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Finite-element-based Faedo–Galerkin weak solutions to the Navier–Stokes equations in the three-dimensional torus are suitable

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Abstract

It is shown in this paper that Faedo–Galerkin weak solutions to the Navier–Stokes equations in the three-dimensional torus are suitable provided they are constructed using finite-dimensional spaces having a discrete commutator property and satisfying a proper inf–sup condition. Low order mixed finite element spaces appear to be acceptable for this purpose. This question was open since the notion of suitable solution was introduced.

Résumé

Dans cet article il est montré que les solutions faibles de Faedo–Galerkin des équations de Navier–Stokes, en dimension trois dans le tore, sont acceptables si elles sont construites à partir d'espaces de dimension finie possédant une propriété de commutateur discret et satisfaisant une certaine condition de compatibilité. Les espaces d'éléments finis de bas degré satisfont ces hypothèses. Cette question était ouverte depuis l'introduction de la notion de solution faible acceptable.

MSC: 35Q30; 65N35; 76M05

Keywords: Navier–Stokes equations; Suitable weak solutions; Faedo–Galerkin approximation; Finite element approximation

1. Introduction

1.1. Position of the problem

This paper is concerned with the regularity of weak solutions of the Navier–Stokes equation in the three-dimensional torus Ω:

$$\begin{cases} \partial_t u + u\cdot\nabla u + \nabla p - \nu\nabla^2 u = f & \text{in } Q_T, \\ \nabla\cdot u = 0 & \text{in } Q_T, \\ u|_{t=0} = u_0, & u \text{ is periodic}, \end{cases} \tag{1.1}$$

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where $Q_T = \Omega \times (0, T)$. Henceforth we assume $f \in L^2(0, T; [H^{-1}(\Omega)]^3)$ and $u_0 \in H = \{v \in L^2(\Omega)^3;\ \nabla \cdot v = 0;\ v \cdot n \text{ is periodic}\}$.

To the present time, the best partial regularity result is the so-called Caffarelli–Kohn–Nirenberg theorem [4,9] proving that the one-dimensional Hausdorff measure of the set of singularities of a suitable weak solution is zero. One intriguing hypothesis on which this result is based is that the weak solution must be suitable. The notion of suitable weak solution has been introduced by Scheffer [12] and boils down to the following:

Definition 1.1 *(Scheffer).* A weak solution to the Navier–Stokes equation (u, p) is suitable if $u \in L^2(0, T; [H^1(\Omega)]^3) \cap L^\infty(0, T; [L^2(\Omega)]^3)$, $p \in L^{5/4}(Q_T)$ and the local energy balance,

$$\partial_t\left(\frac{1}{2}u^2\right) + \nabla \cdot \left(\left(\frac{1}{2}u^2 + p\right)u\right) - \nu\nabla^2\left(\frac{1}{2}u^2\right) + \nu(\nabla u)^2 - f \cdot u \leqslant 0, \tag{1.2}$$

is satisfied in the distributional sense.

Although it has been proved recently by He Cheng [8] that the result of the CKN theorem also holds for weak solutions it is not known whether indeed weak solutions are suitable.

Two important questions arise a this points: (1) Are suitable weak solutions unique? (2) Are the solution constructed by the Faedo–Galerkin method suitable? (see, e.g., [1], [2, p. 77], [9, p. 245]). The purpose of the present work is to give a partial answer to the second question which seems to have been open since Scheffer introduced the notion of suitable solution. The main result of this paper is that, yes indeed, in the three-dimensional torus the Faedo–Galerkin weak solutions to the Navier–Stokes equations are suitable provided the finite-dimensional spaces involved in the construction have a discrete commutator property and satisfy a proper inf–sup condition. It is shown that, contrary to high order Fourier-based spectral methods, low order mixed finite element spaces are acceptable for this purpose.

The paper is organized as follows. In Section 2 we introduce the discrete setting and we define the Galerkin approximation to (1.1). In Section 3 we derive a priori estimates. A key estimate is derived for the pressure in Lemma 3.2. This estimate is intimately linked to the fact that we are working in the three-dimensional torus. Generalizing this estimate or a similar one with Dirichlet boundary conditions and using finite elements still seems to be challenging at the present time. The main result of this paper is reported in Section 4 where we show that the Galerkin solution converges (up to sequences) to a suitable weak solution to (1.1), see Theorem 4.1. The key to this result is that, contrary to approximation spaces based on trigonometric polynomials, finite element spaces have a discrete commutator property, see Definition 4.1.

1.2. Notations and conventions

Henceforth Ω denote the three-dimensional torus. As usual, we denote by $W^{s,p}(\Omega)$ the Sobolev spaces of functions in $L^p(\Omega)$ with partial derivatives of order up to s in $L^p(\Omega)$ when s is integer and $W^{s,p}(\Omega)$ is defined by interpolation otherwise. We do not make notational distinctions between vector- and scalar-valued functions. For $s \geqslant 1$, $W^{s,p}_\#(\Omega)$ denotes the functions in $W^{s,p}(\Omega)$ that are periodic.

In the following c is a generic constant which may depend on the data f, u_0, ν, Ω, T. The value of c may vary at each occurrence.

2. The Galerkin approximation

2.1. The discrete setting

For the time being we do not particularize the setting to the torus. Let X be a closed subspace of $[H^1(\Omega)]^3$ (think of $[H^1_0(\Omega)]^3$ if homogeneous Dirichlet boundary conditions are enforced or think of $[H^1_\#(\Omega)]^3$ if periodicity is enforced). Let $M = L^2_{\int=0}(\Omega)$, where $L^2_{\int=0}(\Omega)$ is composed of those functions in $L^2(\Omega)$ that are of zero mean.

To construct a Galerkin approximation of the Navier–Stokes equations, we assume that we have at hand two families of finite-dimensional spaces, $\{X_h\}_{h>0}$, $\{M_h\}_{h>0}$ such that $X_h \subset X$ and $M_h \subset M$. The velocity is approximated in X_h and the pressure in M_h. To avoid irrelevant technicalities we assume $M_h \subset H^1(\Omega) \cap M$.

Let $\pi_h : [L^2(\Omega)]^3 \to X_h$ be the L^2-projection onto X_h. We assume that X_h and M_h are compatible in the sense that there is $c > 0$ independent of h such that

$$\forall q_h \in M_h, \quad \|\pi_h \nabla q_h\|_{L^2} \geqslant c\|\nabla q_h\|_{L^2}. \tag{2.1}$$

A first consequence of this hypothesis is that X_h and M_h satisfy the so-called LBB condition, see, e.g., [7]. That is to say:

Lemma 2.1. *Assume that* (2.1) *holds, X_h and M_h are such that $(q_h, \nabla \cdot v_h) = -(\nabla q_h, v_h)$ for all $q_h \in M_h$ and all $v_h \in X_h$, and there exists $\mathcal{C}_h : [H^1(\Omega)]^3 \to X_h$ an H^1-stable interpolation operator such that $\|\mathcal{C}_h v - v\|_{L^2} \leqslant ch\|v\|_{H^1}$ for all $v \in [H^1(\Omega)]^3$, then there is a constant c independent of h such that*

$$\inf_{0 \neq q_h \in M_h} \sup_{0 \neq v_h \in X_h} \frac{(q_h, \nabla \cdot v_h)}{\|q_h\|_{L^2}\|v_h\|_{H^1}} \geqslant c. \tag{2.2}$$

Proof. See Appendix A.1. The operator $\mathcal{C}_h$ can be, e.g., the Clément interpolation operator [6] or the Scott–Zhang operator [13]. □

Lemma 2.2. *Hypothesis* (2.1) *holds in either one of the following situations*:

(i) *X_h is composed of $\mathbb{P}_1$-Bubble H^1-conforming finite elements and M_h is composed of $\mathbb{P}_1$ H^1-conforming finite elements* (*i.e., the so-called MINI element*).
(ii) *X_h is composed of $\mathbb{P}_2$ H^1-conforming finite elements and M_h is composed of $\mathbb{P}_1$ H^1-conforming finite elements* (*i.e., the so-called Hood–Taylor element*)*, and no tetrahedron has more than* 3 *edges on $\partial\Omega$.*

