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Subgrid stabilization of Galerkin approximations of linear contraction semi-groups of class $\mathcal{C}^0$

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Abstract. This paper presents a stabilized Galerkin technique for approximating linear contraction semi-groups of class $\mathcal{C}^0$ in a Hilbert space. The main result of this paper is that this technique yields optimal error estimates in the graph norm. The key idea is twofold, first it consists in introducing an approximation space that is broken up into resolved scales and subgrid scales so that the generator of the semi-group satisfies a uniform inf-sup condition with respect to this decomposition. Second, the Galerkin approximation is slightly modified by introducing an artificial diffusion acting only on the subgrid scales.

Key words: Finite Elements – Galerkin methods – Stabilization – Linear hyperbolic equations – Semi-groups – Subgrid modeling – Artificial viscosity – Multi-scale methods

1 Introduction

The objective of the present work is to propose a framework to stabilize Galerkin approximations of time dependent linear problems that do not possess a coercivity property (for instance, linear hyperbolic equations).

More precisely, we consider a separable Hilbert space L and $A : D(A) \subset L \longrightarrow L$ the generator of a contraction semi-group of class $\mathcal{C}^0$ in L. Hereafter, we are concerned with the following linear problem: For $f \in \mathcal{C}^1([0, +\infty[; L)$ and $u_0 \in D(A)$,

$$\begin{cases} \text{Find } u \in \mathcal{C}^1([0, +\infty[; L) \cap \mathcal{C}^0([0, +\infty[; D(A)) \text{ s.t.} \\ u_{|t=0} = u_0, \\ \frac{\mathrm{d}u}{\mathrm{d}t} + Au = f, \end{cases} \tag{1}$$

The main goal of the present paper is to present a stabilized Galerkin technique for approximating (1). The main feature of the proposed technique is that it yields optimal convergence estimates in the graph norm. This result seems to be new. To the author's knowledge, the existing stabilization methods cannot be easily extended to time dependent problems (Streamline Upwind Petrov Galerkin, see e.g. Brooks-Hughes [4], or Galerkin Least Square, see e.g. Hughes et al. [9], or residual Free Bubbles, see e.g. Brezzi et al. [2]) without the framework of the discontinuous Galerkin technique (see e.g. Lesaint-Raviart [12] and Johnson-Pitkäranta [11]).

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2 The model problem

Given the Hille-Yosida theorem (see Brezis [1, p. 110] or Yosida [14, p. 248]), since A is the generator of a contraction semi-group, A is necessarily monotone:

$$\forall v \in D(A), \qquad (Av, v)_L \geq 0, \tag{2}$$

and A is maximal

$$\forall f \in L,\ \exists v \in D(A), \qquad v + Av = f. \tag{3}$$

Having in mind a Galerkin approximation of the evolution equation in (1) we want to introduce a bilinear form $a(u, v) = (Au, v)_L$. To this end, we set $V = D(A)$ and we equip V with the graph norm: $\|v\|_V = (\|v\|_L^2 + \|Av\|_L^2)^{1/2}$. Since the graph of A is closed, V is a Banach space. Furthermore, it is clear that equipped with the inner product $(u, v)_L + (Au, Av)_L$, V is a Hilbert space. Since $D(A) = V$ is dense in L, we are in the following classical situation $V \subset L \equiv L' \subset V'$. Hence $a : V \times L \longrightarrow \mathbb{R}$, as defined above, is a continuous bilinear form. Furthermore, we introduce the semi-norm $|v|_V = \|Av\|_L$. From this definition we deduce

$$\forall u \in V, \qquad \sup_{v \in L} \frac{a(u, v)}{\|v\|_L} = |u|_V. \tag{4}$$

This tautology will play a key role hereafter.

The problem (1) can be recast into the following equivalent form: For $f \in \mathcal{C}^1([0, +\infty[; L)$ and $u_0 \in V$, find u in

$\mathcal{C}^1([0,+\infty[;L)\cap\mathcal{C}^0([0,+\infty[;V)$ so that

$$\begin{cases}(u(0),v)=(u_0,v), & \forall v\in L\\ \left(\frac{\mathrm{d}u}{\mathrm{d}t},v\right)_L+a(u,v)=(f,v)_L, & \forall v\in L,\ \forall t\geq 0.\end{cases} \tag{5}$$

Formally, by using u as a test function in the equation above and by integrating with respect to time, we obtain that $\|u(t)\|_L$ is bounded. Furthermore, by deriving the quation with respect to time and by using u_t as a test function we obtain that, provided $u_0\in V$, the L-norm of the time derivative, $\|u_t(t)\|_L$, is bounded. As result

$$\begin{aligned}|u(t)|_V=\sup_{v\in L}\frac{a(u,v)}{\|v\|_L}&\leq\|u_t\|_L+\|f\|_L\|u(t)\|_L\\ &\leq c(t,u_0,\ f,\ f_t).\end{aligned}$$

That is to say, if the initial data is bounded in V (i.e. in the Graph norm), then $u(t)$ is bounded in the graph norm for all times. The ultimate goal of the present paper is to present a Galerkin method that reproduces this property.

