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Faedo–Galerkin weak solutions of the Navier–Stokes equations with Dirichlet boundary conditions are suitable

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Abstract

Faedo–Galerkin weak solutions of the three-dimensional Navier–Stokes equations supplemented with Dirichlet boundary conditions in bounded domains are suitable in the sense of Scheffer [V. Scheffer, Hausdorff measure and the Navier–Stokes equations, Comm. Math. Phys. 55 (2) (1977) 97–112] provided they are constructed using finite-dimensional approximation spaces having a discrete commutator property and satisfying a proper inf-sup condition. Finite element and wavelet spaces appear to be acceptable for this purpose. This result extends that of [J.-L. Guermond, Finite-element-based Faedo–Galerkin weak solutions to the Navier–Stokes equations in the three-dimensional torus are suitable, J. Math. Pures Appl. (9) 85 (3) (2006) 451–464] where periodic boundary conditions were assumed.

Résumé

On s'intéresse aux solutions faibles des équations de Navier–Stokes avec conditions aux limites de Dirichlet homogènes en dimension trois qui sont construites comme limites d'approximation de Faedo–Galerkin. Ces solutions sont admissibles (*suitable*) au sens de Scheffer [V. Scheffer, Hausdorff measure and the Navier–Stokes equations, Comm. Math. Phys. 55 (2) (1977) 97–112] si les espaces d'approximation jouissent d'une propriété de commutateur discret et satisfont une certaine condition de compatibilité. Les éléments finis et les ondelettes satisfont ces hypothèses. Ce résultat étend celui de [J.-L. Guermond, Finite-element-based Faedo–Galerkin weak solutions to the Navier–Stokes equations in the three-dimensional torus are suitable, J. Math. Pures Appl. (9) 85 (3) (2006) 451–464] qui a été démontré pour des conditions aux limites périodiques.

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1. Introduction

1.1. Position of the problem

Let Ω be a connected, open, bounded domain in $\mathbb{R}^3$. This paper continues the study (initiated in [15]) of the suitable weak solutions to the Navier–Stokes equation in Ω:

$$\begin{cases} \partial_t \mathsf{u} + \mathsf{u} \cdot \nabla \mathsf{u} + \nabla \mathsf{p} - \nu \Delta \mathsf{u} = \mathsf{f} & \text{in } Q_T, \\ \nabla \cdot \mathsf{u} = 0 \quad \text{in } Q_T, \\ \mathsf{u}|_{t=0} = \mathsf{u}_0, \quad \mathsf{u}|_\Gamma = 0, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0,T)$, Γ is the boundary of Ω. Henceforth we assume $\mathsf{f} \in L^2((0,T);\mathbf{H}^{-1}(\Omega))$ and $\mathsf{u}_0 \in \mathbf{V}^0 = \{v \in \mathbf{L}^2(\Omega);\ \nabla \cdot v = 0;\ v \cdot n|_\Gamma = 0\}$. Additional regularity requirements on f and u_0 will be added in Section 3, see (3.14).

The notion of suitable weak solution has been introduced by Scheffer [25] and boils down to the following:

Definition 1.1 *(Scheffer).* Let (u,p), $\mathsf{u} \in L^2((0,T);\mathbf{H}^1(\Omega)) \cap L^\infty((0,T);\mathbf{L}^2(\Omega))$, $\mathsf{p} \in \mathcal{D}'((0,T);L^2(\Omega))$, be a weak solution to the Navier–Stokes equation (1.1).

The pair (u,p) is said to be suitable if the local energy balance,

$$\partial_t\left(\frac{1}{2}\mathsf{u}^2\right) + \nabla \cdot \left(\left(\frac{1}{2}\mathsf{u}^2 + \mathsf{p}\right)\mathsf{u}\right) - \nu\Delta\left(\frac{1}{2}\mathsf{u}^2\right) + \nu(\nabla \mathsf{u})^2 - \mathsf{f} \cdot \mathsf{u} \leqslant 0, \tag{1.2}$$

is satisfied in the distributional sense, i.e., in $\mathcal{D}'(Q_T;\mathbb{R}^+)$.

To the present time, the best partial regularity result for the Navier–Stokes equations, i.e., the so-called Caffarelli–Kohn–Nirenberg theorem [8,21], holds for suitable weak solutions. In a nutshell, this result asserts that the one-dimensional Hausdorff measure of the set of singularities of a suitable weak solution is zero.

The questions we want to investigate in this work is the following: Is the class of suitable weak solutions a proper subclass of weak solutions? This problem seems to have been open since Scheffer introduced the notion of suitable solution. The techniques that are commonly used to construct suitable weak solutions mainly consist of regularizing the Navier–Stokes equations by adding hyperviscosity [3] or regularizing the nonlinear term [2,8,10]. It is remarkable that the weak solutions constructed by Leray [20] are actually suitable. Until recently it was not known whether other ways of constructing suitable solutions existed besides the above explicit regularization tricks. For instance, what can be said of a weak solution that is constructed by means of the Faedo–Galerkin method? In other words, does discretization introduces enough regularization (numericists would say artificial dissipation) to select a suitable solution? In this context, the purpose of the present paper is to partially answer a question raised by Beirão da Veiga [2, p. 321] which asks whether there are "*evidence that solutions obtained by the Faedo–Galerkin method verify the local energy estimate*". A partial positive answer to this question has been given in [15] when Ω is the three-dimensional torus. It is shown in [15] that periodic weak solutions that are limits of Faedo–Galerkin approximations are suitable provided the approximation spaces satisfy a discrete commutator property. We go further in this paper and, assuming again that the discrete commutator property holds, the same conclusion is shown to hold if homogeneous Dirichlet boundary conditions are enforced.

The two main stumbling blocks for proving the local energy estimate are in the passage to the limit in the nonlinear terms $\nabla \cdot (\mathsf{u}^2\mathsf{u})$ and $\nabla \cdot (\mathsf{p}\mathsf{u})$. Whereas the discrete commutator property together with standard a priori estimates is just what it takes to take care of $\nabla \cdot (\mathsf{u}^2\mathsf{u})$, passing to the limit on $\nabla \cdot (\mathsf{p}\mathsf{u})$ requires non-trivial estimates on the pressure. This can be taken care of quite easily in the torus, since the pressure solves a Poisson equation and no boundary conditions are required (see [15, Lemma 3.2]). The matter is significantly more complex when Dirichlet boundary conditions are enforced. The trick consists of reproducing for the discrete pressure a priori estimates that are similar to the $L^p(\mathbf{L}^q)$ estimates of Sohr and von Wahl [26] or Solonnikov [27] (note that p is the pressure and $p \in [1,+\infty]$ is an exponent). But the non-Hilbertian setting being pretty awkward to handle at the discrete level, this program is carried out by making use of the fractional exponents of the discrete Stokes operator and deriving estimates in the $H^\tau(\mathbf{H}^{-\alpha})$-norm.

The paper is organized as follows. The rest of this section is devoted to introducing notation and recalling the definitions of the Leray projector and the Stokes operator. The discrete finite-element-like setting and the Galerkin approximation of (1.1) alluded to above is introduced in Section 2. A priori estimates are derived in Section 3. Velocity estimates are stated in Theorem 3.1 and key estimates on the pressure are stated in Lemma 3.5. The approximate Galerkin solution is shown to converge to a weak solution of (1.1) in Section 4. The main result of this paper is reported in Section 5, where we show that the Galerkin solution converges (up to sequences) to a suitable weak solution of (1.1), see Theorem 5.1. Besides the pressure estimate, the key to this result is that, contrary to spectral bases, finite element and wavelet spaces have a discrete commutator property, see Definition 5.1.

1.2. Notations and conventions

Spaces of $\mathbb{R}^3$-valued functions on Ω are denoted in bold fonts. No notational distinction is made between $\mathbb{R}$-valued and $\mathbb{R}^3$-valued functions. The Euclidean norm in $\mathbb{R}^3$ is denoted by $|\cdot|$. In the following c is a generic constant which may depend on the data f, u_0, ν, Ω, T. The value of c may vary at each occurrence. Whenever E is a normed space, $\|\cdot\|_E$ denotes a norm in E. The scalar product in $L^2(\Omega)$ is denoted with parentheses, i.e., $(v,w) := \int_\Omega v(x)w(x)\,\mathrm{d}x$; the same notation is used for the scalar product in $\mathbf{L}^2(\Omega)$.

For $0<s<1$, the space $H^s(\Omega)$ is defined by the real method of interpolation between $H^1(\Omega)$ and $L^2(\Omega)$, i.e., the so-called K-method of Lions and Peetre [24], see also [23] or [1], [6, Appendix A]. To define $H^s(\Omega)$, we interpolate between $H^1(\Omega)$ and $H^2(\Omega)$ if $1<s<2$. We denote $H_0^s(\Omega)$ to be the closure of $\mathcal{D}(\Omega)$ in $H^s(\Omega)$ for $0<s<1$ and $\widetilde{H}_0^s(\Omega)$ to be the interpolation space $[L^2(\Omega), H_0^1(\Omega)]_s$ for $0 \leqslant s \leqslant 1$ ($\mathcal{D}(\Omega)$ is the space of $\mathcal{C}^\infty$ functions that are compactly supported in Ω). For $s \in (1,2]$, $\widetilde{H}_0^s(\Omega)$ is defined to be $H^s(\Omega) \cap H_0^1(\Omega)$. Note that the spaces $H^s(\Omega)$ and $H_0^s(\Omega)$ coincide for $0 \leqslant s \leqslant \frac{1}{2}$ with uniformly equivalent norms (see [23, Theorem 11.1]). The spaces $H^s(\Omega)$ and $\widetilde{H}_0^s(\Omega)$ coincide for $0 \leqslant s < \frac{1}{2}$ and their norms are equivalent; i.e., there is $c_1>0$ and a non-decreasing function c_u such that

$$c_1\|v\|_{\mathbf{H}^s} \leqslant \|v\|_{\widetilde{\mathbf{H}}_0^s} \leqslant c_u(s)\|v\|_{\mathbf{H}^s}, \quad \forall v \in \widetilde{\mathbf{H}}_0^s,\ \forall s \in \left[0, \frac{1}{2}\right), \tag{1.3}$$

with $\lim_{s\to\frac{1}{2}} c_u(s) = \infty$, see [23, Theorem 11.7].

For negative s, $\widetilde{H}_0^s(\Omega)$ is the dual of $\widetilde{H}_0^{-s}(\Omega)$. The space $H^{-s}(\Omega)$ for $s>0$ is defined by duality, i.e.,

$$\|v\|_{H^{-s}} = \sup_{0\neq w\in\mathcal{D}(\Omega)} \frac{(v,w)}{\|w\|_{H^s}}.$$

For $s \in [0,\frac{1}{2}) \cup (\frac{1}{2},\frac{3}{2})$, H^{-s} coincides with $\widetilde{H}_0^{-s}(\Omega)$. Duality pairing is denoted with brackets, e.g., $\langle f, v\rangle_{H^{-1}(\Omega),H_0^1(\Omega)}$ denotes $f(v)$ for all $f \in H^{-1}(\Omega)$ and all $v \in H_0^1(\Omega)$.

We define $L^2_{\int=0}(\Omega)$ (resp. $H^s_{\int=0}(\Omega)$) to be composed of those functions in $L^2(\Omega)$ (resp. $H^s(\Omega)$, $s\in[0,1]$) that are of zero mean.

We denote by $-\Delta : \mathbf{D}(\Delta) := \mathbf{H}_0^1(\Omega) \cap \mathbf{H}^2(\Omega) \to \mathbf{L}^2(\Omega)$ the unbounded vector-valued Laplace operator supplemented with homogeneous Dirichlet boundary conditions. The boundary of Ω is assumed to be such that the $\mathbf{H}^2$-regularity property of the Laplace operator holds, i.e., there is $c>0$ such that

$$\forall v \in \mathbf{D}(\Delta), \quad \|v\|_{\mathbf{H}^2} \leqslant c\|\Delta v\|_{\mathbf{L}^2}. \tag{1.4}$$

For instance, Ω convex or Ω of class $\mathcal{C}^{1,1}$ are sufficient conditions for this property to hold, cf. e.g. [14]. The boundary of Ω is denoted by Γ.

