Omega := \frac{1}{|\Omega|}\int_\Omega \phi$, where $|\Omega|$ denotes the measure of Ω). Hence, we modify (3.7) as follows:

$$\begin{cases} \kappa^{-1}\sigma + \nabla u = 0, \\ \nabla\cdot\sigma + \beta\cdot\nabla u + \langle u\rangle_\Omega = f, \end{cases} \tag{3.15}$$

equipped with the boundary condition $\sigma\cdot n|_{\partial\Omega} = 0$. Since $\langle\nabla\cdot\sigma + \beta\cdot\nabla u\rangle_\Omega = \langle f\rangle_\Omega = 0$, the second PDE implies $\langle u\rangle_\Omega = 0$, i.e., (3.15) is equivalent to (3.7) together with $\langle u\rangle_\Omega = 0$. Moreover, for all $z \in W$, $a(z,z) \geq |\Omega|\langle z^u\rangle_\Omega^2$, so that (A3B$''$) results from the fact that for all $\phi \in H^1(\Omega)$, $\|\phi\|_{L_u} \lesssim \langle\phi\rangle_\Omega + \|\nabla\phi\|_{L_\sigma}$.

3.3.2. Two-field DG approximation. When Dirichlet boundary conditions are enforced, let $\eta_1 > 0$, $\eta_2 > 0$ (these parameters can vary from face to face), and set

$$M_F^{uu}(v) = \eta_1 h_F^{-1} v, \qquad S_F^{uu}(v) = \eta_2 h_F^{-1} v, \qquad R_F^{uu} \equiv 0. \tag{3.16}$$

Since $\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u} = 1$ and $\mathcal{D}^{uu} = \beta\cdot n$, properties (DG2A)–(DG2F) hold. Many other choices can be considered for M_F^{uu}, S_F^{uu}, and R_F^{uu}; see Part II [16] for details.

PROPOSITION 3.6. *Property* (3.4) *holds.*

Proof. The proof is a direct consequence of the fact that for all $v_h \in U_h$,

$$\|v_h\|_{L_u}^2 \lesssim \sum_{K\in\mathcal{T}_h} \|\nabla v_h\|_{L_\sigma,K}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{-1}\|[\![v_h]\!]\|_{L_u,F}^2 + \sum_{F\in\mathcal{F}_h^{\partial}} h_F^{-1}\|v_h\|_{L_u,F}^2.$$

See [1, 6] or [14, p. 134] for the proof. ∎

When mixed Robin–Neumann boundary conditions are enforced, (DG2C) holds for $M_F^{uu}(v) = \mathcal{M}^{uu}v = (2\varrho + \beta\cdot n)v$, provided $\varrho \geq -\min(\beta\cdot n, 0)$, while S_F^{uu} can be chosen as in (3.16).

PROPOSITION 3.7. *Assume that either $\mu - \frac{1}{2}\nabla\cdot\beta \neq 0$ or $\varrho + \frac{1}{2}\beta\cdot n \neq 0$ and that Ω is such that $H^{\frac{3}{2}+\epsilon}$-elliptic regularity holds, $\epsilon > 0$. Then property* (3.4) *holds.*

Proof. Let v_h be an arbitrary function in U_h. Let $\psi \in H^1(\Omega)$ solve

$$(\mu - \tfrac{1}{2}\nabla\cdot\beta)\psi - \Delta\psi = v_h, \qquad \partial_n\psi|_{\partial\Omega} = -(\rho + \tfrac{1}{2}\beta\cdot n)\psi.$$

This problem is well-posed (proceed as in the proof of Proposition 3.4), and the $H^{\frac{3}{2}+\epsilon}$-elliptic regularity hypothesis means that $\|\psi\|_{H^{\frac{3}{2}+\epsilon}} \lesssim \|v_h\|_{L_u}$. Testing with v_h yields

$$\begin{aligned}\|v_h\|_{L_u}^2 &= ((\mu - \tfrac{1}{2}\nabla\cdot\beta)\psi, v_h)_{L_u} + \sum_{K\in\mathcal{T}_h} (\nabla\psi, \nabla v_h)_{L_\sigma,K} \\ &\quad - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F 2\nabla\psi\cdot \{n v_h\} + \sum_{F\in\mathcal{F}_h^{\partial}} \int_F (\varrho + \tfrac{1}{2}\beta\cdot n)\psi v_h \\ &\lesssim \|\psi\|_{H^{\frac{3}{2}+\epsilon}} \Bigg(\|(\mu - \tfrac{1}{2}\nabla\cdot\beta)^{\frac{1}{2}} v_h\|_{L_u} + \Big(\sum_{K\in\mathcal{T}_h} \|\nabla v_h\|_{L_\sigma,K}^2\Big)^{\frac{1}{2}} + |v_h|_J \\ &\qquad + \|(\varrho + \tfrac{1}{2}\beta\cdot n)^{\frac{1}{2}} v_h\|_{L_u,\partial\Omega}\Bigg).\end{aligned}$$

The conclusion follows readily. ∎

3.4. Example 2: Compressible linear continuum mechanics. Let $\beta \in [L^\infty(\Omega)]^d$ with $\nabla\cdot\beta \in L^\infty(\Omega)$, let λ, $\delta \in L^\infty(\Omega)$, and let $f \in [L^2(\Omega)]^d$. Consider the set of PDEs

$$\begin{cases} \sigma - \frac{1}{d+\delta}\mathrm{tr}(\sigma)\mathcal{I}_d - \frac{1}{2}(\nabla u + (\nabla u)^t) = 0, \\ -\frac{1}{2}\nabla\cdot(\sigma + \sigma^t) + \beta\cdot\nabla u + \lambda u = f, \end{cases} \tag{3.17}$$

where σ is $\mathbb{R}^{d,d}$-valued and u is $\mathbb{R}^d$-valued. Assuming $\delta \neq 0$, the first equation in (3.17) implies $\sigma = \frac{1}{2}(\nabla u + (\nabla u)^t) + \delta^{-1}(\nabla\cdot u)\mathcal{I}_d$. Equations (3.17) are a linearized version of the compressible Navier–Stokes equations and model also linear continuum mechanics (when $\lambda = 0$ and $\beta = 0$). In the context of linear continuum mechanics these equations are usually referred to as the displacement-stress formulation. The tensor field σ with values in $\mathbb{R}^{d,d}$ can be identified with the vector field $\overline{\sigma}$ with values in $\mathbb{R}^{d^2}$ by setting $\overline{\sigma}_{[ij]} = \sigma_{ij}$ with $1 \le i,j \le d$ and $[ij] := d(j-1) + i$. To alleviate the notation, we henceforth use the same symbol for both fields.

Set $m = d^2 + d$, $m_\sigma = d^2$, and $m_u = d$. The mixed formulation (3.17) fits in the framework of two-field Friedrichs' systems by setting, for all $z \in L$ and for all $k \in \{1,\dots,d\}$,

$$K(z) = \left[\begin{array}{c:c} \mathcal{I}_{d^2} - \frac{1}{d+\delta}\mathcal{Z} & 0 \\ \hdashline 0 & \lambda\mathcal{I}_d \end{array}\right] z, \qquad \mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{E}^k \\ \hdashline (\mathcal{E}^k)^t & \mathcal{C}^k \end{array}\right], \tag{3.18}$$

where $\mathcal{Z} \in \mathbb{R}^{d^2,d^2}$ is such that $\mathcal{Z}_{[ij][kl]} = \delta_{ij}\delta_{kl}$ with $1 \le i,j,k,l \le d$, and for all $k \in \{1,\dots,d\}$, $\mathcal{C}^k = \beta^k\mathcal{I}_d \in \mathbb{R}^{d,d}$ and $\mathcal{E}^k \in \mathbb{R}^{d^2,d}$ is such that $\mathcal{E}^k_{[ij],l} = -\frac{1}{2}(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk})$ with $1 \le i,j,l \le d$; here, the δ's are Kronecker symbols. Clearly, hypotheses (A1), (A2), and (A4) hold. We further assume that

$$\lambda_0 := \inf_\Omega \mathrm{ess}(\lambda - \tfrac{1}{2}\nabla\cdot\beta) \ge 0, \qquad \delta_0 := \inf_\Omega \mathrm{ess}\,\delta > 0. \tag{3.19}$$

Then observing that

$$\begin{aligned}\mathcal{K}^{\sigma\sigma}\sigma\cdot\sigma = (\sigma - \tfrac{1}{d+\delta}\mathrm{tr}(\sigma)\mathcal{I}_d) : \sigma &= \tfrac{\delta}{d+\delta}\|\sigma\|^2 + \tfrac{d}{d+\delta}\|\sigma - \tfrac{1}{d}\mathrm{tr}(\sigma)\mathcal{I}_d\|^2 \\ &= \tfrac{\delta}{(d+\delta)^2}|\mathrm{tr}(\sigma)|^2 + \|\sigma - \tfrac{1}{d+\delta}\mathrm{tr}(\sigma)\mathcal{I}_d\|^2,\end{aligned} \tag{3.20}$$

where $\|\cdot\|$ is the Frobenius norm, we deduce that (A5) also holds. The incompressible limit $\delta_0 = 0$ is treated in sections 3.5 and 4.3. Note that (A3) does not hold ((A3) would hold if $\lambda_0 > 0$. The case $\lambda_0 \geq 0$ covers the usual compressible solid mechanics problems for which $\lambda = 0$ and $\beta = 0$.