The key idea of the present work is twofold, first it consists in introducing an approximation space that is broken up into resolved scales and subgrid scales so that the bilinear form a satisfies a discrete counterpart of the inf-sup condition (4) with respect to this decomposition. Second, the Galerkin approximation is slightly modified by introducing an artificial diffusion that is restricted to the subgrid scales. The present work follows an idea that is, to some extent, similar to the spectral viscosity method proposed by Tadmor [13].

3 The discrete approximation

3.1 The discrete setting

We introduce a sequence of finite dimensional subspaces of V, say $(X_H)_{0<H\leq 1}$, where the index H denotes a positive parameter tending to zero.

The space X_H is assumed to have the following approximation property: there are W, a dense subspace of V, a linear operator $I_H\in\mathcal{L}(W,X_H)$ and $k>0$, $c>0$ so that

$$\forall v\in W,\quad \|v-I_Hv\|_L+H\|v-I_Hv\|_V\leq cH^{k+1}\|v\|_W. \tag{6}$$

From now on, c denotes a generic constant that does not depend on H and the value of which may change on different occurrences.

In general (4) has no uniform discrete counterpart. That is to say, in general there is $c(H)$ so that

$$\forall u_H\in X_H,\quad \sup_{v_H\in X_H}a(u_H,v_H)/\|v_H\|_L\geq c(H)|u_H|_V,$$

but either $c(H)$ is zero or depends on the meshsize H. The solution that we propose to cure to this problem consists in enlarging the test space. Hence, we introduce a new sequence of finite dimensional subspaces of V, say $(X_h)_{0\leq h\leq 1}$, so that there are $c_a>0$ and $c_\delta\geq 0$, independent of (H,h) so that

$$\forall v_h\in X_h,\quad \sup_{\phi_h\in X_h}\frac{a(v_H,\phi_h)}{\|\phi_h\|_L}\geq c_a|v_H|_V-c_\delta\|v_h\|_L. \tag{7}$$

This hypothesis is the keystone of the theory to be developed hereafter. When $c_\delta=0$, this inequality is the discrete counterpart of the tautology (4). Since the algorithm that we shall propose preserves the contraction property of the semi-group at the discrete level, the L-norm of the approximate solution $u_h(t)$ will always be bounded in time. Hence, we shall weaken (4) by allowing c_δ not to be zero in (7).

Furthermore, we assume that $X_H\subset X_h$, and there is a linear projection operator $P_H:X_h\longrightarrow X_H$ that is stable with respect to the L-norm:

$$\exists c>0,\ \forall(H,h),\ \forall v_h\in X_h\qquad \|P_Hv_h\|_L\leq c\|v_h\|_L. \tag{8}$$

For further references, we denote

$$X_h^H=(1-P_H)X_h,$$

and for all v_h in X_h we set $v_H=P_Hv_h$ and $v_h^H=v_h-v_H$; that is to say,

$$X_h=X_H\oplus X_h^H.$$

Hereafter the two parameters H and h are assumed to be equivalent:

$$c_1H\leq h\leq c_2H. \tag{9}$$

This hypothesis amounts to saying that X_H and X_h have about the same interpolation properties, i.e. X_h is not significantly larger than X_H, but X_h is large enough for (7) to be satisfied uniformly.

Since X_h is a finite dimensional normed space, we shall use an inverse inequality. For this purpose, we set

$$\lambda(H)^{-1}=\sup_{v_h\in X_h}\frac{|v_h|_V}{\|v_h\|_L}. \tag{10}$$

In the case of a finite element approximation, $\lambda(H)$ is proportional to the meshsize H if the mesh is quasi-uniform (see e.g. Girault–Raviart [6, p. 103]).

3.2 The artificial viscosity

Looking back at (7), we see that the subgrid scales (i.e. the space X_h^H) have been introduced to help control the graph norm of the resolved scales (i.e. the elements of X_H). The question we are left with now is: how can we control the subgrid scales? The answer we propose is: control the graph norm of the subgrid scales by means of an artificial diffusion mechanism. Note that by grossly diffusing the subgrid scales we shall not spoil the accuracy of the Galerkin method, since by adding X_h^H to X_H we did not improve significantly the approximation property of X_H (see the equivalence inequality (9)).

More precisely, we define a bilinear form $b_h:X_h^H\times X_h^H\longrightarrow\mathbb{R}$ that satisfies the following continuity and coercivity properties: There are a semi-norm $|\cdot|_b$ and $c_B>0$ so that

$$\begin{cases}b_h(v_h^H,v_h^H)\geq\lambda(H)|v_h^H|_b^2,\\ b_h(v_h^H,w_h^H)\leq c_B\lambda(H)|v_h^H|_b\,|w_h^H|_b,\end{cases} \tag{11}$$

where the semi-norm $|\cdot|_b$ is such that there are two constants $c_{e1} > 0$ and $c_{e2} > 0$ so that for all v_h^H in X_h^H we have

$$c_{e1}|v_h^H|_V \leq |v_h^H|_b \leq c_{e2}\lambda(H)^{-1}\|v_h^H\|_L, \tag{12}$$