To account for solenoidal vector fields we set as in [29],

$$\mathbf{V}^0 = \{v \in \mathbf{L}^2(\Omega);\ \nabla\cdot v = 0;\ v\cdot n|_\Gamma = 0\}, \tag{1.5}$$

$$\mathbf{V}^1 = \{v \in \mathbf{H}^1(\Omega);\ \nabla\cdot v = 0;\ v|_\Gamma = 0\}, \tag{1.6}$$

$$\mathbf{V}^2 = \mathbf{V}^1 \cap \mathbf{H}^2(\Omega). \tag{1.7}$$

We denote by $P : \mathbf{L}^2(\Omega) \to \mathbf{V}^0$ the $\mathbf{L}^2$-projection onto $\mathbf{V}^0$ (i.e., the so-called Leray projection). We introduce the Stokes operator $A : \mathbf{D}(A) := \mathbf{V}^2 \to \mathbf{V}^0$ by setting $A = -P\Delta|_{\mathbf{V}^2}$. We assume that the domain Ω is such that there is $c>0$ so that

$$\forall v \in \mathbf{V}^2, \quad \|v\|_{\mathbf{H}^2} \leqslant c\|Av\|_{\mathbf{L}^2}. \tag{1.8}$$

This property holds in two and three dimensions ($d = 2, 3$) whenever Ω is convex or of class $\mathcal{C}^{1,1}$, see [9, Theorem 6.3]. We shall also make use of the discrete counterpart of the following generalization of (1.8),

$$\forall v \in \mathbf{V}^s, \quad c_1\|v\|_{\widetilde{\mathbf{H}}_0^s} \leqslant \left\|A^{\frac{s}{2}} v\right\|_{\mathbf{L}^2} \leqslant c_2\|v\|_{\widetilde{\mathbf{H}}_0^s}, \quad \forall s \in \left(-\frac{1}{2}, 2\right]. \tag{1.9}$$

The reader is referred to [12, Lemma 4.5, Chapter 3] for a proof.

2. The Galerkin approximation

In this section we introduce the discrete setting and we formulate the discrete problem.

2.1. The discrete setting

We assume that we have at hand two families of finite-dimensional spaces, $\{\mathbf{X}_h\}_{h>0}$, $\{M_h\}_{h>0}$ such that $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$ and $M_h \subset L^2_{\int=0}(\Omega)$. The velocity is approximated in $\mathbf{X}_h$ and the pressure in M_h. To avoid irrelevant technicalities we assume $M_h \subset H^1_{\int=0}(\Omega)$.

To characterize the approximation properties of the spaces $\{\mathbf{X}_h\}_{h>0}$, $\{M_h\}_{h>0}$ we assume that

$$\forall v \in \mathbf{H}_0^1(\Omega), \quad \inf_{v_h \in \mathbf{X}_h} \|v - v_h\|_{\mathbf{H}^1} \xrightarrow{h\to 0} 0, \tag{2.1}$$

$$\forall q \in L^2(\Omega), \quad \inf_{q_h \in M_h} \|q - q_h\|_{L^2} \xrightarrow{h\to 0} 0. \tag{2.2}$$

These hypotheses are standard in the case of finite elements.

2.2. The discrete Stokes operator and H^s-stability

We define the discrete Laplace operator $\Delta_h : \mathbf{X}_h \to \mathbf{X}_h$ as follows:

$$(\Delta_h x_h, y_h) = -(\nabla x_h, \nabla y_h), \quad \forall x_h, y_h \in \mathbf{X}_h.$$

To account for the solenoidality constraint we set:

$$\mathbf{V}_h = \left\{v_h \in \mathbf{X}_h;\ (v_h, \nabla q_h) = 0,\ \forall q_h \in M_h\right\}. \tag{2.3}$$

$\mathbf{V}_h$ is composed of the fields of $\mathbf{X}_h$ that are discretely divergence free. This allows us to define the discrete Stokes operator $A_h : \mathbf{V}_h \to \mathbf{V}_h$ as follows: For all $u_h \in \mathbf{V}_h$, $A_h u_h$ is the element of $\mathbf{V}_h$ such that

$$(A_h u_h, v_h) = (\nabla u_h, \nabla v_h), \quad \forall v_h \in \mathbf{V}_h. \tag{2.4}$$

Since A_h is self-adjoint and positive definite, the operator A_h^s is well defined for all $s \in \mathbb{R}$. We equip the vector space $\mathbf{V}_h$ with the norm:

$$\|v_h\|_{\mathbf{V}_h^s} = \left(A_h^s v_h, v_h\right)^{\frac{1}{2}}, \tag{2.5}$$

and we denote by $\mathbf{V}_h^s$ the corresponding normed (Hilbert) space. It is clear that $\{\mathbf{V}_h^s\}_{s\in\mathbb{R}}$ is a Hilbert scale in the sense of the K-interpolation method.

We now assume that the discrete setting is such that there is a positive non-increasing function $c_1 > 0$ uniform in h so that

$$c_1(s)\|\Delta_h v_h\|_{\widetilde{\mathbf{H}}_0^{-s}} \leqslant \|A_h v_h\|_{\mathbf{V}_h^{-s}}, \quad \forall v_h \in \mathbf{V}_h,\ s \in \left[0, \frac{3}{2}\right). \tag{2.6}$$

Observe that for $s = 0$ this a discrete counterpart of (1.8). This inequality can be proved to hold for a fairly general set of finite element spaces (see e.g. [18, Corollary 4.4] for the case $s = 0$ or [17] for the general case).

We also assume that there is a positive function $c_2 > 0$ uniform in h, non-decreasing for negative arguments and non-increasing for positive arguments, so that

$$c_2(s)\|v_h\|_{\widetilde{\mathbf{H}}_0^s} \leqslant \|v_h\|_{\mathbf{V}_h^s}, \quad \forall v_h \in \mathbf{V}_h,\ \forall s \in \left(-\frac{1}{2}, \frac{3}{2}\right). \tag{2.7}$$

This is the discrete counterpart of (1.9) but it is slightly less trivial to prove than (2.6). It is shown to hold for various finite-element-like settings in [17].

2.3. The LBB condition in H^s

We finally assume that the pair $(\mathbf{X}_h, M_h)$ is compatible in the sense that it satisfies a generalized LBB condition: There is c, independent of h, such that for all $s \in [0,1]$,

$$\sup_{0\neq v_h\in \mathbf{X}_h} \frac{(\nabla q_h, v_h)}{\|v_h\|_{\widetilde{\mathbf{H}}_0^{1-s}}} \geqslant c\|q_h\|_{H^s}, \quad \forall q_h \in M_h. \tag{2.8}$$

When $s=0$, the above inequality is standard and is often referred to in the literature as the Ladyzhenskaya–Babuška–Brezzi condition, see e.g. [7,13]. The more general case, $s \in [0,1]$, is quite new and has been shown to hold in [16, Theorem 3.1] for various pairs of finite element spaces, e.g. the MINI finite element and the Hood–Taylor finite element.

2.4. The discrete problem

Since $\mathbf{V}_h$ is not a subspace of $\mathbf{V}$, i.e., $\mathbf{V}_h$ is not composed of solenoidal vector-fields, we modify the nonlinear term in the Navier–Stokes equations as follows. We introduce a bilinear operator $nl_h \in \mathcal{L}([\mathbf{H}_0^1(\Omega)]^2; \mathbf{H}^{-1}(\Omega))$, and we define the trilinear form $b_h \in \mathcal{L}([\mathbf{H}_0^1(\Omega)]^3;\mathbb{R})$ such that $b_h(u,v,w) = \langle (nl_h(u,v), w\rangle_{\mathbf{H}^{-1},\mathbf{H}_0^1}$. We assume that b_h satisfies the following property:

$$b_h(u,v,v) = 0, \quad \forall v \in \mathbf{V}+\mathbf{V}_h. \tag{2.9}$$

For instance, an admissible form of the nonlinear term is as follows (see e.g. [28])

$$nl_h(u,v) = u\cdot\nabla v + \frac{1}{2} v\nabla\cdot u. \tag{2.10}$$

Let $\mathcal{K}_h : L^2(\Omega) \to M_h \oplus \operatorname{span}\{1\}$ be a linear L^2-stable interpolation operator (i.e., $\mathcal{K}_h z \to z$ for all $z \in L^2(\Omega)$), then another admissible form of the nonlinear term is:

$$nl_h(u,v) = (\nabla\times u)\times v + \frac{1}{2}\nabla\big(\mathcal{K}_h(u\cdot v)\big). \tag{2.11}$$

The discrete problem we henceforth consider is as follows: Seek $u_h \in \mathcal{C}^0([0,T];\mathbf{X}_h)$ with $\partial_t u_h \in L^2((0,T);\mathbf{X}_h)$ and $p_h \in L^2((0,T);M_h)$ such that for all $v_h \in \mathbf{X}_h$, all $q_h \in M_h$, a.e. $t \in [0,T]$:

$$\begin{cases} (\partial_t u_h, v_h) + b_h(u_h,u_h,v_h) - (p_h, \nabla\cdot v_h) + \nu(\nabla u_h, \nabla v_h) = \langle \mathsf{f}, v_h\rangle, \\ (\nabla\cdot u_h, q) = 0, \\ u_h|_{t=0} = P_h \mathsf{u}_0. \end{cases} \tag{2.12}$$

Note that for all v_h in $\mathbf{X}_h$ the approximate momentum equation holds in $L^2(0,T)$.

3. A priori estimates

In this section we derive a priori estimates on the velocity and the pressure. The main result of this section are the velocity estimates in Theorem 3.1 and the pressure estimates stated in Lemma 3.5.

3.1. Energy estimates

Owing to the skew-symmetry property (2.9), the following standard a priori energy estimates on u_h are easily deduced:

$$\max_{0\leqslant t\leqslant T} \big\|u_h(t)\big\|_{\mathbf{L}^2} + \|u_h\|_{L^2(\mathbf{H}^1)} \leqslant c. \tag{3.1}$$

The following immediately follows:

Lemma 3.1. *Under the above assumptions on* f *and* u_0*, there is c, independent of h, such that*

$$\|u_h\|_{L^r(\mathbf{H}^{\frac{2}{r}})} + \|u_h\|_{L^r(\mathbf{L}^k)} \leqslant c, \quad \text{with } \frac{3}{k} + \frac{2}{r} = \frac{3}{2},\ 2 \leqslant r,\ 2 \leqslant k \leqslant 6. \tag{3.2}$$

Proof. This result is standard and is a consequence of the interpolation inequality (see e.g. Lions and Peetre [24]), $\|v\|_{\mathbf{H}^{\frac{2}{r}}} \leqslant c\|v\|_{\mathbf{L}^2}^{1-\frac{2}{r}} \|v\|_{\mathbf{H}^1}^{\frac{2}{r}}$, when $2 \leqslant r$, and the embedding $\mathbf{H}^{\frac{2}{r}}(\Omega) \subset \mathbf{L}^k(\Omega)$ for $\frac{1}{k} = \frac{1}{2} - \frac{2}{3r}$. □

3.2. More estimates

At variance with what has been done for the periodic situation in [15], it is not possible to immediately infer from the above velocity estimates a bound on the approximate pressure by solving an approximate Poisson equation, since no (easily controllable) boundary condition on the pressure is at hand.

The alternative path we are going to follow is to take inspiration from Solonnikov [27] and Sohr and von Wahl [26]. The idea is to put the nonlinear term in the right-hand side and deduce estimates by using properties of the time-dependent Stokes equations. For instance, in the space continuous situation (3.2) implies that $\mathsf{u} \cdot \nabla \mathsf{u}$ is in $L^p((0,T); \mathbf{L}^q(\Omega))$ where p and q satisfy the equality $\frac{2}{p} + \frac{3}{q} = 4$, $1 \leqslant p \leqslant 2$, $1 \leqslant q \leqslant \frac{3}{2}$. Hence, provided $\mathsf{f} \in L^p((0,T); \mathbf{L}^q(\Omega))$, the right-hand side of the time-dependent Stokes problem, say $\mathsf{f} - \mathsf{u} \cdot \nabla \mathsf{u}$, is in $L^p((0,T); \mathbf{L}^q(\Omega))$. Then under the additional condition $p > 1$ and $q > 1$, Sohr and von Wahl [26] have shown that

$$\|\nabla \mathsf{p}\|_{L^p((0,T);\mathbf{L}^q)} + \|\partial_t \mathsf{u}\|_{L^p((0,T);\mathbf{L}^q)} + \|\Delta \mathsf{u}\|_{L^p((0,T);\mathbf{L}^q)} \leqslant c\|\mathsf{f}\|_{L^p((0,T);\mathbf{L}^q)}, \tag{3.3}$$

$$\|\mathsf{p}\|_{L^p((0,T);\mathbf{L}^\ell)} \leqslant c\|\mathsf{f}\|_{L^p((0,T);\mathbf{L}^q)}, \tag{3.4}$$

where $\frac{1}{\ell} := \frac{1}{q} - \frac{1}{3}$. Then, using $\ell = 2$, one infers that p is in $L^{\frac{4}{3}}((0,T); L^2(\Omega))$. This together with u being in $L^\infty((0,T); \mathbf{L}^2(\Omega))$ implies that the term $\nabla \cdot (\mathsf{p}\mathsf{u})$ is meaningful in $\mathcal{D}'(\Omega_T)$ and if we were able to reproduce (3.4) for the discrete pressure p_h, we could pass to the limit on $\nabla \cdot (p_h u_h)$. Unfortunately, obtaining a discrete version of (3.4) requires an $L^p(\mathbf{L}^q)$ theory of the resolvent of the discrete Stokes operator which is not available at the present time, to the best of the author's knowledge. To go around this difficulty, we are going to work with the Hilbertian setting and use the theory that has been developed in [17]. The idea is to use the Fourier transform in time as done in Lions [22, p. 77] to evaluate regularity in time.