The graph space is $W = H_\sigma \times [H^1(\Omega)]^d$ with $H_\sigma = \{\sigma \in [L^2(\Omega)]^{d,d}; \nabla\cdot(\sigma + \sigma^t) \in [L^2(\Omega)]^d\}$ and the boundary operator D is such that for all $z, y \in W$ with $z = (\sigma, u)$ and $y = (\tau, v)$,

$$\langle Dz, y\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau+\tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma+\sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta\cdot n)uv, \tag{3.21}$$

where $\langle , \rangle_{-\frac{1}{2},\frac{1}{2}}$ denotes the duality pairing between $[H^{-\frac{1}{2}}(\partial\Omega)]^d$ and $[H^{\frac{1}{2}}(\partial\Omega)]^d$. An homogeneous Dirichlet boundary condition on the u-component is obtained by setting

$$\langle Mz, y\rangle_{W',W} = \langle \tfrac{1}{2}(\tau+\tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tfrac{1}{2}(\sigma+\sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}}, \tag{3.22}$$

yielding $V = V^* = H_\sigma \times [H^1_0(\Omega)]^d$. Similarly, a mixed Robin–Neumann boundary condition is obtained by setting

$$\langle Mz, y\rangle_{W',W} = -\langle \tfrac{1}{2}(\tau+\tau^t)\cdot n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \langle \tfrac{1}{2}(\sigma+\sigma^t)\cdot n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho+\beta\cdot n)uv, \tag{3.23}$$

where $\varrho \in L^\infty(\partial\Omega)$ is such that $2\varrho + \beta\cdot n \geq 0$ a.e. on $\partial\Omega$. Then $V = \{(\sigma, u) \in W; \frac{1}{2}(\sigma+\sigma^t)\cdot n - \varrho u|_{\partial\Omega} = 0\}$ and $V^* = \{(\sigma, u) \in W; \frac{1}{2}(\sigma+\sigma^t)\cdot n + (\varrho+\beta\cdot n)u|_{\partial\Omega} = 0\}$. A standard choice is $\varrho = -\min(\beta\cdot n, 0)$.

In terms of boundary fields, letting $\mathcal{N} = \sum_{k=1}^d n_k \mathcal{E}^k \in \mathbb{R}^{d^2,d}$, (3.21), (3.22), and (3.23), respectively, yields

$$\mathcal{D} = \left[\begin{array}{c:c} 0 & \mathcal{N} \\ \hdashline \mathcal{N}^t & (\beta\cdot n)\mathcal{I}_d \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right], \quad \mathcal{M} = \left[\begin{array}{c:c} 0 & \mathcal{N} \\ \hdashline -\mathcal{N}^t & (2\varrho+\beta\cdot n)\mathcal{I}_d \end{array}\right]. \tag{3.24}$$

Observe that $\mathrm{Ker}(\mathcal{D}^{\sigma u}) = \{0\}$ and that $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ in the Robin–Neumann case, provided $\varrho = -\min(\beta\cdot n, 0)$.

3.4.1. Well-posedness. For the sake of brevity, we restrict ourselves to homogeneous Dirichlet boundary conditions. The case of mixed Robin–Neumann boundary conditions can be treated by proceeding as in section 3.3.1.

PROPOSITION 3.8. *Assume $\lambda_0 = 0$ and $\delta_0 > 0$ and that Dirichlet boundary conditions are enforced. Then assumptions* (A3A), (A3B$'$)*, and* (A3B$''$) *hold.*

Proof. Assumptions (A3A)–(A3B$'$) are evident. Moreover, since Dirichlet boundary conditions are enforced on the u-component, $z \in V = V^*$ implies $z^u \in [H^1_0(\Omega)]^d$; hence, (A3B$''$) results from Korn's first inequality. ∎

3.4.2. Two-field DG approximation. We assume again for simplicity that homogeneous Dirichlet boundary conditions are enforced. Let $\eta_1 > 0$, $\eta_2 > 0$ (these parameters can vary from face to face), and

$$M^{uu}_F(v) = \eta_1 h_F^{-1} v, \qquad S^{uu}_F(v) = \eta_2 h_F^{-1} v, \qquad R^{uu}_F \equiv 0. \tag{3.25}$$

Since $\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u} = \frac{1}{2}(\mathcal{I}_d + n\otimes n)$ and $\mathcal{D}^{uu} = (\beta\cdot n)\mathcal{I}_d$, properties (DG2A)–(DG2D) hold. Many other choices can be considered for M^{uu}_F, S^{uu}_F, and R^{uu}_F.

PROPOSITION 3.9. *Property* (3.4) *holds.*

Proof. The proof is a direct consequence of the fact that for all $v_h \in U_h$,

$$\|v_h\|_{L_u}^2 \lesssim \sum_{K\in\mathcal{T}_h} \|\nabla v_h + (\nabla v_h)^t\|_{L_\sigma,K}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{-1}\|[\![v_h]\!]\|_{L_u,F}^2 + \sum_{F\in\mathcal{F}_h^{\partial}} h_F^{-1}\|v_h\|_{L_u,F}^2.$$

See [7, 13] for a proof. ∎

3.5. Remarks on the incompressible limit. The above analysis is not uniform with respect to the compressibility factor δ, i.e., the analysis fails as $\delta \to 0$; this is the well-known locking phenomenon. The origin of the problem is that the operator $K^{\sigma\sigma}$ is not invertible at the limit $\delta = 0$; indeed, since $K^{\sigma\sigma}\sigma = \sigma - \frac{1}{d+\delta}\mathrm{tr}(\sigma)\mathcal{I}_d$, it is clear that $K^{\sigma\sigma}\sigma = 0$ when $\delta = 0$ and σ is proportional to the identity. In other words, it is our insisting on eliminating σ that is responsible for locking (i.e., the primal formulation locks).

Actually the above two-field DG approximation is robust with respect to δ if one *abandons* the idea of eliminating σ; in other words, the mixed formulation is robust. This fact has already been recognized in Franca and Stenberg [20] using the GaLS technique, and this is also true for conforming mixed approximations; see Brezzi and Fortin [8] and Schwab and Suri [28]. (A DG method (IP) where σ is eliminated and which does not lock for small δ is analyzed in Hansbo and Larson [23]. This method, though, yields a linear system with a condition number that grows unboundedly as $\delta \to 0$.)

Let us now substantiate the above claim. Since it is not more difficult to analyze the general case, let us forget problem (3.17) and let us deal with the general DG approximation (2.16). We now weaken the L_σ-coercivity requirement used in section 3.1 as follows. We replace (A3A) by the following: Assume that there exists an operator $\pi \in \mathcal{L}(L_\sigma, L_\sigma)$ such that

$$\text{(A3A}'\text{)} \qquad \forall z \in L, \qquad ((K + K^* - \nabla\cdot A)z, z)_L \gtrsim \|\mathcal{K}^{\sigma\sigma} z^\sigma\|_{L_\sigma}^2 + \|\pi z^\sigma\|_{L_\sigma}^2.$$

We also modify (DG2A) as follows:

$$\text{(DG2A}'\text{)} \qquad M_F = \left[\begin{array}{c|c} 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hline \alpha\mathcal{D}^{u\sigma} & M_F^{uu} \end{array}\right],\ \alpha\in\{-1,+1\}, \quad S_F = \left[\begin{array}{c|c} S_F^{\sigma\sigma} & 0 \\ \hline 0 & S_F^{uu} \end{array}\right], \quad R_F \equiv 0,$$

where $S_F^{\sigma\sigma}$ is defined by

$$\text{(DG2A}''\text{)} \qquad (S_F^{\sigma\sigma}(z^\sigma), y^\sigma)_{L_\sigma,F} = \eta h_F (z^\sigma - \pi z^\sigma, y^\sigma - \pi y^\sigma)_{L_\sigma,F},$$

with user-dependent parameter $\eta > 0$. Note that owing to the presence of $S_F^{\sigma\sigma}$, z_h^σ can no longer be locally eliminated. We introduce the notation $|z^\sigma|_{J'}^2 := \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |z^\sigma|_{J',F}^2$, $|z^\sigma|_{J',F}^2 = (S_F^{\sigma\sigma}([\![z^\sigma]\!]), [\![z^\sigma]\!])_{L_\sigma,F}$, and we redefine the $\|\cdot\|_{h,B}$ discrete norm as

$$\|y\|_{h,B}^2 := \|y\|_L^2 + |y^u|_J^2 + |y^\sigma|_{J'}^2 + |y^u|_M^2 + \|B_h y^u\|_{L_\sigma}^2. \tag{3.26}$$

To ascertain control over $z_h^\sigma - \pi z_h^\sigma$ we make the two following assumptions:

$$\forall z_h^\sigma \in \Sigma_h, \quad \|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}^2 \lesssim \|\pi z_h^\sigma\|_{L_\sigma}^2 + |z_h^\sigma|_{J'}^2 + \left(\sup_{0\neq v_h^u\in U_h} \frac{\mathbb{B}_h(z_h^\sigma, v_h^u)}{\|(0, v_h^u)\|_{h,B}}\right)^2, \tag{3.27}$$

$$\forall z_h^u \in U_h, \quad \|C_h z_h^u\|_{L_u} \lesssim \|B_h z_h^u\|_{L_\sigma} + \|z_h^u\|_{L_u}, \tag{3.28}$$

where $\mathbb{B}_h(z_h^\sigma, v_h^u) := a_h((z_h^\sigma, 0), (0, v_h^u))$, i.e.,

$$\mathbb{B}_h(z_h^\sigma, v_h^u) := (B_h^\dagger z_h^\sigma, v_h^u)_{L_u} - 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{v_h^u\})_{L_u,F} + \tfrac{\alpha-1}{2} \sum_{F\in\mathcal{F}_h^{\partial}} (\mathcal{D}^{u\sigma} z_h^\sigma, v_h^u)_{L_u,F}. \tag{3.29}$$

THEOREM 3.10. *The statement of Theorem* 2.3 *remains true under the assumptions* (A3A′)–(A3B′), (DG2A′)–(DG2A″), (DG2B)–(DG2F), *and* (3.4), (3.27), *and* (3.28).

