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DISCONTINUOUS GALERKIN METHODS FOR FRIEDRICHS' SYSTEMS. I. GENERAL THEORY*

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Abstract. This paper presents a unified analysis of discontinuous Galerkin methods to approximate Friedrichs' systems. An abstract set of conditions is identified at the continuous level to guarantee existence and uniqueness of the solution in a subspace of the graph of the differential operator. Then a general discontinuous Galerkin method that weakly enforces boundary conditions and mildly penalizes interface jumps is proposed. All the design constraints of the method are fully stated, and an abstract error analysis in the spirit of Strang's Second Lemma is presented. Finally, the method is formulated locally using element fluxes, and links with other formulations are discussed. Details are given for three examples, namely, advection-reaction equations, advection-diffusion-reaction equations, and the Maxwell equations in the so-called elliptic regime.

Key words. Friedrichs' systems, finite elements, partial differential equations, discontinuous Galerkin method

AMS subject classifications. 65N30, 65M60, 35F15

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1. Introduction. Discontinuous Galerkin (DG) methods were introduced in the 1970s, and their development has since followed two somewhat parallel routes depending on whether the PDE is hyperbolic or elliptic.

For hyperbolic PDEs, the first DG method was introduced by Reed and Hill in 1973 [28] to simulate neutron transport, and the first analysis of DG methods for hyperbolic equations in an already rather general and abstract form was performed by Lesaint and Raviart in 1974 [23, 24]. The analysis was subsequently improved by Johnson, Nävert, and Pitkäranta who established that the optimal order of convergence in the L^2-norm is $p + \frac{1}{2}$ if polynomials of degree p are used [21]. More recently, DG methods for hyperbolic and nearly hyperbolic equations experienced a significant development based on the ideas of numerical fluxes, approximate Riemann solvers, and slope limiters; see, e.g., Cockburn et al. [9] and references therein for a thorough review. This renewed interest in DG methods is stimulated by several factors including the flexibility offered by the use of nonmatching grids and the possibility to use high-order hp-adaptive finite element methods; see, e.g., Süli et al. [30].

For elliptic PDEs, DG methods originated from the early work of Nitsche on boundary-penalty methods [25] and the use of interior penalties (IP) to weakly enforce continuity conditions imposed on the solution or its derivatives across the interfaces between adjoining elements; see, e.g., Babuška [4], Babuška and Zlámal [3], Douglas and Dupont [13], Baker [6], Wheeler [31], and Arnold [2]. DG methods for elliptic problems in mixed form were introduced more recently. Initially, a discontinuous approximation was used solely for the primal variable, the flux being still discretized in a conforming fashion; see, e.g., Dawson [11, 12]. Then, a discontinuous approximation

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of both the primal variable and its flux has been introduced by Bassi and Rebay [7] and further extended by Cockburn and Shu [10] leading to the so-called local discontinuous Galerkin (LDG) method. Around the same time, Baumann and Oden [8] proposed a nonsymmetric variant of DG for elliptic problems. This method was further developed and analyzed by Oden, Babuška, and Baumann [26] and by Rivière, Wheeler, and Girault [29].

The fact that several DG methods (including IP methods) share common features and can be tackled by similar analysis tools called for a unified analysis. A first important step in that direction has been recently accomplished by Arnold et al. [1] for elliptic equations. It is shown in [1] that it is possible to cast many DG methods for the Poisson equation with homogeneous Dirichlet boundary conditions into a single framework amenable to a unified error analysis. The main idea consists of using the mixed formulation of the Poisson equation to define numerical fluxes and to locally eliminate these fluxes so as to derive a method involving only the primal variable.

The goal of the present paper is to propose a unified analysis of DG methods that goes beyond the traditional hyperbolic/elliptic classification of PDEs by making systematic use of the theory of Friedrichs' systems [17] to formulate DG methods and to perform the convergence analysis. This paper, which concentrates on first-order PDEs, is the first part of a more comprehensive study on DG methods for Friedrichs' systems. The forthcoming second part will deal more specifically with Friedrichs' systems associated with second-order PDEs. Some preliminary results on Friedrichs' systems related to this work can be found in [15, p. 227].

The paper is organized as follows. In section 2 we investigate the well posedness of Friedrichs' systems in graph spaces. Originally, Friedrichs addressed the question of the uniqueness of strong solutions in $\mathfrak{C}^1$ and that of the existence of weak solutions in L^2 [17]. The analysis of Friedrichs' systems in graph spaces has been undertaken by Rauch [27] and more recently by Jensen [20]. The main novelty of the present approach is that we avoid invoking traces at the boundary by introducing a bounded linear operator from the graph space to its dual that satisfies sufficient conditions ensuring well posedness. In section 3 we illustrate the abstract results of section 2 on three important examples of Friedrichs' systems, namely, advection-reaction equations, advection-diffusion-reaction equations, and a simplified version of the Maxwell equations in the so-called elliptic regime. Drawing on earlier ideas by Lesaint and Raviart [23, 24] and Johnson et al. [21], we propose in section 4 a general framework for DG methods. This section contains three main contributions. First, the generic DG method is formulated in terms of a boundary operator enforcing boundary conditions weakly and in terms of an interface operator penalizing the jumps of the solution across the mesh interfaces. Second, the convergence analysis is performed in the spirit of Strang's Second Lemma by using two different norms, namely, a stability norm for which a discrete inf-sup condition holds and an approximability norm ensuring the continuity of the DG bilinear form. All the design constraints to be fulfilled by the boundary and the interface operators for the error analysis to hold are clearly stated. Finally, using integration by parts, the DG method is reinterpreted locally by introducing the concept of element fluxes and element adjoint-fluxes, thus providing a direct link with engineering practice where approximation schemes are often designed by specifying such fluxes. Finally, section 5 reviews various DG approximations for the model problems investigated in section 3. In all the cases, the degrees of freedom in the design of the DG method are underlined.

2. Friedrichs' systems. The goal of this section is to reformulate Friedrichs' theory by giving special care to the meaning of the boundary conditions. The main results of this section are Theorems 2.5 and 2.8. Theorem 2.8 will be the starting point of the DG method developed in section 4.

2.1. The setting. Let Ω be a bounded, open, and connected Lipschitz domain in $\mathbb{R}^d$. We denote by $\mathfrak{D}(\Omega)$ the space of $\mathfrak{C}^\infty$ functions that are compactly supported in Ω. Let m be a positive integer. Let $\mathcal{K}$ and $\{\mathcal{A}^k\}_{1\le k\le d}$ be $(d+1)$ functions on Ω with values in $\mathbb{R}^{m,m}$. Following Friedrichs [17], we assume that

(A1) $\qquad \mathcal{K} \in [L^\infty(\Omega)]^{m,m}$,

(A2) $\qquad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m} \quad \text{and} \quad \sum_{k=1}^{d} \partial_k \mathcal{A}^k \in [L^\infty(\Omega)]^{m,m}$,

(A3) $\qquad \forall k \in \{1,\dots,d\},\ \mathcal{A}^k = (\mathcal{A}^k)^t$ a.e. in Ω,

(A4) $\qquad \mathcal{Z} := \mathcal{K} + \mathcal{K}^t - \sum_{k=1}^{d} \partial_k \mathcal{A}^k \ge 2\mu_0 \mathcal{I}_m$ a.e. on Ω,

where $\mathcal{I}_m$ is the identity matrix in $\mathbb{R}^{m,m}$. Set $L = [L^2(\Omega)]^m$. We say that a function u in L has an A-weak derivative in L if the linear form

$$[\mathfrak{D}(\Omega)]^m \ni \varphi \longmapsto -\int_\Omega \sum_{k=1}^{d} u^t \partial_k(\mathcal{A}^k\varphi) \in \mathbb{R}, \tag{2.1}$$

is bounded on L, and we denote by Au the function in L that can be associated with the above linear form by means of the Riesz representation theorem. Clearly, if u is smooth enough, e.g., $u \in [\mathfrak{C}^1(\Omega)]^m$,

$$Au = \sum_{k=1}^{d} \mathcal{A}^k \partial_k u. \tag{2.2}$$

Define the graph space

$$W = \{w \in L;\ Aw \in L\}, \tag{2.3}$$

and equip W with the graph norm

$$\|w\|_W = \|Aw\|_L + \|w\|_L, \tag{2.4}$$

and the associated scalar product. W is a Hilbert space. Indeed, let v_n be a Cauchy sequence in W; i.e., v_n and Av_n are Cauchy sequences in L. Let v and w be the corresponding limits in L. Let $\varphi \in [\mathfrak{D}(\Omega)]^m$. Then, using the symmetry of $\mathcal{A}^k$ and an integration by parts yields

$$\int_\Omega \sum_{k=1}^{d} v^t \partial_k(\mathcal{A}^k\varphi) \leftarrow \int_\Omega \sum_{k=1}^{d} v_n^t \partial_k(\mathcal{A}^k\varphi) = -\int_\Omega \varphi^t A v_n \to -\int_\Omega \varphi^t w,$$

which means that v has an A-weak derivative in L and $Av = w$. Since $[\mathcal{D}(\Omega)]^m \subset W$ and $[\mathcal{D}(\Omega)]^m$ is dense in L, W is dense in L; as a result, we shall henceforth use L as a pivot space, i.e., $W \subset L \equiv L' \subset W'$. Note that owing to (A2), $[H^1(\Omega)]^m$ is a subspace of W.