The simplest choice for b_h is

$$b_h(v_h^H, w_h^H) = \lambda(H)(A\,v_h^H, A\,w_h^H)_L,$$

but other choices are possible. For instance, assume that X is a dense subpace of V with continuous embedding. Assume that there is a symmetric positive bilinear form $(\cdot,\cdot)_X$ on $X \times X$ so that the semi-norm $|\cdot|_X$ associated with this form satisfies:

$$c_{e1}|v_h^H|_V \leq |v_h^H|_X \leq c_{e2}\lambda(H)^{-1}\|v_h^H\|_L$$

for all v_h^H in X_h^H. This hypothesis means that X and V are associated with differential operators of the same order. Then, by assuming $X_h \subset X$, one can set

$$b_h(v_h^H, w_h^H) = \lambda(H)(v_h^H, w_h^H)_X.$$

We can illustrate these definitions on the scalar transport equation $\mathrm{d}u/\mathrm{d}t + \beta{\cdot}\nabla\,u = f$ in Ω. With suitable assumptions on the vector field β, we have

$$\begin{aligned} Au &= \beta{\cdot}\nabla\,u, \\ L &= L^2(\Omega), \\ V &= \{v \in L^2(\Omega) \mid \beta{\cdot}\nabla\,v \in L^2(\Omega),\ v_{|\Gamma^-} = 0\}, \end{aligned}$$

where Γ^- is the inflow boundary. By assuming $X_h \subset H^1(\Omega) \subset V$, the following two definitions are possible for b_h:

$$b_h(v_h^H, w_h^H) = \begin{cases} \lambda(H)\int_\Omega(\beta{\cdot}\nabla\,v_h^H)(\beta{\cdot}\nabla\,w_h^H), \\ \lambda(H)\int_\Omega(\nabla v_h^H)\cdot(\nabla w_h^H). \end{cases} \tag{13}$$

The second model may be helpful in practice for two reasons: First, it may help damping cross-wind oscillations when approximating very stiff problems; second, if in practice β is time-dependent (though the present theory assumes β to be time-independent), this model is time-independent (hence the assembling of the matrix is done only once).

3.3 The discrete problem

For the sake of simplicity we assume that u_0 is in W, and we approximate the initial data by $I_H u_0$. The discrete problem we consider hereafter consists in finding u_h in $\mathcal{C}^1([0,+\infty[; X_h)$ so that

$$\begin{cases} u_{h|t=0} = I_H u_0, \\ \left(\frac{\mathrm{d}u_h}{\mathrm{d}t}, v_h\right)_L + a(u_h, v_h) + b_h(u_h^H, v_h^H) \\ \qquad\qquad = (f, v_h), \qquad \forall v_h \in X_h. \end{cases} \tag{14}$$

The discrete problem (14) has clearly a unique solution since it is a system of linear ODE's.

4 The error analysis

4.1 Preliminaries

In the following we shall need the following stability result.

Lemma 1. *There is $c_b > 0$ so that*

$$\forall v_h^H \in X_h^H, \qquad \sup_{w_h \in X_h} \frac{b_h(v_h^H, w_h^H)}{\|w_h\|_L} \leq c_b|v_h^H|_b. \tag{15}$$

Proof. The stability hypothesis (8) on P_H together with the inverse stability property (12) yields

$$\begin{aligned} b_h(v_h^H, w_h^H) &\leq c_B\lambda(H)|v_h^H|_b|w_h^H|_b, \\ &\leq c_B c_{e2}|v_h^H|_b\|w_h^H\|_L, \\ &\leq c_B c_{e2}|v_h^H|_b\|(1-P_H)w_h\|_L, \\ &\leq c_B c_{e2}\|1-P_H\||v_h^H|_b\|w_h\|_L. \end{aligned}$$

The desired result follows readily. □

Let T be a strictly positive real number. We shall make use of the following version of the Gronwall lemma, the proof of which is left to the reader.

Lemma 2. *Let ϕ be in $W^{1,1}((0,T);\mathbb{R})$ (the derivative being understood in the usual distribution sense) and ψ be in $L^1((0,T);\mathbb{R})$ so that $\phi \geq 0$, $\psi \geq 0$. Assume there are $a, b \geq 0$ two real numbers so that $\phi^{(1)}(t) + \psi \leq a\phi^{1/2} + b$, then*

$$\|\phi\|_{L^\infty(0,T)} + \|\psi\|_{L^1(0,T)} \leq e\left(\frac{a^2}{4}T^2 + bT + \phi(0)\right).$$

4.2 The main convergence result

The main convergence result of this section is

Theorem 1. *Assume u is in $W^{2,\infty}([0,T];W)$, then the discrete solution u_h of (14) satisfies*

$$\|u-u_h\|_{L^\infty([0,T];L)} + \left[\int_0^T a(u-u_h, u-u_h)\right]^{1/2} \leq c_1 H^{k+1/2}, \tag{16}$$

$$\left[\frac{1}{T}\int_0^T \|u-u_h\|_V^2\right]^{1/2} \leq c_2 H^k, \tag{17}$$

where constants c_1 and c_2 can be bounded from above as follows

$$c_1 \leq c\left[H + T\left[\frac{H}{\lambda(H)} + 1 + T\right]\right]^{1/2}\|u\|_{W^{2,\infty}([0,T];W)},$$

$$c_2 \leq c\left[\frac{H}{\lambda(H)} + 1 + T\right]\|u\|_{W^{2,\infty}([0,T];W)}.$$