Proof. We just sketch the proof since most of the arguments will be repeated with more details in section 4.2. Let us prove the inf-sup inequality

$$\forall z_h \in W_h, \qquad \|z_h\|_{h,B} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,B}}. \tag{3.30}$$

Let $z_h \in W_h$ and define $\mathbb{S} := \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,B}}$. Owing to the definition of a_h, (DG2A′)–(DG2A″), and (A3A′),

$$\|\mathcal{K}^{\sigma\sigma} z_h^\sigma\|_{L_\sigma}^2 + \|\pi z_h^\sigma\|_{L_\sigma}^2 + |z_h^u|_J^2 + |z_h^\sigma|_{J'}^2 + |z_h^u|_M^2 \lesssim a_h(z_h, z_h) \lesssim \mathbb{S}\,\|z_h\|_{h,B}.$$

Set $\varpi_h = (B_h z_h^u, 0)$ and observe that $\varpi_h \in W_h$ since the fields $\mathcal{B}^k$ are constant over Ω (see (A3B′)) and $p_u - 1 \le p_\sigma$. Then

$$\begin{aligned}\|B_h z_h^u\|_{L_\sigma}^2 &= a_h(z_h, \varpi_h) - (\mathcal{K}^{\sigma\sigma} z_h^\sigma, B_h z_h^u)_{L_\sigma} - \sum_{F\in\mathcal{F}_h^{\partial}} \tfrac{\alpha+1}{2} (\mathcal{D}^{\sigma u} z_h^u, B_h z_h^u)_{L_\sigma,F} \\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{B_h z_h^u\})_{L_\sigma,F} - (S_F^{\sigma\sigma}(z_h^\sigma), B_h z_h^u)_{L_\sigma,F}.\end{aligned}$$

Proceeding as in the proof of Lemma 3.2 and observing that $\|\varpi_h\|_{h,B} \lesssim \|B_h z_h^u\|_{L_\sigma} + |B_h z_h^u|_{J'} \lesssim \|z_h\|_{h,B}$, we infer

$$\|B_h z_h^u\|_{L_\sigma}^2 \lesssim a_h(z_h, \varpi_h) + a_h(z_h, z_h) \lesssim \mathbb{S}\,\|z_h\|_{h,B}.$$

Using (3.4) then yields $\|z_h^u\|_{L_u}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$. Finally, to control $\|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}$, observe that $\mathbb{B}_h(z_h^\sigma, v_h^u) = a_h(z_h, (0, v_h^u)) - a_h(0, z_h^u), (0, v_h^u))$ and using (3.27)–(3.28) proceed as in the proof of Lemma 4.2, step (4c). This yields $\|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$. The rest of the proof is standard. ∎

Coming back to our compressible linear elasticity problem (3.17), observe that the above analysis applies by setting $\pi\sigma = \sigma - \frac{1}{d}\mathrm{tr}(\sigma)\mathcal{I}_d$. Inequality (A3A′) holds *uniformly* with respect to δ, as can be seen by taking the mean of the two equations in (3.20). Assumption (3.28) results from Korn's second inequality. A proof for (3.27) is given in the proof of Proposition 4.5. In conclusion, the present two-field DG formulation is robust with respect to δ. However, owing to the presence of the stabilization operator $S_F^{\sigma\sigma}$ which penalizes the jumps of $\mathrm{tr}(\sigma_h)$, σ_h cannot be eliminated locally. This motivates the introduction of a new scalar variable (the pressure) whose jumps will be penalized instead of those of $\mathrm{tr}(\sigma_h)$.

4. Three-field theory. The goal of this section is to weaken even further the set of hypotheses (A3A), (A3B′), and (A3B″). We have in mind the linear elasticity equations, and we want to come up with a method which is robust in the incompressible limit and for which the σ-component can be eliminated locally. To this purpose, we introduce a three-field theory of Friedrichs' systems and adapt the DG approximation to this setting. Thus, we assume that z can be decomposed into three fields, z^σ, z^p, and z^u, and we have in mind to locally eliminate z^σ.

4.1. Three-field Friedrichs' systems. We now assume that $L = L_\sigma \times L_p \times L_u$, where $L_\sigma = [L^2(\Omega)]^{m_\sigma}$, $L_p = [L^2(\Omega)]^{m_p}$, and $L_u = [L^2(\Omega)]^{m_u}$, with $m = m_\sigma + m_p + m_u$. We also assume that the operator K and the matrices $\mathcal{A}^k$, $1 \le k \le d$, admit the following 3×3 structure:

$$K = \left[\begin{array}{c:c:c} K^{\sigma\sigma} & K^{\sigma p} & 0 \\ \hdashline K^{p\sigma} & K^{pp} & 0 \\ \hdashline 0 & 0 & K^{uu} \end{array}\right] \quad \text{and} \quad \mathcal{A}^k = \left[\begin{array}{c:c:c} 0 & 0 & \mathcal{B}^k \\ \hdashline 0 & 0 & 0 \\ \hdashline [\mathcal{B}^k]^t & 0 & \mathcal{C}^k \end{array}\right]. \tag{4.1}$$

We assume that the block $K^{\sigma\sigma}$ has a local representation, i.e., there is $\mathcal{K}^{\sigma\sigma} \in [L^\infty(\Omega)]^{m_\sigma, m_\sigma}$ such that $K^{\sigma\sigma}(y^\sigma) = \mathcal{K}^{\sigma\sigma} y^\sigma$ for all $y^\sigma \in L_\sigma$ (this local representation is needed in the three-field DG method described below to locally eliminate the discrete σ-component). We denote by K^σ (resp., K^p) the canonical projection of K onto L_σ (resp., L_p). Finally, we assume that there exists $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$ such that

$$\forall z \in L, \quad ((K + K^* - \nabla\cdot A)z, z)_L \gtrsim \|\pi z^\sigma\|^2_{L_\sigma} + \|K^\sigma z\|^2_{L_\sigma} + \|K^p z\|^2_{L_p}, \tag{A3c}$$

$$\forall z \in V \cup V^*, \quad \|z^u\|_{L_u} \lesssim a(z,z)^{\frac{1}{2}} + \|Bz^u\|_{L_\sigma}, \tag{A3d}$$

$$\forall z \in W, \quad \|Cz^u\|_{L_u} \lesssim \|z^u\|_{L_u} + \|Bz^u\|_{L_\sigma}, \tag{A3e}$$

$$\forall z \in V \cup V^*, \quad \|z^\sigma - \pi z^\sigma\|_{L_\sigma} \lesssim \|\pi z^\sigma\|_{L_\sigma} + \|B^\dagger z^\sigma\|_{L_u}, \tag{A3f}$$

$$\forall z \in L, \quad \|z^p\|^2_{L_p} \lesssim ((K + K^* - \nabla\cdot A)z, z)_L + \|K^{pp} z^p\|^2_{L_p}, \tag{A3g}$$

$$\forall k \in \{1, \dots, k\}, \quad \mathcal{B}^k \text{ is constant over } \Omega, \tag{A3h}$$

and that (A3c) and (A3g) also hold when K^σ, K^p, and K^{pp} are substituted by the corresponding terms in K^*, say $K^{*\sigma}$, K^{*p}, and K^{*pp}.

THEOREM 4.1. *The conclusions of Theorem* 2.1 *still hold if assumption* (A3) *is replaced by assumptions* (A3c)–(A3h).

Proof. The proof, which is similar to that of Theorem 3.1, is only sketched. We only prove that $T : V \longrightarrow L$ is an isomorphism, since the rest of the proof is unchanged or goes along the same lines.

(1) Proof of (3.2). Let $z \in V$. Recall that $(Tz, z)_L = a(z,z) \ge ((K + K^* - \nabla\cdot A)z, z)_L$. Hence, property (A3c) implies

$$\|K^p z\|_{L_p} + \|K^\sigma z\|_{L_\sigma} + \|\pi z^\sigma\|_{L_\sigma} \lesssim (Tz, z)_L^{\frac{1}{2}}.$$

Since $Bz^u + K^\sigma z = (Tz)^\sigma$, we infer $\|Bz^u\|_{L_\sigma} \lesssim \|Tz\|_L + (Tz,z)_L^{\frac{1}{2}}$. Then, owing to (A3d)–(A3e),

$$\|z^u\|_{L_u} + \|Cz^u\|_{L_u} \lesssim \|Tz\|_L + (Tz,z)_L^{\frac{1}{2}}.$$

Now we use (A3f), the above bounds, and the fact that $B^\dagger z^\sigma = (Tz)^u - Cz^u - K^{uu} z^u$ to deduce

$$\|z^\sigma - \pi z^\sigma\|_{L_\sigma} \lesssim \|B^\dagger z^\sigma\|_{L_u} + (Tz,z)_L^{\frac{1}{2}} \lesssim \|Tz\|_L + (Tz,z)_L^{\frac{1}{2}},$$

and the same bound holds for $\|z^\sigma\|_{L_\sigma}$ by the triangle inequality. To derive a bound on $\|z^p\|_{L_p}$ we use the fact that $K^p z = (Tz)^p$ to infer

$$\|K^{pp} z^p\|_{L_p} \le \|K^p z\|_{L_p} + \|K^{p\sigma} z^\sigma\|_{L_p} \lesssim \|Tz\|_L + \|z^\sigma\|_{L_\sigma}.$$

Hence, (A3G) and the above bounds yield $\|z^p\|_{L_p} \lesssim \|Tz\|_L + (Tz,z)_L^{\frac{1}{2}}$. Combining the previous estimates yields $\|z\|_L \lesssim \|Tz\|_L + (Tz,z)_L^{\frac{1}{2}}$ and we conclude as usual.

(2) Proof of (3.3). Assume that $y \in L$ is such that $(Tz,y)_L = 0$ for all $z \in V$. Then $y \in V^*$ and $\tilde{T}y = 0$. Hence (A3C) implies $\pi y^\sigma = 0$ and $K^{*\sigma}y = 0$. Then, observing that $0 = (\tilde{T}y)^\sigma = K^{*\sigma}y - By^u$ since the fields $\mathcal{B}^k$ are constant, we infer $By^u = 0$. Then (A3D) and (A3E) yield $y^u = 0$ and $Cy^u = 0$. Using $0 = (\tilde{T}y)^u$ and the fact that the fields $\mathcal{B}^k$ are constant, we then infer $B^\dagger y^\sigma = 0$ so that (A3F) implies $y^\sigma - \pi y^\sigma = 0$, and hence $y^\sigma = 0$. Finally, since $0 = (\tilde{T}y)^p = K^{*pp}y^p$, using (A3G) we infer $y^p = 0$, thus completing the proof. ◻

4.2. Three-field DG approximation. We analyze in this section a DG method to approximate the three-field Friedrichs' systems introduced in section 4.1. We assume that hypotheses (A3C)–(A3H) hold so that the continuous problem is well-posed.