Let $K \in \mathcal{L}(L;L)$ be defined such that $K : L \ni v \mapsto \mathcal{K}v \in L$ and set

$$T = A + K. \tag{2.5}$$

Then, $T \in \mathcal{L}(W;L)$. Let $K^* \in \mathcal{L}(L;L)$ be the adjoint operator of K, i.e., for all $v \in L$, $K^*v = \mathcal{K}^t v$. Let $\tilde{T} \in \mathcal{L}(W;L)$ be the formal adjoint of T,

$$\tilde{T}w = -\sum_{k=1}^{d} \partial_k(\mathcal{A}^k w) + K^* w \quad \forall w \in W. \tag{2.6}$$

In this definition $\sum_{k=1}^d \partial_k(\mathcal{A}^k w)$ is understood in the weak sense. It can easily be verified that this weak derivative exists in L whenever w is in W. Moreover, the usual rule for differentiating products applies. In particular, upon introducing the operator $\nabla{\cdot}A \in \mathcal{L}(L;L)$ such that $(\nabla{\cdot}A)w = (\sum_{k=1}^d \partial_k \mathcal{A}^k)w$ for all $w \in L$, the following holds

$$\forall w \in W, \quad Tw + \tilde{T}w = (K + K^* - \nabla{\cdot}A)\, w. \tag{2.7}$$

Observe that (A4) means that

$$\forall w \in W, \quad (Tw, w)_L + (w, \tilde{T}w)_L \geq 2\mu_0 \|w\|_L^2. \tag{2.8}$$

Definition 2.1. *Let $D \in \mathcal{L}(W;W')$ be the operator such that*

$$\forall (u,v) \in W \times W, \quad \langle Du, v\rangle_{W',W} = (Tu, v)_L - (u, \tilde{T}v)_L. \tag{2.9}$$

This definition makes sense since both T and $\tilde{T}$ are in $\mathcal{L}(W;L)$. Note that D is a boundary operator in the sense that $[\mathfrak{D}(\Omega)]^m \subset \mathrm{Ker}(D)$; see also Remark 2.1. A more precise result (see [14]) is that $\mathrm{Ker}(D) = W_0$ and $\mathrm{Im}(D) = W_0^\perp$, where W_0 is the closure of $[\mathfrak{D}(\Omega)]^m$ in W and for any subset $E \subset W'$ we denote by $E^\perp$ the polar set of E, i.e., the set of the continuous linear forms in $W'' \equiv W$ that are zero on E.

Lemma 2.2. *The operator D is self-adjoint.*

Proof. Let $(u,v) \in W \times W$ and set $Z = K + K^* - \nabla{\cdot}A$. A straightforward calculation yields

$$\begin{aligned}\langle Du, v\rangle_{W',W} - \langle Dv, u\rangle_{W',W} &= (Tu, v)_L - (u, \tilde{T}v)_L - (Tv, u)_L + (v, \tilde{T}u)_L \\ &= (Zu, v)_L - (u, Zv)_L = 0,\end{aligned}$$

since Z is self-adjoint. ∎

Remark 2.1. Let $n = (n_1, \ldots, n_d)^t$ be the unit outward normal to $\partial\Omega$. The usual way of presenting Friedrichs' systems consists of assuming that the fields $\{\mathcal{A}^k\}_{1\leq k\leq d}$ are smooth enough so that the matrix $\mathcal{D} = \sum_{k=1}^d n_k \mathcal{A}^k$ is meaningful at the boundary. Then, the operator D can be represented as follows

$$\langle Du, v\rangle_{W',W} = \int_{\partial\Omega} \sum_{k=1}^{d} v^t n_k \mathcal{A}^k u = \int_{\partial\Omega} v^t \mathcal{D} u,$$

whenever u and v are smooth functions. Provided $[\mathfrak{C}^1(\overline{\Omega})]^m$ is dense in $[H^1(\Omega)]^m$ and in W, it can be shown that $\mathcal{D}u \in [H^{-\frac{1}{2}}(\partial\Omega)]^m$. Further characterization and regularity results on $\mathcal{D}u$ can be found in [27] and in [20].

2.2. The well posedness result. Consider the following problem: For f in L, seek $u \in W$ such that $Tu = f$. In general, boundary conditions must be enforced for this problem to be well posed. In other words, one must find a closed subspace V of W such that the restricted operator $T : V \to L$ is an isomorphism.

The key hypothesis introduced by Friedrichs to select boundary conditions consists of assuming that there exists a matrix-valued field at the boundary, say, $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$, such that a.e. on $\partial\Omega$,

$$\mathcal{M} \text{ is positive, i.e., } (\mathcal{M}\xi, \xi)_{\mathbb{R}^m} \geq 0 \text{ for all } \xi \text{ in } \mathbb{R}^m, \tag{2.10}$$

$$\mathbb{R}^m = \mathrm{Ker}(\mathcal{D} - \mathcal{M}) + \mathrm{Ker}(\mathcal{D} + \mathcal{M}), \tag{2.11}$$

where $\mathcal{D}$ is defined in Remark 2.1. Then, it is possible to prove uniqueness of the so-called strong solution $u \in [\mathfrak{C}^1(\overline{\Omega})]^m$ of the PDE system $Tu = f$ supplemented with the boundary condition $(\mathcal{D} - \mathcal{M})u|_{\partial\Omega} = 0$. Moreover, it is also possible to prove existence of a weak solution in L, namely, of a function $u \in L$ such that the relation $(u, \tilde{T}v)_L = (f, v)_L$ holds for all $v \in [\mathfrak{C}^1(\overline{\Omega})]^m$ such that $(\mathcal{D} + \mathcal{M}^t)v|_{\partial\Omega} = 0$; see [27]. In this paper, we want to investigate the bijectivity of T in a subspace V of the graph W, and it is not possible to set $V = \{v \in W; (\mathcal{D} - \mathcal{M})v|_{\partial\Omega} = 0\}$ since the meaning of traces is not clear.

To overcome this difficulty, we modify Friedrichs' hypothesis by the following assumption: there exists an operator $M \in \mathcal{L}(W; W')$ such that

$$M \text{ is positive, i.e., } \langle Mw, w\rangle_{W',W} \geq 0 \text{ for all } w \text{ in } W, \tag{M1}$$

$$W = \mathrm{Ker}(D - M) + \mathrm{Ker}(D + M). \tag{M2}$$

Let $M^* \in \mathcal{L}(W; W')$ be the adjoint operator of M, i.e., for all $(u, v) \in W \times W$, $\langle M^*u, v\rangle_{W',W} = \langle Mv, u\rangle_{W',W}$. Then, one can prove (see [14]) that (M1)–(M2) imply that $\mathrm{Ker}(D) = \mathrm{Ker}(M)$, $\mathrm{Im}(D) = \mathrm{Im}(M)$, and

$$W = \mathrm{Ker}(D - M^*) + \mathrm{Ker}(D + M^*). \tag{2.12}$$

Since $\mathrm{Ker}(D) = \mathrm{Ker}(M)$, M is a boundary operator. Set

$$V = \mathrm{Ker}(D - M) \quad \text{and} \quad V^* = \mathrm{Ker}(D + M^*), \tag{2.13}$$

and equip V and V^* with the graph norm (2.4). The following result is proven in [14].