Proof. To simplify the notation, let us set $\eta_h(t) = u(t) - I_H u(t)$, and $e_h(t) = I_H u(t) - u_h(t)$. Note that we have $u -$

$u_h = \eta_h + e_h$. From the definition of $\eta_h(t)$ we deduce that for all $j \in \{0, 1, 2\}$,

$$\|\eta_h^{(j)}\|_{L^\infty([0,T];H)} + H\|\eta_h^{(j)}\|_{L^\infty([0,T];V)} \leq cH^{k+1}\|u\|_{W^{2,\infty}([0,T];W)}.$$

The equation that controls e_h is obtained by subtracting (14) from (5) where the test functions span X_h:

$$\forall v_h \in X_h, \quad \left(\frac{\mathrm{d}e_h}{\mathrm{d}t}, v_h\right)_L + a(e_h, v_h) - b_h(u_h^H, v_h^H) = \left(\frac{\mathrm{d}\eta_h}{\mathrm{d}t}, v_h\right)_L - a(\eta_h, v_h).$$

Since X_H is invariant by the projection P_H and P_H is linear, we infer

$$\begin{aligned} u_h^H &= u_h - P_H u_h \\ &= u_h - I_H u - P_H(u_h - I_H u) \\ &= -e_h + P_H e_h \\ &= -e_h^H. \end{aligned}$$

As a result, the equation that controls e_h can be recast into the form: for all v_h in X_h,

$$\left(\frac{\mathrm{d}e_h}{\mathrm{d}t}, v_h\right)_L + a(e_h, v_h) + b_h(e_h^H, v_h^H) = -\left(\frac{\mathrm{d}\eta_h}{\mathrm{d}t}, v_h\right)_L - a(\eta_h, v_h). \tag{18}$$

Furthermore, $\mathrm{d}e_h/\mathrm{d}t = e_h^{(1)}$ is controlled by: for all v_h in X_h,

$$(e_h^{(2)}, v_h)_L + a(e_h^{(1)}, v_h) + b_h(e_h^{(1)H}, v_h^H) = -(\eta_h^{(2)}, v_h)_L - a(\eta_h^{(1)}, v_h). \tag{19}$$

Let us derive some bounds on the initial data. It is clear that $e_h(0) = 0$; furthermore, by using (18) at $t = 0$, we infer $\|e_h^{(1)}(0)\|_L \leq \|\eta_h^{(1)}(0)\|_L + \|\eta_h(0)\|_V$. As a result, we have the following error estimates at $t = 0$

$$\begin{cases} \|e_h(0)\|_L = 0, \\ \|e_h^{(1)}(0)\|_L \leq cH^k\|u\|_{W^{2,\infty}([0,T];W)}. \end{cases}$$

Now, we seek a bound on $\|e_h^{(1)}\|_{L^\infty([0,T];L)}$. We take $e_h^{(1)}$ as test function in (19). Owing to the coercivity property of b_h, we obtain:

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h^{(1)}\|_L^2 + a(e_h^{(1)}, e_h^{(1)}) + \lambda(H)\|e_h^{H(1)}\|_b^2 \leq (\|\eta_h^{(2)}\|_L + |\eta_h^{(1)}|_V)\|e_h^{(1)}\|_L.$$

Since the bilinear form a is positive, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\|e_h^{(1)}\|_L^2 \leq 2(\|\eta_h^{(2)}\|_L + |\eta_h^{(1)}|_V)\|e_h^{(1)}\|_L.$$

By using Lemma 2, we infer

$$\|e_h^{(1)}\|^2_{L^\infty([0,T];L)} \leq c(\|e_h^{(1)}(0)\|_L^2 + T^2\|\eta_h\|^2_{W^{2,\infty}([0,T];V)}).$$

As a result we have

$$\|e_h^{(1)}\|_{L^\infty([0,T];L)} \leq cH^k(1+T)\|u\|_{W^{2,\infty}([0,T];W)}.$$

To obtain a bound on e_h, we use e_h as a test function in (18). Let us introduce the symmetric part of a:

$$a_s(x, y) = \frac{1}{2}(a(x, y) + a(y, x)).$$

Since a_s is symmetric and positive, we infer

$$\begin{aligned} \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_L^2 + a_s(e_h, e_h) + \lambda(H)\|e_h^H\|_b^2 &\leq \|\eta_h^{(1)}\|_L\|e_h\|_L + a(e_h, \eta_h) - 2a_s(e_h, \eta_h), \\ &\leq \|\eta_h^{(1)}\|_L\|e_h\|_L + |e_h|_V\|\eta_h\|_L \\ &\quad + \gamma a_s(e_h, e_h) + c_\gamma a_s(\eta_h, \eta_h), \end{aligned}$$

where we have used the inequality

$$2a_s(x, y) \leq \gamma a_s(x, x) + a_s(y, y)/\gamma,$$

which is valid for any positive constant γ. Hereafter, γ denotes a generic constant that can be chosen as small as needed and c_γ is a constant that depends on γ. The value of γ and c_γ may change on different occurrences. By choosing $\gamma = 1/2$, we obtain

$$\begin{aligned} \frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_L^2 + a_s(e_h, e_h) + 2\lambda(H)\|e_h^H\|_b^2 &\leq 2\|e_h\|_L\|\eta_h^{(1)}\|_L \\ &\quad + c(|e_h|_V\|\eta_h\|_L + \|\eta_h\|_V\|\eta_h\|_L). \end{aligned} \tag{20}$$