Let $p_u > 0$ be a *positive* integer and let p_σ and p_p be such that

$$p_u - 1 \le p_\sigma \le p_u + 1, \qquad p_\sigma \le p_p. \tag{4.2}$$

Consider the finite elements spaces

$$\Sigma_h = [P_{h,p_\sigma}]^{m_\sigma}, \quad P_h = [P_{h,p_p}]^{m_p}, \quad U_h = [P_{h,p_u}]^{m_u}, \quad W_h = \Sigma_h \times P_h \times U_h. \tag{4.3}$$

Consider the discrete problem (2.16) with the bilinear form still defined by (2.15).

The key property of the three-field DG approximation developed hereafter is that the operators M_F and S_F are designed in such a way that the discrete σ-component can be locally eliminated. We consider either Dirichlet boundary conditions or mixed Robin–Neumann boundary conditions enforced by setting $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$; see sections 3.3 and 3.4. The design conditions of the three-field DG method are the following:

(DG3A) $$M_F = \left[\begin{array}{c:c:c} 0 & 0 & -\alpha\mathcal{D}^{\sigma u} \\ \hdashline 0 & 0 & 0 \\ \hdashline \alpha\mathcal{D}^{u\sigma} & 0 & M_F^{uu} \end{array}\right], \quad S_F = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & S_F^{pp} & 0 \\ \hdashline 0 & 0 & S_F^{uu} \end{array}\right], \quad R_F = \left[\begin{array}{c:c:c} 0 & 0 & 0 \\ \hdashline 0 & 0 & 0 \\ \hdashline 0 & 0 & R_F^{uu} \end{array}\right];$$

(DG3B) if $\alpha = +1$, $\begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \operatorname{Ker}(\mathcal{D}^{\sigma u}) \subset \operatorname{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F^{-1}|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}; \end{cases}$

(DG3C) if $\alpha = -1$, $M_F^{uu}(v) = \mathcal{M}^{uu}v$;

(DG3D) $S_F^{uu} = (S_F^{uu})^*$, and $h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F^{-1}|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim h_F^{-1}\mathcal{I}_{m_u}$;

(DG3E) $S_F^{pp} = (S_F^{pp})^*$, and $h_F\mathcal{I}_{m_p} \lesssim S_F^{pp} \lesssim h_F\mathcal{I}_{m_p}$,

where $\alpha \in \{-1,+1\}$ in the definition of M_F in (DG3A).

Our aim is to control the approximation error in the norm $\|\cdot\|_{h,B}$ defined by

$$\|y\|_{h,B}^2 = \|y\|_L^2 + |y^u|_M^2 + |y^u|_{J^u}^2 + |y^p|_{J^p}^2 + \|B_h y^u\|_{L_\sigma}^2, \tag{4.4}$$

with $|y^u|_{J^u}^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |y^u|_{J^u,F}^2$, $|y^p|_{J^p}^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |y^p|_{J^p,F}^2$, $|y^u|_M^2 = \sum_{F\in\mathcal{F}_h^\partial} |y^u|_{M,F}^2$,

$$|y^u|_{J^u,F}^2 = (S_F^{uu}([\![y^u]\!]), [\![y^u]\!])_{L_u,F}, \quad |y^p|_{J^p,F}^2 = (S_F^{pp}([\![y^p]\!]), [\![y^p]\!])_{L_p,F}, \tag{4.5}$$

$$|y^u|_{M,F}^2 = (M_F^{uu}(y^u), y^u)_{L_u,F}. \tag{4.6}$$

The user-dependent operator R_F^{uu} must be designed so that for all $z_h^u \in U_h$ and all $(z^u, y_h^u) \in U(h) \times U_h$,

$$\text{(DG3F)} \qquad \sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z_h^u]\!]), [\![z_h^u]\!])_{L_u,F} \geq -\frac{1}{4}(|z_h^u|_{J^u}^2 + |z_h^u|_M^2),$$

$$\text{(DG3G)} \qquad \sum_{F\in\mathcal{F}_h} (R_F^{uu}([\![z^u]\!]), [\![y_h^u]\!])_{L_u,F} \lesssim (|z^u|_{J^u} + |z^u|_M)(|y_h^u|_{J^u} + |y_h^u|_M).$$

The discrete counterpart of assumption (A3D) is still (3.4), while the discrete counterpart of assumption (A3F) is the following: For all $z_h^\sigma \in \Sigma_h$,

$$\|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}^2 \lesssim \|\pi z_h^\sigma\|_{L_\sigma}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F \|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L_\sigma,F}^2 + \left(\sup_{0\neq v_h^u\in U_h} \frac{\mathbb{B}_h(z_h^\sigma, v_h^u)}{\|(0,0,v_h^u)\|_{h,B}}\right)^2, \tag{4.7}$$

where $\mathbb{B}_h(z_h^\sigma, v_h^u)$ is defined by (3.29). Finally, the discrete counterpart of assumption (A3E) is

$$\forall z_h^u \in U_h, \qquad \|C_h z_h^u\|_{L_u} \lesssim \|B_h z_h^u\|_{L_\sigma} + \|z_h^u\|_{L_u}. \tag{4.8}$$

Since the jumps of the σ-component are not controlled in the three-field DG method (so as to eliminate this component locally), stability must come from the control on the jumps of the p-component. The link between the jumps of the σ- and p-components is provided by the equation for the p-component. This motivates the following additional assumptions:

$$\forall z_h \in W_h, \quad K^p z_h \in P_h, \tag{4.9}$$

$$\forall z_h \in W_h, \quad \forall F \in \mathcal{F}_h^{\mathrm{i}}, \quad \|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L_\sigma,F} \lesssim \|[\![K^p z_h]\!]\|_{L_p,F} + \|[\![z_h^p]\!]\|_{L_p,F}. \tag{4.10}$$

LEMMA 4.2. *Assume that* (A3C)–(A3H) *and the discrete assumptions* (DG3A)–(DG3G), (3.4), (4.2), (4.7), (4.8), (4.9), *and* (4.10) *hold. Then the following holds:*

$$\forall z_h \in W_h, \qquad \|z_h\|_{h,B} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h, y_h)}{\|y_h\|_{h,B}}. \tag{4.11}$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} = \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h,B}}$.

(1) Owing to the definition of a_h, (DG3A), (DG3F), and (A3C),

$$\|K^\sigma z_h\|_{L_\sigma}^2 + \|K^p z_h\|_{L_p}^2 + |\pi z_h^\sigma\|_{L_\sigma}^2 + |z_h^u|_M^2 + |z_h^u|_{J^u}^2 + |z_h^p|_{J^p}^2 \lesssim a_h(z_h, z_h) \leq \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.12}$$

(2) Control on z_h^u, $C_h z_h^u$, and $B_h z_h^u$. Set $y_h = (B_h z_h^u, 0, 0)$ and observe that $y_h \in W_h$ owing to (A3H) and the fact that $p_u - 1 \leq p_\sigma$. Moreover, $\|y_h\|_{h,B} \lesssim \|z_h\|_{h,B}$. Furthermore,

$$\|B_h z_h^u\|_{L_\sigma}^2 = a_h(z_h, y_h) - (K^\sigma z_h, B_h z_h^u)_{L_\sigma} - \tfrac{\alpha+1}{2} \sum_{F\in\mathcal{F}_h^\partial} (\mathcal{D}^{\sigma u} z_h^u, B_h z_h^u)_{L_\sigma,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{\sigma u} z_h^u\}, \{B_h z_h^u\})_{L_\sigma,F} := T_1 + T_2 + T_3 + T_4.$$

Clearly, $|T_1| \lesssim \mathbb{S}\,\|z_h\|_{h,B}$ and owing to (4.12),

$$|T_2| \lesssim \|K^\sigma z_h\|_{L_\sigma}\|B_h z_h^u\|_{L_\sigma} \lesssim \mathbb{S}\,\|z_h\|_{h,B} + \gamma\|B_h z_h^u\|_{L_\sigma}^2,$$

where $\gamma > 0$ can be chosen as small as needed. Similarly, using (DG3B), $\{\mathcal{D}^{\sigma u}\} = 0$, (DG3D), and a trace inverse inequality leads to

$$|T_3| + |T_4| \lesssim (|z_h^u|_M + |z_h^u|_{J^u})\|B_h z_h^u\|_{L_\sigma} \lesssim \mathbb{S}\,\|z_h\|_{h,B} + \gamma\|B_h z_h^u\|_{L_\sigma}^2.$$

Collecting the above bounds and choosing the γ's small enough, it is inferred that $\|B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$. Then, owing to (3.4), (4.8), and (4.12), this in turn implies

$$\|z_h^u\|_{L_u}^2 + \|C_h z_h^u\|_{L_u}^2 + \|B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.13}$$

(3) Control on z_h^σ. It remains to control $z_h^\sigma - \pi z_h^\sigma$. The idea is to use (4.7) by estimating the three terms on the right-hand side, say R_1–R_3.

(3a) Clearly, $R_1 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$.

(3b) To control R_2, use (4.10) and the fact that $\|[\![K^p z_h]\!]\|_{L_p,F} \lesssim h_F^{-1/2}\|K^p z_h\|_{L_p,\mathcal{T}(F)}$ owing to a trace inverse inequality (which can be used by (4.9)) to infer

$$\|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L\sigma,F}^2 \lesssim h_F^{-1}\|K^p z_h\|_{L_p,\mathcal{T}(F)}^2 + h_F^{-1}|z_h^p|_{J^p,F}^2.$$

Hence, owing to the estimates for $\|K^p z_h\|_{L_p}$ and $|z_h^p|_{J^p,F}$ resulting from step (1), $R_2 \lesssim \mathbb{S}\,\|z_h\|_{h,B}$.