LEMMA 2.3. *Assume* (M1)–(M2). *Then,*

$$D(V)^\perp = V^* \quad \text{and} \quad D(V^*)^\perp = V. \tag{2.14}$$

LEMMA 2.4. *Assume* (A1)–(A4) *and* (M1)–(M2). *Then, T is L-coercive on V and $\tilde{T}$ is L-coercive on V^*.*

Proof. Using (2.8) and (2.9) yields

$$(Tw, w)_L \geq \mu_0 \|w\|_L^2 + \frac{1}{2}\langle Dw, w\rangle_{W',W},$$
$$(\tilde{T}w, w)_L \geq \mu_0 \|w\|_L^2 - \frac{1}{2}\langle Dw, w\rangle_{W',W}.$$

Use (2.13) and (M1) to conclude. ∎

THEOREM 2.5. *Assume* (A1)–(A4) *and* (M1)–(M2). *Let V and V^* be defined in* (2.13). *Then,*

(i) *$T : V \to L$ is an isomorphism.*
(ii) *$\tilde{T} : V^* \to L$ is an isomorphism.*

Proof. We only prove (i) since the proof of (ii) is similar.

(1) Owing to (2.13), V is closed in W; hence, V is a Hilbert space. As a result, showing that $T : V \to L$ is an isomorphism amounts to proving statement (ii) in Theorem 2.6 below with $L \equiv L'$.

(2) Proof of (2.15). Let $u \in V$. Observe that $\sup_{v \in L\setminus\{0\}} \frac{(Tu,v)_L}{\|v\|_L} = \|Tu\|_L$. Lemma 2.4 implies $\|Tu\|_L \geq \mu_0 \|u\|_L$. Furthermore,

$$\|Tu\|_L \geq \|Au\|_L - \|K\|_{\mathcal{L}(L;L)}\|u\|_L \geq \|Au\|_L - \frac{\|K\|_{\mathcal{L}(L;L)}}{\mu_0}\|Tu\|_L.$$

This readily yields $\|Au\|_L \leq c\|Tu\|_L$ and thus $\|u\|_W \leq c\|Tu\|_L$.

(3) Proof of (2.16). Assume that $v \in L$ is such that $(Tu, v)_L = 0$ for all $u \in V$. Since $[\mathfrak{D}(\Omega)]^m \subset V$, a standard distribution argument shows that $\tilde{T}v = 0$ in $[\mathfrak{D}'(\Omega)]^m$. Still in the distribution sense, this means that $\sum_{k=1}^d \mathcal{A}^k \partial_k v = K^* v - (\nabla \cdot A)v$. Since the right-hand side is a bounded linear functional on L, v has an A-weak derivative in L, i.e., $v \in W$. As a result, $\langle Du, v\rangle_{W',W} = 0$ for all $u \in V$, i.e., $v \in D(V)^\perp$. Owing to Lemma 2.3, $v \in V^*$. Finally, since $(\tilde{T}v, v)_L = 0$ and $v \in V^*$, Lemma 2.4 implies that v is zero. □

THEOREM 2.6 (Banach–Nečas–Babuška (BNB)). *Let* V, L *be two Banach spaces, and denote by* $\langle\cdot,\cdot\rangle_{L',L}$ *the duality pairing between* L' *and* L. *The following statements are equivalent:*

(i) $T \in \mathcal{L}(V; L)$ *is bijective.*

(ii) *There exists a constant* $\alpha > 0$ *such that*

$$\forall u \in V, \quad \sup_{v \in L'\setminus\{0\}} \frac{\langle v, Tu\rangle_{L',L}}{\|v\|_{L'}} \geq \alpha \|u\|_V, \tag{2.15}$$

$$\forall v \in L', \quad (\langle v, Tu\rangle_{L',L} = 0 \ \forall u \in V) \implies (v = 0). \tag{2.16}$$

As an immediate consequence of Theorem 2.5, the following problems are well posed: For f in L,

$$\text{seek } u \in V \text{ such that } Tu = f, \tag{2.17}$$

$$\text{seek } u^* \in V^* \text{ such that } \tilde{T}u^* = f. \tag{2.18}$$

Remark 2.2. To guarantee that $T : V \to L$ and $\tilde{T} : V^* \to L$ are isomorphisms, it is also possible to specify assumptions on the spaces V and V^* without using the boundary operator M. Introduce the cones $C^\pm = \{w \in W;\ \pm\langle Dw, w\rangle_{W',W} \geq 0\}$. Then, under the following assumptions:

$$V \subset C^+ \text{ and } V^* \subset C^-, \tag{V1}$$

$$V^* = D(V)^\perp \text{ and } V = D(V^*)^\perp, \tag{V2}$$

$T : V \to L$ and $\tilde{T} : V^* \to L$ are isomorphisms [14]. This way of introducing Friedrichs' systems seems to be new. We think that assumptions (V1)–(V2) are more natural than (M1)–(M2) since they do not involve the somewhat ad hoc operator M.

2.3. Boundary conditions weakly enforced. As we have in mind to solve (2.17) by means of DG methods with the boundary conditions weakly enforced, we now propose alternative formulations of (2.17) and (2.18). Define the bilinear forms

$$a(u,v) = (Tu,v)_L + \frac{1}{2}\langle (M-D)u, v\rangle_{W',W}, \tag{2.19}$$

$$a^*(u,v) = (\tilde{T}u,v)_L + \frac{1}{2}\langle (M^*+D)u, v\rangle_{W',W}. \tag{2.20}$$

It is clear that a and a^* are in $\mathcal{L}(W\times W;\mathbb{R})$. A remarkable property is the following lemma.

LEMMA 2.7. *Under assumption* (A4)*, the following holds for all* $w\in W$*,*

$$a(w,w) \geq \mu_0\|w\|_L^2 + \frac{1}{2}\langle Mw, w\rangle_{W',W}, \tag{2.21}$$

$$a^*(w,w) \geq \mu_0\|w\|_L^2 + \frac{1}{2}\langle Mw, w\rangle_{W',W}. \tag{2.22}$$

As a result, a and a^* are L-coercive on W whenever (A4) *and* (M1) *hold.*

Proof. Let $w\in W$. Owing to (2.9),

$$\begin{aligned} a(w,w) &= (Tw,w)_L - \frac{1}{2}\langle Dw,w\rangle_{W',W} + \frac{1}{2}\langle Mw,w\rangle_{W',W} \\ &= \frac{1}{2}((T+\tilde{T})w,w)_L + \frac{1}{2}\langle Mw,w\rangle_{W',W}. \end{aligned}$$

Hence, (2.21) follows from (2.8). The proof of (2.22) is similar. ◻

Consider the following problems: For $f\in L$,

$$\text{seek } u\in W \text{ such that } a(u,v) = (f,v)_L \ \forall v\in W, \tag{2.23}$$

$$\text{seek } u^*\in W \text{ such that } a^*(u^*,v) = (f,v)_L \ \forall v\in W. \tag{2.24}$$

THEOREM 2.8. *Assume* (A1)–(A4) *and* (M1)–(M2)*. Then,*

(i) *There is a unique solution to* (2.23) *and this solution solves* (2.17)*;*

(ii) *There is a unique solution to* (2.24) *and this solution solves* (2.18).

Owing to Theorem 2.5, there is a unique $u\in V$ solving $Tu=f$. Moreover, since u is in V, $(D-M)u=0$. Hence, $a(u,v)=(f,v)_L$ for all $v\in W$, i.e., u solves (2.23). In addition, since a is L-coercive on W owing to Lemma 2.7, it is clear that the solution to (2.23) is unique. This proves statement (i). The proof of the second statement is similar. ◻

Remark 2.3. Neither the bilinear form a nor the bilinear form a^* induce an isomorphism between W and W'. In particular, there is no guarantee that (2.23) or (2.24) has a solution if the right-hand side is replaced by $\langle f,v\rangle_{W',W}$ whenever $f\in W'$.

3. Examples. This section discusses admissible boundary conditions for three important examples of Friedrichs' systems: advection-reaction equations, advection-diffusion-reaction equations, and a simplified version of the Maxwell equations in the elliptic regime. We stress the fact that the existence of an operator $M\in\mathcal{L}(W;W')$ such that (M1)–(M2) hold provides sufficient conditions for well posedness. Although the existence of $M\in\mathcal{L}(W;W')$ may not be granted in all cases (this is reflected, for instance, in the necessity to make assumption (H2) to treat advection–reaction equations; see section 3.1), the formalism appears to be general enough to treat advection-diffusion-reaction equations, and Maxwell's equations in the elliptic regime; see sections 3.2 and 3.3.