Note that the term $|e_h|_V\|\eta_h\|_L$ in the right-hand side of (20) is not controlled yet; it is the most critical one in this error analysis. It is at this point that the inf-sup inequality (7) plays its role.

$$\begin{aligned} c_a|e_H|_V &\leq \sup_{\phi_h \in X_h}\frac{a(e_H, \phi_h)}{\|\phi_h\|_L} + c_\delta\|e_h\|_L, \\ &\leq \sup_{\phi_h \in X_h}\frac{-(e_h^{(1)}, \phi_h) - a(e_h^H, \phi_h) - b_h(e_h^H, \phi_h^H)}{\|\phi_h\|_L} \\ &\quad + \sup_{\phi_h \in X_h}\frac{-(\eta_h^{(1)}, \phi_h) - a(\eta_h, \phi_h)}{\|\phi_h\|_L} + c_\delta\|e_h\|_L, \\ &\leq \|e_h^{(1)}\|_L + |e_h^H|_V + \|\eta_h^{(1)}\|_L + |\eta_h|_V \\ &\quad + c_b|e_h^H|_b + c_\delta\|e_h\|_L, \\ &\leq c(\|e_h^{(1)}\|_L + |e_h^H|_b + \|e_h\|_L + \|\eta_h^{(1)}\|_L + |\eta_h|_V). \end{aligned}$$

By using the bound already obtained on $\|e_h^{(1)}\|_L$ together with the triangular inequality, we infer

$$\begin{aligned} |e_h|_V &\leq |e_H|_V + |e_h^H|_V, \\ &\leq c(|e_h^H|_b + \|e_h\|_L + (1+T)H^k\|u\|_{W^{2,\infty}([0,T];W)}). \end{aligned}$$

Coming back to (20), we have

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_L^2 &+ a_s(e_h,e_h) + 2\lambda(H)\|e_h^H\|_b^2 \\ &\le 2\|e_h\|_L\|\eta_h^{(1)}\|_L + cH^{2k+1}\|u\|^2_{W^{2,\infty}([0,T];W)} \\ &\quad + c'\left[|e_h^H|_b + \|e_h\|_L \right. \\ &\quad \left. +(1+T)H^k\|u\|_{W^{2,\infty}([0,T];W)}\right]\|\eta_h\|_L, \\ &\le c\|e_h\|_L(\|\eta_h\|_L + \|\eta_h^{(1)}\|_L) \\ &\quad + c'(1+T)H^{2k+1}\|u\|^2_{W^{2,\infty}([0,T];W)} \\ &\quad + \gamma\lambda(H)|e_h^H|_b^2 + c_\gamma\lambda(H)^{-1}\|\eta_h\|_L^2.\end{aligned}$$

By choosing $\gamma = 1$, we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\|e_h\|_L^2 &+ a_s(e_h,e_h) + \lambda(H)\|e_h^H\|_b^2 \\ &\le c\|e_h\|_L H^{k+1}\|u\|_{W^{2,\infty}([0,T];W)} \\ &\quad + c'(1+\lambda(H)^{-1}H + T)H^{2k+1}\|u\|^2_{W^{2,\infty}([0,T];W)}.\end{aligned}$$

Owing to Lemma 2, we infer

$$\begin{aligned}\|e_h\|^2_{L^\infty([0,T];L)} &+ \int_0^T (a_s(e_h,e_h) + \lambda(H)\|e_h^H\|_b^2) \\ &\le c\left[\|e_h(0)\|_L^2 + T^2H^{2k+2}\|u\|^2_{W^{2,\infty}([0,T];W)}\right. \\ &\quad \left. + T(1+\lambda(H)^{-1}H+T)H^{2k+1}\|u\|^2_{W^{2,\infty}([0,T];W)}\right] \\ &\le cT\left[1+\lambda(H)^{-1}H+T\right]H^{2k+1}\|u\|^2_{W^{2,\infty}([0,T];W)},\end{aligned}$$

which yields

$$\|u-u_h\|_{L^\infty([0,T];L)} \le c_1 H^{k+1/2},$$

where

$$c_1 \le c\left[H + T(1+\lambda(H)^{-1}H+T)\right]^{1/2}\|u\|_{W^{2,\infty}([0,T];W)}.$$

Now we derive an error estimate in the graph norm. By using the bound already obtained on $|e_h|_V$, we have

$$\begin{aligned}\int_0^T |e_h|_V^2 &\le c\int_0^T \left[|e_h^H|_b^2 + \|e_h\|_L^2 \right. \\ &\quad \left. + (1+T)^2H^{2k}\|u\|^2_{W^{2,\infty}([0,T];W)}\right], \\ [4pt] &\le cTH^{2k}\left[H(\lambda(H)^{-1}+T)(1+\lambda(H)^{-1}H+T)\right. \\ &\quad \left. +(1+T)^2\right]\|u\|^2_{W^{2,\infty}([0,T];W)}, \\ &\le cTH^{2k}\left[\lambda(H)^{-2}H^2 + (1+T)^2\right]\|u\|^2_{W^{2,\infty}([0,T];W)}.\end{aligned}$$