Let H be a Hilbert space with norm $\|\cdot\|_H$. Let $\delta \in [1,\infty)$, and define $L^\delta(\mathbb{R}; H) = \{\psi\colon\ \mathbb{R} \ni t \mapsto \psi(t) \in H;\ \int_{-\infty}^{+\infty} \|\psi(t)\|_H^\delta \,\mathrm{d}t < \infty\}$. For all $\psi \in L^1(\mathbb{R}; H)$, denote by $\hat{\psi}(k) = \int_{-\infty}^{+\infty} \psi(t) \mathrm{e}^{-2\mathrm{i}\pi k t} \,\mathrm{d}t$ for all $k \in \mathbb{R}$ the Fourier transform of ψ with respect to t. The notion of Fourier transform is extended to the space of tempered distributions with values in H, say $\mathcal{S}'(\mathbb{R}; H)$. We shall make use of the following

Lemma 3.2 *(Hausdorff–Young inequality). There is* $c > 0$ *such that for all* $p, \delta \in [1,2]$*, and for all* $\psi \in L^\delta(\mathbb{R}; H) \cap L^1(\mathbb{R}; H)$*,*

$$\|\hat{\psi}\|_{L^{\delta'}(\mathbb{R};H)} \leqslant c\|\psi\|_{L^\delta(\mathbb{R};H)}, \quad \frac{1}{\delta} + \frac{1}{\delta'} = 1. \tag{3.5}$$

Following [23, p. 21], we now define:

$$H^\gamma(\mathbb{R}; H) = \left\{ v \in \mathcal{S}'(\mathbb{R}; H);\ \int_{-\infty}^{+\infty} \big(1 + |k|\big)^{2\gamma} \|\hat{v}\|_H^2 \,\mathrm{d}k < +\infty \right\}. \tag{3.6}$$

The space $H^\gamma((0,T); H)$ is composed of those tempered distributions in $\mathcal{S}'((0,T); H)$ that can be extended to $\mathcal{S}'(\mathbb{R}; H)$ and whose extension is in $H^\gamma(\mathbb{R}; H)$. The norm in $H^\gamma((0,T); H)$ is the quotient norm, i.e.,

$$\|v\|_{H^\gamma((0,T);H)} = \inf_{\substack{\tilde{v}=u \\ \text{a.e. on}(0,T)}} \|\tilde{v}\|_{H^\gamma(\mathbb{R};H)}. \tag{3.7}$$

Observing that $q \in [1, \frac{3}{2}]$ and $p \in [1, 2]$, we set, $s = d(\frac{1}{q} - \frac{1}{2})$, $\bar{r} = \frac{1}{p} - \frac{1}{2}$. Then the embedding $\widetilde{\mathbf{H}}_0^s(\Omega) \subset \mathbf{H}^s(\Omega) \subset \mathbf{L}^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} = 1$, and the Hausdorff–Young inequality imply:

$$\mathsf{f} - \mathsf{u} \cdot \nabla \mathsf{u} \in L^p\big((0,T); \mathbf{L}^q(\Omega)\big) \subset H^{-r}\big((0,T); \widetilde{\mathbf{H}}_0^{-s}(\Omega)\big), \quad \forall r > \bar{r}. \tag{3.8}$$

Hence our goal is to derive estimates in spaces like $H^{-r}((0,T); \widetilde{\mathbf{H}}_0^{-s}(\Omega))$.

Henceforth p, q, $\bar{r}$, and s are real numbers satisfying:

$$\frac{2}{p} + \frac{3}{q} = 4, \quad p \in [1,2], \quad q \in \left[1, \frac{3}{2}\right], \quad \frac{s}{3} := \frac{1}{q} - \frac{1}{2}, \quad \bar{r} := \frac{1}{p} - \frac{1}{2}. \tag{3.9}$$

The condition $q \in [1, \frac{3}{2}]$ implies that

$$s \in \left[\frac{1}{2}, \frac{3}{2}\right]. \tag{3.10}$$

Note in particular that this implies $s \geqslant \frac{1}{2}$. This fact will have important consequences in the sequel.

We now make the following continuity hypothesis on nl_h: There is c, independent of h, such that

$$\big\|nl_h(v,v)\big\|_{\widetilde{\mathbf{H}}_0^{-s}} \leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k}, \quad \frac{1}{k} + \frac{1}{2} = \frac{1}{q}, \ \forall v \in \mathbf{H}^1(\Omega). \tag{3.11}$$

This is justified by the following:

Lemma 3.3. *The continuity property* (3.11) *holds for definition* (2.10) *and also for* (2.11) *provided the operator $\mathcal{K}_h$ is uniformly stable in $H^{1-s}(\Omega)$.*

Proof. (1) Let us assume that definition (2.10) holds. Using the embedding $\widetilde{\mathbf{H}}_0^s(\Omega) \subset \mathbf{H}^s(\Omega) \subset \mathbf{L}^{q'}(\Omega)$, $\frac{1}{q} + \frac{1}{q'} := 1$, we infer that definition (2.10) yields

$$\big\|nl_h(v,v)\big\|_{\widetilde{\mathbf{H}}_0^{-s}} \leqslant c\big\|nl_h(v,v)\big\|_{\mathbf{L}^q} \leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k}, \quad \frac{1}{k} + \frac{1}{2} = \frac{1}{q}.$$

(2) Let us now assume that definition (2.11) holds. Using the same argument as above, the fact that $\mathcal{K}_h$ is uniformly stable in $H^{1-s}(\Omega)$, and the embedding $\mathbf{W}^{1,q}(\Omega) \subset \mathbf{H}^{1-s}(\Omega)$, definition (2.11) yields:

$$\begin{aligned}
\big\|nl_h(v,v)\big\|_{\widetilde{\mathbf{H}}_0^{-s}} &\leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k} + c'\big\|\nabla\mathcal{K}^h(v\cdot v)\big\|_{\widetilde{\mathbf{H}}_0^{-s}}, \quad \frac{1}{k} + \frac{1}{2} = \frac{1}{q} \\
&\leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k} + c'\big\|\mathcal{K}^h(v\cdot v)\big\|_{\mathbf{H}^{1-s}}, \quad -\frac{1}{2} \leqslant s - 1 \leqslant \frac{1}{2} \\
&\leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k} + c'\|v\cdot v\|_{\mathbf{H}^{1-s}} \\
&\leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k} + c'\|v\cdot v\|_{\mathbf{W}^{1,q}} \leqslant c\|v\|_{\mathbf{H}^1}\|v\|_{\mathbf{L}^k}.
\end{aligned}$$

This completes the proof. □

A simple application of the $L^{\bar{r}}(\mathbf{L}^k)$ estimate in (3.2) yields the uniform bound,

$$\big\|nl_h(u_h,u_h)\big\|_{L^p(\widetilde{\mathbf{H}}_0^{-s})} \leqslant c. \tag{3.12}$$

As an immediate consequence of (3.11) and (3.12), we have:

Lemma 3.4. *Let u_h solve* (2.12)*, then under the hypothesis* (3.11)*, there is c uniform in h such that*

$$\big\|nl_h(u_h,u_h)\big\|_{H^{-r}(\widetilde{\mathbf{H}}_0^{-s})} \leqslant c, \quad \forall r > \bar{r}. \tag{3.13}$$

Proof. Extend $nl_h(v,v)$ by zero outside $[0,T]$. Using the Hausdorff–Young inequality, we have:

$$\begin{aligned}\left\|nl_h(u_h,u_h)\right\|^2_{H^{-r}(\widetilde{\mathbf{H}}_0^{-s})} &= \int_{-\infty}^{+\infty} \left(1+|k|\right)^{-2r}\left\|\widehat{nl_h}(u_h,u_h)\right\|^2_{\widetilde{\mathbf{H}}_0^{-s}}\,\mathrm{d}k \\ &\leqslant \left\|\left(1+|k|\right)^{-2r}\right\|_{L^\beta}\left\|\widehat{nl_h}(u_h,u_h)\right\|^2_{L^{p'}(\widetilde{\mathbf{H}}_0^{-s})}, \quad \frac{1}{2\beta}=\bar{r} \\ &\leqslant c\left\|nl_h(u_h,u_h)\right\|^2_{L^p(\widetilde{\mathbf{H}}_0^{-s})},\end{aligned}$$

then conclude using (3.12). □

Remark 3.1. The hypothesis $\mathcal{K}_h$ being stable in $H^{1-s}(\Omega)$ in Lemma 3.3 is not restrictive. For finite elements for instance, the L^2-projection onto M_h is known to be $H^1(\Omega)$-stable under weak assumptions on the mesh, see e.g. [5].

3.3. Estimate on $\partial_t u_h$ and $A_h u_h$

To avoid unnecessary additional technicalities we henceforth assume

$$\mathsf{f} \in L^2\big((0,T+1);\mathbf{H}^{-1}(\Omega)\big)\cap L^p\big((0,T+1);\mathbf{L}^q(\Omega)\big), \quad \text{and} \quad \mathsf{u}_0\in\mathbf{V}^2. \tag{3.14}$$

Obviously, the estimates (3.1), (3.2), and (3.13) uniformly hold on the interval $[0,T+1]$. We set $u_{h0} := P_h\mathsf{u}_0$.

The main result of this section is the following:

Theorem 3.1. *For all $s\in[\frac{1}{2},\frac{3}{2})$, there is c independent of h so that,*

$$\|\partial_t u_h\|_{H^{\tau-1}(\widetilde{\mathbf{H}}_0^{-\alpha})} + \|u_h\|_{H^\tau(\widetilde{\mathbf{H}}_0^{-\alpha})} \leqslant c, \tag{3.15}$$

for all α, $0\leqslant\alpha\leqslant s\leqslant 1+2\alpha<2$, and for all $\tau<\bar{\tau}:=\frac{1+\alpha}{1+s}(\frac{s}{2}+\frac{1}{4})$. And

$$\|\Delta_h u_h\|_{H^{-r}(\widetilde{\mathbf{H}}_0^{-s})}\leqslant c, \quad \forall r>\bar{r}=\frac{3}{4}-\frac{s}{2}=\frac{1}{p}-\frac{1}{2}, \tag{3.16}$$

Proof. (1) *Extension*. We extend f by zero on $(-\infty,0]$ and $[T+1,+\infty)$, and we slightly abuse the notation by still denoting this extension by f. We extend u_h on $[-1,0]$ by $(t+1)u_{h0}$ and we extend u_h on $[T+1,+\infty]$ by zero. We still denote this extension by u_h. Let $\varphi\in\mathcal{C}^\infty(\mathbb{R})$ be an infinitely smooth function compactly supported on $(-1,T+1)$ and equal to 1 on $[0,T]$. We now set:

$$\tilde{u}_h=\varphi u_h, \quad \text{and} \quad \tilde{\mathsf{f}}=\begin{cases}(1+t)\varphi' u_{h0}+\varphi u_{h0}+\nu(1+t)\varphi A_h u_{h0}, & t\in(-1,0),\\ \varphi(\mathsf{f}-nl_h(u_h,u_h))+\varphi' u_h, & \text{otherwise.}\end{cases}$$