3.1. Advection-reaction. Let β be a vector field in $\mathbb{R}^d$, assume $\beta \in [L^\infty(\Omega)]^d$, $\nabla{\cdot}\beta \in L^\infty(\Omega)$, and define

$$\partial\Omega^\pm = \{x \in \partial\Omega;\ \pm\,\beta(x){\cdot}n(x) > 0\}, \tag{3.1}$$

as well as $\partial\Omega^0 = \partial\Omega\backslash(\overline{\partial\Omega^-} \cup \overline{\partial\Omega^+})$; $\partial\Omega^-$ is the inflow boundary, $\partial\Omega^+$ the outflow boundary, and $\partial\Omega^0$ the interior of the set $\{x \in \partial\Omega;\ \beta(x){\cdot}n(x) = 0\}$.

Let μ be a function in $L^\infty(\Omega)$ such that

$$\mu(x) - \frac{1}{2}\nabla{\cdot}\beta(x) \geq \mu_0 > 0 \quad \text{a.e. in } \Omega, \tag{3.2}$$

and consider the advection-reaction equation

$$\mu u + \beta{\cdot}\nabla u = f. \tag{3.3}$$

This PDE falls into the category studied above by setting $Kv = \mu v$ for all $v \in L^2(\Omega)$, and $\mathcal{A}^k = \beta^k$ for $k \in \{1, \dots, d\}$. It is clear that (A1)–(A4) hold with $m = 1$. The graph space is $W = \{w \in L^2(\Omega);\ \beta{\cdot}\nabla w \in L^2(\Omega)\}$.

Henceforth, we assume that

(H1) $\mathfrak{C}_0^1(\mathbb{R}^d)$ is dense in W,

(H2) $\partial\Omega^-$ and $\partial\Omega^+$ are well separated, i.e., $\mathrm{dist}(\partial\Omega^-, \partial\Omega^+) > 0$.

Hypothesis (H1) is a regularity assumption on Ω. It can be shown to hold by using Friedrichs' mollifier whenever Ω and β are smooth. Let $L^2(\partial\Omega; |\beta{\cdot}n|)$ be the space of real-valued functions that are square integrable with respect to the measure $|\beta{\cdot}n|\,\mathrm{d}x$ where $\mathrm{d}x$ is the Lebesgue measure on $\partial\Omega$.

LEMMA 3.1. *Provided* (H1)–(H2) *hold,*

(i) *The trace operator* $\gamma : \mathfrak{C}_0^1(\mathbb{R}^d) \ni v \longrightarrow v \in L^2(\partial\Omega; |\beta{\cdot}n|)$ *extends uniquely to a continuous operator on* W*;*

(ii) *The operator* D *has the following representation: for all* $u, v \in W$,

$$\langle Du, v\rangle_{W',W} = \int_{\partial\Omega} uv(\beta{\cdot}n). \tag{3.4}$$

Proof. Since $\partial\Omega^-$ and $\partial\Omega^+$ are well separated, there are two nonnegative functions ψ^- and ψ^+ in $\mathfrak{C}_0^1(\mathbb{R}^d)$ such that

$$\psi^- + \psi^+ = 1 \ \text{ on } \overline{\Omega}, \qquad \psi^-|_{\partial\Omega^+} = 0, \qquad \psi^+|_{\partial\Omega^-} = 0. \tag{3.5}$$

Let u be a function in $\mathfrak{C}_0^1(\mathbb{R}^d)$. Then,

$$\begin{aligned}\int_{\partial\Omega} u^2|\beta{\cdot}n| &= \int_{\partial\Omega} u^2(\psi^- + \psi^+)|\beta{\cdot}n| = \int_{\partial\Omega^-\cup\partial\Omega^0} u^2\psi^-|\beta{\cdot}n| + \int_{\partial\Omega^+\cup\partial\Omega^0} u^2\psi^+|\beta{\cdot}n| \\ &= -\int_{\partial\Omega} u^2\psi^-(\beta{\cdot}n) + \int_{\partial\Omega} u^2\psi^+(\beta{\cdot}n) = -\int_\Omega \nabla{\cdot}(u^2\psi^-\beta) + \int_\Omega \nabla{\cdot}(u^2\psi^+\beta).\end{aligned}$$

Hence, $0 \leq \int_{\partial\Omega} u^2|\beta{\cdot}n| \leq c(\psi^+, \psi^-)\|u\|_W^2$. Statement (i) follows from the density of $\mathfrak{C}_0^1(\mathbb{R}^d)$ in W. The proof of (ii) is an immediate consequence of the existence of traces in $L^2(\partial\Omega; |\beta{\cdot}n|)$. ∎

To specify boundary conditions, define for $u, v \in W$,

$$\langle Mu, v\rangle_{W',W} = \int_{\partial\Omega} uv|\beta{\cdot}n|. \tag{3.6}$$

LEMMA 3.2. *Let $M \in \mathcal{L}(W;W')$ be defined in* (3.6). *Then,*

(i) (M1)–(M2) *hold;*

(ii) $V = \{v \in W;\ v|_{\partial\Omega^-} = 0\}$ *and* $V^* = \{v \in W;\ v|_{\partial\Omega^+} = 0\}$.

Proof of (i). (M1) directly results from (3.6). Let ψ^+, ψ^- be the partition of unity introduced in (3.5). Let $w \in W$ and write $w = \psi^+ w + \psi^- w$. It is clear that $\psi^+ w \in \mathrm{Ker}(D-M)$ since for all $v \in W$, $\langle (D-M)\psi^+ w, v\rangle_{W',W} = \int_{\partial\Omega^+} \psi^+ vw(\beta\cdot n - |\beta\cdot n|) = 0$. Similarly, $\psi^- w \in \mathrm{Ker}(D+M)$. Hence, (M2) holds.

Proof of (ii). Let $v \in \mathrm{Ker}(D-M)$. Then, for all $w \in W$, $-2\int_{\partial\Omega^-} |\beta\cdot n| vw = 0$. Take $w = v$ to infer $v|_{\partial\Omega^-} = 0$; thus, $\mathrm{Ker}(D-M) \subset V$. Conversely, if $v|_{\partial\Omega^-} = 0$, it is clear that for all $w \in W$, $\langle (D-M)v, w\rangle_{W',W} = -2\int_{\partial\Omega^-} |\beta\cdot n| vw = 0$, i.e., $v \in \mathrm{Ker}(D-M)$. Proceed similarly to prove that $V^* = \{v \in W;\ v|_{\partial\Omega^+} = 0\}$. ◻

3.2. Advection-diffusion-reaction equations. Let $\beta : \Omega \longrightarrow \mathbb{R}^d$ be a vector field such that $\beta \in [L^\infty(\Omega)]^d$ and $\nabla\cdot\beta \in L^\infty(\Omega)$. Let μ be a function in $L^\infty(\Omega)$ such that (3.2) holds, and consider the advection-diffusion-reaction equation

$$-\Delta u + \beta\cdot\nabla u + \mu u = f. \tag{3.7}$$

This equation can be written as a system of first-order PDEs in the form

$$\begin{cases} \sigma + \nabla u = 0, \\ \mu u + \nabla\cdot\sigma + \beta\cdot\nabla u = f. \end{cases} \tag{3.8}$$

The above differential operator can be cast into the form of a Friedrichs' operator by setting $K(\sigma,u) = (\sigma, \mu u)$ for all $(\sigma,u) \in [L^2(\Omega)]^{d+1}$, and for $k \in \{1,\dots,d\}$,

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0 & e^k \\ \hdashline (e^k)^t & \beta^k \end{array}\right], \tag{3.9}$$

where e^k is the kth vector in the canonical basis of $\mathbb{R}^d$. It is clear that hypotheses (A1)–(A4) hold with $m = d+1$. Upon observing the norm equivalence

$$\begin{aligned} c_1(\|\nabla u\|_{L^2(\Omega)} + \|\nabla\cdot\sigma\|_{L^2(\Omega)}) \le \|\nabla u\|_{L^2(\Omega)} &+ \|\beta\cdot\nabla u + \nabla\cdot\sigma\|_{L^2(\Omega)} \\ &\le c_2(\|\nabla u\|_{L^2(\Omega)} + \|\nabla\cdot\sigma\|_{L^2(\Omega)}), \end{aligned}$$

it is inferred that the graph space is $W = H(\mathrm{div};\Omega) \times H^1(\Omega)$. Moreover, the boundary operator D is such that for all $(\sigma,u), (\tau,v) \in W$,

$$\langle D(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac12,\frac12} + \langle \tau\cdot n, u\rangle_{-\frac12,\frac12} + \int_{\partial\Omega} (\beta\cdot n)uv, \tag{3.10}$$

where $\langle,\rangle_{-\frac12,\frac12}$ denotes the duality pairing between $H^{-\frac12}(\partial\Omega)$ and $H^{\frac12}(\partial\Omega)$. Note that (3.10) makes sense since functions in $H^1(\Omega)$ have traces in $H^{\frac12}(\partial\Omega)$ and vector fields in $H(\mathrm{div};\Omega)$ have normal traces in $H^{-\frac12}(\partial\Omega)$.