Proof. See Appendix A.2. □

We now particularize the functional setting to the torus. We assume that $X = [H^1_\#(\Omega)]^3$, i.e., $X_h \subset [H^1_\#(\Omega)]^3$, and to minimize technicalities we assume $M_h \subset H^1_\#(\Omega) \cap L^2_{\int=0}(\Omega)$. Moreover, we assume that there is an interpolation operator $\mathcal{J}_h : H^2_\#(\Omega) \to M_h$ such that for all $q \in H^2_\#(\Omega)$,

$$\big\|\nabla(q - \mathcal{J}_h q)\big\|_{L^2} \leqslant ch\|q\|_{H^2}. \tag{2.3}$$

We also make the following key hypotheses: There is c independent of h such that for all $v \in [H^1_\#(\Omega)]^3$,

$$\|v - \pi_h v\|_{L^2} = \inf_{w_h \in X_h} \|v - w_h\|_{L^2} \leqslant ch\|v\|_{H^1}, \tag{2.4}$$

$$\|\pi_h v\|_{H^1} \leqslant c\|v\|_{H^1}. \tag{2.5}$$

In addition to the above interpolation properties, we assume that the following inverse inequality holds in X_h: There is $c > 0$ independent of h such that

$$\|v_h\|_{H^1} \leqslant c\,h^{-1}\|v_h\|_{L^2}, \quad \forall v_h \in X_h. \tag{2.6}$$

Remark 2.1. (i) The above interpolation and stability results (2.4), (2.5) hold only with periodic boundary conditions. In the case of Dirichlet boundary conditions, i.e., $X_h \subset [H^1_0(\Omega)]^3$, the above results are not true; in this case we only have $\|v - \pi_h v\|_{L^2} \leqslant ch^{1/2}\|v\|_{H^1}$ and $\|\pi_h v\|_{H^1} \leqslant ch^{-1/2}\|v\|_{H^1}$ for all $v \in [H^1(\Omega)]^3$. This limitation is the main obstacle to the extension of the results stated in the remainder of the paper to more general boundary conditions.

(ii) The inequality (2.6) holds whenever the family of spaces $\{X_h\}_{h>0}$ is composed of finite element spaces based on mesh families that are quasi-uniform, see, e.g., [5].

We define the map $\psi_h : H^2_\#(\Omega) \to M_h$ such that for all q in $H^2_\#(\Omega)$, $\psi_h(q)$ solves:

$$\big(\pi_h \nabla \psi_h(q), \nabla r_h\big) = (\nabla q, \nabla r_h), \quad \forall r_h \in M_h. \tag{2.7}$$

Observe that the above problem has a unique solution since the bilinear form $(\pi_h \nabla q_h, \nabla r_h)$ is coercive owing to hypothesis (2.1).

Lemma 2.3. *There exists $c > 0$ independent of h such that for all q in $H^2_\#(\Omega)$,*

$$\left\|\nabla\big(\psi_h(q) - q\big)\right\|_{L^2} \leqslant c\,h\|q\|_{H^2}, \tag{2.8}$$

$$\left\|\pi_h \nabla\psi_h(q)\right\|_{H^1} \leqslant c\,\|q\|_{H^2}. \tag{2.9}$$

Proof. See Appendix A.3. □

2.2. The discrete problem

Denote by V the closed subspace of $[H^1_\#(\Omega)]^3$ that is composed of the vector fields in $[H^1_\#(\Omega)]^3$ that are solenoidal. Define the space:

$$V_h = \big\{v_h \in X_h;\ (\nabla\cdot v_h, q_h) = 0,\ \forall q_h \in L^2(\Omega)\big\}. \tag{2.10}$$

Since V_h is not a subspace of V, i.e., V_h is not composed of solenoidal vector-fields, we modify the nonlinear term as follows. We introduce a bilinear operator $nl_h \in \mathcal{L}(([H^1_\#(\Omega)]^3)^2; [H^{-1}_\#(\Omega)]^3)$. We assume that nl_h satisfies the following continuity property:

$$\left\|nl_h(v,v)\right\|_{H^{-1}} \leqslant c\|v\|_{H^1}\|v\|_{L^3}. \tag{2.11}$$

We define the trilinear form $b_h \in \mathcal{L}(([H^1_0(\Omega)]^3)^3; \mathbb{R})$ such that $b_h(u,v,w) = \langle(nl_h(u,v), w\rangle_{H^{-1},H^1}$. We assume that b_h satisfies the following property:

$$b_h(u,v,v) = 0, \quad \forall v \in V + V_h. \tag{2.12}$$

For instance, an admissible form of the nonlinear term is as follows (see, e.g., [14]),

$$nl_h(u,v) = u\cdot\nabla v + \frac{1}{2}v\nabla\cdot u. \tag{2.13}$$

Let $\mathcal{K}_h : L^2(\Omega) \to M_h$ be a linear L^2-stable interpolation operator (i.e., $\mathcal{K}_h z \to z$ for all $z \in L^2(\Omega)$), then another admissible form of the nonlinear term is:

$$nl_h(u,v) = (\nabla\times u)\times v + \frac{1}{2}\nabla\big(\mathcal{K}_h(u\cdot v)\big). \tag{2.14}$$

The discrete problem we henceforth consider is as follows: Seek $u_h \in \mathcal{C}^0([0,T]; X_h)$ with $\partial_t u_h \in L^2(0,T; X_h)$ and $p_h \in L^2([0,T]; M_h)$ such that for all $v_h \in X_h$, all $q_h \in M_h$, a.e. $t \in [0,T]$:

$$\begin{cases} (\partial_t u_h, v_h) + b_h(u_h, u_h, v_h) - (p_h, \nabla\cdot v_h) + \nu(\nabla u_h, \nabla v_h) = \langle f, v_h\rangle, \\ (\nabla\cdot u_h, q) = 0, \\ u_h|_{t=0} = \mathcal{I}_h u_0, \end{cases} \tag{2.15}$$

where $\mathcal{I}_h : L^2(\Omega) \to V_h$ is a L^2-stable interpolation operator; that is to say, $\mathcal{I}_h z \to z$ for all $z \in [L^2(\Omega)]^3$ (actually, weak convergence is enough). Note that for all v_h in X_h the approximate momentum equation holds in $L^2(0,T)$.

3. A priori estimates

3.1. Energy estimates

Owing to (2.12), we have the usual a priori energy estimates on u_h, namely

$$\max_{0\leqslant t\leqslant T}\left\|u_h(t)\right\|_{L^2} + \|u_h\|_{L^2(H^1)} \leqslant c, \tag{3.1}$$

from which we deduce the following:

Lemma 3.1. *Under the above assumptions on f and u_0, there is c, independent of h, such that*

$$\|u_h\|_{L^r(H^{2/r})} + \|u_h\|_{L^r(L^q)} \leqslant c, \quad \text{with } \frac{3}{q} + \frac{2}{r} = \frac{3}{2},\ 2 \leqslant r,\ 2 \leqslant q \leqslant 6. \tag{3.2}$$

Proof. This result is standard and is a consequence of the interpolation inequality (see, e.g., Lions and Peetre [11]), $\|v\|_{H^{2/r}} \leqslant c\,\|v\|_{L^2}^{1-2/r}\,\|v\|_{H^1}^{2/r}$, when $2 \leqslant r$, and the embedding $H^{2/r}(\Omega) \subset L^q(\Omega)$ for $1/q = 1/2 - 2/(3r)$. $\square$

3.2. Pressure estimate

Now we want to deduce *a priori* estimates on the pressure p_h. The main tool we are going to use is a duality argument. We define $q = (-\nabla^2)^{-1} p_h$ and we test the momentum equation with $\pi_h \nabla(\psi_h(q))$.

Lemma 3.2. *Under the above assumptions, there is c, independent of h, such that*

$$\|p_h\|_{L^{4/3}(0,T;L^2)} \leqslant c. \tag{3.3}$$

Proof. (1) Let $q \in H^2_\#(\Omega)$ solve:

$$(\nabla q, \nabla \phi) = (p_h, \phi), \quad \forall \phi \in H^1_\#(\Omega).$$

Owing to standard regularity results,

$$\|q\|_{H^2} \leqslant c\|p_h\|_{L^2}. \tag{3.4}$$

(2) Let us test the momentum equation with $\pi_h \nabla(\psi_h(q))$; note that $\pi_h \nabla(\psi_h(q))$ is an admissible test function since $\pi_h \nabla(\psi_h(q)) \in X_h$.