It is clear that $\tilde{u}_h$ and $\tilde{\mathsf{f}}$ are well defined on the time interval $(-\infty,+\infty)$. Moreover, the estimate (3.13) and the hypothesis (3.14) imply that $\|\tilde{\mathsf{f}}\|_{H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s})}$ is uniformly bounded. The approximate problem takes the following form in $\mathcal{S}'(\mathbb{R};\mathbf{V}_h)$:

$$\partial_t\tilde{u}_h+\nu A_h\tilde{u}_h=P_h\tilde{\mathsf{f}}.$$

Then, denoting by $\hat{u}_h$ and $\hat{\mathsf{f}}$ the Fourier transform of $\tilde{u}_h$ and $\tilde{\mathsf{f}}$, respectively, and upon taking the Fourier transform of the above equation, we obtain:

$$2\mathrm{i}\pi k\hat{u}_h+\nu A_h\hat{u}_h=P_h\hat{\mathsf{f}}. \tag{3.17}$$

(2) *Bound on u_h*. Let $\alpha\in\mathbb{R}^+$. Testing the above equation with the complex conjugate of $A_h^{-\alpha}\hat{u}_h$ and taking the imaginary part of the result yields:

$$2\pi|k|\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}}\leqslant\|\hat{\mathsf{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}}\left\|A_h^{-\alpha}u_h\right\|_{\widetilde{\mathbf{H}}_0^{s}}.$$

Using the bound in (2.7) for $s\in[0,\frac{3}{2})$, we obtain:

$$|k|\|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}}\leqslant c\|\hat{\mathsf{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}}\left\|A_h^{-\alpha}\hat{u}_h\right\|_{\widetilde{\mathbf{H}}_0^{s}}\leqslant c'\|\hat{\mathsf{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}}\left\|A_h^{-\alpha}\hat{u}_h\right\|_{\mathbf{V}_h^{s}}\leqslant c\|\hat{\mathsf{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}}\|\hat{u}_h\|_{\mathbf{V}_h^{s-2\alpha}}.$$

Assume $\alpha \leqslant s \leqslant 1+2\alpha$, then by interpolation we obtain:

$$\|\hat{u}_h\|_{\mathbf{V}_h^{s-2\alpha}} \leqslant \|\hat{u}_h\|^{\gamma}_{\mathbf{V}_h^{-\alpha}} \|\hat{u}_h\|^{1-\gamma}_{\mathbf{V}_h^1},$$

where $\gamma = \frac{2\alpha+1-s}{1+\alpha}$. Inserting this inequality in the previous estimate yields:

$$|k| \|\hat{u}_h\|^{2-\gamma}_{\mathbf{V}_h^{-\alpha}} \leqslant c' \|\hat{\mathrm{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}} \|\hat{u}_h\|^{1-\gamma}_{\mathbf{V}_h^1}.$$

This in turn implies:

$$|k|^{\frac{2}{2-\gamma}-\mu} \|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \leqslant c\big(1+|k|\big)^{-\mu} \|\hat{\mathrm{f}}\|^{\frac{2}{2-\gamma}}_{\widetilde{\mathbf{H}}_0^{-s}} \|\hat{u}_h\|^{\frac{2(1-\gamma)}{2-\gamma}}_{\mathbf{V}_h^1},$$

and we observe that if we take $\mu = \frac{2r}{2-\gamma}$, by integrating over $\mathbb{R}$ with respect to k, we obtain:

$$\int_{-\infty}^{+\infty} |k|^{\frac{2}{2-\gamma}-\mu} \|\hat{u}_h\|^2_{\mathbf{V}_h^{-\alpha}} \,\mathrm{d}k \leqslant c \|\hat{\mathrm{f}}\|^{\frac{2}{2-\gamma}}_{H^{-r}(\widetilde{\mathbf{H}}_0^{-s})} \|\hat{u}_h\|^{\frac{2(1-\gamma)}{2-\gamma}}_{L^2(\mathbf{V}_h^1)}.$$

Assume now that $\alpha \in [0, \frac{1}{2})$ so as to be able to use the norm equivalence (2.7). The fact that u_h is uniformly bounded in $L^2((0,T);\mathbf{H}^1(\Omega))$ then implies:

$$\|\partial_t u_h\|_{H^{\tau-1}((0,T);\widetilde{\mathbf{H}}_0^{-\alpha})} + \|u_h\|_{H^{\tau}((0,T);\widetilde{\mathbf{H}}_0^{-\alpha})} \leqslant c, \tag{3.18}$$

where

$$\tau < \bar{\tau} = \frac{1+\alpha}{1+s}(1-\bar{r}), \quad \text{and} \quad \alpha \leqslant s \leqslant 1+2\alpha < 2.$$

(3) Bound on $A_h u_h$. Multiply (3.17) by $A_h^{1-s}\hat{u}_h$ and take the real part to obtain:

$$\nu \|A_h \hat{u}_h\|^2_{\mathbf{V}_h^{-s}} \leqslant \|\hat{\mathrm{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}} \|A_h^{1-s}\hat{u}_h\|_{\widetilde{\mathbf{H}}_0^{s}} \leqslant c\|\hat{\mathrm{f}}\|_{\widetilde{\mathbf{H}}_0^{-s}} \|A_h \hat{u}_h\|_{\mathbf{V}_h^{-s}}.$$

Note that we used again the lower bound in (2.7) for $s \in [0, \frac{3}{2})$. This in turn implies:

$$\frac{1}{(1+|k|)^{2r}} \|\hat{u}_h\|^2_{\mathbf{V}_h^{-s}} \leqslant c \frac{1}{(1+|k|)^{2r}} \|\hat{\mathrm{f}}\|^2_{\widetilde{\mathbf{H}}_0^{-s}}.$$

We now use (2.6) to deduce:

$$\|\Delta_h u_h\|_{H^{-r}((0,T);\widetilde{\mathbf{H}}_0^{-s})} \leqslant c, \tag{3.19}$$

for all $r > \bar{r}$. □

Corollary 3.1. *For all $\alpha \in [\frac{1}{4}, \frac{1}{2})$, there is c independent of h so that*

$$\|\partial_t u_h\|_{H^{\tau-1}(\widetilde{\mathbf{H}}_0^{-\alpha})} + \|u_h\|_{H^{\tau}(\widetilde{\mathbf{H}}_0^{-\alpha})} \leqslant c, \tag{3.20}$$

and for all $\tau < \bar{\tau} := \frac{2}{5}(1+\alpha)$.

Proof. Observe that $s \in [\frac{1}{2}, \frac{3}{2})$ is a free parameter in (3.15) and $\frac{1}{1+s}(\frac{s}{2}+\frac{1}{4})$ is maximum at $s = \frac{3}{2}$, the maximum being $\frac{2}{5}$. The condition $s \leqslant 1+2\alpha$ then yields $\alpha \geqslant \frac{1}{4}$. □

Remark 3.2. It is a remarkable fact that the present analysis does not yield estimates on $\partial_t u_h$ in $H^{\tau-1}((0,T);\mathbf{H}^{-\alpha}(\Omega))$ with $\alpha \in [\frac{1}{2}, 1]$. The fundamental reason being that the inequality (2.7) does not hold for negative exponents in the interval $[-1, -\frac{1}{2}]$. This is a bit strange, since if we were nevertheless to apply (3.15) with $\alpha = s = 1$, that would give $\bar{r} = \frac{3}{4}$, i.e., $\bar{r} - 1 = -\frac{1}{4}$. In other words, we would obtain an estimate on $\partial_t u_h$ in $H^{-\frac{1}{4}-\varepsilon}((0,T);\mathbf{H}^{-1}(\Omega))$, which would be compatible with the estimate in $L^{\frac{4}{3}}((0,T);\mathbf{L}^{\frac{6}{5}}(\Omega))$ which is expected to hold from Sohr and von

Wahl [26]. Nevertheless, whether (3.20) might be true for $\alpha \in [\frac{1}{2}, 1]$ is not clear at all. For instance it is shown in [19] that the ratio $\|\partial_t u_h\|_{\mathbf{H}^{-1}}/\|\Delta u_h\|_{\mathbf{H}^{-1}}$ can be arbitrarily large, i.e., $\|\partial_t u_h\|_{\mathbf{H}^{-1}}$ and $\|\Delta u_h\|_{\mathbf{H}^{-1}}$ do not behave similarly. This result seems to point in the direction that (3.20) may not hold for $-\alpha \in [-1, -\frac{1}{2}]$.

Remark 3.3. Note that the estimates (3.20) are in some sense slightly stronger than what can be deduced from the Sohr and Von Wahl estimates (3.3) using embeddings and the Hausdorff–Young inequality. For instance $q = \frac{3}{2}$ and $p = 1$ in (3.3) gives a bound on $\partial_t \mathsf{u}$ in $H^{-\frac{1}{2}-\varepsilon}((0,T); \mathbf{H}^{-\frac{1}{2}-\varepsilon'}(\Omega))$ whereas (3.20) gives a bound in $H^{-\frac{2}{5}-\varepsilon}((0,T); \mathbf{H}^{-\frac{1}{2}-\varepsilon'}(\Omega))$, which is better since $\frac{2}{5} < \frac{1}{2}$. This leads us to conjecture that it may be possible to prove an estimate on $\partial_t \mathsf{u}$ in $L^{\frac{10}{9}}((0,T); \mathbf{L}^{\frac{3}{2}}(\Omega))$ (note $[H^{-\frac{2}{5}}(0,T)]' = H^{\frac{2}{5}}(0,T) \subset L^{10}(0,T) = [L^{\frac{10}{9}}(0,T)]'$), which would be a slight improvement over the estimate in $L^{1+\varepsilon}((0,T); \mathbf{L}^{\frac{3}{2}-\varepsilon'}(\Omega))$ given by (3.3).

3.4. Pressure estimate

As usual, an estimate on the pressure is obtained by using the equation:

$$(\nabla p_h, v_h) = -(\partial_t u_h, v_h) - \big(nl_h(u_h, u_h), v_h\big) - \nu(\nabla u_h, \nabla v_h) + \langle \mathsf{f}, v_h\rangle, \tag{3.21}$$

which holds for all $v_h \in \mathbf{X}_h$.

Lemma 3.5. *There is c independent of h such that for $s \in [\frac{1}{2}, \frac{7}{10}]$,*

$$\|p_h\|_{H^{-r}(H^{1-s})} \leqslant c, \tag{3.22}$$

for all $r > \bar{r} = \frac{3}{4} - \frac{s}{2}$.

Proof. Let $s \in [\frac{1}{2}, \frac{7}{10}]$ and let $\varepsilon > 0$ be a small positive real number. Using (2.8) together with (3.21) we deduce:

$$\begin{aligned}\|p_h\|_{H^{1-s}} &\leqslant c \sup_{0 \neq v_h \in \mathbf{X}_h} \frac{(\nabla p_h, v_h)}{\|v_h\|_{\widetilde{\mathbf{H}}_0^s}} \\ &\leqslant c\big(\|\partial_t u_h\|_{\widetilde{\mathbf{H}}_0^{-s}} + \|\Delta u_h\|_{\widetilde{\mathbf{H}}_0^{-s}} + \big\|nl_h(u_h,u_h)\big\|_{\widetilde{\mathbf{H}}_0^{-s}} + \|\mathsf{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\big) \\ &\leqslant c\big(\|\partial_t u_h\|_{\widetilde{\mathbf{H}}_0^{-\frac{1}{2}+\frac{5}{4}\varepsilon}} + \|\Delta u_h\|_{\widetilde{\mathbf{H}}_0^{-s}} + \big\|nl_h(u_h,u_h)\big\|_{\widetilde{\mathbf{H}}_0^{-s}} + \|\mathsf{f}\|_{\widetilde{\mathbf{H}}_0^{-s}}\big).\end{aligned}$$

Let $r > \bar{r} = \frac{3}{4} - \frac{s}{2}$. We can choose $\varepsilon > 0$ such that $r \geqslant \frac{3}{4} - \frac{s}{2} + \varepsilon$. Moreover, since $\frac{7}{10} \geqslant s$, we have $r \geqslant \frac{2}{5} + \varepsilon$. Then using (3.13) and (3.16) we deduce:

$$\|p_h\|_{H^{-r}(H^{1-s})} \leqslant c_1 + c_2\|\partial_t u_h\|_{H^{-(\frac{2}{5}+\varepsilon)}(\widetilde{\mathbf{H}}_0^{-\frac{1}{2}+\frac{5}{4}\varepsilon})}.$$

Using $\alpha = \frac{1}{2} - \frac{5}{4}\varepsilon$ in (3.20) we obtain that $\|\partial_t u_h\|_{H^{-(\frac{2}{5}+\varepsilon)}(\widetilde{\mathbf{H}}_0^{-\frac{1}{2}+\frac{5}{4}\varepsilon})}$ is bounded since $\frac{2}{5} + \varepsilon > \frac{2}{5} + \frac{\varepsilon}{2}$ (note that we can always take $\varepsilon \leqslant \frac{1}{5}$ to guaranty $\frac{1}{4} \leqslant \frac{1}{2} - \frac{5}{4}\varepsilon < \frac{1}{2}$). □

At this point the end is in sight. The main difficulty that remains in the way consists of proving that the pair (u_h, p_h) converges to a suitable weak solution by passing to the limit on the product $p_h u_h$. With the estimates we have at hand, this is now possible. Indeed, set $\alpha = 1 - s$ with $s \in [\frac{1}{2}, \frac{7}{10}]$, then $\widetilde{\mathbf{H}}_0^{-\alpha}(\Omega) = [\widetilde{\mathbf{H}}_0^{1-s}(\Omega)]'$. Moreover, u_h is uniformly bounded in $H^\tau(\widetilde{\mathbf{H}}_0^{-\alpha}(\Omega))$, $\tau < \frac{2}{5}(2-s)$, and p_h is uniformly bounded in $H^{-r}(H^{1-s}(\Omega))$, $r > \frac{3}{4} - \frac{s}{2}$. Observing that $\frac{2}{5}(2-s) > \frac{3}{4} - \frac{s}{2}$, it is now clear that the product $u_h p_h$ is bounded uniformly in $L^1(\mathbf{L}^1)$ and it should be possible to pass to the limit modulo a compactness argument on u_h.