3.2.1. Dirichlet boundary conditions. A suitable operator M to weakly enforce Dirichlet boundary conditions is such that for all $(\sigma,u),(\tau,v) \in W$,

$$\langle M(\sigma,u),(\tau,v)\rangle_{W',W} = \langle \sigma\cdot n, v\rangle_{-\frac12,\frac12} - \langle \tau\cdot n, u\rangle_{-\frac12,\frac12}. \tag{3.11}$$

LEMMA 3.3. *Let* $M \in \mathcal{L}(W; W')$ *be defined in* (3.11)*. Then,*
(i) (M1)–(M2) *hold;*
(ii) $V = V^* = \{(\sigma, u) \in W;\ u|_{\partial\Omega} = 0\}$.

Proof of (i). (M1) clearly holds since $M + M^* = 0$. Let $w = (\sigma, u) \in W$ and write $w = w^+ + w^-$ with $w^+ = (-\frac{1}{2}\beta u, u)$ and $w^- = (\sigma + \frac{1}{2}\beta u, 0)$. By assumption on β, the vector-valued field βu is in $H(\mathrm{div};\Omega)$ if $u \in H^1(\Omega)$; hence, $w^\pm$ are in W. Moreover, a straightforward calculation shows that $w^\pm \in \mathrm{Ker}(D \pm M)$. Hence, (M2) holds.

Proof of (ii). The identity $V = V^*$ results from the fact that $M + M^* = 0$. Moreover, observe that for all $(\sigma, u), (\tau, v) \in W$,

$$\langle (D - M)(\sigma, u), (\tau, v)\rangle_{W',W} = 2\langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (\beta{\cdot}n) uv.$$

Let $(\sigma, u) \in \mathrm{Ker}(D - M)$. Let $\gamma \in H^{-\frac{1}{2}}(\partial\Omega)$. There exists $\tau \in H(\mathrm{div};\Omega)$ such that $\tau{\cdot}n = \gamma$ in $H^{-\frac{1}{2}}(\partial\Omega)$. Then, using $(\tau, 0)$ in the above equation yields $\langle \gamma, u\rangle_{-\frac{1}{2},\frac{1}{2}} = 0$. Since γ is arbitrary, this implies $u|_{\partial\Omega} = 0$. Hence, $V \subset \{(\sigma, u) \in W;\ u|_{\partial\Omega} = 0\}$. Conversely, let $(\sigma, u) \in W$ be such that $u|_{\partial\Omega} = 0$. Then, the above equation shows that $(\sigma, u) \in \mathrm{Ker}(D - M) = V$. ∎

Remark 3.1. The choice of the operator M to enforce homogeneous Dirichlet boundary conditions is not unique. For instance, one can take $\langle M(\sigma, u), (\tau, v)\rangle_{W',W} = \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} \varsigma uv$, where ς is a nonnegative real number.

3.2.2. Neumann and Robin boundary conditions. Let $\varrho \in L^\infty(\partial\Omega)$ be such that $2\varrho + \beta{\cdot}n \geq 0$ a.e. on $\partial\Omega$. Neumann and Robin boundary conditions are treated simultaneously, the choice $\varrho = 0$ yielding a Neumann boundary condition (in this case, $\beta{\cdot}n \geq 0$ a.e. on $\partial\Omega$ corresponding to an outflow boundary). A suitable operator M to weakly enforce Neumann or Robin boundary conditions is such that for all $(\sigma, u), (\tau, v) \in W$,

$$\langle M(\sigma, u), (\tau, v)\rangle_{W',W} = \langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + \int_{\partial\Omega} (2\varrho + \beta{\cdot}n) uv. \tag{3.12}$$

LEMMA 3.4. *Let* $M \in \mathcal{L}(W; W')$ *be defined in* (3.12)*. Then,*
(i) (M1)–(M2) *hold;*
(ii) $V = \{(\sigma, u) \in W;\ \sigma{\cdot}n = \varrho u|_{\partial\Omega}\}$ *and* $V^* = \{(\sigma, u) \in W;\ \sigma{\cdot}n = -(\varrho + \beta{\cdot}n) u|_{\partial\Omega}\}$.

Proof. (M1) holds since $2\varrho + \beta{\cdot}n \geq 0$ a.e. on $\partial\Omega$. Furthermore, observe that

$$\begin{aligned}
\langle (D - M)(\sigma, u), (\tau, v)\rangle_{W',W} &= 2\langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} - 2\textstyle\int_{\partial\Omega} \varrho uv,\\
\langle (D + M)(\sigma, u), (\tau, v)\rangle_{W',W} &= 2\langle \tau{\cdot}n, u\rangle_{-\frac{1}{2},\frac{1}{2}} + 2\textstyle\int_{\partial\Omega} (\varrho + \beta{\cdot}n) uv,\\
\langle (D + M^*)(\sigma, u), (\tau, v)\rangle_{W',W} &= 2\langle \sigma{\cdot}n, v\rangle_{-\frac{1}{2},\frac{1}{2}} + 2\textstyle\int_{\partial\Omega} (\varrho + \beta{\cdot}n) uv.
\end{aligned}$$

Let $w = (\sigma, u) \in W$. Since $\varrho u|_{\partial\Omega} \in H^{-\frac{1}{2}}(\partial\Omega)$, there is $\sigma_0 \in H(\mathrm{div};\Omega)$ such that $\sigma_0{\cdot}n = \varrho u|_{\partial\Omega}$. Then, setting $w^+ = (\sigma - \sigma_0, 0)$ and $w^- = (\sigma_0, u)$, it is easily verified that $w^\pm \in \mathrm{Ker}(D \pm M)$ and, hence, (M2) holds. Finally, proceed as in the proof of Lemma 3.3 to prove (ii). ∎

3.3. Maxwell's equations in the elliptic regime. We close this series of examples by considering a simplified form of Maxwell's equations in $\mathbb{R}^3$ in the elliptic regime, i.e., when displacement currents are negligible. Let σ and μ be two positive functions in $L^\infty(\Omega)$ uniformly bounded away from zero. Consider the following problem

$$\begin{cases} \mu H + \nabla\times E = f, \\ \sigma E - \nabla\times H = g. \end{cases} \tag{3.13}$$

This problem can be cast into the form of a Friedrichs' system by setting $K(H,E) = (\mu H, \sigma E)$ for all $(H,E) \in [L^2(\Omega)]^3 \times [L^2(\Omega)]^3$ and for $k \in \{1,2,3\}$,

$$\mathcal{A}^k = \left[\begin{array}{c:c} 0 & \mathcal{R}^k \\ \hdashline (\mathcal{R}^k)^t & 0 \end{array}\right]. \tag{3.14}$$

The entries of the matrices $\mathcal{R}^k \in \mathbb{R}^{3,3}$ are those of the Levi–Civita permutation tensor, i.e., $\mathcal{R}^k_{ij} = \epsilon_{ikj}$ for $1 \le i,j,k \le 3$. Hypotheses (A1)–(A4) hold with $m = 6$. The graph space is $W = H(\text{curl};\Omega)\times H(\text{curl};\Omega)$, and the boundary operator D is such that for all $(H,E),(h,e) \in W$,

$$\begin{aligned} \langle D(H,E),(h,e)\rangle_{W',W} = {} & (\nabla\times E, h)_{[L^2(\Omega)]^3} - (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ & + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}. \end{aligned} \tag{3.15}$$

When H and E are smooth the above duality product can be interpreted as the boundary integral $\int_{\partial\Omega} (n\times E)\cdot h + (n\times e)\cdot H$.

Let us now define acceptable boundary conditions for (3.13). One possibility (among many others) consists of setting for all $(H,E),(h,e) \in W$,

$$\begin{aligned} \langle M(H,E),(h,e)\rangle_{W',W} = {} & -(\nabla\times E, h)_{[L^2(\Omega)]^3} + (E, \nabla\times h)_{[L^2(\Omega)]^3} \\ & + (H, \nabla\times e)_{[L^2(\Omega)]^3} - (\nabla\times H, e)_{[L^2(\Omega)]^3}. \end{aligned} \tag{3.16}$$

LEMMA 3.5. *Let M be defined in* (3.16)*. Then,*
(i) (M1)–(M2) *hold;*
(ii) $V = V^* = \{(H,E) \in W;\ (E\times n)|_{\partial\Omega} = 0\}$.