As a result,

$$\left[\frac{1}{T}\int_0^T |u-u_h|_V^2\right]^{1/2} \le c_2 H^k,$$

where

$$c_2 \le \left[\frac{H}{\lambda(H)} + 1 + T\right]\|u\|_{W^{2,\infty}([0,T];W)},$$

The final estimate in the graph norm is obtained by combining this bound and that in the L-norm. □

Note that for finite elements $\lambda(H) \sim H$. Hence, for finite elements, the bound (17) is optimal in the graph norm. The estimate (16) is identical to the one that could be obtained by applying the counterpart of the discontinuous Galerkin method to the present problem (see Johnson-Pitkäranta [11]).

Note also that for large time T, $c_1 = \mathcal{O}(T)$ and $c_2 = \mathcal{O}(T)$; that is, in the most unfavourable case, the error grows linearly with respect to T.

5 Numerical implementation

5.1 The finite element framework

To illustrate the possibilities of the present method, we apply it to a 2D advection model problem:

$$\begin{cases}\frac{\mathrm{d}u}{\mathrm{d}t} + \beta\cdot\nabla u = f, & \text{in } \Omega \\ u_{|t=0} = u_0, \end{cases} \tag{21}$$

where $\beta : \Omega \subset \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ is a smooth vector field satisfying reasonable assumptions so that $\beta\cdot\nabla(\cdot)$ is the generator of a $\mathcal{C}^0$ contraction semi-group.

Let $\mathcal{T}_H$ be a quasi-uniform triangulation of Ω composed of simplexes. We use $\mathbb{P}_1$ finite elements. The resolved scales space X_H is defined by

$$X_H = \{v_h \in \mathcal{C}^0(\overline{\Omega}) \mid v_{h|K} \in \mathbb{P}_1(K),\ \forall K \in \mathcal{T}_H\}$$

To define X_h^H, we proceed locally as follows. We partition each simplex K into 3 subsimplexes by inserting a node at the barycenter of K. We define ψ_K as being the Lagrange basis function of degree 1 associated with the barycenter of K (see Fig. 1), and we set

$$X_h^H = \bigoplus_{K\in\mathcal{T}_H} \mathrm{span}(\psi_K).$$

It is clear that $X_H \cap X_h^H = \emptyset$ and the decomposition $X_h = X_H \oplus X_h^H$ is L^2 stable.

The bilinear form associated with the artificial viscosity is defined by

$$b_h(v_h^H, w_h^H) = \sum_{K\in\mathcal{T}_H} \mathrm{mes}(K)^{1/2}\int_K \nabla v_h^H \cdot \nabla w_h^H.$$

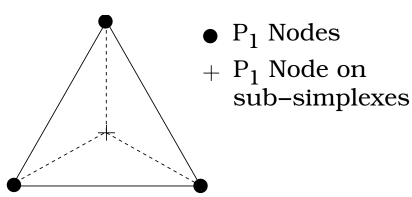

Fig. 1. Representation of the subgrid space X_h^H: a node is inserted at the barycenter of simplex K and K is divided into 3 $\mathbb{P}_1$ subsimplexes

Let us introduce the semi-norm $|v|_{1,\beta} = \|\beta{\cdot}\nabla\, v\|_0$. It is shown in Guermond [7, 8] that the following results hold:

Theorem 2. *If β is piecewise constant on each simplex K of $\mathcal{T}_H$, there is $c_\beta > 0$ independent of H, so that*

$$\inf_{u_H \in X_H} \sup_{v_h \in X_h} \frac{\int_\Omega (\beta{\cdot}\nabla\, u_H) v_h}{|u_H|_{1,\beta} \|v_h\|_0} \geq c_\beta. \tag{22}$$

For a general vector field β we have

Theorem 3. *If β is in $\mathcal{C}^1(\overline{\Omega})^2$, there are $c_\beta > 0$ and $c_\delta \geq 0$, both independent of H, so that for all $u_H \in X_H$,*

$$\sup_{v_h \in X_h} \frac{\int_\Omega (\beta{\cdot}\nabla\, u_H) v_h}{\|v_h\|_0} \geq c_\beta |u_H|_{1,\beta} - c_\delta \|u_H\|_0. \tag{23}$$

5.2 Time stepping

The approximate solution is approximated in time by means of a fully implicit time stepping strategy based on the second order, three level, backward differentiation formula

$$\frac{\partial u}{\partial t}(t^{k+1}) = \frac{3u(t^{k+1}) - 4u(t^k) + u(t^{k-1})}{2\delta t} + \mathcal{O}(\delta t^2).$$

The time step δt is chosen small enough so that the time error is much smaller than the space error.