4. Convergence to a weak solution

To simplify notation we henceforth identify the spaces $\mathbf{H}^\gamma(\Omega)$ and $\widetilde{\mathbf{H}}_0^\gamma(\Omega)$ whenever $\gamma \in (-\frac{1}{2}, \frac{1}{2})$.

4.1. Consistency of the nonlinear term

Before proving that the sequence of pairs (u_h, p_h) converges to a weak solution, up to subsequences, we make sure that we are solving the right problem, i.e., we now formulate a consistency hypothesis on the nonlinear term.

In this section x denotes a real number such that $x > \frac{1}{2}$. We assume that the nonlinear term has the following consistency property: For all functions w in $L^2((0,T);\mathbf{V}^1)$ and all sequences of functions $(w_h)_{h>0}$ in $\mathcal{C}^0([0,T];\mathbf{X}_h)$ converging weakly to w in $L^2((0,T);\mathbf{H}^1_0(\Omega))$ and strongly in $L^2((0,T);\mathbf{L}^3(\Omega))$, the following holds true,

$$\int_0^T \bigl(nl_h(w_h,w_h),v_h\bigr) \to \int_0^T (w\cdot\nabla w, v), \tag{4.1}$$

for all sequence of functions v_h in $H^x((0,T);\mathbf{X}_h)$ strongly converging to v in $H^x((0,T);\mathbf{H}^1_0(\Omega))$.

Lemma 4.1. *The consistency property* (4.1) *holds for definitions* (2.10) *and for* (2.11).

Proof. Let v be a function in $H^x((0,T);\mathbf{H}^1_0(\Omega))$ and $(v_h)_{h>0}$ be a sequence in $H^x((0,T);\mathbf{X}_h)$ converging to v.

(1) Assume that nl_h is defined as in (2.10). Observe that $H^x((0,T);\mathbf{H}^1_0(\Omega)) \subset \mathcal{C}^0([0,T];\mathbf{L}^6(\Omega))$ (see Lemma 4.3). Then $w_h \to w$ in $L^2((0,T);\mathbf{L}^3(\Omega))$ and $v_h \to v$ in $\mathcal{C}^0([0,T];\mathbf{L}^6(\Omega))$ implies $v_h \otimes w_h \to v\otimes w$ and $v_h\cdot w_h \to v\cdot w$ in $L^2((0,T);\mathbf{L}^2(\Omega))$ and $L^2((0,T);L^2(\Omega))$, respectively. As a result $\int_0^T (v_h\otimes w_h, \nabla w_h) \to \int_0^T (v\otimes w,\nabla w)$ and $\int_0^T (v_h\cdot w_h, \nabla\cdot w_h) \to \int_0^T (v\cdot w,\nabla\cdot w)$. Moreover, since $\nabla\cdot w = 0$, a.e. in Q_T, we infer $\int_0^T (v\cdot w_h,\nabla\cdot w_h)\to 0$. The conclusion follows readily.

(2) Assume that nl_h is defined as in (2.11). The only term that poses a difficulty is $\int_0^T (\nabla(\mathcal{K}_h(|w_h|^2)), v_h)$. Integrating by parts, we rewrite this term as follows $-\int_0^T (\mathcal{K}_h(|w_h|^2),\nabla\cdot v_h)$. Then,

$$\begin{aligned}\int_0^T \bigl(\mathcal{K}_h\bigl(|w_h|^2\bigr),\nabla\cdot v_h\bigr) &= \int_0^T \bigl(\mathcal{K}_h\bigl(|w_h|^2\bigr) - \mathcal{K}_h\bigl(|w|^2\bigr),\nabla\cdot v_h\bigr) + \int_0^T \bigl(\mathcal{K}_h\bigl(|w|^2\bigr),\nabla\cdot v_h\bigr)\\ &:= R + \int_0^T \bigl(\mathcal{K}_h\bigl(|w|^2\bigr),\nabla\cdot v_h\bigr).\end{aligned}$$

Banach–Steinhaus' theorem implies that $\|\mathcal{K}_h\|$ is uniformly bounded. Then linearity implies:

$$\begin{aligned}|R| &\leqslant c\bigl\||w_h|^2 - |w|^2\bigr\|_{L^1(\mathbf{L}^2)} \max_{0\leqslant t\leqslant T}\|v_h\|_{\mathbf{H}^1}\\ &\leqslant c\bigl\|(w_h - w)\cdot(w_h+w)\bigr\|_{L^1(\mathbf{L}^2)}\\ &\leqslant c\|w_h - w\|_{L^2(\mathbf{L}^3)}\bigl(\|w_h\|_{L^2(\mathbf{L}^6)} + \|w\|_{L^2(\mathbf{L}^6)}\bigr).\end{aligned}$$

Note that $\|w_h\|_{L^2(\mathbf{L}^6)}$ is bounded since w_h converges weakly to w in $L^2((0,T);\mathbf{L}^6(\Omega))$. The above inequality implies $|R|\to 0$. Moreover, $\mathcal{K}_h(|w|^2)\nabla\cdot v_h \to |w|^2\nabla\cdot v$ in $L^1(\Omega)$ a.e. on $(0,T)$, and the function $\|\mathcal{K}_h(|w|^2)\nabla\cdot v_h\|_{L^1}$ is uniformly bounded by $c\||w|^2\|_{L^2}\max_{0\leqslant t\leqslant T}\|v(t)\|_{\mathbf{H}^1} \in L^1(0,T)$; hence, Lebesgue's Dominated Convergence Theorem implies $\mathcal{K}_h(|w|^2)\nabla\cdot v_h \to |w|^2\nabla\cdot v$ in $L^1(Q_T)$. As a result we obtain $-\int_0^T (\mathcal{K}_h(|w_h|^2),\nabla\cdot v) \to -\int_0^T (|w|^2,\nabla\cdot v)$. Finally,

$$\int_0^T \bigl\langle nl_h(w_h,w_h),v\bigr\rangle \to \int_0^T \left((\nabla\times w)\times w + \frac{1}{2}\nabla\bigl(|w|^2\bigr), v\right) = \int_0^T (w\cdot\nabla w, v).$$

Hence (4.1) holds. □

4.2. Convergence to a weak solution

Before stating the convergence result, let us state Aubin–Lions-like compactness results

Lemma 4.2. *Let $H_0 \subset H \subset H_1$ three Hilbert spaces with dense and continuous embedding. Assume that the embedding $H_0 \subset H$ is compact and let $\gamma > 0$ be a positive real number. Then, the injection $L^2((0,T);H_0) \cap H^\gamma((0,T);H_1) \to L^2((0,T);H)$ is compact.*

Proof. See Lions [22, p. 61, Theorem 5.2]. □

Lemma 4.3. *Let $X \subset Y$ be two Hilbert spaces with compact embedding and let $\tau > \frac{1}{2}$. The injection $H^\tau((0,T);X) \to \mathcal{C}^0([0,T];X)$ is continuous and the injection $H^\tau((0,T);X) \to \mathcal{C}^0([0,T];Y)$ is compact.*

Proof. See Appendix A.1. □

Lemma 4.4. *Let $H_0 \subset H_1$ be two Hilbert spaces with compact embedding. Let $\gamma > 0$ and $\gamma > \mu$, then the injection $H^\gamma((0,T);H_0) \subset H^\mu((0,T);H_1)$ is compact.*

Proof. See Appendix A.2. □

Theorem 4.1. *Under the above hypotheses, the pair (u_h, p_h) convergences, up to subsequences, to a weak solution to* (1.1)*, say* (u,p). *u_h converges to u in $L^2((0,T);\mathbf{H}^1_0(\Omega))$ weak and in any $L^2((0,T);\mathbf{H}^\beta(\Omega))$ strong, $\beta < 1$, and p_h converges to p in $H^{-r}((0,T);H^\delta(\Omega))$ weak, $\delta \in [\frac{3}{10},\frac{1}{2}]$, $r > \frac{1}{4}+\frac{\delta}{2}$.*

Proof. We briefly outline the main steps of the proof for the arguments are quite standard.

(1) Since u_h is uniformly bounded in $L^2((0,T);\mathbf{H}^1_0(\Omega))$ and in $H^\tau((0,T);\mathbf{H}^{-\alpha}(\Omega))$, $\alpha \in [\frac{1}{4},\frac{1}{2})$, $0 < \tau < \frac{2}{5}(1+\alpha)$. Lemma 4.2 implies that there exists a subsequence (u_{h_l}) converging to some u in $L^2((0,T);\mathbf{H}^1_0(\Omega))$ weak and in $L^2((0,T);\mathbf{H}^\beta(\Omega))$ strong, $\beta < 1$. Moreover $(\partial_t u_{h_l})$ converges weakly to $\partial_t\mathsf{u}$ in $H^{-\mu}((0,T);\mathbf{H}^{-\alpha}(\Omega))$, $\alpha \in [\frac{1}{4},\frac{1}{2})$, $\mu > \frac{3}{5}-\frac{2}{5}\alpha$. Since (p_h) is bounded uniformly in $H^{-r}((0,T);H^\delta(\Omega))$, $\delta \in [\frac{3}{10},\frac{1}{2}]$, $r > \frac{1}{4}+\frac{\delta}{2}$, there exists a subsequence (p_{h_l}) converging weakly in $H^{-r}((0,T);H^\delta(\Omega))$ to some p.

(2) Let $q \in L^2((0,T);L^2(\Omega))$ and let $(q_{h_l})_{h_l>0}$ be a sequence of functions in $L^2((0,T);M_h)$ strongly converging to q in $L^2((0,T);L^2(\Omega))$. (Note that the approximability property (2.2) implies that such a sequence can always be constructed for every test function q.) Then $0 = \int_0^T(\nabla\cdot u_{h_l}, q_{h_l}) \to \int_0^T(\nabla\cdot\mathsf{u}, q)$ since $\nabla\cdot u_{h_l} \rightharpoonup \nabla\cdot\mathsf{u}$ in $L^2((0,T);L^2(\Omega))$. As a result, $\nabla\cdot\mathsf{u} = 0$, a.e. in Q_T; that is to say u is in $L^2((0,T);\mathbf{V}^1)$.

(3) Let x be a real number such that $\frac{1}{2} < x < \infty$. Let v be any function in $H^x((0,T);\mathbf{H}^1_0(\Omega))$ and let $(v_{h_l})_{h_l>0}$ be a sequence of functions in $H^x((0,T);\mathbf{X}_h)$ strongly converging to v in $H^x((0,T);\mathbf{H}^1_0(\Omega))$. (Note again that the approximability property (2.1) implies that such a sequence can always be constructed for every test function v.) Then

(4) $\int_{Q_T}\partial_t u_{h_l}\cdot v_{h_l} \to \int_{Q_T}\partial_t\mathsf{u}\cdot v$, since $\partial_t u_{h_l} \rightharpoonup \partial_t\mathsf{u}$ in $H^{-x}((0,T);\mathbf{H}^{-\frac{1}{4}}(\Omega))$ and $v_{h_l} \to v$ in $H^x((0,T);\mathbf{H}^1_0(\Omega))$. Here and after we abuse the notation by using $\int_{Q_T}$ to represent duality parings using $L^2(Q_T)$ as pivot space.