(3c) For R_3, observe that $\mathbb{B}_h(z_h^\sigma, v_h^u) = a_h(z_h,(0,0,v_h^u)) - a_h((0,0,z_h^u),(0,0,v_h^u))$ and that for all $v_h^u \in U_h$, the following holds:

$$a_h((0,0,z_h^u),(0,0,v_h^u)) \lesssim \mathbb{A}_h := (\mathbb{S}\,\|z_h\|_{h,B})^{\frac{1}{2}}\,\|(0,0,v_h^u)\|_{h,B}. \tag{4.14}$$

Indeed,

$$\begin{aligned} a_h((0,0,z_h^u),(0,0,v_h^u)) &= (K^{uu}z_h^u, v_h^u)_{L_u} + (C_h z_h^u, v_h^u)_{L_u} \\ &\quad + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{1}{2}(M_F^{uu}(z_h^u) - \mathcal{D}^{uu}z_h^u, v_h^u)_{L_u,F} - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}^{uu}z_h^u\},\{v_h^u\})_{L_u,F} \\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} ((S_F^{uu} + R_F^{uu})([\![z_h^u]\!]), [\![v_h^u]\!])_{L_u,F} := T_1 + T_2 + T_3 + T_4 + T_5. \end{aligned}$$

Using (4.13) yields $|T_1| + |T_2| \lesssim \mathbb{A}_h$, while (4.12) together with (DG3G) readily yields $|T_5| \lesssim \mathbb{A}_h$. Since $\{\mathcal{D}^{uu}\} = 0$, using (DG3D) and a trace inverse inequality leads to

$$|T_4| \lesssim \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{\frac{1}{2}}|z_h^u|_{J^u,F}h_F^{-\frac{1}{2}}\|v_h^u\|_{L_u,\mathcal{T}(F)} \lesssim \mathbb{A}_h.$$

Finally, to control T_3, we proceed similarly using (DG3B) if $\alpha = +1$ to infer $|T_3| \lesssim |z_h^u|_M(|v_h^u|_M + \|v_h^u\|_{L_u}) \lesssim \mathbb{A}_h$, while if $\alpha = -1$, we use (DG3C) and the assumption $\mathcal{M}^{uu} = |\mathcal{D}^{uu}|$ to infer $|T_3| \lesssim |z_h^u|_M|v_h^u|_M \lesssim \mathbb{A}_h$. Collecting the bounds for T_1–T_5 yields (4.14), whence the bound $R_3 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}$ is readily inferred.

(3d) Collecting the bounds for R_1–R_3 yields

$$\|z_h^\sigma - \pi z_h^\sigma\|_{L_\sigma}^2 \le R_1 + R_2 + R_3 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.15}$$

The same bound is inferred for $\|z_h^\sigma\|_{L_\sigma}^2$ by the triangle inequality.

(4) Control on z_h^p. Using (A3G), the fact that $K^{pp}z_h^p = K^p z_h - K^{p\sigma}z_h^\sigma$, and the derived bounds on $\|K^p z_h\|_{L_p}$ and $\|z_h^\sigma\|_{L_\sigma}$ leads to

$$\|z_h^p\|_{L_p}^2 \lesssim a_h(z_h, z_h) + \|K^p z_h\|_{L_p}^2 + \|K^{p\sigma}z_h^\sigma\|_{L_p}^2 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}. \tag{4.16}$$

(5) Conclusion. Collecting the above bounds yields $\|z_h\|_{h,B}^2 \lesssim \mathbb{S}^2 + \mathbb{S}\,\|z_h\|_{h,B}$, whence (4.11) readily follows. □

Remark 4.1. Assumption (DG3D) (resp., (DG3B)) requires a stronger control on $|\mathcal{D}^{uu}|$ with respect to (DG2D) (resp., (DG2B)). This stronger control is needed to prove (4.14) in step (3c) of the above proof. This is not really a restriction, since in practice $|\mathcal{D}^{uu}| \lesssim (\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac12}$ so that $h_F^{-1}(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac12} \lesssim S_F^{uu}$ already yields $h_F^{-1}|\mathcal{D}^{uu}| \lesssim S_F^{uu}$ (see the examples in sections 3.3 and 3.4).

It is now straightforward to verify the following convergence result.

THEOREM 4.3. *The statement of Theorem* 2.3 *remains valid under the assumptions of Lemma* 4.2.

4.3. Example: Linear continuum mechanics. Let us consider again problem (3.17) in section 3.4 and let us introduce a third field $p := -\frac{1}{d+\delta}\mathrm{tr}(\sigma)$ so that (3.17) can be recast into

$$\begin{cases} \sigma + p\mathcal{I}_d - \frac12(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + (d+\delta)p = 0, \\ -\frac12\nabla\cdot(\sigma+\sigma^t) + \beta\cdot\nabla u + \lambda u = f; \end{cases} \tag{4.17}$$

p is often referred to as the pressure. This type of problem arises in the modeling of non-Newtonian fluids. (4.17) is the limit of the so-called upper convected Maxwell model for small relaxation times; see Fortin and Pierre [19], Fortin, Guénette, and Pierre [18], and Schwab and Suri [28]. Although three-field models may not be easy to implement or may lead to increased computational costs, they appear to be more robust for high Weissenberg numbers; see Baaijens for a review on this question [3].

Instead of (3.19), we now assume that

$$\lambda_0 \geq 0, \qquad \delta_0 = 0. \tag{4.18}$$

This setting covers incompressible materials (i.e., $\lambda = 0$, $\beta = 0$, and $\delta = 0$) and incompressible fluid flows. For the sake of simplicity, we henceforth restrict ourselves to homogeneous Dirichlet boundary conditions, i.e., $\alpha = 1$. Then the pressure is defined up to a constant, and to avoid this arbitrariness, we choose the representative of the pressure which is of zero mean, i.e., $\langle p\rangle_\Omega = 0$. Accordingly, we modify slightly the equations as follows:

$$\begin{cases} \sigma + p\mathcal{I}_d - \frac12(\nabla u + (\nabla u)^t) = 0, \\ \mathrm{tr}(\sigma) + (d+\delta)p + \langle p\rangle_\Omega = 0, \\ -\frac12\nabla\cdot(\sigma+\sigma^t) + \beta\cdot\nabla u + \lambda u = f. \end{cases} \tag{4.19}$$

Note that by taking the trace of the first equation and integrating over Ω, we obtain $\langle \mathrm{tr}(\sigma) + dp\rangle_\Omega = \langle \nabla\cdot u\rangle_\Omega = 0$ for homogeneous Dirichlet boundary conditions on u. Accounting for the second equation in (4.19) yields $\langle \mathrm{tr}(\sigma)\rangle_\Omega = \langle p\rangle_\Omega = 0$. Hence, we are really solving (4.17) with $\langle p\rangle_\Omega = 0$. Note that K is self-adjoint and that K^{pp} is the only nonlocal block in the operator K.

4.3.1. Well-posedness. Define $\pi \in \mathcal{L}(L_\sigma; L_\sigma)$ such that for all $\sigma \in L_\sigma$,

$$\pi\sigma = \sigma - \tfrac{1}{d}(\mathrm{tr}(\sigma) - \langle \mathrm{tr}(\sigma)\rangle_\Omega)\mathcal{I}_d. \tag{4.20}$$

PROPOSITION 4.4. *Assumptions* (A3C)–(A3H) *hold.*

Proof. Let us first prove (A3C). A straightforward calculation yields, for all $z = (\sigma, p, 0) \in W$,

$$\begin{aligned}(Kz, z)_L &= \|\sigma + p\mathcal{I}_d\|^2_{L_\sigma} + \delta\|p\|^2_{L_p} + \langle p\rangle^2_\Omega \\ &= \tfrac{d}{d+\delta}\|\sigma - \tfrac{1}{d}\mathrm{tr}(\sigma)\mathcal{I}_d\|^2_{L_\sigma} + \tfrac{1}{d+\delta}\|\mathrm{tr}(\sigma) + (d+\delta)p\|^2_{L_p} + \tfrac{\delta}{d+\delta}\|\sigma\|^2_{L_\sigma} + \langle p\rangle^2_\Omega.\end{aligned}$$

Hence, taking the mean of the two above equations and using triangle inequalities,

$$\gamma(Kz, z)_L \geq \|\pi\sigma\|^2_{L_\sigma} + \|K^\sigma z\|^2_{L_\sigma} + \|K^p z\|^2_{L_p} + \langle p\rangle^2_\Omega,$$

with γ independent of δ_0. Since K is self-adjoint, $\lambda_0 \geq 0$, and the fields $\mathcal{B}^k$ are constant over Ω, it is inferred that for all $z = (\sigma, p, u) \in W$,

$$\tfrac{1}{2}((K + K^* - \nabla\cdot A)z, z)_L \geq (K(\sigma, p, 0), (\sigma, p, 0))_L.$$