(3) We first take care of the pressure term. The definition of q together with that of $\psi_h(q)$ yield:

$$-\big(p_h, \nabla \cdot \big(\pi_h \nabla\big(\psi_h(q)\big)\big)\big) = \big(\nabla p_h, \pi_h \nabla\big(\psi_h(q)\big)\big) = (\nabla p_h, \nabla q) = \|p_h\|^2_{L^2}. \tag{3.5}$$

(4) The contribution of the time derivative is zero since

$$\big(\partial_t u_h, \pi_h \nabla\big(\psi_h(q)\big)\big) = \big(\partial_t u_h, \nabla\big(\psi_h(q)\big)\big) = -\big(\nabla \cdot (\partial_t u_h), \psi_h(q)\big) = 0, \tag{3.6}$$

owing to the fact that $\partial_t u_h \in V_h$ and $\psi_h(q) \in M_h$.

(5) We take care of the viscous term as follows. Using the stability estimate (2.9) we infer:

$$\big|\big(\nabla u_h, \nabla\big(\pi_h \nabla\big(\psi_h(q)\big)\big)\big)\big| \leqslant \|\nabla u_h\|_{L^2} \big\|\pi_h \nabla\big(\psi_h(q)\big)\big\|_{H^1} \leqslant c\|\nabla u_h\|_{L^2}\|q\|_{H^2}.$$

Then the stability estimate (3.4) implies:

$$\big(\nabla u_h, \nabla\big(\pi_h \nabla\big(\psi_h(q)\big)\big)\big) \leqslant c\|\nabla u_h\|_{L^2}\|p_h\|_{L^2}. \tag{3.7}$$

(6) For the nonlinear term we proceed as follows:

$$\big|b_h\big(u_h, u_h, \psi_h(q)\big)\big| = \big|\big\langle nl_h(u_h, u_h), \pi_h \nabla \psi_h(q)\big\rangle\big| \leqslant \big\|nl_h(u_h, u_h)\big\|_{H^{-1}} \big\|\pi_h \nabla \psi_h(q)\big\|_{H^1}.$$

Using the bound (2.11) together with the estimates (2.9), (3.4), we obtain:

$$\big|b_h\big(u_h, u_h, \psi_h(q)\big)\big| \leqslant c\|u_h\|_{L^3}\|u_h\|_{H^1}\|p_h\|_{L^2}. \tag{3.8}$$

(7) We proceed similarly as above for the source term,

$$\big|\big\langle f, \pi_h \nabla\big(\psi_h(q)\big)\big\rangle\big| \leqslant \|f\|_{H^{-1}} \big\|\pi_h \nabla\big(\psi_h(q)\big)\big\|_{H^1} \leqslant c\|f\|_{H^{-1}}\|q\|_{H^2}.$$

That is to say:

$$\big|\big\langle f, \pi_h \nabla\big(\psi_h(q)\big)\big\rangle\big| \leqslant c\|f\|_{H^{-1}}\|p_h\|_{L^2}. \tag{3.9}$$

(8) Combining (3.5)–(3.9), we deduce:

$$\|p_h\|^2_{L^2} \leqslant c\big(\nu\|u_h\|_{H^1} + \|u_h\|_{L^3}\|u_h\|_{H^1} + \|f\|_{H^{-1}}\big)\|p_h\|_{L^2}.$$

Then, as a consequence of the bound (3.2), we infer:

$$\int_0^T \|p_h\|^{4/3}_{L^2} \leqslant c\left(\int_0^T \|u_h\|^4_{L^3} + \|u_h\|^2_{H^1} + \|f\|^2_{H^{-1}}\right) \leqslant c.$$

This completes the proof. $\square$

3.3. Estimate on $\partial_t u_h$

As a consequence of Lemma 3.2 we infer:

Corollary 3.1. *Under the above assumptions, there is c independent of h such that*

$$\|\partial_t u_h\|_{L^{4/3}(0,T;H^{-1})} \leqslant c. \tag{3.10}$$

Proof. Using the H^1-stability of π_h, we infer:

$$\|\partial_t u_h\|_{H^{-1}} = \sup_{v\in[H^1_\#(\Omega)]^3} \frac{(\partial_t u_h, v)}{\|v\|_{H^1}} = \sup_{v\in[H^1_\#(\Omega)]^3} \frac{(\partial_t u_h, \pi_h v)}{\|v\|_{H^1}} \leqslant c \sup_{v\in[H^1_\#(\Omega)]^3} \frac{(\partial_t u_h, \pi_h v)}{\|\pi_h v\|_{H^1}} \leqslant c \sup_{v_h\in X_h} \frac{(\partial_t u_h, v_h)}{\|v_h\|_{H^1}}$$
$$\leqslant c\big(\nu\|u_h\|_{H^1} + \big\|nl_h(u_h,u_h)\big\|_{H^{-1}} + \|p_h\|_{L^2} + \|f\|_{H^{-1}}\big).$$

Using the bound (2.11), we deduce:

$$\|\partial_t u_h\|_{H^{-1}} \leqslant c\big(\nu\|u_h\|_{H^1} + \|u_h\|_{L^3}\|u_h\|_{H^1} + \|p_h\|_{L^2} + \|f\|_{H^{-1}}\big).$$

Then, the conclusion follows readily as a consequence of the bound (3.2) together with the pressure estimate (3.3). □

3.4. Convergence to a weak solution

Before proving that subsequences of (u_h) converge to a weak solution, we make sure that we are solving the right problem, i.e., we now formulate consistency hypotheses on the nonlinear term.

In this section s denote a real number such that $4 < s < \infty$. We denote by s' and s^* the two real numbers such that $1/s + 1/s' = 1$ and $1/s + 1/s^* = 1/2$, respectively. We assume that the nonlinear term has the following consistency property: For all functions w in $L^2(0,T;V) \cap L^4(0,T;[L^3(\Omega)]^3)$ and all sequences of functions $(w_h)_{h>0}$ in $\mathcal{C}^0([0,T];X_h)$ converging weakly to w in $L^2(0,T;[H^1_\#(\Omega)]^3)$ and strongly in $L^{s^*}(0,T;[L^3(\Omega)]^3)$, the following holds:

$$nl_h(w_h,w_h) \rightharpoonup w\cdot\nabla w, \quad \text{in } L^{s'}\big(0,T;\big[H^{-1}_\#(\Omega)\big]^3\big). \tag{3.11}$$

Lemma 3.3. *The consistency property* (3.11) *holds for definition* (2.13) *and for definition* (2.14).

Proof. Let v be a function in $L^s(0,T;[H^1_\#(\Omega)]^3)$.

(1) Assume that nl_h is defined as in (2.13). Observing that $v \in L^s(0,T;[L^6(\Omega)]^3)$, we deduce that $v\otimes w_h \to v\otimes w$ and $v\cdot w_h \to v\cdot w$ in $L^2(0,T;[L^2(\Omega)]^{3\times3})$ and $L^2(0,T;L^2(\Omega))$, respectively. As a result $\int_0^T (v\otimes w_h, \nabla w_h) \to \int_0^T (v\otimes w, \nabla w)$ and $\int_0^T (v\cdot w_h, \nabla\cdot w_h) \to \int_0^T (v\cdot w, \nabla\cdot w)$. Moreover, since $\nabla\cdot w = 0$, a.e. in Q_T, we infer $\int_0^T (v\cdot w_h, \nabla\cdot w_h) \to 0$. The conclusion follows readily.