(5) $\int_{Q_T}\nabla u_{h_l}{:}\nabla v_{h_l} \to \int_{Q_T}\nabla\mathsf{u}{:}\nabla v$, since $\nabla u_{h_l} \rightharpoonup \nabla\mathsf{u}$ in $L^2((0,T);\mathbf{L}^2(\Omega))$ and $\nabla v_{h_l} \to \nabla v$ in $H^x((0,T);\mathbf{L}^2(\Omega))$.

(6) $\int_{Q_T}p_{h_l}\nabla\cdot v_{h_l} \to \int_{Q_T}\mathsf{p}\nabla\cdot v$, since $p_{h_l} \rightharpoonup \mathsf{p}$ in $H^{-x}((0,T);H^{\frac{1}{2}}(\Omega))$ and $\nabla\cdot v_{h_l} \to \nabla\cdot v$ in $H^x((0,T);L^2(\Omega))$.

(7) Since u_{h_l} converges weakly to u in $L^2((0,T);\mathbf{H}^1_0(\Omega))$ and strongly in $L^2((0,T);\mathbf{L}^3(\Omega))$, the consistency hypothesis (4.1) holds; hence, $\int_0^T b_h(u_{h_l},u_{h_l},v_{h_l}) \to \int_0^T(\mathsf{u}\cdot\nabla\mathsf{u}, v)$.

(8) Finally, Lemma 4.3 implies that u_{h_l} converges in $\mathcal{C}^0([0,T];\mathbf{H}^{-\alpha}(\Omega))$, $\alpha \in (\frac{1}{4},\frac{1}{2})$. In other words $\mathsf{u}_0 \leftarrow \mathcal{I}_{h_l}\mathsf{u}_0 = u_{h_l}(0) \to \mathsf{u}(0)$ in $\mathbf{H}^{-\alpha}(\Omega)$; hence, $\mathsf{u}(0) = \mathsf{u}_0$.

(9) That u satisfies Leray's energy inequality is standard, [29]. It is a consequence of the inequality $2\nabla(u_{h_l}-\mathsf{u})\cdot\nabla\mathsf{u} + |\nabla\mathsf{u}|^2 \leqslant |\nabla u_{h_l}|^2$. The theorem is proved. □

5. Convergence to a suitable solution

The main issue we address in this section is to determine whether the weak solutions (u, p) we have constructed using the Galerkin method are suitable in the sense of Definition 1.1. To answer this question we assume that the discrete framework satisfies the following property that we henceforth refer to as the discrete commutator property (see Bertoluzza [4], [15, Appendix B], or [11, Chapter I.7]).

Definition 5.1. We say that $\mathbf{X}_h$ (resp. M_h) has the discrete commutator property if there is an operator $P_h \in \mathcal{L}(\mathbf{H}_0^1(\Omega); \mathbf{X}_h)$ (resp. $Q_h \in \mathcal{L}(H^1(\Omega); M_h)$) such that for all ϕ in $W_0^{2,\infty}(\Omega)$ (resp. all ϕ in $W_0^{2,\infty}(\Omega)$) and all $v_h \in \mathbf{X}_h$ (resp. all $q_h \in M_h$),

$$\begin{aligned}
&\left\|\phi v_h - P_h(\phi v_h)\right\|_{\mathbf{H}^l} \leqslant ch^{1+m-l}\|v_h\|_{\mathbf{H}^m}\|\phi\|_{W^{m+1,\infty}}, \quad 0 \leqslant l \leqslant m \leqslant 1,\\
&\left\|\phi q_h - Q_h(\phi q_h)\right\|_{H^l} \leqslant ch^{1+m-l}\|q_h\|_{H^m}\|\phi\|_{W^{m+1,\infty}}.
\end{aligned}$$

Remark 5.1. When P_h (resp. Q_h) is a projector, the above definition is an estimate of the operator norm of the commutator $[\Phi, P_h] := \Phi \circ P_h - P_h \circ \Phi$ where $\Phi \circ v = \phi v$.

Remark 5.2. The discrete commutator property is known to hold in discrete spaces where there exist projectors that have local approximation properties, see Bertoluzza [4]. It is known to hold for finite elements and wavelets. The key property is localization. To understand how the discrete commutator property can be proved let us assume that P_h is a linear projector and let $x \in \Omega$. For every y in a ball of radius h centered at x, we formally have $P_h(\phi v_h)(y) \approx P_h((\phi(x) + \mathcal{O}(h))v_h)(y) \approx (\phi(x) + \mathcal{O}(h))P_h(v_h)(y) = (\phi(x) + \mathcal{O}(h))v_h(y) + \mathcal{O}(h)$, that is to say $P_h(\phi v_h)(y) - (\phi v_h)(y) \approx \mathcal{O}(h)v_h(y)$, where $\mathcal{O}(h)$ depends on the gradient of ϕ.

Remark 5.3. Spectral-based approximation spaces do not have the discrete commutator property since spectral expansions do not have local interpolation properties. Spectral methods are very accurate but they only have global interpolation properties.

We moreover assume that the following inverse inequality holds: There is c uniform in h such that for all $s \in [0, 1]$

$$\forall v_h \in \mathbf{X}_h, \quad \|v_h\|_{\widetilde{\mathbf{H}}_0^s} \leqslant c\,h^{-s}\|v_h\|_{\mathbf{L}^2}, \quad \|v_h\|_{\mathbf{L}^2} \leqslant c\,h^{-s}\|v_h\|_{\widetilde{\mathbf{H}}_0^{-s}}. \tag{5.1}$$

The above hypotheses are usually satisfied when $\mathbf{X}_h$ and M_h are constructed by using finite elements [13].

We also assume that the following consistency property holds for the nonlinear term: For all functions w in $L^2((0,T); \mathbf{V}^1) \cap L^\infty((0,T); \mathbf{L}^2(\Omega))$ and all sequences of functions $(w_h)_{h>0}$ in $\mathcal{C}^0([0,T]; \mathbf{V}_h)$ uniformly bounded in $L^2((0,T); \mathbf{H}_0^1(\Omega)) \cap L^\infty((0,T); L^2(\Omega))$ and strongly converging to w in $L^2((0,T); \mathbf{L}^3(\Omega))$, the following holds true,

$$b_h(w_h, w_h, \phi w_h) \to -\left(\frac{1}{2}w|w|^2, \nabla\phi\right), \quad \forall \phi \in \mathcal{D}(Q_T). \tag{5.2}$$

Lemma 5.1. *The consistency property* (4.1) *holds for definitions* (2.10) *and for* (2.11) *provided M_h has the discrete commutator property.*

Proof. (1) The situation for definition (2.10) is quite simple, since

$$\begin{aligned}
b_h(w_h, w_h, \phi w_h) &= (w_h \cdot \nabla w_h, \phi w_h) + \frac{1}{2}(w_h \nabla \cdot w_h, \phi w_h)\\
&= \left(w_h \cdot \nabla\left(\frac{1}{2}|w_h|^2\right) + \frac{1}{2}|w_h|^2 \nabla \cdot w_h, \phi\right)\\
&= \left(\nabla \cdot \left(w_h \frac{1}{2}|w_h|^2\right), \phi\right) = -\left(w_h \frac{1}{2}|w_h|^2, \nabla\phi\right).
\end{aligned}$$

Now observe,

$$\begin{aligned}\int_{Q_T}\big[w_h|w_h|^2 - w|w|^2\big] &= \int_{Q_T}\big[(w_h-w)|w_h|^2 + w(w_h+w)\cdot(w_h-w)\big]\\ &\leqslant c\int_0^T \|w_h-w\|_{\mathbf{L}^3}\big(\|w_h\|^2_{\mathbf{L}^3}+\|w\|^2_{\mathbf{L}^3}\big)\\ &\leqslant c\|w_h-w\|_{L^2(\mathbf{L}^3)}\big(\|w_h\|^2_{L^4(\mathbf{L}^3)}+\|w\|^2_{L^4(\mathbf{L}^3)}\big).\end{aligned}$$

Then, $\|w_h|w_h|^2 - w|w|^2\|_{L^1(Q_T)} \to 0$ since $\|w_h-w\|_{L^2(\mathbf{L}^3)}\to 0$ and $\|w_h\|_{L^4(\mathbf{L}^3)}$, $\|w\|_{L^4(\mathbf{L}^3)}$ are bounded. This immediately implies that $b_h(w_h,w_h,\phi w_h)\to -(\frac{1}{2}w|w|^2,\nabla\phi)$.

(2) For definition (2.11) we proceed as follows:

$$\begin{aligned}b_h(w_h,w_h,\phi w_h) &= \big((\nabla\times w_h)\times w_h,\phi w_h\big) + \frac{1}{2}\big(\nabla(\mathcal{K}_h\big(|w_h|^2\big),\phi w_h)\big)\\ &= -\frac{1}{2}\big(\mathcal{K}_h\big(|w_h|^2\big),\nabla\cdot(\phi w_h)\big)\\ &= -\frac{1}{2}\big(w_h\mathcal{K}_h\big(|w_h|^2\big),\nabla\phi\big) - \frac{1}{2}\big(\phi\mathcal{K}_h\big(|w_h|^2\big),\nabla\cdot w_h\big)\\ &= -\frac{1}{2}\big(w_h|w_h|^2\nabla\phi\big) + R_1 + R_2,\end{aligned}$$

where $R_1 := -\frac{1}{2}(w_h(\mathcal{K}_h(|w_h|^2) - |w_h|^2),\nabla\phi)$ and $R_2 := -\frac{1}{2}(\phi\mathcal{K}_h(|w_h|^2),\nabla\cdot w_h)$. To control R_1, we further decompose the integrand as follows:

$$\begin{aligned}w_h\big(\mathcal{K}_h\big(|w_h|^2\big) - |w_h|^2\big) &= (w_h-w)\big(\mathcal{K}_h\big(|w_h|^2\big)-|w_h|^2\big) + w\big(\mathcal{K}_h\big(|w_h|^2\big)-\mathcal{K}_h\big(|w|^2\big)\big)\\ &\quad + w\big(\mathcal{K}_h\big(|w|^2\big)-|w|^2\big) + w\big(|w|^2-|w_h|^2\big)\\ &:= R_{11}+R_{12}+R_{13}+R_{14}.\end{aligned}$$

Proceeding as in step (1), we infer:

$$\|R_{11}\|_{L^1(Q_T)} + \|R_{12}\|_{L^1(Q_T)} + \|R_{14}\|_{L^1(Q_T)} \leqslant c\|w_h-w\|_{L^2(\mathbf{L}^3)}\big(\|w_h\|^2_{L^4(\mathbf{L}^3)}+\|w\|^2_{L^3(\mathbf{L}^3)}\big).$$

Furthermore, observe that $R_{13}\to 0$ a.e. in Q_T and $|R_{13}|\leqslant c|w|^3\in L^1(Q_T)$. Lebesgue's Dominated Convergence Theorem implies $\|R_{13}\|_{L^1(Q_T)}\to 0$. In conclusion $\int_0^T|R_1|\to 0$ as $h\to 0$. For R_2 we use the fact that $w_h(t)\in\mathbf{V}_h$ and M_h has the discrete commutator property as follows:

$$\begin{aligned}|R_2| &= \frac{1}{2}\big|\big(\phi\mathcal{K}_h\big(|w_h|^2\big) - Q_h\big(\phi\mathcal{K}_h\big(|w_h|^2\big)\big),\nabla\cdot w_h\big)\big|\\ &\leqslant \frac{1}{2}\big\|\phi\mathcal{K}_h\big(|w_h|^2\big) - Q_h\big(\phi\mathcal{K}_h\big(|w_h|^2\big)\big)\big\|_{\mathbf{L}^2}\|w_h\|_{\mathbf{H}^1}\\ &\leqslant ch\big\|\mathcal{K}_h\big(|w_h|^2\big)\|_{\mathbf{L}^2}\|w_h\|_{\mathbf{H}^1} \leqslant c\,h\big\||w_h|^2\big\|_{\mathbf{L}^2}\|w_h\|_{\mathbf{H}^1}\\ &\leqslant ch\|w_h\|^2_{\mathbf{L}^4}\|w_h\|_{\mathbf{H}^1} \leqslant ch\|w_h\|^{\frac{1}{2}}_{\mathbf{L}^2}\|w_h\|^{\frac{3}{2}}_{\mathbf{L}^6}\|w_h\|_{\mathbf{H}^1}\\ &\leqslant ch\|w_h\|^{\frac{1}{2}}_{\mathbf{L}^2}\|w_h\|^{\frac{1}{2}}_{\mathbf{H}^1}\|w_h\|^2_{\mathbf{H}^1} \leqslant c\,h^{\frac{1}{2}}\|w_h\|_{\mathbf{L}^2}\|w_h\|^2_{\mathbf{H}^1},\end{aligned}$$

where we have used (5.1) to derive the last inequality. Hence

$$\int_0^T |R_2| \leqslant c\,h^{\frac{1}{2}}\|w_h\|^2_{L^2(\mathbf{H}^1)}\|w_h\|_{L^\infty(\mathbf{L}^2)}.$$