Proof of (i). Observe that $M + M^* = 0$; hence, M is positive. Let $w = (H,E) \in W$. Write $w = w^+ + w^-$ with $w^+ = (0,E)$ and $w^- = (H,0)$. One easily verifies that $w^\pm \in \text{Ker}(D \pm M)$, i.e., (M2) holds.

Proof of (ii). The identity $V = V^*$ results from the fact that $M + M^* = 0$. Let $(H,E) \in \text{Ker}(D - M)$. Then, for all $(h,e) \in W$,

$$\langle (D-M)(H,E),(h,e)\rangle_{W',W} = 2(\nabla\times E, h)_{[L^2(\Omega)]^3} - 2(E,\nabla\times h)_{[L^2(\Omega)]^3} = 0.$$

Since vector fields in $H(\text{curl};\Omega)$ have tangential traces in $[H^{-\frac12}(\partial\Omega)]^3$, we infer that for all $h \in [H^1(\Omega)]^3$, $\langle (E\times n), h\rangle_{-\frac12,\frac12} = 0$. Since h is arbitrary and the traces of vector fields in $[H^1(\Omega)]^3$ span $[H^{\frac12}(\partial\Omega)]^3$, we conclude that $(E\times n)|_{\partial\Omega} = 0$. Conversely, let $(H,E) \in W$ be such that $(E\times n)|_{\partial\Omega} = 0$. Then, it is clear that $\langle (D - M)(H,E),(h,e)\rangle_{W',W} = 0$ for all $h \in [H^1(\Omega)]^3$ and all $e \in H(\text{curl};\Omega)$. Since $[H^1(\Omega)]^3$ is dense in $H(\text{curl};\Omega)$ and both D and M are in $\mathcal{L}(W;W')$, it follows that $(H,E) \in \text{Ker}(D - M)$. ◻

4. Discontinuous Galerkin. The goal of this section is to introduce a generic DG method to approximate the abstract problem (2.23). The fact that the boundary conditions are enforced weakly through the boundary operator M is a key to the theory. The discrete problem is stated in (4.12)–(4.13). The design constraints of the method are (DG1) to (DG8). The main convergence result is stated in Theorem 4.6.

4.1. The discrete setting. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of meshes of Ω. The meshes are assumed to be affine to avoid unnecessary technicalities, i.e., Ω is assumed to be a polyhedron. However, we do not make any assumption on the matching of element interfaces.

Let p be a nonnegative integer. Define

$$W_h = \{v_h \in [L^2(\Omega)]^m;\, \forall K \in \mathcal{T}_h,\, v_h|_K \in [\mathbb{P}_p]^m\}, \tag{4.1}$$

$$W(h) = [H^1(\Omega)]^m + W_h. \tag{4.2}$$

We denote by $\mathcal{F}_h^{\mathrm{i}}$ the set of interior faces (or interfaces), i.e., $F \in \mathcal{F}_h^{\mathrm{i}}$ if F is a $(d-1)$-manifold and there are $K_1(F), K_2(F) \in \mathcal{T}_h$ such that $F = K_1(F) \cap K_2(F)$. We denote by $\mathcal{F}_h^{\partial}$ the set of the faces that separate the mesh from the exterior of Ω, i.e., $F \in \mathcal{F}_h^{\partial}$ if F is a $(d-1)$-manifold and there is $K(F) \in \mathcal{T}_h$ such that $F = K(F) \cap \partial\Omega$. Finally, we set $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^{\partial}$. Since every function v in $W(h)$ has a (possibly two-valued) trace almost everywhere on $F \in \mathcal{F}_h^{\mathrm{i}}$, it is meaningful to set

$$v^1(x) = \lim_{\substack{y\to x\\ y\in K_1(F)}} v(y), \qquad v^2(x) = \lim_{\substack{y\to x\\ y\in K_2(F)}} v(y), \qquad \text{for a.e. } x \in F, \tag{4.3}$$

$$[\![v]\!] = v^1 - v^2, \qquad \{v\} = \frac{1}{2}(v^1 + v^2), \qquad \text{a.e. on } F. \tag{4.4}$$

The arbitrariness in the choice of $K_1(F)$ and $K_2(F)$ could be avoided by choosing an intrinsic notation that would, however, unnecessarily complicate the presentation. For instance, we could have chosen to set $[\![v]\!] = v^1 \otimes n^1 + v^2 \otimes n^2$ where n^1, n^2 are the unit outward normals of $K_1(F)$ and $K_2(F)$, respectively. Although having to choose $K_1(F)$ and $K_2(F)$ may seem cumbersome, nothing that is said hereafter depends on the choice that is made.

For any measurable subset of Ω or $\mathcal{F}_h$, say E, $(\cdot,\cdot)_{L,E}$ denotes the scalar product induced by $[L^2(\Omega)]^m$ or $[L^2(\mathcal{F}_h)]^m$ on E, respectively, and $\|\cdot\|_{L,E}$ the associated norm. Similarly, $\|\cdot\|_{L^d,E}$ denotes the norm induced by $[L^2(\Omega)]^{m\times d}$ or $[L^2(\mathcal{F}_h)]^{m\times d}$ on E. For $K \in \mathcal{T}_h$ (resp., $F \in \mathcal{F}_h$), h_K (resp., h_F) denotes the diameter of K (resp., F).

The mesh family $\{\mathcal{T}_h\}_{h>0}$ is assumed to be shape-regular so that there is a constant c, independent of $h = \max_{K\in\mathcal{T}_h} h_K$, such that for all $v_h \in W_h$ and for all $K \in \mathcal{T}_h$,

$$\|\nabla v_h\|_{L^d,K} \le c\, h_K^{-1} \|v_h\|_{L,K}, \tag{4.5}$$

$$\|v_h\|_{L,F} \le c\, h_K^{-\frac{1}{2}} \|v_h\|_{L,K} \quad \forall F \subset \partial K. \tag{4.6}$$

4.2. Boundary operators. Henceforth we denote $\mathcal{D}_{\partial\Omega} = \sum_{k=1}^d n_k \mathcal{A}^k$ and we assume that the boundary operator M is associated with a matrix-valued field $\mathcal{M} : \partial\Omega \longrightarrow \mathbb{R}^{m,m}$. Hence, for all functions u, v smooth enough (e.g., $u, v \in [H^1(\Omega)]^m$), the following holds:

$$\langle Du, v\rangle_{W',W} = \int_{\partial\Omega} v^t \mathcal{D}_{\partial\Omega} u, \qquad \langle Mu, v\rangle_{W',W} = \int_{\partial\Omega} v^t \mathcal{M} u. \tag{4.7}$$

To enforce boundary conditions weakly, we introduce for all $F \in \mathcal{F}_h^{\partial}$ a linear operator $M_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$. The design of the boundary operators $\{M_F\}_{F\in\mathcal{F}_h^{\partial}}$

must comply with the following conditions: For all $F \in \mathcal{F}_h^\partial$ and for all $v, w \in [L^2(F)]^m$,

$$(M_F(v), v)_{L,F} \geq 0, \tag{DG1}$$

$$(\mathcal{M}v = \mathcal{D}_{\partial\Omega}v) \implies (M_F(v) = \mathcal{D}_{\partial\Omega}v), \tag{DG2}$$

$$|(M_F(v) - \mathcal{D}_{\partial\Omega}v, w)_{L,F}| \leq c|v|_{M,F}\|w\|_{L,F}, \tag{DG3}$$

$$|(M_F(v) + \mathcal{D}_{\partial\Omega}v, w)_{L,F}| \leq c\|v\|_{L,F}|w|_{M,F}, \tag{DG4}$$

where c is a mesh-independent constant and where we have introduced for all $v \in W(h)$ the following seminorms:

$$|v|_M^2 = \sum_{F \in \mathcal{F}_h^\partial} |v|_{M,F}^2 \qquad \text{with} \qquad |v|_{M,F}^2 = (M_F(v), v)_{L,F}. \tag{4.8}$$

Remark 4.1.

(i) Examples of boundary operators M_F are presented in section 5 for all the model problems introduced in section 3.

(ii) Assumption (DG2) is a consistency assumption while assumptions (DG3) and (DG4) are related to the stability and continuity of the discrete bilinear form; see the analysis in section 4.5.