5.3 Algebraic elimination of the subgrid d.o.f.

At each time step, the following linear system has to be solved

$$\begin{pmatrix} \mathcal{A} & \mathcal{B} \\ \mathcal{C} & \mathcal{D} \end{pmatrix} \cdot \begin{pmatrix} \mathcal{U}_H \\ \mathcal{U}_h^H \end{pmatrix} = \begin{pmatrix} \mathcal{F}_H \\ \mathcal{F}_h^H \end{pmatrix}$$

where $\mathcal{U}_H$ and $\mathcal{U}_h^H$ are the nodal values of u_H and u_h^H. Note that the matrix $\mathcal{D}$ is diagonal since the subgrid base functions have disjoint supports. As a result, $\mathcal{D}$ is easily invertible. In practice we eliminate the subgrid unknowns as follows.

$$(\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}) \cdot \mathcal{U}_H = \mathcal{F}_H - \mathcal{B}\mathcal{D}^{-1}\mathcal{F}_h^H.$$

As a result, the dimension of the linear system to be solved is equal to $\dim(X_H)$. The sparsity of $\mathcal{B}\mathcal{D}^{-1}\mathcal{C}$ being exactly the same as that of $\mathcal{A}$, the assembling of $\mathcal{A} - \mathcal{B}\mathcal{D}^{-1}\mathcal{C}$ does not pose more problems than that of $\mathcal{A}$. The subgrid unknowns are subsequently given by

$$\mathcal{U}_h^H = \mathcal{D}^{-1}(\mathcal{F}_h^H - \mathcal{C}\,\mathcal{U}_H).$$

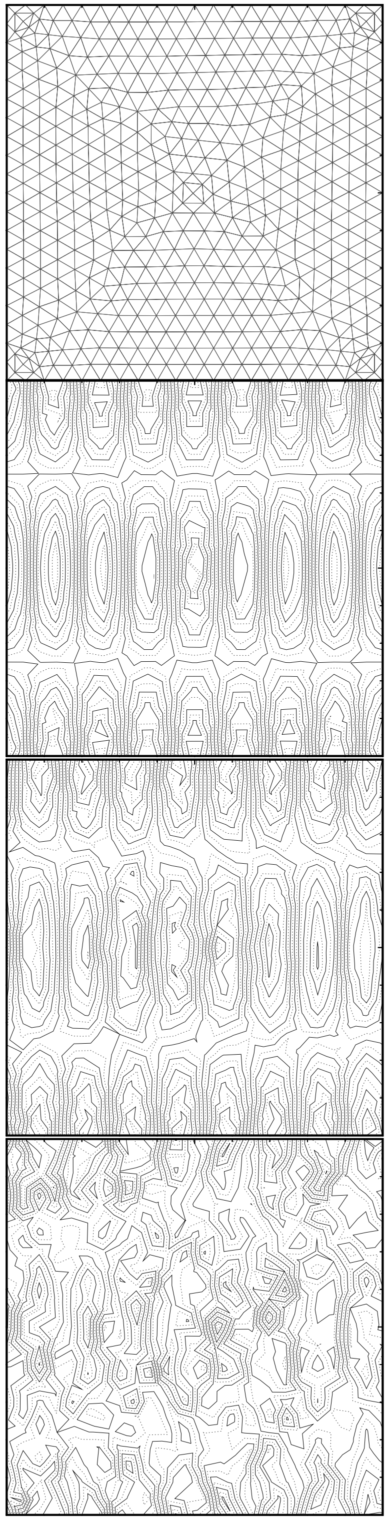

Fig. 2. Problem 1. Solution at $T = 5$: (*top*) $\mathbb{P}_1$ mesh of domain $\Omega =]0,1[^2$, (*top centre*) isovalues of $\mathbb{P}_1$ interpolate of solution, (*bottom centre*) stabilized approximate solution, (*bottom*) Galerkin solution

5.4 Model problem 1

The first problem we consider is

$$\begin{cases} u_{|t=0} = \cos(8\pi x)\cos(2\pi y), \\ \frac{\partial u}{\partial t} + \frac{\partial u}{\partial x} = 0, \qquad \text{in } \Omega =]0,1[^2, \\ \text{periodic boundary conditions.} \end{cases}$$

This problem falls within the general framework of contraction semi-groups with

$$L = L^2(\Omega),$$
$$V = \{v \in L^2(\Omega) \mid \partial_x v \in L^2(\Omega),\, v_{|x=0} = v_{|x=1}\},$$