This proves (A3C). Assumption (A3D) is a simple consequence of Korn's first inequality since $V = V^* = H_\sigma \times L^2(\Omega) \times [H^1_0(\Omega)]^d$. Assumption (A3E) results from Korn's second inequality. Let us prove (A3F). Let $z \in V$ be such that $\sigma \neq \pi\sigma$; then $\mathrm{tr}(\sigma - \pi\sigma) \neq 0$. Since $\mathrm{tr}(\sigma - \pi\sigma) = \mathrm{tr}(\sigma) - \langle \mathrm{tr}(\sigma)\rangle_\Omega \in L^2_0(\Omega)$, there is $0 \neq v \in [H^1_0(\Omega)]^d$ such that $\nabla\cdot v = \mathrm{tr}(\sigma - \pi\sigma)$ and $\|v\|_{[H^1(\Omega)]^d} \lesssim \|\sigma - \pi\sigma\|_{L_\sigma}$. Since $\langle \nabla\cdot v\rangle_\Omega = 0$, we have $Bv - \pi Bv = \sigma - \pi\sigma$. Hence,

$$\begin{aligned}\|\sigma - \pi\sigma\|_{L_\sigma} &\lesssim \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv - \pi Bv, \sigma - \pi\sigma)_{L_\sigma} \lesssim \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, \sigma - \pi\sigma)_{L_\sigma} \\ &\lesssim \|\pi\sigma\|_{L_\sigma} + \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, \sigma)_{L_\sigma} \lesssim \|\pi\sigma\|_{L_\sigma} + \|B^\dagger\sigma\|_{L_u},\end{aligned}$$

whence (A3F). To prove (A3G), observe that $\|z^p\|_{L_p} \lesssim \|K^{pp}z^p\|_{L_p} + \langle p\rangle_\Omega$ and that $\langle p\rangle_\Omega \lesssim (Kz, z)^{1/2}_L$. Finally, assumption (A3H) is evident. ∎

4.3.2. Three-field DG approximation. Let $\eta_1 > 0$, $\eta_2 > 0$, $\eta_3 > 0$ (these parameters can vary from face to face) and

$$M^{uu}_F(v) = \eta_1 h_F^{-1} v, \qquad S^{uu}_F(v) = \eta_2 h_F^{-1} v, \qquad S^{pp}_F(q) = \eta_3 h_F q, \qquad R^{uu}_F \equiv 0. \tag{4.21}$$

Clearly, assumptions (DG3B)–(DG3G) hold. Other choices can be considered for M^{uu}_F, S^{uu}_F, and S^{pp}_F; see, e.g., [10, 9] for a similar DG method to approximate the Stokes and the Oseen equations.

PROPOSITION 4.5. *The discrete assumptions* (3.4)*,* (4.7)*,* (4.8)*,* (4.9)*, and* (4.10) *hold.*

Proof. The discrete Poincaré inequality (3.4) has already been shown to hold in section 3.4. Furthermore, assumptions (4.8), (4.9), and (4.10) are evident. It remains to prove (4.7). Let $z_h \in W_h$ be such that $z^\sigma_h \neq \pi z^\sigma_h$. Proceeding as in the proof of (A3E) in Proposition 4.4, there is $0 \neq v \in [H^1_0(\Omega)]^d$ such that $\nabla\cdot v = \mathrm{tr}(z^\sigma_h - \pi z^\sigma_h)$, $\|v\|_{[H^1(\Omega)]^d} \lesssim \|z^\sigma_h - \pi z^\sigma_h\|_{L_\sigma}$, and

$$\|z^\sigma_h - \pi z^\sigma_h\|_{L_\sigma} \lesssim \|\pi z^\sigma_h\|_{L_\sigma} + \tfrac{1}{\|v\|_{[H^1(\Omega)]^d}}(Bv, z^\sigma_h)_{L_\sigma}.$$

Since $v \in [H_0^1(\Omega)]^d$, integration by parts yields

$$(Bv, z_h^\sigma \pi z_h^\sigma)_{L_\sigma} = -(v, B_h^\dagger z_h^\sigma)_{L_u} + 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, v)_{L_u,F}.$$

Let v_h be the L_u-orthogonal projection of v onto U_h. Then

$$\begin{aligned}(Bv, z_h^\sigma)_{L_\sigma} = & -(v - v_h, B_h^\dagger z_h^\sigma)_{L_u} + 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{v - v_h\})_{L_u,F} \\ & - (v_h, B_h^\dagger z_h^\sigma)_{L_u} + 2 \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{v_h\})_{L_u,F} := T_1 + T_2 + T_3 + T_4.\end{aligned}$$

By construction, $T_1 = 0$ and by definition $T_3 + T_4 = -\mathbb{B}_h(z_h^\sigma, v_h)$. Moreover, adding and subtracting πz_h^σ in T_2, using trace and inverse inequalities and the facts that $p_\sigma \le p_u + 1$ and $\|v - v_h\|_{L_u,\partial K} \lesssim h_K^{\frac{1}{2}} \|v\|_{[H^1(K)]^d}$ for all $K \in \mathcal{T}_h$, leads to

$$|T_2| \lesssim \left(\|\pi z_h^\sigma\|_{L_\sigma}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F \|[\![z_h^\sigma - \pi z_h^\sigma]\!]\|_{L_\sigma,F}^2 \right)^{\frac{1}{2}} \|v\|_{[H^1(\Omega)]^d},$$

whence (4.7). □

5. Singular perturbation analysis. The situation we want to analyze in this section is that of two-field Friedrichs' systems where the off-diagonal term $\mathcal{B}^k$ coupling the σ- and u-components and the diagonal term $\mathcal{C}^k$ may have different magnitude. The model situation we have in mind is that of an elliptic/hyperbolic-type change. We show that this situation is properly accounted for by appropriately defining (tuning) the stabilizing parameters controlling the interface jumps of z_h^u.

To avoid irrelevant technicalities, we henceforth assume that (A1)–(A5) hold, i.e., full L-coercivity holds. Hypothesis (A3) can be replaced by the weaker hypotheses introduced in section 3, but these developments are omitted for brevity. The singular perturbation analysis for the three-field DG approximation will be reported elsewhere.

5.1. The setting. Let $1 \ge \epsilon > 0$ be a positive real number. The setting of section 2.2 is modified by considering the following two-field structure:

$$K = \left[\begin{array}{c|c} K^{\sigma\sigma} & K^{\sigma u} \\ \hline K^{u\sigma} & K^{uu} \end{array}\right], \qquad \mathcal{A}^k = \left[\begin{array}{c|c} 0 & \epsilon^{\frac{1}{2}} \mathcal{B}^k \\ \hline \epsilon^{\frac{1}{2}} [\mathcal{B}^k]^t & \mathcal{C}^k \end{array}\right], \tag{5.1}$$

where it is assumed that all the blocks of the operator K as well as the fields $\mathcal{B}^k$ and $\mathcal{C}^k$ are independent of the parameter ϵ.

Owing to (5.1), the definitions (2.11) and (2.12) are now replaced by

$$\mathcal{D} = \left[\begin{array}{c|c} 0 & \epsilon^{\frac{1}{2}} \mathcal{D}^{\sigma u} \\ \hline \epsilon^{\frac{1}{2}} \mathcal{D}^{u\sigma} & \mathcal{D}^{uu} \end{array}\right], \qquad \mathcal{M} = \left[\begin{array}{c|c} 0 & -\alpha\epsilon^{\frac{1}{2}} \mathcal{D}^{\sigma u} \\ \hline \alpha\epsilon^{\frac{1}{2}} \mathcal{D}^{u\sigma} & \mathcal{M}^{uu} \end{array}\right]. \tag{5.2}$$

The discrete problem we consider is (2.16) with the bilinear form a_h still defined by (2.15). As in section 2.5, we assume that p_u is a *positive* integer and that $p_u - 1 \le p_\sigma \le p_u$.

Henceforth the notation $\xi \lesssim \zeta$ now means that there is a positive c, *independent of* h *and* ϵ, such that $\xi \le c\zeta$.

5.2. Design of the boundary and jump operators. To avoid unnecessary technicalities we assume that the user-dependent operator R_F^{uu} is zero. Everything that is said hereafter extends to IP-like methods provided the assumptions (DG2E)–(DG2F) are localized. The details are left to the reader. To account for the presence of ϵ, we modify the design conditions (DG2A)–(DG2D) for the operators M_F and S_F as follows:

$$(\text{DG2}_\epsilon\text{A})\qquad M_F = \left[\begin{array}{c|c} 0 & -\alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{\sigma u} \\ \hline \alpha\epsilon^{\frac{1}{2}}\mathcal{D}^{u\sigma} & M_F^{uu}\end{array}\right],\ \alpha\in\{-1,+1\},\qquad S_F = \left[\begin{array}{c|c} 0 & 0 \\ \hline 0 & S_F^{uu}\end{array}\right];$$

$$(\text{DG2}_\epsilon\text{B})\qquad \text{if } \alpha=+1, \begin{cases} M_F^{uu} = (M_F^{uu})^* \quad \text{and} \quad \mathrm{Ker}(\mathcal{D}^{\sigma u}) \subset \mathrm{Ker}(M_F^{uu} - \mathcal{D}^{uu}), \\ \epsilon(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim h_F M_F^{uu} \lesssim \theta_F \mathcal{I}_{m_u}; \end{cases}$$

$$(\text{DG2}_\epsilon\text{C})\qquad \text{if } \alpha=-1,\ M_F^{uu}(v) = \mathcal{M}^{uu}v \text{ and } |\mathcal{D}^{uu}| \lesssim \mathcal{M}^{uu} \lesssim \mathcal{I}_{m_u};$$

$$(\text{DG2}_\epsilon\text{D})\qquad S_F^{uu} = (S_F^{uu})^* \text{ and } \epsilon(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}} + h_F|\mathcal{D}^{uu}| \lesssim h_F S_F^{uu} \lesssim \theta_F\mathcal{I}_{m_u},$$

where we have set

$$\theta_S = \max(\epsilon, h_S) \qquad \forall S \in \mathcal{F}_h \cup \mathcal{T}_h. \tag{5.3}$$

If $\epsilon \ll h$, (DG2$_\epsilon$D) amounts to $|\mathcal{D}^{uu}| \lesssim S_F^{uu} \lesssim \mathcal{I}_{m_u}$, that is, assumption (DG1E) for one-field Friedrichs' systems is recovered for the (uu)-blocks. If $\epsilon \sim 1$, (DG2$_\epsilon$D) leads to assumption (DG2D) for two-field Friedrichs' systems concerning the control on $(\mathcal{D}^{u\sigma}\mathcal{D}^{\sigma u})^{\frac{1}{2}}$ and leads to a slightly stronger control on $\mathcal{D}^{uu}$, namely, $|\mathcal{D}^{uu}| \lesssim M_F^{uu}$. The reason for this difference is that in the present analysis, we aim at obtaining a sharper convergence result for the u-component.