4.3. Interface operators. For $K \in \mathcal{T}_h$, define the matrix-valued field $\mathcal{D}_{\partial K} : \partial K \to \mathbb{R}^{m,m}$ as

$$\mathcal{D}_{\partial K}(x) = \sum_{k=1}^{d} n_{K,k}\mathcal{A}^k(x) \quad \text{a.e. on } \partial K, \tag{4.9}$$

where $n_K = (n_{K,1}, \ldots, n_{K,d})^t$ is the unit outward normal to K on ∂K. Note that this definition is compatible with that of $\mathcal{D}_{\partial\Omega}$ in (4.7) if $\partial K \cap \partial\Omega \neq \emptyset$. Moreover, observe that for all u, v in $W(h)$ and for all $K \in \mathcal{T}_h$,

$$(\mathcal{D}_{\partial K}u, v)_{L,\partial K} = (Tu, v)_{L,K} - (u, \tilde{T}v)_{L,K}. \tag{4.10}$$

We denote by $\mathcal{D}$ the matrix-valued field defined on $\mathcal{F}_h = \mathcal{F}_h^{\mathrm{i}} \cup \mathcal{F}_h^\partial$ as follows. On $\mathcal{F}_h^\partial$, $\mathcal{D}$ is single-valued and coincides with $\mathcal{D}_{\partial\Omega}$. On $\mathcal{F}_h^{\mathrm{i}}$, $\mathcal{D}$ is two-valued and for all $F \in \mathcal{F}_h^{\mathrm{i}}$, its two values are $\mathcal{D}_{\partial K_1(F)}$ and $\mathcal{D}_{\partial K_2(F)}$. Note that $\{\mathcal{D}\} = 0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$.

To control the jumps of functions in W_h across mesh interfaces, we introduce for all $F \in \mathcal{F}_h^{\mathrm{i}}$ a linear operator $S_F \in \mathcal{L}([L^2(F)]^m; [L^2(F)]^m)$. The analysis below will show that the design of the interface operators $\{S_F\}_{F \in \mathcal{F}_h^{\mathrm{i}}}$ must comply with the following conditions. For all $F \in \mathcal{F}_h^{\mathrm{i}}$ and for all $v, w \in [L^2(F)]^m$,

$$(S_F(v), v)_{L,F} \geq 0, \tag{DG5}$$

$$\|S_F(v)\|_{L,F} \leq c\|v\|_{L,F}, \tag{DG6}$$

$$|(S_F(v), w)_{L,F}| \leq c|v|_{S,F}|w|_{S,F}, \tag{DG7}$$

$$|(\mathcal{D}_{\partial K(F)}v, w)_{L,F}| \leq c|v|_{S,F}\|w\|_{L,F}, \tag{DG8}$$

where c is a mesh-independent constant, $K(F)$ denotes any of the two elements sharing F and $\partial K(F)$ its boundary, and where we have introduced for all $v \in W(h)$ the following seminorms:

$$|v|_S^2 = \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} |v|_{S,F}^2 \qquad \text{with} \qquad |v|_{S,F}^2 = (S_F(v), v)_{L,F}. \tag{4.11}$$

Remark 4.2.

(i) Examples of interface operators S_F are presented in section 5 for all the model problems introduced in section 3.

(ii) Since S_F is positive, a sufficient condition for (DG7) to hold with $c = 1$ is S_F be self-adjoint.

4.4. The discrete problem. We now turn our attention to the construction of a discrete counterpart of (2.23). To this end we introduce the bilinear form a_h such that for all v, w in $W(h)$,

$$\begin{aligned} a_h(v,w) = \sum_{K\in\mathcal{T}_h} (Tv,w)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \frac{1}{2}(M_F(v) - \mathcal{D}v, w)_{L,F} \\ - \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} 2(\{\mathcal{D}v\}, \{w\})_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![v]\!]), [\![w]\!])_{L,F}. \end{aligned} \tag{4.12}$$

Then, we construct an approximate solution to (2.23) as follows. For $f \in L$,

$$\begin{cases} \text{seek } u_h \in W_h \text{ such that} \\ a_h(u_h, v_h) = (f, v_h)_L \quad \forall v_h \in W_h. \end{cases} \tag{4.13}$$

Remark 4.3. In the definition of a_h, the second term weakly enforces the boundary conditions. The purpose of the third term is to ensure that a coercivity property holds, see Lemma 4.1. The last term controls the jump of the discrete solution across interfaces. Some user-dependent arbitrariness appears in the second and fourth term through the definition of the operators M_F and S_F. The design constraints on M_F and S_F are (DG1)–(DG4) and (DG5)–(DG8), respectively.

4.5. Convergence analysis. To perform the error analysis we introduce the following discrete norms on $W(h)$,

$$\|v\|_{h,A}^2 = \|v\|_L^2 + |v|_J^2 + |v|_M^2 + \sum_{K\in\mathcal{T}_h} h_K \|Av\|_{L,K}^2, \tag{4.14}$$

$$\|v\|_{h,\frac{1}{2}}^2 = \|v\|_{h,A}^2 + \sum_{K\in\mathcal{T}_h} [h_K^{-1}\|v\|_{L,K}^2 + \|v\|_{L,\partial K}^2], \tag{4.15}$$

where we have introduced the jump seminorms

$$|v|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |v|_{J,F}^2 \qquad \text{with} \qquad |v|_{J,F} = |[\![v]\!]|_{S,F}. \tag{4.16}$$

The norm $\|\cdot\|_{h,A}$ is used to measure the approximation error, and the norm $\|\cdot\|_{h,\frac{1}{2}}$ serves to measure the interpolation properties of the discrete space W_h.

Throughout this section, we assume that:

• Problem (2.23) is well-posed.

• The mesh family $\{\mathcal{T}_h\}_{h>0}$ is shape-regular so that (4.5) and (4.6) hold.

• The design assumptions (DG1)–(DG8) on M_F and S_F hold.

• For all $k \in \{1,\dots,d\}$, $\mathcal{A}^k \in [\mathfrak{C}^{0,\frac{1}{2}}(\overline{\Omega})]^{m,m}$.

LEMMA 4.1 (L-coercivity). *For all h and for all v in $W(h)$,*

$$a_h(v,v) \geq \mu_0\|v\|_L^2 + |v|_J^2 + \frac{1}{2}|v|_M^2. \tag{4.17}$$

Proof. Let v in $W(h)$. Using (4.10) and summing over the mesh elements yields

$$\sum_{F\in\mathcal{F}_h^\partial}\frac{1}{2}(\mathcal{D}v,v)_{L,F}+\sum_{F\in\mathcal{F}_h^{\mathrm{i}}}\int_F\{v^t\mathcal{D}v\}=\frac{1}{2}\sum_{K\in\mathcal{T}_h}[(Tv,v)_{L,K}-(v,\tilde{T}v)_{L,K}].$$

Subtracting this equation from (4.12) and using the fact that $\{v^t\mathcal{D}v\}=2\{v^t\}\{\mathcal{D}v\}$ leads to

$$a_h(v,v)=\frac{1}{2}\sum_{K\in\mathcal{T}_h}\Big[(Tv,v)_{L,K}+(v,\tilde{T}v)_{L,K}\Big]+|v|_J^2+\frac{1}{2}|v|_M^2.$$

Then, the desired result follows using (A4). □

LEMMA 4.2. *There is $c>0$, independent of h, such that for all F in $\mathcal{F}_h^{\mathrm{i}}$ and for all $v,w\in W(h)$,*

$$|(S_F([\![v]\!]),[\![w]\!])_{L,F}|+|(\{\mathcal{D}v\},\{w\})_{L,F}|\le c|v|_{J,F}(\|\{w\}\|_{L,F}+\|[\![w]\!]\|_{L,F}). \tag{4.18}$$

Proof.

(1) Owing to (DG7), $(S_F([\![v]\!]),[\![w]\!])_{L,F}\le c|v|_{J,F}|w|_{J,F}$, and owing to (DG6), $|w|_{J,F}\le c\|[\![w]\!]\|_{L,F}$. Hence, $(S_F([\![v]\!]),[\![w]\!])_{L,F}\le c|v|_{J,F}\|[\![w]\!]\|_{L,F}$.

(2) Let $K_1(F)$ and $K_2(F)$ be the two mesh elements such that $F=K_1(F)\cap K_2(F)$. Then, $2\{\mathcal{D}v\}=\mathcal{D}_{K_1(F)}[\![v]\!]$ since $\{\mathcal{D}\}=0$. Using (DG8) yields

$$|(\{\mathcal{D}v\},\{w\})_{L,F}|=|(\mathcal{D}_{K_1(F)}[\![v]\!],\{w\})_{L,F}|\le c|v|_{J,F}\|\{w\}\|_{L,F}.$$

The proof is complete. □

LEMMA 4.3 (stability). *There is $c>0$, independent of h, such that*

$$\inf_{v_h\in W_h\setminus\{0\}}\ \sup_{w_h\in W_h\setminus\{0\}}\frac{a_h(v_h,w_h)}{\|v_h\|_{h,A}\|w_h\|_{h,A}}\ge c. \tag{4.19}$$

Proof.