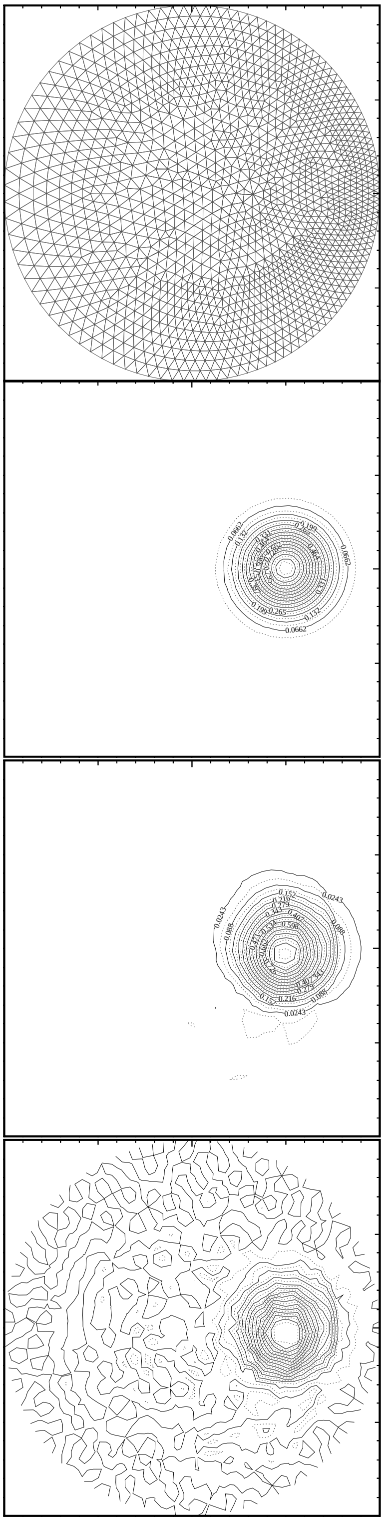

Fig. 3. Cone problem. Solution at $T = 5$: (*top*) $\mathbb{P}_1$ mesh of domain, (*top centre*) isovalues of $\mathbb{P}_1$ interpolate of solution, (*bottom centre*) stabilized approximate solution, (*bottom*) Galerkin solution

and $A = \partial_x$. By setting $\beta = (1, 0)$, Theorem 2 ensures that the discrete inf-sup condition (7) is satisfied with $c_\delta = 0$.

We have performed the numerical tests on a quasi-uniform triangulation composed of 932 simplexes ($H \sim 1/20$, 507 $\mathbb{P}_1$ nodes). The mesh is shown in Fig. 2 (top). To guaranty that the error induced by the time stepping strategy is significantly smaller than that introduced by the space approximation, we have used $\delta t = 10^{-3}$. Tests performed with smaller time steps have confirmed this choice. The total time of integration is $T = 5$; that is to say, the solution has crossed Ω five times. Note that this test is quite demanding since for each wavelength in the x direction there are only 5 to 6 $\mathbb{P}_1$ nodes; that is $\lambda_x/H \approx 5$ or 6.

On Fig. 2, we have plotted the isovalues of the solution at time $T = 5$: (top centre) the isovalues of the $\mathbb{P}_1$ interpolate of the solution; (bottom centre) approximate solution obtained by means of the subgrid stabilized Galerkin technique; (bottom) isovalues of the Galerkin solution. Except for an overall small phase error, the main feature of the stabilized solution are very well preserved at $T = 5$. It is clear that the Galerkin solution has lost all the features of the exact solution.

5.5 *Model problem 2*

The last problem we consider is the rotating cone in the disk: $\Omega = \{(x, y) \in \mathbb{R}^2 \mid \sqrt{x^2 + y^2} < 1\}$,

$$\begin{cases} u_{|t=0} = \exp(-|r - r_0|^2/(0.2)^2), \\ \frac{\partial u}{\partial t} + \beta \cdot \nabla u = 0, \qquad \text{in } \Omega, \end{cases}$$

where $\beta(x, y) = 2\pi \times (-y, x)$, $r = \sqrt{x^2 + y^2}$, and $r_0 = (0.5, 0)$. The solution to this problem consists in a smooth cone rotating about the origin at constant rotation speed. The period of the solution is 1. This problem falls within the $\mathcal{C}^0$ contraction semi-groups framework developed above with

$$A = \beta \cdot \nabla(\cdot),$$
$$L = L^2(\Omega),$$
$$V = \{v \in L^2(\Omega) \mid \beta \cdot \nabla v \in L^2(\Omega)\}.$$

Owing to Theorem 3, the discrete inf-sup condition (7) is satisfied with $c_\delta \neq 0$.

The same time stepping as in the preceding test is used with $\delta t = 10^{-3}$. In Fig. 3, we have plotted the $\mathbb{P}_1$ mesh that we used in the computation (top) together with the isovalues of the solution at $T = 5$: $\mathbb{P}_1$ interpolate of the exact solution (top centre), stabilized solution (bottom centre), Galerkin solution (bottom). This figure shows clearly that the Galerkin technique does not control the graph norm of the solution for it generates very small scales throughout the domain as time evolves. As expected, the stabilized technique preserves the coherence of the cone in time.

6 Concluding remarks

A stabilized Galerkin approximation of linear contraction semi-groups of class $\mathcal{C}^0$ has been proposed. The main result of this paper is that the proposed technique yields an optimal approximation in the graph norm. The convergence proofs given in the present paper assume that the grid is quasi-uniform since uniform inverse inequalities have been used. This hypothesis can be weakened by using the local meshsize in the definition of the artificial diffusion bilinear form b_h and by proceeding as in Guermond [8].

Though the importance of bubble functions for stabilizing PDE's has been recognized for some time (see the paper on multiscale phenomena by [10], the method of the Residual-Free Bubbles by [2, 3, 5]), the importance of the inequality (7) together with the role of bubble functions to prove (7) seems to be new, up to the author's knowledge.

The generalization of the present technique to non-linear conservation laws is being investigated.

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