(2) Assume that nl_h is defined as in (2.14). The only term that poses a difficulty is $\int_0^T (\nabla(\mathcal{K}_h(|w_h|^2), v)$. Integrating by parts, we rewrite this term as follows $-\int_0^T (\mathcal{K}_h(|w_h|^2), \nabla\cdot v)$. Banach–Steinhaus theorem implies that $\|\mathcal{K}_h\|$ is uniformly bounded, then using linearity:

$$\big\|\mathcal{K}_h\big(|w_h|^2\big) - \mathcal{K}_h\big(|w|^2\big)\big\|_{L^{s'}(L^2)} \leqslant c\big\||w_h|^2 - |w|^2\big\|_{L^{s'}(L^2)} \leqslant c\big\|(w_h - w)\cdot(w_h + w)\big\|_{L^{s'}(L^2)}$$
$$\leqslant c\|w_h - w\|_{L^{s^*}(L^3)}\big(\|w_h\|_{L^2(L^6)} + \|w\|_{L^2(L^6)}\big).$$

In the last inequality we used the fact $1/s^* + 1/2 = 1/s'$. Note that $\|w_h\|_{L^2(L^6)}$ is bounded since w_h converges weakly to w in $L^2(0,T;L^6(\Omega))$. The above inequality implies $\mathcal{K}_h(|w_h|^2) \to \mathcal{K}_h(|w|^2)$ in $L^{s'}(0,T;L^2(\Omega))$. Moreover, $\mathcal{K}_h(|w|^2) \to |w|^2$ in $L^2(\Omega)$ a.e. on $(0,T)$, $\|\mathcal{K}_h(|w|^2)\|^{s'}_{L^2}$ is uniformly bounded by $c\||w|^2\|^{s'}_{L^2} \in$

$L^1(0,T)$; hence, Lebesgues' Dominated Convergence theorem implies $\mathcal{K}_h(|w|^2) \to |w|^2$ in $L^{s'}(0,T;L^2(\Omega))$. As a result we obtain $-\int_0^T(\mathcal{K}_h(|w_h|^2), \nabla\cdot v) \to -\int_0^T(|w|^2, \nabla\cdot v)$. Finally,

$$\int_0^T \langle nl_h(w_h,w_h), v\rangle \to \int_0^T \left((\nabla\times w)\times w + \frac{1}{2}\nabla\big(|w|^2\big), v\right) = \int_0^T (w\cdot\nabla w, v).$$

Hence (3.11) holds since v is arbitrary. □

We have the following classical result:

Corollary 3.2. *Under the above hypotheses, u_h convergences, up to subsequences, to a weak solution to* (1.1) *in $L^2(0,T;[H^1_\#(\Omega)]^3)$ weak and in any $L^r(0,T;L^q(\Omega)^3)$ strong* ($1 \leqslant q < 6r/(3r-4)$, $2 \leqslant r < \infty$)*, and, up to subsequences, p_h converges to p in $L^{4/3}(0,T;L^2(\Omega))$ weak.*

Proof. We briefly outline the main steps of the proof for the arguments are quite standard.

(1) Since u_h is uniformly bounded in $L^2(0,T;[H^1_\#(\Omega)]^3)$ and $\partial_t u_h$ is bounded uniformly in $L^{4/3}(0,T;[H^{-1}_\#(\Omega)]^3)$, the Aubin–Lions Compactness lemma (see Lions [10, p. 57] or [15]) implies that there exists a subsequence (u_{h_l}) converging weakly in $L^2(0,T;[H^1_\#(\Omega)]^3)$ and strongly in any $L^r(0,T;L^q(\Omega))$, such that $1 \leqslant q < 6r/(3r-4)$, $2 \leqslant r < \infty$, and that $(\partial_t u_{h_l})$ converges weakly in $L^{4/3}(0,T;[H^{-1}_\#(\Omega)]^3)$. Moreover, since (p_h) is bounded uniformly in $L^{4/3}(0,T;L^2(\Omega))$, there exists a subsequence (p_{h_l}) converging weakly in $L^{4/3}(0,T;L^2(\Omega))$. Let u and p denote these limits.

(2) Let $q \in L^2(0,T;L^2(\Omega))$ and let $(q_{h_l})_{h_l>0}$ be a sequence of functions in $L^2(0,T;M_h)$ strongly converging to q in $L^2(0,T;L^2(\Omega))$. Then $0 = \int_0^T(\nabla\cdot u_{h_l}, q_{h_l}) \to \int_0^T(\nabla\cdot u, q)$ since $\nabla\cdot u_{h_l} \rightharpoonup \nabla\cdot u$ in $L^2(0,T;L^2(\Omega))$. As a result, $\nabla\cdot u = 0$, a.e. in Q_T; that is to say u is in $L^2(0,T;V)$.

(3) Let s be a real number such that $4 < s < \infty$. Let v be any function in $L^s(0,T;[H^1_\#(\Omega)]^3)$ and let $(v_{h_l})_{h_l>0}$ be a sequence of functions in $L^s(0,T;X_h)$ strongly converging to v in $L^s(0,T;[H^1_\#(\Omega)]^3)$. Then

(4) $\int_{Q_T}\partial_t u_{h_l}\cdot v_{h_l} \to \int_{Q_T}\partial_t u\cdot v$, since $\partial_t u_{h_l} \rightharpoonup \partial_t u$ in $L^{4/3}(0,T;[H^{-1}_\#(\Omega)]^3)$.

(5) $\int_{Q_T}\nabla u_{h_l}{:}\nabla v_{h_l} \to \int_{Q_T}\nabla u{:}\nabla v$, since $\nabla u_{h_l} \rightharpoonup \nabla u$ in $L^2(0,T;[L^2(\Omega)]^3) \subset L^{4/3}(0,T;[L^2(\Omega)]^3)$.

(6) $\int_{Q_T} p_{h_l}\nabla\cdot v_{h_l} \to \int_{Q_T} p\nabla\cdot v$, since $p_{h_l} \rightharpoonup p$ in $L^{4/3}(0,T;L^2(\Omega))$.

(7) Since u_{h_l} converges weakly to u in $L^2(0,T;[H^1_\#(\Omega)]^3)$ and strongly in $L^{s^*}(0,T;[L^3(\Omega)]^3)$, the hypotheses of (3.11) hold; hence, $\int_0^T b_h(u_{h_l}, u_{h_l}, v_{h_l}) \to \int_0^T(u\cdot\nabla u, v)$.

(8) Finally, since u_{h_l} converges in $\mathcal{C}^0([0,T];L^2_w(\Omega))$ (functions that are continuous over $[0,T]$ with value in $L^2(\Omega)$ equipped with the weak topology) $u_0 \leftarrow \mathcal{I}_{h_l}u_0 = u_{h_l}(0) \rightharpoonup u(0)$ in $L^2(\Omega)$; hence, $u(0) = u_0$.

(9) That u satisfies Leray's energy inequality is standard. It is a consequence of the inequality $2\nabla(u_{h_l}-u){\cdot}\nabla u + |\nabla u|^2 \leqslant |\nabla u_{h_l}|^2$. The theorem is proved. □

4. Convergence to a suitable solution

The main issue we address in the present work is to determine whether weak solutions are suitable in the sense of Definition 1.1. To answer this question we assume that the discrete framework satisfies the following property that we henceforth refer to as the discrete commutator property (see Bertoluzza [3]).

Definition 4.1. We say that X_h (resp. M_h) has the discrete commutator property if there is an operator $P_h \in \mathcal{L}([H^1_\#(\Omega)]^3; X_h)$ (resp. $Q_h \in \mathcal{L}(L^2(\Omega); M_h)$) such that for all ϕ in $W^{2,\infty}_\#(\Omega)$ (resp. all ϕ in $W^{1,\infty}_\#(\Omega)$) and all $v_h \in X_h$ (resp. all $q_h \in M_h$),

$$\big\|\phi v_h - P_h(\phi v_h)\big\|_{H^l} \leqslant c h^{1+m-l}\|v_h\|_{H^m}\|\phi\|_{W^{m+1,\infty}}, \quad 0 \leqslant l \leqslant m \leqslant 1,$$
$$\big\|\phi q_h - Q_h(\phi q_h)\big\|_{L^2} \leqslant c h\|q_h\|_{L^2}\|\phi\|_{W^{1,\infty}}.$$

Remark 4.1. Fourier-based approximation spaces do not have the discrete commutator property since Fourier series do not have local interpolation properties. Fourier series are very accurate but they only have global interpolation properties.