Then clearly $\int_0^T|R_2|\to 0$ as $h\to 0$. In conclusion $b_h(w_h,w_h,\phi w_h)\to -(\frac{1}{2}w|w|^2,\nabla\phi)$ since $w_h\frac{1}{2}|w_h|^2\to w\frac{1}{2}|w|^2$ in $L^1(Q_T)$ and $\int_0^T|R_1| + \int_0^T|R_2|\to 0$. That concludes the proof. □

Before stating the main result of this paper let us now collect the various hypotheses that have been used so far. The source term $\mathbf{f}$ and the initial data u_0 are assumed to satisfy (3.14) where the exponents p and q are defined in (3.9). For instance $\mathbf{f} \in L^2_{\text{loc}}([0,\infty); \mathbf{L}^2(\Omega))$ is sufficient for (3.14) to hold. For the two families of approximation spaces $\{\mathbf{X}_h\}_{h>0}$ and $\{M_h\}_{h>0}$ we assume that the approximation properties (2.1)–(2.2) hold. These are standard evident hypotheses. We further assume that $\mathbf{X}_h$ and M_h are compatible in the sense that (2.8) holds. This is a generalization of the so-called LBB condition and it has been shown in [16] to hold for reasonable pairs of finite element spaces. We further assume the norm equivalences (2.6)–(2.7). These are the discrete counterparts of (1.8)–(1.9) and have been shown to hold for various finite-element-like settings in [17] (the proofs therein invokes a standard quasi-uniformity hypothesis on the mesh when finite elements are considered). The next nontrivial hypotheses are the inverse inequalities (5.1). These say in some sense that h is the smallest scale that can be represented in $\mathbf{X}_h$. These hypotheses invoke again a quasi-uniformity hypothesis on the mesh when finite elements or wavelets are considered. Constructing families $\{\mathbf{X}_h\}_{h>0}$ satisfying these nonrestrictive hypotheses is a standard exercise. We finish this list by mentioning the two consistency hypotheses (4.1)–(5.2). Since we allow ourselves some freedom on how to compute the nonlinear term to account for the fact that the approximate velocity field may not be solenoidal (i.e., it is solenoidal in a weak discrete sense only), these two hypotheses constrain the way that can be done (i.e., it must be done reasonably). (4.1)–(5.2) have been shown to hold in Lemmas 4.1 and 5.1 under the above mentioned structure hypotheses on the discrete spaces families $\{\mathbf{X}_h\}_{h>0}$ and $\{M_h\}_{h>0}$ if the nonlinear term is computed using either definitions (2.10)–(2.11). It is important to note that no regularization is performed on the nonlinear term.

The main result of the paper is now stated in the following theorem.

Theorem 5.1. *Under the above hypotheses, if $\mathbf{X}_h$ and M_h have the discrete commutator property, the pair (u_h, p_h) convergences, up to subsequences, to a suitable solution to* (1.1)*, say* (u, p).

Proof. To simplify notation we still denote by (u_h) and (p_h) the subsequence that converges to u and p, respectively.

(1) Let ϕ be a non negative function in $\mathcal{D}(Q_T)$. Testing the momentum equation in (2.12) by $P_h(u_h\phi)$, we obtain:

$$\int_0^T \big[\big(\partial_t u_h, P_h(u_h\phi)\big) + b_h\big(u_h, u_h, P_h(u_h\phi)\big) - \big(p_h, \nabla\cdot P_h(u_h\phi)\big) + \nu\big(\nabla u_h, \nabla P_h(u_h\phi)\big) - \big(\mathsf{f}, P_h(u_h\phi)\big)\big]\,\mathrm{d}t = 0.$$

Each of the terms on the left-hand side of the equation are now treated separately in the following steps:

(2) For the time derivative we have:

$$\int_0^T \big(\partial_t u_h, P_h(u_h\phi)\big) = \int_0^T (\partial_t u_h, u_h\phi) + \int_0^T R = -\frac{1}{2}\int_0^T \big(|u_h|^2, \partial_t\phi\big) + \int_0^T R,$$

where we have set $R = (u_{h,t}, P_h(u_h\phi) - u_h\phi)$. It is clear that $-\frac{1}{2}\int_0^T (|u_h|^2, \partial_t\phi) \to -\frac{1}{2}\int_0^T (\mathsf{u}^2, \partial_t\phi)$ since $|u_h|^2 \to |\mathsf{u}|^2$ in $L^r(\mathbf{L}^1)$ for any $1 \leqslant r \leqslant 2$. Let us introduce a small positive number $\varepsilon > 0$ and set $x = \frac{1}{2} + \frac{\varepsilon}{5}$. To control the residual, we use the discrete commutator property as follows:

$$\int_0^T |R| = \int_0^T \big(u_{h,t}, P_h(u_h\phi) - u_h\phi\big) \leqslant \|u_{h,t}\|_{H^{-x}(\mathbf{H}^{-\frac{1}{4}})} \big\|P_h(u_h\phi) - u_h\phi\big\|_{H^x(\mathbf{H}^{\frac{1}{4}})} \leqslant ch\|u_{h,t}\|_{H^{-x}(\mathbf{H}^{-\frac{1}{4}})}\|u_h\|_{H^x(\mathbf{H}^{\frac{1}{4}})}.$$

Owing to Corollary 3.1, $\|u_{h,t}\|_{H^{-x}(\mathbf{H}^{-\frac{1}{4}})}$ is bounded since $x > \frac{1}{2}$. Moreover,

$$\frac{2}{5}\left(1 + \frac{1}{4} + \varepsilon\right) = \frac{1}{2} + \frac{2}{5}\varepsilon > \frac{1}{2} + \frac{\varepsilon}{5} = x,$$

that is, $\|u_h\|_{H^x(\mathbf{H}^{-(\frac{1}{4}+\varepsilon)})}$ is bounded. Using the inverse inequality (5.1), we infer:

$$\int_0^T |R| \leqslant ch^{\frac{1}{2}-\varepsilon} \|u_{h,t}\|_{H^{-x}(\mathbf{H}^{-\frac{1}{4}})} \|u_h\|_{H^x(\mathbf{H}^{-\frac{1}{4}-\varepsilon})}.$$

Now, it is clear that $\int_0^T |R| \to 0$ as $h \to 0$ provided $\varepsilon \in (0, \frac{1}{2})$.

(3) For the viscous term, we integrate by parts to obtain:

$$\begin{aligned}\big(\nabla u_h, \nabla P_h(u_h\phi)\big) &= \big(\nabla u_h, \nabla(u_h\phi)\big) + R \\ &= \big(|\nabla u_h|^2, \phi\big) - \left(\frac{1}{2}|u_h|^2, \Delta\phi\right) + R\end{aligned}$$

where $R := (\nabla u_h, \nabla(P_h(u_h\phi) - u_h\phi))$. For the first term we proceed as follows:

$$\begin{aligned}\int_0^T \big(|\nabla u_h|^2, \phi\big) &= \int_0^T \big(\big|\nabla(u_h - \mathsf{u} + \mathsf{u})\big|^2, \phi\big) \\ &= \int_0^T \big(\big|\nabla(u_h - \mathsf{u})\big|^2 + 2\nabla(u_h - \mathsf{u}) : \nabla \mathsf{u} + |\nabla \mathsf{u}|^2, \phi\big).\end{aligned}$$

Since $u_h \rightharpoonup \mathsf{u}$ in $L^2((0,T); \mathbf{H}^1)$ and ϕ is non negative, we infer $\liminf \int_0^T (|\nabla u_h|^2, \phi) \geqslant \int_0^T (|\nabla \mathsf{u}|^2, \phi)$. For the second term we have $\int_0^T -(\frac{1}{2}|u_h|^2, \Delta\phi) \to -\int_0^T (\frac{1}{2}|\mathsf{u}|^2, \Delta\phi)$ since $|u_h|^2 \to |\mathsf{u}|^2$ in $L^r(\mathbf{L}^1)$ for any $1 \leqslant r \leqslant 2$. Now we control the residual as follows:

$$|R| = \big|\big(\nabla u_h, \nabla\big(P_h(u_h\phi) - u_h\phi\big)\big)\big| \leqslant ch\|u_h\|^2_{\mathbf{H}^1}.$$

Then it is clear that $\int_0^T |R| \to 0$ as $h \to 0$. In conclusion,

$$\liminf_{h\to 0} \int_0^T \big(\nabla u_h, \nabla P_h(u_h\phi)\big) \geqslant \int_0^T \big(|\nabla \mathsf{u}|^2, \phi\big) - \left(\frac{1}{2}|\mathsf{u}|^2, \Delta\phi\right).$$

(4) For the pressure term we have:

$$\big(p_h, \nabla\cdot\big(P_h(u_h\phi)\big)\big) = \big(p_h, \nabla\cdot(u_h\phi)\big) + R_1 = (p_h u_h, \nabla\phi) + R_1 + R_2,$$

where $R_1 := (p_h, \nabla\cdot(P_h(u_h\phi) - u_h\phi))$ and $R_2 := (\phi p_h \nabla\cdot u_h)$. Let us take care of R_1 first. Let $\varepsilon > 0$ be a small positive real number and set $r = \frac{1}{2} - \frac{4}{9}\varepsilon$ and $s = \frac{1}{2} - \frac{16}{9}\varepsilon$. Then

$$r = \frac{1}{2} - \frac{4}{9}\varepsilon > \frac{1}{2} - \frac{8}{9}\varepsilon = \frac{1}{4} + \frac{1}{2}s,$$

meaning that $\|p_h\|_{H^{-r}(H^s)}$ is uniformly bounded according to Lemma 3.5. Using the discrete commutator property, we deduce:

$$\begin{aligned}\int_0^T |R_1| &\leqslant c\|p_h\|_{H^{-r}(H^s)} \big\|\nabla\cdot\big(P_h(u_h\phi) - u_h\phi\big)\big\|_{H^r(H^{-s})} \\ &\leqslant c\|p_h\|_{H^{-r}(H^s)} \big\|P_h(u_h\phi) - u_h\phi\big\|_{H^r(\mathbf{H}^{1-s})} \\ &\leqslant ch\|p_h\|_{H^{-r}(H^s)} \|u_h\|_{H^r(\mathbf{H}^{1-s})}.\end{aligned}$$

Now let us set $\alpha = \frac{1}{4} - \frac{5}{9}\varepsilon$. Observe that

$$r = \frac{1}{2} - \frac{4}{9}\varepsilon < \frac{1}{2} - \frac{2}{9}\varepsilon = \frac{2}{5}\left(\frac{5}{4} - \frac{5}{9}\varepsilon\right) = \frac{2}{5}\left(1 + \frac{1}{4} - \frac{5}{9}\varepsilon\right) = \frac{2}{5}(1+\alpha),$$

meaning that $\|u_h\|_{H^r(\mathbf{H}^{-\alpha})}$ is uniformly bounded. Then, using the inverse inequality (5.1), we infer:

$$\int_0^T |R_1| \leqslant ch^{1-(1-s+\alpha)} \|p_h\|_{H^{-r}(H^s)} \|u_h\|_{H^r(\mathbf{H}^{-\alpha})} \leqslant c' h^{\frac{1}{4}-\varepsilon}.$$

Then clearly $|\int_0^T R_1| \to 0$ as $h \to 0$, provided $\varepsilon \in (0, \frac{1}{4})$. We proceed similarly for R_2 using the fact that u_h take its values in $\mathbf{V}_h$,

$$\begin{aligned}\int_0^T R_2 &= \int_0^T \big(\phi p_h - Q_h(\phi p_h), \nabla \cdot u_h\big) \\ &\leqslant \big\|\phi p_h - Q_h(\phi p_h)\big\|_{H^{-r}(H^s)} \|\nabla \cdot u_h\|_{H^r(H^{-s})} \\ &\leqslant ch\|p_h\|_{H^{-r}(H^s)} \|u_h\|_{H^r(\mathbf{H}^{1-s})} \leqslant c' h^{\frac{1}{4}-\varepsilon}.\end{aligned}$$