5.3. Convergence analysis. For all $z \in W(h)$, we introduce the following norms:

$$\|z\|_{h\epsilon,A}^2 = \|z\|_L^2 + |z^u|_J^2 + |z^u|_M^2 + \|\epsilon^{\frac{1}{2}}B_h z^u\|_{L_\sigma}^2 + \|\mathfrak{h}^{\frac{1}{2}}C_h z^u\|_{L_u}^2, \tag{5.4}$$

$$\|z\|_{h\epsilon,1}^2 = \|z\|_{h\epsilon,A}^2 + \sum_{K\in\mathcal{T}_h}[\theta_K h_K^{-1}(h_K^{-1}\|z^u\|_{L_u,K}^2 + \|z^u\|_{L_u,\partial K}^2) + h_K\|z^\sigma\|_{L_\sigma,\partial K}^2]. \tag{5.5}$$

We denote by $\mathcal{T}_h^+$ the set of mesh cells K such that $h_K \geq \epsilon$. We also denote by $\mathcal{F}_h^{i+}$ the set of faces F such that $\max_{K\in\mathcal{T}(F)} h_K \geq \epsilon$; observe that $h_F \gtrsim \epsilon$ whenever $F \in \mathcal{F}_h^{i+}$. The same definition applies for $\mathcal{F}_h^{\partial+}$.

LEMMA 5.1. *Assume $\mathcal{B}^k \in [\mathfrak{C}^{0,1}(K)]^{m_\sigma,m_u}$ and $\mathcal{C}^k \in [\mathfrak{C}^{0,\frac{1}{2}}(K)]^{m_u,m_u}$ for all $K \in \mathcal{T}_h$ and all $1 \leq k \leq d$, and that*

$$\forall z_h \in W_h, \quad \|C_h z_h^u\|_{L_u} \lesssim \|B_h z_h^u\|_{L_\sigma} + \|z_h^u\|_{L_u}. \tag{5.6}$$

Then

$$\forall z_h \in W_h, \qquad \|z_h\|_{h\epsilon,A} \lesssim \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h\epsilon,A}}. \tag{5.7}$$

Proof. Let $z_h \in W_h$ and set $\mathbb{S} = \sup_{y_h\in W_h\setminus\{0\}} \frac{a_h(z_h,y_h)}{\|y_h\|_{h\epsilon,A}}$.

(1) Using the definition of a_h together with (DG2$_\epsilon$A) and (A3) yields

$$\|z_h\|_L^2 + |z_h^u|_J^2 + |z_h^u|_M^2 \lesssim a_h(z_h,z_h) \leq \mathbb{S}\,\|z_h\|_{h\epsilon,A}. \tag{5.8}$$

(2) Define the field π_h^σ such that for all $K \in \mathcal{T}_h$, $\pi_h^\sigma|_K = \sum_{k=1}^d \epsilon^{\frac12}\overline{\mathcal{B}_K^k}\partial_k z_h^u$, where $\overline{\mathcal{B}_K^k}$ denotes the mean-value of $\mathcal{B}^k$ over K. Owing to the regularity of the $\mathcal{B}^k$'s, a standard inverse inequality, and the fact that $\epsilon \le 1$,

$$\|(\pi_h^\sigma, 0)\|_{h\epsilon,A} = \|\pi_h^\sigma\|_{L_\sigma} \lesssim \|\epsilon^{\frac12} B_h z_h^u\|_{L_\sigma} + \epsilon^{\frac12}\|z_h^u\|_{L_u} \lesssim \|z_h\|_{h\epsilon,A}. \tag{5.9}$$

From the definition of a_h and (DG2$_\epsilon$A), it follows that

$$\begin{aligned}\|\epsilon^{\frac12} B_h z_h^u\|_{L_\sigma}^2 = a_h(z_h,(\pi_h^\sigma,0)) &+ (\epsilon^{\frac12} B_h z_h^u, \epsilon^{\frac12} B_h z_h^u - \pi_h^\sigma)_{L_\sigma} \\ &- (K^{\sigma\sigma} z_h^\sigma + K^{\sigma u} z_h^u, \pi_h^\sigma)_{L_\sigma} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac{\alpha+1}{2}\epsilon^{\frac12}(\mathcal{D}^{\sigma u} z_h^u, \pi_h^\sigma)_{L_\sigma,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2\epsilon^{\frac12}(\{\mathcal{D}^{\sigma u} z_h^u\}, \{\pi_h^\sigma\})_{L_\sigma,F}.\end{aligned}$$

Then proceeding as in the proof of [16, lemma 5.5] in Part II yields

$$\|\epsilon^{\frac12} B_h z_h^u\|_{L_\sigma}^2 \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}. \tag{5.10}$$

(3) Let $\overline{\mathcal{C}_K^k}$ denote the mean-value of $\mathcal{C}^k$ over K and define the field π_h^u such that $\pi_h^u|_K = \sum_{k=1}^d h_K \overline{\mathcal{C}_K^k}\partial_k z_h^u$ if $h_K \ge \epsilon$, and $\pi_h^u|_K = 0$ otherwise. Owing to the regularity of the $\mathcal{C}^k$'s and a standard inverse inequality,

$$h_K^{-\frac12}\|\pi_h^u\|_{L_u,K} \lesssim h_K^{\frac12}\|C_h z_h^u\|_{L_u,K} + \|z_h^u\|_{L_u,K}, \tag{5.11}$$

whence it is inferred, using inverse inequalities, the fact that $h_K \ge \epsilon$ in the support of π_h^u, and the upper bounds in (DG2$_\epsilon$B)–(DG2$_\epsilon$D), that $\|(0,\pi_h^u)\|_{h\epsilon,A} \lesssim \|z_h\|_{h\epsilon,A}$. Set $\mathtt{C} = \sum_{K\in\mathcal{T}_h^+} h_K\|C_h z_h^u\|_{L_u,K}^2$. From the definition of a_h, it follows that

$$\begin{aligned}\mathtt{C} &= a_h(z_h,(0,\pi_h^u)) - (K^{u\sigma} z_h^\sigma + K^{uu} z_h^u, \pi_h^u)_{L_u} \\ &- \sum_{K\in\mathcal{T}_h^+}(\epsilon^{\frac12} B_h^\dagger z_h^\sigma, \pi_h^u)_{L_u,K} + \sum_{K\in\mathcal{T}_h^+}(C_h z_h^u, h_K C_h z_h^u - \pi_h^u)_{L_u,K} \\ &- \sum_{F\in\mathcal{F}_h^{\partial+}} \tfrac{\alpha-1}{2}\epsilon^{\frac12}(\mathcal{D}^{u\sigma} z_h^\sigma, \pi_h^u)_{L_u,F} - \sum_{F\in\mathcal{F}_h^{\partial+}} \tfrac12(M_F^{uu}(z_h^u) - \mathcal{D}^{uu} z_h^u, \pi_h^u)_{L_u,F} \\ &+ \sum_{F\in\mathcal{F}_h^{i+}} 2\epsilon^{\frac12}(\{\mathcal{D}^{u\sigma} z_h^\sigma\}, \{\pi_h^u\})_{L_u,F} + \sum_{F\in\mathcal{F}_h^{i+}} 2(\{\mathcal{D}^{uu} z_h^u\}, \{\pi_h^u\})_{L_u,F} \\ &- \sum_{F\in\mathcal{F}_h^{i+}} (S_F^{uu}([\![z_h^u]\!]), [\![\pi_h^u]\!])_{L_u,F} = a_h(z_h,(0,\pi_h^u)) + R_1 + \cdots + R_8.\end{aligned}$$

Let us estimate the remainder terms R_i, $1 \le i \le 8$, on the right-hand side. Clearly, $|R_1| \lesssim \|z_h\|_L\|\pi_h^u\|_{L_u}$. Furthermore, using an inverse inequality and the fact that $\epsilon \le h_K$ for $K \in \mathcal{T}_h^+$,

$$|R_2| \lesssim \sum_{K\in\mathcal{T}_h^+} \epsilon^{\frac12} h_K^{-1}\|z_h^\sigma\|_{L_\sigma,K}\|\pi_h^u\|_{L_u,K} \lesssim \sum_{K\in\mathcal{T}_h^+}\|z_h^\sigma\|_{L_\sigma,K} h_K^{-\frac12}\|\pi_h^u\|_{L_u,K},$$

and $|R_3| \lesssim \sum_{K\in\mathcal{T}_h^+} h_K^{\frac12}\|C_h z_h^u\|_{L_u,K}\|z_h^u\|_{L_u,K}$. If $\alpha=+1$, then $R_4=0$, while if $\alpha=-1$, then

$$
\begin{aligned}
|R_4| &\lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} \epsilon^{\frac12} h_F^{-\frac12}\|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}\\
&\lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} \|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}.
\end{aligned}
$$

Moreover, since $|\mathcal{D}^{uu}| \lesssim M_F^{uu} \lesssim \mathcal{I}_{m_u}$ for all $F\in\mathcal{F}_h^{\partial+}$ in both cases for α,

$$
|R_5| \lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} |z_h^u|_{M,F}(|\pi_h^u|_{M,F}+\|\pi_h^u\|_{L_u,F}) \lesssim \sum_{F\in\mathcal{F}_h^{\partial+}} |z_h^u|_{M,F} h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}.
$$

Similarly,

$$
|R_6|+|R_7| \lesssim \sum_{F\in\mathcal{F}_h^{i+}} (\|z_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} + |z_h^u|_{J,F}) h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)},
$$

and $|R_8| \le \sum_{F\in\mathcal{F}_h^{i+}} |z_h^u|_{J,F}|\pi_h^u|_{J,F} \lesssim \sum_{F\in\mathcal{F}_h^{i+}} |z_h^u|_{J,F} h_F^{-\frac12}\|\pi_h^u\|_{L_u,\mathcal{T}(F)}$. Collecting the above bounds and using (5.8) and (5.11), we deduce $|R_1|+\cdots+|R_8| \lesssim \gamma\mathtt{C}+a_h(z_h,z_h)$, where γ can be chosen as small as needed. Hence, $\mathtt{C} \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}$, and using (5.6) and (5.10) leads to the same bound for $\|\mathfrak{h}^{\frac12} C_h z^u\|_{L_u}^2$.