We also assume that the following consistency property holds for the nonlinear term: For all functions w in $L^2(0,T;V)\cap L^\infty(0,T;[L^2(\Omega)]^3)$ and all sequences of functions $(w_h)_{h>0}$ in $\mathcal{C}^0([0,T];X_h)$ uniformly bounded in $L^2(0,T;[H^1_\#(\Omega)]^3)\cap L^\infty(0,T;[L^2(\Omega)]^3)$ and strongly converging to w in $L^{s^*}(0,T;[L^3(\Omega)]^3)$, where $3\leqslant s^*<4$ (i.e., $4<s\leqslant 6$), the following holds:

$$\int_0^T b_h(w_h,w_h,\phi w_h)\to -\int_0^T\left(\frac{1}{2}w|w|^2,\nabla\phi\right),\quad \forall\phi\in\mathcal{D}\big(0,T;\mathcal{C}^\infty_\#\big(\overline{\Omega}\big)\big). \tag{4.1}$$

Lemma 4.1. *The consistency property* (4.1) *holds for definition* (2.13) *and also for definition* (2.14) *provided M_h has the discrete commutator property.*

Proof. (1) The situation for Definition (2.13) is quite simple since

$$\begin{aligned}b_h(w_h,w_h,\phi w_h)&=\Big(w_h\cdot\nabla w_h,\phi w_h\Big)+\frac{1}{2}(w_h\nabla\cdot w_h,\phi w_h)=\left(w_h\cdot\nabla\left(\frac{1}{2}|w_h|^2\right)+\frac{1}{2}|w_h|^2\nabla\cdot w_h,\phi\right)\\&=\left(\nabla\cdot\left(w_h\frac{1}{2}|w_h|^2\right),\phi\right)=-\left(w_h\frac{1}{2}|w_h|^2,\nabla\phi\right).\end{aligned}$$

Then, clearly $\int_0^T b_h(w_h,w_h,\phi w_h)\to-\int_0^T(\frac{1}{2}w|w|^2,\nabla\phi)$ since $w_h\frac{1}{2}|w_h|^2\to w\frac{1}{2}|w|^2$ in $L^{s^*/3}(0,T;L^1(\Omega))\subset L^1(Q_T)$.

(2) For definition (2.14) we proceed as follows:

$$\begin{aligned}b_h(w_h,w_h,\phi w_h)&=\big((\nabla\times w_h)\times w_h,\phi w_h\big)+\frac{1}{2}\big(\nabla\big(\mathcal{K}_h\big(|w_h|^2\big),\phi w_h\big)\big)=-\frac{1}{2}\big(\mathcal{K}_h\big(|w_h|^2\big),\nabla\cdot(\phi w_h)\big)\\&=-\frac{1}{2}\big(w_h\mathcal{K}_h\big(|w_h|^2\big),\nabla\phi\big)-\frac{1}{2}\big(\phi\mathcal{K}_h\big(|w_h|^2\big),\nabla\cdot w_h\big)=-\frac{1}{2}\big(w_h|w_h|^2\nabla\phi\big)+R_1+R_2,\end{aligned}$$

where $R_1=-\frac{1}{2}(w_h(\mathcal{K}_h(|w_h|^2)-|w_h|^2),\nabla\phi)$ and $R_2=-\frac{1}{2}(\phi\mathcal{K}_h(|w_h|^2),\nabla\cdot w_h)$. By using the same arguments as in the second part of the proof of Lemma 3.3 we infer $\mathcal{K}_h(|w_h|^2)\to|w|^2$ in $L^{s'}(0,T;L^2(\Omega))$; that is to say, $\mathcal{K}_h(|w_h|^2)-|w_h|^2\to 0$ in $L^{s'}(0,T;L^2(\Omega))$. Since $w_h\cdot\nabla\phi\to w\cdot\nabla\phi$ in $L^s(0,T;L^2(\Omega))$, we infer $\int_0^T|R_1|\to 0$ as $h\to 0$. For R_2 we use the fact that M_h has the discrete commutator property as follows:

$$\begin{aligned}|R_2|&=\frac{1}{2}\big|\big(\phi\mathcal{K}_h\big(|w_h|^2\big)-Q_h\big(\phi\mathcal{K}_h\big(|w_h|^2\big)\big),\nabla\cdot w_h\big)\big|\leqslant\frac{1}{2}\big\|\phi\mathcal{K}_h\big(|w_h|^2\big)-Q_h\big(\phi\mathcal{K}_h\big(|w_h|^2\big)\big)\big\|_{L^2}\|w_h\|_{H^1}\\&\leqslant ch\big\|\mathcal{K}_h\big(|w_h|^2\big)\big\|_{L^2}\|w_h\|_{H^1}\leqslant ch\big\||w_h|^2\big\|_{L^2}\|w_h\|_{H^1}\leqslant ch\|w_h\|^2_{L^4}\|w_h\|_{H^1}\leqslant ch\|w_h\|^{1/2}_{L^2}\|w_h\|^{3/2}_{L^6}\|w_h\|_{H^1}\\&\leqslant ch\|w_h\|^{1/2}_{L^2}\|w_h\|^{1/2}_{H^1}\|w_h\|^2_{H^1}\leqslant ch^{1/2}\|w_h\|_{L^2}\|w_h\|^2_{H^1}.\end{aligned}$$

Hence

$$\int_0^T|R_2|\leqslant ch^{1/2}\|w_h\|^2_{L^2(H^1)}\|w_h\|_{L^\infty(L^2)}.$$

Then clearly $\int_0^T|R_2|\to 0$ as $h\to 0$. In conclusion $\int_0^T b_h(w_h,w_h,\phi w_h)\to-\int_0^T(\frac{1}{2}w|w|^2,\nabla\phi)$ since $w_h\frac{1}{2}|w_h|^2\to w\frac{1}{2}|w|^2$ in $L^{s^*/3}(0,T;L^1(\Omega))\subset L^1(Q_T)$ and $\int_0^T|R_1|+\int_0^T|R_2|\to 0$. That concludes the proof. □

The main result of the paper is stated in the following theorem:

Theorem 4.1. *Under the aboves hypotheses, if X_h and M_h have the discrete commutator property, the couple (u_h,p_h) convergences, up to subsequences, to a suitable solution to* (1.1)*, say (u,p).*

Proof. To alleviate notations we still denote by (u_h) and (p_h) the subsequences that converge to u and p, respectively.

(1) Let ϕ be a non-negative function in $\mathcal{D}(0,T;\mathcal{C}^\infty_\#(\overline{\Omega}))$. Testing the momentum equation in (2.15) by $P_h(u_h\phi)$, we obtain:

$$\big(\partial_t u_h, P_h(u_h\phi)\big) + b_h\big(u_h, u_h, P_h(u_h\phi)\big) - \big(p_h, \nabla\cdot P_h(u_h\phi)\big) + \nu\big(\nabla u_h, \nabla P_h(u_h\phi)\big) - \big(f, P_h(u_h\phi)\big) = 0.$$

Each of the terms on the left-hand side of the equation are now treated separately in the following steps:

(2) For the time derivative we have:

$$\int_0^T \big(\partial_t u_h, P_h(u_h\phi)\big) = \int_0^T (\partial_t u_h, u_h\phi) + \int_0^T R = -\frac{1}{2}\int_0^T \big(|u_h|^2, \partial_t\phi\big) + \int_0^T R,$$

where we have set $R = (u_{h,t}, P_h(u_h\phi) - u_h\phi)$. It is clear that $-\frac{1}{2}\int_0^T (|u_h|^2, \partial_t\phi) \to -\frac{1}{2}\int_0^T (u^2, \partial_t\phi)$ since $|u_h|^2 \to |u|^2$ in $L^r(L^1)$ for any $1 \leqslant r < \infty$. To control the residual we use the discrete commutator property and the inverse inequality (2.6) as follows:

$$\begin{aligned}\int_0^T |R| &= \int_0^T \big(u_{h,t}, P_h(u_h\phi) - u_h\phi\big) \leqslant \int_0^T \|u_{h,t}\|_{H^{-1}} \big\|P_h(u_h\phi) - u_h\phi\big\|_{H^1} \\ &\leqslant ch\|u_{h,t}\|_{L^{4/3}(H^{-1})}\|u_h\|_{L^4(H^1)} \leqslant ch^{1/2}\|u_{h,t}\|_{L^{4/3}(H^{-1})}\|u_h\|^{1/2}_{L^\infty(L^2)}\|u_h\|^{1/2}_{L^2(H^1)}.\end{aligned}$$

Now, it is clear that $\int_0^T |R| \to 0$ as $h \to 0$.