Then again $|\int_0^T R_2| \to 0$ as $h \to 0$. Now we have to pass to the limit on $\int_0^T (p_h u_h, \nabla\phi)$. We are going to use Lemma 4.4. Let $\varepsilon > 0$ be a small positive real number and set $r = \frac{2}{5} + \varepsilon$, $s = \frac{3}{10}$. This choice implies that $\|p_h\|_{H^{-r}(H^s)}$ is uniformly bounded, i.e., $p_h \rightharpoonup \mathsf{p}$ in $H^{-r}((0,T); H^s(\Omega))$. Now let $\varepsilon' \in (0, \frac{1}{20}]$ be an other small positive real number and set $\alpha = \frac{3}{10} - \varepsilon' \geqslant \frac{1}{4}$, $\tau = \frac{2}{5}(1 + \alpha - \varepsilon')$. This choice implies that $\|u_h\|_{H^\tau(\mathbf{H}^{-\alpha})}$ is uniformly bounded, i.e., $u_h \rightharpoonup \mathsf{u}$ in $H^\tau((0,T); \mathbf{H}^{-\alpha}(\Omega))$. Using Lemma 4.4, we infer that $u_h \to \mathsf{u}$ in $H^r((0,T); \mathbf{H}^{-s}(\Omega))$ provided $\tau > r$ and $s > \alpha$. Clearly $s > \alpha$ provided $\varepsilon' > 0$; moreover $\tau > r$ if,

$$\frac{2}{5} + \frac{1}{25}(3 - 20\varepsilon') > \frac{2}{5} + \varepsilon.$$

This holds true if we set $\varepsilon = \frac{1}{50}(3 - 20\varepsilon')$, since $\varepsilon' \in (0, \frac{1}{20}]$. Assuming the above choices are made on ε' and ε, it now becomes evident that $\int_0^T (p_h u_h, \nabla\phi) \to \int_0^T (\mathsf{pu}, \nabla\phi)$.

(5) The source term does not pose any particular difficulty,

$$\big\langle \mathsf{f}, P_h(\phi u_h)\big\rangle = \langle \mathsf{f}, \phi u_h\rangle + R,$$

where $R := \langle \mathsf{f}, P_h(\phi u_h) - \phi u_h\rangle$. Clearly $\int_0^T \langle \mathsf{f}, \phi u_h\rangle \to \int_0^T \langle \mathsf{f}, \phi \mathsf{u}\rangle$ since $u_h \rightharpoonup u$ in $L^2((0,T); \mathbf{H}_0^1(\Omega))$ and $f \in L^2((0,T); \mathbf{H}^{-1}(\Omega))$. Moreover,

$$\begin{aligned}\int_0^T |R| &\leqslant \|\mathsf{f}\|_{L^2(\mathbf{H}^{-1})} \big\|P_h(\phi u_h) - \phi u_h\big\|_{L^2(\mathbf{H}^1)} \\ &\leqslant ch\|\mathsf{f}\|_{L^2(\mathbf{H}^{-1})} \|u_h\|_{L^2(\mathbf{H}^1)}.\end{aligned}$$

Then $\int_0^T |R| \to 0$ as $h \to 0$.

(6) Now we pass to the limit in the nonlinear term:

$$b_h\big(u_h, u_h, P_h(\phi u_h)\big) = b_h(u_h, u_h, \phi u_h) + R,$$

where $R = b_h(u_h, u_h, P_h(\phi u_h) - \phi u_h)$. Then

$$\begin{aligned}|R| &\leqslant \big\|nl_h(u_h, u_h)\big\|_{\mathbf{H}^{-1}} \big\|P_h(\phi u_h) - \phi u_h\big\|_{\mathbf{H}^1} \\ &\leqslant ch\|u_h\|_{\mathbf{L}^3} \|u_h\|_{\mathbf{H}^1} \|u_h\|_{\mathbf{H}^1} \leqslant c\,h\|u_h\|_{\mathbf{L}^2}^{\frac{1}{2}} \|u_h\|_{\mathbf{H}^1}^{\frac{1}{2}} \|u_h\|_{\mathbf{H}^1}^2.\end{aligned}$$

That is to say

$$\int_0^T |R| \leqslant ch^{\frac{1}{2}} \|u_h\|_{L^\infty(\mathbf{L}^2)} \|u_h\|_{L^2(\mathbf{H}^1)}^2.$$

This in turn implies $\int_0^T |R| \to 0$ as $h \to 0$. Then conclude using hypothesis (5.2). □

Appendix A. Proof of compactness lemmas

A.1. Proof of Lemma 4.3

Lemma 4.3. *Let $X \subset Y$ be two Hilbert spaces with compact embedding. The injection $H^\tau((0,T);X) \to \mathcal{C}^0([0,T];X)$ is continuous and the injection $H^\tau((0,T);X) \to \mathcal{C}^0([0,T];Y)$ is compact, $\tau > \frac{1}{2}$.*

Proof. (1) Indeed, let v be in the unit ball in $H^\tau((0,T);X)$. Then, owing to the fact that $\tau > \frac{1}{2}$, we have:

$$\int_{-\infty}^{\infty} \big\|\hat{v}(k)\big\|_X \,\mathrm{d}k \leqslant \|v\|_{H^\tau(X)} \big\|(1+|k|)^{-\tau}\big\|_{L^2} \leqslant c\|v\|_{H^\tau(X)}.$$

This in turns means that the inverse Fourier transform formula holds for a.e. t in $[0,T]$,

$$v(t) = \int_{-\infty}^{+\infty} \hat{v}(k)\mathrm{e}^{2\mathrm{i}\pi kt}\,\mathrm{d}k.$$

(2) Let v_n be a bounded sequence in $H^\tau((0,T);X)$. Let $t \in [0,T]$, then using the above representation of $v_n(t)$,

$$\big\|v_n(t)\big\|_X \leqslant \|\hat{v}_n\|_{L^1(X)} \leqslant c\,\|v_n\|_{H^\tau(X)} \leqslant c'.$$

In other words the sequence $(\|v_n(t)\|_X)$ is bounded uniformly. This means that the sequence $(v_n(t))$ is relatively compact in Y.

(3) Let $t, t' \in [0,T]$. Then

$$\begin{aligned}
\big\|v_n(t) - v_n(t')\big\|_X &\leqslant \int_{-\infty}^{+\infty} \big\|\hat{v}_n(k)\big\|_X \big|\mathrm{e}^{2\mathrm{i}\pi kt} - \mathrm{e}^{2\mathrm{i}\pi kt'}\big|\,\mathrm{d}k \\
&\leqslant c\int_{-\infty}^{+\infty} \big\|\hat{v}_n(k)\big\|_X \frac{|k(t-t')|}{1+|k(t-t')|}\,\mathrm{d}k \\
&\leqslant c\int_{-\infty}^{+\infty} \big\|\hat{v}_n(k)\big\|_X \big(1+|k|\big)^\tau \frac{|k^{1-\tau}(t-t')|}{1+|k(t-t')|}\,\mathrm{d}k \\
&\leqslant c|t-t'|^{\tau-\frac{1}{2}} \|v_n\|_{H^\tau(X)} \big\|k^{1-\tau}\big(1+|k|\big)^{-1}\big\|_{L^2} \\
&\leqslant c'\,|t-t'|^{\tau-\frac{1}{2}}.
\end{aligned}$$

This means that the sequence v_n is equi-continuous in $\mathcal{C}^0([0,T];X)$. The sequence is also obviously equi-continuous in $\mathcal{C}^0([0,T];Y)$. Then the Ascoli–Arzelà theorem implies that the sequence v_n is relatively compact in $\mathcal{C}^0([0,T];Y)$, which concludes the proof. □

A.2. Proof of Lemma 4.4

Lemma 4.4. *Let $H_0 \subset H_1$ be two Hilbert spaces with compact embedding. Let $\gamma > 0$ and $\gamma > \mu$, then the injection $H^\gamma((0,T);H_0) \subset H^\mu((0,T);H_1)$ is compact.*

Proof. We adapt the proof of [22, Theorem 5.2]. The result amounts to proving that if v_n is a sequence converging weakly to 0 in $H^\gamma((0,T);H_0)$, then v_n converges strongly to 0 in $H^\mu((0,T);H_1)$. Since $[0,T]$ is a bounded, v_n being in $H^\gamma((0,T);H_0)$ means that there is a function $w_n \in H^\gamma(\mathbb{R};H_0)$ with support in $[-1,T+1]$ such that $w_n = v_n$ a.e. on $[0,T]$, w_n converges weakly to 0 in $H^\gamma((0,T);H_0)$, and $\|v_n\|_{H^\gamma(H_0)} \leqslant \|w_n\|_{H^\gamma(H_0)} \leqslant 2\|v_n\|_{H^\gamma(H_0)}$.

Let $\varepsilon > 0$ be an arbitrarily small positive real number. Let $M > 0$ be an other real number yet to be fixed. Then,

$$\|w_n\|_{H^\mu(H_1)}^2 = \int_{-\infty}^{+\infty} \left(1+|k|\right)^{2\mu} \|\hat{w}_n\|_{H_1}^2$$
$$= \int_{|k|\leqslant M} \left(1+|k|\right)^{2\mu} \|\hat{w}_n\|_{H_1}^2 + \int_{|k|>M} \left(1+|k|\right)^{2\gamma} \|\hat{w}_n\|_{H_1}^2 \left(1+|k|\right)^{2(\mu-\gamma)}$$
$$= \int_{|k|\leqslant M} \left(1+|k|\right)^{2\mu} \|\hat{w}_n\|_{H_1}^2 + c\left(1+|M|\right)^{2(\mu-\gamma)}.$$

Since $\mu-\gamma>0$, it is possible to choose M such that $c(1+|M|)^{2(\mu-\gamma)} \leqslant \varepsilon$. Let us now evaluate the other term in the right-hand side. Let $\psi \in \mathcal{D}(\mathbb{R})$ such that $\psi|_{[-1,T+1]}=1$. For all $\phi \in H_0$ we have:

$$\left(\hat{w}_n(k),\phi\right)_{H_0} = \int_{-\infty}^{+\infty} \left(w_n(t), \left(\psi(t)\mathrm{e}^{-2\mathrm{i}\pi kt}\right)\phi\right)_{H_0} \mathrm{d}t.$$

Since $(\psi(t)\mathrm{e}^{-2\mathrm{i}\pi kt})\phi \in H^\gamma((0,T);H_0)$ and w_n converges weakly to 0 in $H^\gamma((0,T);H_0)$, we infer $\lim_{n\to\infty}(\hat{w}_n(k),\phi)_{H_0}=0$, i.e., $\hat{w}_n(k)$ converges weakly to zero in H_0, which in turns implies that $\hat{w}_n(k)$ converges strongly to zero in H_1. In other words, for all $k\in[-M,+M]$,

$$\lim_{n\to+\infty} \left(1+|k|\right)^{2\mu} \left\|\hat{w}_n(k)\right\|_{H_1} = 0.$$

Moreover, owing to $\gamma \geqslant 0$, it is clear that

$$\left\|\hat{w}_n(k)\right\|_{H_1} \leqslant c\|w_n\|_{L^1((0,T);H_1)} \leqslant c'\|w_n\|_{L^2((0,T);H_1)} \leqslant c'\|w_n\|_{H^\gamma(H_1)} \leqslant c'',$$

meaning that $(1+|k|)^{2\mu}\|\hat{w}_n(k)\|_{H_1}^2 \leqslant c\,(1+|k|)^{2\mu} \in L^1((-M,+M))$. We can apply Lebesgue's Dominated Convergence Theorem to conclude,

$$\lim_{n\to+\infty} \int_{|k|\leqslant M} \left(1+|k|\right)^{2\mu} \left\|w_n(k)\right\|_{H_1}^2 \mathrm{d}k = 0.$$

Then we have:

$$\limsup_{n\to+\infty} \|w_n\|_{H^\mu(H_1)}^2 \leqslant \varepsilon,$$

which means $\|w_n\|_{H^\mu(H_1)} \to 0$ since ε is arbitrary. This also means $\|v_n\|_{H^\mu(H_1)} \to 0$ since w_n is an extension of v_n. □

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