(4) Collecting the above bounds yields $\|z_h\|_{h\epsilon,A}^2 \lesssim \mathbb{S}\,\|z_h\|_{h\epsilon,A}$ and hence (5.7). ∎

LEMMA 5.2. *The following holds:*

$$
\forall (z,y_h)\in W(h)\times W_h, \quad a_h(z,y_h) \lesssim \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}. \tag{5.12}
$$

Proof. Use integration by parts to infer

$$
\begin{aligned}
a_h(z,y_h) = &\sum_{K\in\mathcal{T}_h} (z,\tilde{T}y_h)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \tfrac12 (M_F(z)+\mathcal{D}z, y_h)_{L,F}\\
&+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \tfrac12 ([\![\mathcal{D}z]\!],[\![y_h]\!])_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F^{uu}([\![z^u]\!]),[\![y_h^u]\!])_{L_u,F}.
\end{aligned}
$$

Let R_1 to R_4 be the four terms on the right-hand side. Observe that

$$
\begin{aligned}
(z,\tilde{T}y_h)_{L,K} \lesssim\ & \|z\|_{L,K}\|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K}\epsilon^{\frac12}\|B_h y_h^u\|_{L_\sigma,K}\\
&+ \|z^u\|_{L_u,K}\epsilon^{\frac12}\|B_h^\dagger y_h^\sigma\|_{L_u,K} + \|z^u\|_{L_u,K}\|C_h y_h^u\|_{L_u,K}\\
\lesssim\ & \|z\|_{L,K}\|y_h\|_{L,K} + \|z^\sigma\|_{L_\sigma,K}\epsilon^{\frac12}\|B_h y_h^u\|_{L_\sigma,K}\\
&+ \epsilon^{\frac12} h_K^{-1}\|z^u\|_{L_u,K}\|y_h^\sigma\|_{L_\sigma,K} + h_K^{-\frac12}\|z^u\|_{L_u,K} h_K^{\frac12}\|C_h y_h^u\|_{L_u,K}.
\end{aligned}
$$

Hence, $|R_1| \lesssim \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$. Furthermore, if $\alpha=+1$,

$$
\begin{aligned}
|R_2| &\le \sum_{F\in\mathcal{F}_h^\partial} [|\epsilon^{\frac12}(\mathcal{D}^{u\sigma}z^\sigma, y_h^u)_{L_u,F}| + \tfrac12|(M_F^{uu}(z^u)+\mathcal{D}^{uu}z^u, y_h^u)_{L_u,F}|]\\
&\lesssim \sum_{F\in\mathcal{F}_h^\partial} [\|z^\sigma\|_{L_\sigma,F} h_F^{\frac12}|y_h^u|_{M,F} + \theta_F^{\frac12} h_F^{-\frac12}\|z^u\|_{L_u,F}|y_h^u|_{M,F}],
\end{aligned}
$$

while if $\alpha = -1$, $|R_2| \lesssim \sum_{F\in\mathcal{F}_h^\partial}[\epsilon^{\frac{1}{2}}\|z^u\|_{L_u,F}h_F^{-\frac{1}{2}}\|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)} + \|z^u\|_{L_u,F}|y_h^u|_{M,F}]$. Hence, in both cases, $|R_2| \lesssim \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$. Similarly,

$$\begin{aligned}
\epsilon^{\frac{1}{2}}([\![\mathcal{D}^{\sigma u}z^u]\!],[\![y_h^\sigma]\!])_{L_\sigma,F} &\lesssim \epsilon^{\frac{1}{2}}\|\{z^u\}\|_{L_u,F}h_F^{-\frac{1}{2}}\|y_h^\sigma\|_{L_\sigma,\mathcal{T}(F)},\\
\epsilon^{\frac{1}{2}}([\![\mathcal{D}^{u\sigma}z^\sigma]\!],[\![y_h^u]\!])_{L_u,F} &\lesssim \|\{z^\sigma\}\|_{L_\sigma,F}h_F^{\frac{1}{2}}|y_h^u|_{J,F},\\
([\![\mathcal{D}^{uu}z^u]\!],[\![y_h^u]\!])_{L_u,F} &\lesssim \|\{z^u\}\|_{L_u,F}|y_h^u|_{J,F}.
\end{aligned}$$

Hence, $|R_3| \lesssim \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$. Finally, it is clear that $\sum_{F\in\mathcal{F}_h^{\mathrm{i}}}(S_F^{uu}([\![z^u]\!]),[\![y_h^u]\!])_{L_u,F} \leq \|z\|_{h\epsilon,1}\|y_h\|_{h\epsilon,A}$, thereby completing the proof. □

It is now straightforward to derive the following result.

THEOREM 5.3. *Keep the hypotheses of Lemma* 5.1. *Assume that* $z \in [H^1(\Omega)]^m$. *Then*

$$\|z - z_h\|_{h\epsilon,A} \lesssim \inf_{y_h\in W_h}\|z - y_h\|_{h\epsilon,1}. \tag{5.13}$$

In particular, if $z \in [H^{p_u+1}(\Omega)]^m$,

$$\|z - z_h\|_{h\epsilon,A} \lesssim \theta^{\frac{1}{2}}h^{p_u}\|z\|_{[H^{p_u+1}(\Omega)]^m}. \tag{5.14}$$

The convergence estimate in Theorem 5.3 is consistent with that from the two-field DG theory when $1 \sim \epsilon \geq h$, and it degenerates into that from the one-field theory for the u-component when $h \geq \epsilon$. Indeed, if $1 \sim \epsilon \geq h$,

$$\|z^u - z_h^u\|_{L_u} + \|B_h(z^u - z_h^u)\|_{L_\sigma} \lesssim h^{p_u}\|z\|_{[H^{p_u+1}(\Omega)]^m}, \tag{5.15}$$

and the L_u-norm error estimate can be improved if elliptic regularity holds, while if $h \geq \epsilon$,

$$\|z^u - z_h^u\|_{L_u} + \|\mathfrak{h}^{\frac{1}{2}}C_h(z^u - z_h^u)\|_{L_u} \lesssim h^{p_u+\frac{1}{2}}\|z\|_{[H^{p_u+1}(\Omega)]^m}. \tag{5.16}$$

Remark 5.1. A similar result to that of Theorem 5.3 has been proved in Gopalakrishnan and Kanschat [22, Thm. 5.1] for the advection–diffusion equation (5.17).

5.4. Example: Advection dominated advection–diffusion. Consider the advection–diffusion equation introduced in section 3.3 with a diffusion coefficient $\epsilon > 0$. The PDE $-\epsilon\Delta u + \beta\cdot\nabla u + \mu u = f$ in mixed form becomes

$$\begin{cases} \sigma + \epsilon^{\frac{1}{2}}\nabla u = 0,\\ \mu u + \epsilon^{\frac{1}{2}}\nabla\cdot\sigma + \beta\cdot\nabla u = f, \end{cases} \tag{5.17}$$

so that the off-diagonal blocks of $\mathcal{A}^k$ in (3.8) are rescaled by $\epsilon^{\frac{1}{2}}$, while the operator K is unchanged.

In the case of a Dirichlet boundary condition, the boundary and interface operators can be redesigned to fit the above analysis by modifying (3.16) as follows:

$$M_F^{uu}(v) = \eta_1(|\beta\cdot n_F| + \epsilon h_F^{-1})v, \qquad S_F^{uu}(v) = \eta_2(|\beta\cdot n_F| + \epsilon h_F^{-1})v, \tag{5.18}$$

where n_F is a unit normal vector to F and $\eta_1 > 0$, $\eta_2 > 0$ (these two parameters can vary from face to face). It is easily verified that properties (DG2$_\epsilon$B) and (DG2$_\epsilon$D) hold.

Assuming $\varrho \geq -\min(\beta\cdot n, 0)$, mixed Robin–Neumann boundary conditions can be enforced by redesigning the boundary and interface operators as follows:

$$M_F^{uu}(v) = (2\varrho + \beta\cdot n)v, \qquad S_F^{uu}(v) = \eta_2(|\beta\cdot n_F| + \epsilon h_F^{-1})v. \tag{5.19}$$

Again, it is easily verified that properties (DG2$_\epsilon$C) and (DG2$_\epsilon$D) hold.

6. Conclusion. We have analyzed various DG methods in Parts I, II, and III. We have attempted to give a unified analysis for all of these methods. Following the seminal ideas of Lesaint and Raviart [25, 26], we have shown that the framework of symmetric positive Friedrichs' systems is the natural setting for this theory insofar as boundary conditions can be enforced weakly for all of these systems. The first building block of the theory is the bilinear form (2.4) along with the weak formulation (2.6). All the DG methods that we have analyzed can be put into the unified bilinear form (2.15) and the unified formulation (2.16). The differences between all of these methods reside solely in the design of the boundary and interface operators.

The method described in Part I is the most robust in the sense that it is essentially independent of the type of the PDE (which is the main argument for our working with Friedrichs' systems). The price for robustness is the slightly suboptimal convergence rate $\mathcal{O}(h^{p+/2})$ in the L^2-norm. In Part II, two components of the unknown are identified, say (z^σ, z^u), and the generic DG method is set so that the z^σ unknown can be locally eliminated on each mesh cell. The underlying model is that of elliptic equations. Compared to the method of Part I, the convergence rate in the L^2-norm for the z^u unknown is now optimal, i.e., $\mathcal{O}(h^{p+1})$, the downside being a slight loss of robustness with respect to type change, which is addressed in Part III, section 5. Part III revisits the results of Part II in two aspects: (i) weakening of the L^2-coercivity assumption on which Part II is based; (ii) questions regarding robustness with respect to type change.

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