(3) Using the fact that u_h is periodic and the first derivatives of ϕ are also periodic, the viscous term yields:

$$\big(\nabla u_h, \nabla P_h(u_h\phi)\big) = \big(\nabla u_h, \nabla(u_h\phi)\big) + R = \big(|\nabla u_h|^2, \phi\big) - \left(\frac{1}{2}|u_h|^2, \nabla^2\phi\right) + R$$

where $R = (\nabla u_h, P_h(u_h\phi) - u_h\phi)$. For the first term we proceed as follows:

$$\int_0^T \big(|\nabla u_h|^2, \phi\big) = \int_0^T \big(\big|\nabla(u_h - u + u)\big|^2, \phi\big) = \int_0^T \big(\big|\nabla(u_h - u)\big|^2 + 2\nabla(u_h - u) : \nabla u + |\nabla u|^2, \phi\big).$$

Since $u_h \rightharpoonup u$ in $L^2(0,T;H^1)$ and ϕ is non-negative, we infer $\liminf \int_0^T (|\nabla u_h|^2, \phi) \geqslant \int_0^T (|\nabla u|^2, \phi)$. For the second term we have $\int_0^T -(\frac{1}{2}|u_h|^2, \nabla^2\phi) \to -\int_0^T (\frac{1}{2}|u|^2, \nabla^2\phi)$ since $|u_h|^2 \to |u|^2$ in $L^r(L^1)$ for any $1 \leqslant r < \infty$. Now we control the residual as follows:

$$|R| = \big|\big(\nabla u_h, P_h(u_h\phi) - u_h\phi\big)\big| \leqslant ch\|u_h\|^2_{H^1}.$$

Then it is clear that $\int_0^T |R| \to 0$ as $h \to 0$. In conclusion,

$$\liminf_{h\to 0} \int_0^T \big(\nabla u_h, \nabla P_h(u_h\phi)\big) \geqslant \int_0^T \big(|\nabla u|^2, \phi\big) - \left(\frac{1}{2}|u|^2, \nabla^2\phi\right).$$

(4) For the pressure term we have:

$$\big(p_h, \nabla\cdot\big(P_h(u_h\phi)\big)\big) = \big(p_h, \nabla\cdot(u_h\phi)\big) + R_1 = (p_h u_h, \nabla\phi) + R_1 + R_2,$$

where $R_1 = (p_h, \nabla\cdot(P_h(u_h\phi) - u_h\phi))$ and $R_2 = (\phi p_h \nabla\cdot u_h)$. For R_1, using the discrete commutator property together with an inverse inequality (2.6), we have:

$$\begin{aligned}\int_0^T |R_1| &\leqslant c\int_0^T \|p_h\|_{L^2}\big\|P_h(u_h\phi) - u_h\phi\big\|_{H^1} \leqslant ch\int_0^T \|p_h\|_{L^2}\|u_h\|_{H^1} \\ &\leqslant ch\|p_h\|_{L^{4/3}(L^2)}\|u_h\|_{L^4(H^1)} \leqslant ch^{1/2}\|p_h\|_{L^{4/3}(L^2)}\|u_h\|^{1/2}_{L^2(H^1)}\|u_h\|^{1/2}_{L^\infty(L^2)}.\end{aligned}$$

Then clearly $\int_0^T |R_1| \to 0$ as $h \to 0$. We proceed similarly for R_2 using the fact that u_h take its values in V_h,

$$\int_0^T |R_2| = \int_0^T \bigl|\bigl(\phi p_h - Q_h(\phi p_h), \nabla\cdot u_h\bigr)\bigr| \leqslant c\int_0^T \bigl\|\phi p_h - Q_h(\phi p_h)\bigr\|_{L^2}\|u_h\|_{H^1}$$
$$\leqslant ch\|p_h\|_{L^{4/3}(L^2)}\|u_h\|_{L^4(H^1)} \leqslant ch^{1/2}\|p_h\|_{L^{4/3}(L^2)}\|u_h\|_{L^2(H^1)}^{1/2}\|u_h\|_{L^\infty(L^2)}^{1/2}.$$

Then again $\int_0^T |R_2| \to 0$ as $h \to 0$.

(5) The source term does not pose any particular difficulty,

$$\bigl\langle f, P_h(\phi u_h)\bigr\rangle = \langle f, \phi u_h\rangle + R,$$

where $R = \langle f, P_h(\phi u_h) - \phi u_h\rangle$. Clearly $\int_0^T \langle f, \phi u_h\rangle \to \int_0^T \langle f, \phi u\rangle$ since $u_h \rightharpoonup u$ in $L^2(0,T;[H^1_\#(\Omega)]^3)$ and $f \in L^2(0,T;[H^{-1}_\#(\Omega)]^3)$. Moreover,

$$\int_0^T |R| \leqslant \|f\|_{L^2(H^{-1})}\bigl\|P_h(\phi u_h) - \phi u_h\bigr\|_{L^2(H^1)} \leqslant ch\|f\|_{L^2(H^{-1})}\|u_h\|_{L^2(H^1)}.$$

Then $\int_0^T |R| \to 0$ as $h \to 0$.

(6) Now we pass to the limit in the nonlinear term,

$$b_h\bigl(u_h, u_h, P_h(\phi u_h)\bigr) = b_h(u_h, u_h, \phi u_h) + R,$$

where $R = b_h(u_h, u_h, P_h(\phi u_h) - \phi u_h)$. Then

$$|R| \leqslant \bigl\|nl_h(u_h,u_h)\bigr\|_{H^{-1}}\bigl\|P_h(\phi u_h) - \phi u_h\bigr\|_{H^1} \leqslant ch\|u_h\|_{L^3}\|u_h\|_{H^1}\|u_h\|_{H^1} \leqslant ch\|u_h\|_{L^2}^{1/2}\|u_h\|_{H^1}^{1/2}\|u_h\|_{H^1}^2.$$

That is to say,

$$\int_0^T |R| \leqslant ch^{1/2}\|u_h\|_{L^\infty(L^2)}\|u_h\|_{L^2(H^1)}^2.$$

This in turn implies $\int_0^T |R| \to 0$ as $h \to 0$. Then conclude using hypothesis (4.1). □

Appendix A. Proofs from Section 2

A.1. Proof of Lemma 2.1

We start with a standard lemma:

Lemma A.1. *There are $c_1 > 0$, c_2 independent of h such that*

$$\forall q_h \in M_h, \quad c_1\|q_h\|_{L^2} \leqslant c_2 h\|\nabla q_h\|_{L_2} + \sup_{0\neq v_h\in X_h}\frac{(q_h, \nabla\cdot v_h)}{\|v_h\|_{H^1}}. \tag{A.1}$$

Proof. Let q_h be a nonzero function in M_h. Since the linear mapping $\nabla\cdot : [H^1_0(\Omega)]^3 \to L^2_{\int=0}(\Omega)$ is continuous and surjective, there is $\beta > 0$ such that for all $r \in L^2_{\int=0}(\Omega)$ there is $w \in [H^1_0(\Omega)]^3$ verifying $\nabla\cdot w = r$ and $\beta\|w\|_{H^1} \leqslant \|r\|_{L^2}$. Let $v \in [H^1_0(\Omega)]^3$ be such that $\nabla\cdot v = q_h$ and $\beta\|v\|_{H^1} \leqslant \|q_h\|_{L^2}$. Then, using $(\nabla q_h, \mathcal{C}_h v) = -(q_h, \nabla\cdot\mathcal{C}_h v)$,

$$\sup_{0\neq v_h\in X_h}\frac{\int_\Omega q_h\nabla\cdot v_h}{\|v_h\|_{H^1}} \geqslant \frac{\int_\Omega q_h\nabla\cdot\mathcal{C}_h(v)}{\|\mathcal{C}_h(v)\|_{H^1}} \geqslant c\frac{\int_\Omega q_h\nabla\cdot\mathcal{C}_h(v)}{\|v\|_{H^1}} = -c\frac{\int_\Omega \mathcal{C}_h(v)\cdot\nabla q_h}{\|v\|_{H^1}}$$
$$= -c\frac{\int_\Omega v\cdot\nabla q_h}{\|v\|_{H^1}} - c\frac{\int_\Omega(\mathcal{C}_h(v) - v)\cdot\nabla q_h}{\|v\|_{H^1}}.$$

Since $v \in [H_0^1(\Omega)]^3$ we integrate by parts the first term in the right-hand side:

$$\sup_{0\neq v_h\in X_h} \frac{\int_\Omega q_h \nabla\cdot v_h}{\|v_h\|_{H^1}} = c\frac{\int_\Omega q_h \nabla\cdot v}{\|v\|_{H^1}} - c\frac{\int_\Omega (\mathcal{C}_h(v)-v)\cdot\nabla q_h}{\|v\|_{H^1}} \geqslant c_1\beta\|q_h\|_{L^2} - c_2\|\nabla q_h\|_{L^2}\frac{\|\mathcal{C}_h(v)-v)\|_{L^2}}{\|v\|_{H^1}}.$$

Then using $\|\mathcal{C}_h(v)-v)\|_{L^2} \leqslant ch\|v\|_{H^1}$ the results follows easily. $\square$

To prove Lemma 2.1, we use $(\nabla q_h, v_h) = -(q_h, \nabla\cdot v_h)$ and we proceed as follows:

$$\sup_{0\neq v_h\in X_h} \frac{(q_h, \nabla\cdot v_h)}{\|v_h\|_{H^1}} = \sup_{0\neq v_h\in X_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{H^1}} \geqslant \frac{(\nabla q_h, \pi_h\nabla q_h)}{\|\pi_h\nabla q_h\|_{H^1}} = \frac{\|\pi_h\nabla q_h\|_{L^2}^2}{\|\pi_h\nabla q_h\|_{H^1}}.$$

Using the inverse inequality $\|\pi_h\nabla q_h\|_{H^1} \leqslant ch^{-1}\|\pi_h\nabla q_h\|_{L^2}$ together with the hypothesis (2.1), we infer:

$$\sup_{0\neq v_h\in X_h} \frac{(q_h, \nabla\cdot v_h)}{\|v_h\|_{H^1}} \geqslant ch\|\pi_h\nabla q_h\|_{L^2} \geqslant c'h\|\nabla q_h\|_{L^2}.$$

Conclude using (A.1).

A.2. Proof of Lemma 2.2

The technique of proof is adapted from that which is used to prove the standard LBB condition, see, e.g., [16,7].

Let us first prove statement (i). Let q_h be a member of M_h. Let K be an element in the mesh. Let b_K be the bubble function associated with K, i.e., $b_K \in H_0^1(K)$, $0 \leqslant b_K \leqslant 1$, and $\frac{\mathrm{meas}(K)}{|\int_K b_K|^2}\int_K b_K^2 \leqslant c$ where c does not depend on K and h. Set

$$v_h = \sum_{K\in\mathcal{T}_h} \frac{\int_K \nabla q_h}{\int_K b_K} b_K.$$

Observe that $\int_K v_h = \int_K \nabla q_h = \mathrm{meas}(K)\nabla q_h$. Owing to this definition:

$$(v_h, \nabla q_h) = \sum_{K\in\mathcal{T}_h} \int_K v_h\cdot\nabla q_h = \sum_{K\in\mathcal{T}_h} \nabla q_h \cdot \int_K v_h = \sum_{K\in\mathcal{T}_h} \|\nabla q_h\|_{L^2(K)}^2.$$

That is $(v_h, \nabla q_h) = \|\nabla q_h\|_{L^2}^2$. Moreover,

$$\|v_h\|_{L^2}^2 = \sum_{K\in\mathcal{T}_h} \frac{|\nabla q_h|^2\,\mathrm{meas}(K)^2}{|\int_K b_K|^2}\int_K b_K^2.$$

Since bubbles functions are such that $\frac{\mathrm{meas}(K)}{|\int_K b_K|^2}\int_K b_K^2 \leqslant c$ where c does not depend on K and h, we infer:

$$\|v_h\|_{L^2} \leqslant c\|\nabla q_h\|_{L^2}.$$

Then, using the fact that $\pi_h\nabla q_h$ is in X_h and π_h is a projection:

$$\|\pi_h\nabla q_h\|_{L^2} = \sup_{0\neq w_h\in X_h} \frac{(\pi_h\nabla q_h, w_h)}{\|w_h\|_{L^2}} = \sup_{0\neq w_h\in X_h} \frac{(\nabla q_h, w_h)}{\|w_h\|_{L^2}} \geqslant \frac{(\nabla q_h, v_h)}{\|v_h\|_{L^2}} \geqslant c\|\nabla q_h\|_{L^2}.$$

Hence, statement (i) is proved.

(2) Let $A = \{a_n\}$ be the collection of all the vertices in the mesh. Let $E^i = \{e_l\}$ be the collection of all the internal edges in the mesh, $E^\partial = \{e_l\}$ be the collection of all the edges in the mesh that are on $\partial\Omega$. Likewise we denote by $M^i = \{m_l\}$ and $M^\partial = \{m_l\}$ the set of midedges that are internal and the set of those that are at the boundary, respectively. For an edge e_l we denote by τ_l one of the two unit vectors that are aligned with e_l. Let q_h be a member of M_h. Define $v_h \in X_h$ be such that

$$v_h(a_n) = 0, \quad \forall a_n \in A,$$
$$v_h(m_l) = 0, \quad \forall m_l \in M^\partial,$$
$$v_h(m_l) = \tau_l\partial_{\tau_l} q_h, \quad \forall m_l \in M^i.$$

Note that this definition implies that $v_h \in [H_0^1(\Omega)]^3$. Using the quadrature formula:

$$\forall \phi \in \mathbb{P}_2, \quad \int_K \phi = \left(\sum_{m_l \in M_K} \frac{1}{5}\phi(m_l) - \sum_{a_n \in A_K} \frac{1}{20}\phi(a_n) \right) \operatorname{meas}(K),$$

where $M_K = (M^i \cup M^\partial) \cap K$ and $A_K = A \cap K$, we infer:

$$\begin{aligned}(v_h, \nabla q_h) &= \sum_{K \in \mathcal{T}_h} \int_K v_h \cdot \nabla q_h = \frac{1}{5} \sum_{K \in \mathcal{T}_h} \sum_{m_l \in M^i \cap K} \partial_{\tau_l} q_h(m_l)\tau_l \cdot \nabla q_h(m_l) \operatorname{meas}(K) \\ &= \frac{1}{5} \sum_{K \in \mathcal{T}_h} \sum_{m_l \in M^i \cap K} \left|\partial_{\tau_l} q_h(m_l)\right|^2 \operatorname{meas}(K),\end{aligned}$$

and since each element has at least 3 internal edges, we infer:

$$(v_h, \nabla q_h) \geqslant c \sum_{K \in \mathcal{T}_h} |\nabla q_h|^2 \operatorname{meas}(K) \geqslant c \|\nabla q_h\|_{L^2}^2.$$

Moreover it is clear that $\|v_h\|_{L^2} \leqslant c\|\nabla q_h\|_{L^2}$. Then the conclusion follows readily as in part (1) above. This concludes the proof. □

A.3. Proof of Lemma 2.3

(1) Let us first prove the estimate (2.8). Denote $a_h(s, r) = (\pi_h \nabla s, \nabla r)$ and $a(s, r) = (\nabla s, \nabla r)$. It is clear that owing to the L^2-stability of π_h, a_h is continuous over $(H^1(\Omega) + M_h) \times (H^1(\Omega) + M_h)$, i.e.,

$$\left|a_h(s, r)\right| \leqslant \|\nabla s\|_{L^2} \|\nabla r\|_{L^2}. \tag{A.2}$$

It is clear that the hypothesis (2.1) implies the following stability estimate: There is $c > 0$ independent of h such that

$$\inf_{0 \neq q_h \in M_h} \sup_{0 \neq r_h \in M_h} \frac{a_h(q_h, r_h)}{\|q_h\|_{H^1} \|r_h\|_{H^1}} \geqslant c. \tag{A.3}$$

Now let us prove a consistency property. Let q be a member of $H^2_\#(\Omega)$. Observe that

$$\begin{aligned}a(\mathcal{J}_h q, r_h) - a_h(\mathcal{J}_h q, r_h) &= (\nabla \mathcal{J}_h q, \nabla r_h - \pi_h \nabla r_h) = \inf_{w_h \in X_h} (\nabla \mathcal{J}_h q - w_h, \nabla r_h - \pi_h \nabla r_h) \\ &= \inf_{w_h \in X_h} \left(\nabla(\mathcal{J}_h q - q) + \nabla q - w_h, \nabla r_h - \pi_h \nabla r_h\right).\end{aligned}$$

Since $q \in H^2_\#(\Omega)$, ∇q is a member of $[H^1_\#(\Omega)]^3$. Then using the interpolation properties (2.3), (2.4) we infer the following consistency estimate.

$$\sup_{0 \neq r_h \in M_h} \frac{a(\mathcal{J}_h q, r_h) - a_h(\mathcal{J}_h q, r_h)}{\|r_h\|_{H^1}} \leqslant ch\|q\|_{H^2}. \tag{A.4}$$

To conclude we use the First Strang Lemma. In other words, using (A.3), we write

$$\begin{aligned}c\left\|\psi_h(q) - \mathcal{J}_h q\right\|_{H^1} &\leqslant \sup_{0 \neq r_h \in M_h} \frac{a_h(\psi_h(q) - \mathcal{J}_h q, r_h)}{\|r_h\|_{H^1}} \leqslant \sup_{0 \neq r_h \in M_h} \frac{a(q, r_h) - a_h(\mathcal{J}_h q, r_h)}{\|r_h\|_{H^1}} \\ &\leqslant \sup_{0 \neq r_h \in M_h} \frac{a(q - \mathcal{J}_h q, r_h) + a(\mathcal{J}_h q, r_h) - a_h(\mathcal{J}_h q, r_h)}{\|r_h\|_{H^1}}.\end{aligned}$$

The result follows by using (A.4) together with the interpolation property (2.3).

(2) We now prove the estimate (2.9). Using the inverse inequality (2.6) together with (2.8) and the H^1-stability of π_h, (2.5), we infer:

$$\left\|\pi_h \nabla\big(\psi_h(q)\big)\right\|_{H^1} \leqslant \left\|\pi_h \nabla\big(\psi_h(q) - q\big)\right\|_{H^1} + \|\pi_h \nabla q\|_{H^1} \leqslant c_1 h^{-1} \left\|\nabla\big(\psi_h(q) - q\big)\right\|_{L^2} + c_2\|q\|_{H^2} \leqslant c\|q\|_{H^2}.$$

This completes the proof. □

Appendix B. The discrete commutator property

The goal of this section is to show that the discrete commutator property (see Definition 4.1) holds for standard H^1-conforming finite element spaces.

Let $\mathcal{T}_h$ be a regular mesh of simplices and let $Z_h \subset H^1_\#(\Omega)$ be the $\mathbb{P}_k$-Lagrange finite element space based on this mesh. Let $1 \leqslant p < \infty$, and let m be such that $m \geqslant 1$ if $p = 1$ and $m > 1/p$ otherwise. Let $P_h : W^{m,p}_\#(\Omega) \to Z_h$ be the Scott–Zhang interpolation operator [13]. Recall that P_h is linear, is a projection onto Z_h, and satisfies the following interpolation property:

Lemma B.1 *(*Scott–Zhang*). In addition to the above hypotheses, assume $m \leqslant k+1$ then for all $l \in [0,m]$:*

$$\forall v \in W^{m,p}_\#(\Omega),\ \forall K \in \mathcal{T}_h, \quad \|v - P_h v\|_{W^{l,p}(K)} \leqslant c h_K^{m-l} |v|_{W^{m,p}(\Delta_K)},$$

where $h_K = \operatorname{diam}(K)$ and $\Delta_K = \operatorname{interior}(\bigcup\{K' \mid K' \cap K \neq \emptyset\})$.

As a corollary we infer the following so-called discrete commutator property (see, e.g., Bertoluzza [3]).

Lemma B.2 *(*Bertoluzza*). Let m and p be such that the assumptions of Lemma* B.1 *hold, then the following holds for all v_h in Z_h and for all ϕ in $W^{m+1,\infty}(\Omega)$:*

$$\big\|\phi v_h - P_h(\phi v_h)\big\|_{W^{l,p}} \leqslant c h^{1+m-l} \|v_h\|_{W^{m,p}} \|\phi\|_{W^{m+1,\infty}}, \quad 0 \leqslant l \leqslant m \leqslant 1.$$

Proof. We prove the result locally. Let K be a cell in the mesh $\mathcal{T}_h$. Denote by x_K some point in K, say the barycenter of K. Let ϕ be a function in $W^{1,\infty}(\Omega)$. Define $R_K = \phi - \phi(x_K)$. It is clear that $R_K \in W^{1,\infty}(\Omega)$, and

$$\|R_K\|_{L^\infty(\Delta_K)} \leqslant c h_K \|\phi\|_{W^{1,\infty}(\Omega)},$$
$$\|R_K\|_{W^{1,\infty}(\Delta_K)} \leqslant c \|\phi\|_{W^{1,\infty}(\Omega)}.$$

Let $\bar{v}_h$ be the mean value of v_h over Δ_K, then it is clear that

$$\|\bar{v}_h\|_{L^p(\Delta_K)} \leqslant c \|v_h\|_{L^p(\Delta_K)},$$
$$\|v_h - \bar{v}_h\|_{W^{l,p}(\Delta_K)} \leqslant c h_K^{m-l} \|v_h\|_{W^{m,p}(\Delta_K)}, \quad 0 \leqslant l \leqslant m.$$

We have:

$$\big\|\phi v_h - P_h(\phi v_h)\big\|_{W^{l,p}(K)} \leqslant \big\|(1-P_h)(\phi \bar{v}_h)\big\|_{W^{l,p}(K)} + \big\|(1-P_h)\big(\phi(v_h - \bar{v}_h)\big)\big\|_{W^{l,p}(K)}.$$

Let us denote by R_1 and R_2 the two residuals in the right-hand side.

To control R_1 we proceed as follows:

$$R_1 \leqslant c h_K^{1+m-l} \|\phi \bar{v}_h\|_{W^{m+1,p}(K)} \leqslant c h_K^{1+m-l} \|\bar{v}_h\|_{L^p(K)} \|\phi\|_{W^{m+1,\infty}(\Omega)} \leqslant c h_K^{1+m-l} \|v_h\|_{L^p(\Delta_K)} \|\phi\|_{W^{m+1,\infty}(\Omega)}.$$

For the other residual we use the fact that P_h is linear and is a projection as follows:

$$\big\|(1-P_h)\big(\phi(v_h - \bar{v}_h)\big)\big\|_{W^{l,p}(K)} = \big\|(1-P_h)\big(\big(\phi - \phi(x_K)\big)(v_h - \bar{v}_h)\big)\big\|_{W^{l,p}(K)}.$$

As a result

$$\begin{aligned} R_2 &= \big\|(1-P_h)\big(R_K(v_h - \bar{v}_h)\big)\big\|_{W^{l,p}(K)} \leqslant c h_K^{1-l} \big|R_K(v_h - \bar{v}_h)\big|_{W^{1,p}(\Delta_K)} \\ &\leqslant c h_K^{1-l} \big(\|R_K\|_{L^\infty(\Delta_K)} |v_h - \bar{v}_h|_{W^{1,p}(\Delta_K)} + |R_K|_{W^{1,\infty}(\Delta_K)} \|v_h - \bar{v}_h\|_{L^p(\Delta_K)}\big) \\ &\leqslant c h_K^{1-l} \big(h_K |v_h - \bar{v}_h|_{W^{1,p}(\Delta_K)} + \|v_h - \bar{v}_h\|_{L^p(\Delta_K)}\big) \|\phi\|_{W^{1,\infty}(\Omega)} \leqslant c h_K^{1+m-l} \|v_h\|_{W^{m,p}(\Delta_K)} \|\phi\|_{W^{1,\infty}(\Omega)}. \end{aligned}$$

Then, the desired result follows easily owing to the regularity hypothesis on the mesh which implies that $\sup_{K' \in \mathcal{T}_h} \{\operatorname{card}\{K \in \mathcal{T}_h \mid K' \subset \Delta_K\}\}$ can be bounded from above by a constant that does not depend on h. □

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