(1) Let v_h be an arbitrary element in W_h. Let $K\in\mathcal{T}_h$. Denote by $\overline{\mathcal{A}_K^k}$ the mean-value of $\mathcal{A}^k$ on K; then,

$$\|\mathcal{A}^k-\overline{\mathcal{A}_K^k}\|_{[L^\infty(K)]^{m,m}}\le\|\mathcal{A}^k\|_{[\mathfrak{C}^{0,\frac{1}{2}}(\overline{\Omega})]^{m,m}}h_K^{\frac{1}{2}}. \tag{4.20}$$

Set $\overline{A}_Kv_h=\sum_{k=1}^d\overline{\mathcal{A}_K^k}\partial_kv_h$ and $\pi_h=\sum_{K\in\mathcal{T}_h}h_K\overline{A}_Kv_h$. Clearly, $\pi_h\in W_h$. Using (4.20), together with the inverse inequalities (4.5) and (4.6), leads to

$$\begin{cases}\|\overline{A}_Kv_h\|_{L,F}\le c\,h_K^{-\frac{1}{2}}\|\overline{A}_Kv_h\|_{L,K} & \text{if } F\in\mathcal{F}_h^\partial,\\ \|\{\overline{A}_Kv_h\}\|_{L,F}+\|[\![\overline{A}_Kv_h]\!]\|_{L,F}\le c\,h_K^{-\frac{1}{2}}\|\overline{A}_Kv_h\|_{L,K_1\cup K_2} & \text{if } F\in\mathcal{F}_h^{\mathrm{i}},\end{cases} \tag{4.21}$$

$$\|\overline{A}_Kv_h\|_{L,K}\le c\,\min(\|Av_h\|_{L,K}+h_K^{-\frac{1}{2}}\|v_h\|_{L,K},h_K^{-1}\|v_h\|_{L,K}). \tag{4.22}$$

Note that (4.22) implies $\|\pi_h\|_L \le c\|v_h\|_L$. From the definition of a_h it follows that

$$\begin{aligned}\sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}^2 &= a_h(v_h,\pi_h) - (Kv_h,\pi_h)_L - \sum_{F\in\mathcal{F}_h^\partial}\frac{1}{2}(M_F(v_h)-\mathcal{D}v_h,\pi_h)_{L,F}\\ &\quad + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}}[2(\{\mathcal{D}v_h\},\{\pi_h\})_{L,F} - (S_F([\![v_h]\!]),[\![\pi_h]\!])_{L,F}]\\ &\quad + \sum_{K\in\mathcal{T}_h} h_K(Av_h,(A-\overline{A}_K)v_h)_{L,K}\\ &= a_h(v_h,\pi_h) + R_1 + R_2 + R_3 + R_4,\end{aligned}$$

where R_1, R_2, R_3, and R_4 denote the second, third, fourth, and fifth term in the right-hand side of the above equation, respectively. Each of these terms is bounded from above as follows. Using (4.22) yields $\|\pi_h\|_L \le c\|v_h\|_L$ and hence,

$$|R_1| \le c\|v_h\|_L\|\pi_h\|_L \le c\|v_h\|_L^2.$$

Using (DG3) together with (4.21) and (4.22) leads to

$$\begin{aligned}|R_2| &\le \sum_{F\in\mathcal{F}_h^\partial}[c_\gamma(M_F(v_h),v_h)_{L,F} + \gamma\|\pi_h\|_{L,F}^2]\\ &\le c(\|v_h\|_L^2 + |v_h|_M^2) + \gamma\sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}^2,\end{aligned}$$

where $\gamma > 0$ can be chosen as small as needed. For the third term, use Lemma 4.2, together with inequalities (4.21) and (4.22), as follows:

$$\begin{aligned}|R_3| &\le \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} c_\gamma|v_h|_{J,F}^2 + \gamma\sum_{K\in\mathcal{T}_h} h_K\|\overline{A}_Kv_h\|_{L,K}^2\\ &\le c(\|v_h\|_L^2 + |v_h|_J^2) + \gamma\sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}^2.\end{aligned}$$

For the last term, (4.5) and (4.20) yield

$$\begin{aligned}|R_4| &\le \sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}\,c h_K^{\frac{1}{2}}\|\nabla v_h\|_{L^d,K}\\ &\le c\sum_{K\in\mathcal{T}_h} h_K^{\frac{1}{2}}\|Av_h\|_{L,K}\|v_h\|_{L,K} \le c\|v_h\|_L^2 + \gamma\sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}^2.\end{aligned}$$

Using the above four bounds, $\gamma = \frac{1}{6}$, and Lemma 4.1 leads to

$$\frac{1}{2}\sum_{K\in\mathcal{T}_h} h_K\|Av_h\|_{L,K}^2 \le a_h(v_h,\pi_h) + c\,a_h(v_h,v_h). \tag{4.23}$$

(2) Let us now prove that $\|\pi_h\|_{h,A} \le c\,\|v_h\|_{h,A}$. We have already seen that $\|\pi_h\|_L \le c\|v_h\|_L$. Using (4.5), together with inequalities (4.20) and (4.22), leads to

$$\sum_{K\in\mathcal{T}_h} h_K\|A\pi_h\|_{L,K}^2 \le c\sum_{K\in\mathcal{T}_h} h_K^{-1}\|\pi_h\|_{L,K}^2 \le c\sum_{K\in\mathcal{T}_h}[h_K\|Av_h\|_{L,K}^2 + \|v_h\|_{L,K}^2].$$

Moreover, the inverse inequality (4.6), assumption (DG6), and inequalities (4.21) and (4.22) yield

$$|\pi_h|_J^2 = \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} |\pi_h|_{J,F}^2 \le c \sum_{K\in\mathcal{T}_h} h_K^{-1}\|\pi_h\|_{L,K}^2 \le c \sum_{K\in\mathcal{T}_h} [h_K\|Av_h\|_{L,K}^2 + \|v_h\|_{L,K}^2].$$

Proceed similarly to control $|\pi_h|_M$. In conclusion,

$$\|\pi_h\|_{h,A} \le c\,\|v_h\|_{h,A}. \tag{4.24}$$

(3) Owing to (4.17) and (4.23), there is $c_1 > 0$ such that

$$\|v_h\|_{h,A}^2 \le c_1 a_h(v_h, v_h) + a_h(v_h, \pi_h) = a_h(v_h, \pi_h + c_1 v_h).$$

Then, setting $w_h = \pi_h + c_1 v_h$ and using (4.24) yields

$$\|v_h\|_{h,A}\|w_h\|_{h,A} \le c\,\|v_h\|_{h,A}^2 \le c\,a_h(v_h, w_h).$$

The conclusion is straightforward. ▯

Remark 4.4. Note that (4.5) and (4.17) readily imply coercivity in the weaker norm $\|v\|_{h,A^-}^2 = \|v\|_L^2 + |v|_J^2 + |v|_M^2 + \sum_{K\in\mathcal{T}_h} h_K^2\|Av\|_{L,K}^2$, but this property is not sufficient to prove an optimal convergence rate in the broken graph norm; see (4.32).

LEMMA 4.4 (continuity). *There is c, independent of h, such that*

$$\forall (v,w)\in W(h)\times W(h),\quad a_h(v,w) \le c\,\|v\|_{h,\frac{1}{2}}\|w\|_{h,A}. \tag{4.25}$$

Proof. The general principle of the proof is to integrate by parts $a_h(v,w)$ by making use of the formal adjoint $\tilde{T}$. Observing that

$$\sum_{K\in\mathcal{T}_h} [(Tv,w)_{L,K} - (v,\tilde{T}w)_{L,K}] = \sum_{F\in\mathcal{F}_h^\partial} (\mathcal{D}v,w)_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \int_F 2\left\{w^t\mathcal{D}v\right\},$$

and $2\left\{w^t\mathcal{D}v\right\} = 2\left\{w^t\right\}\left\{\mathcal{D}v\right\} + \frac{1}{2}[\![w^t]\!][\![\mathcal{D}v]\!]$, it is clear that

$$\begin{aligned} a_h(v,w) = &\sum_{K\in\mathcal{T}_h} (v,\tilde{T}w)_{L,K} + \sum_{F\in\mathcal{F}_h^\partial} \frac{1}{2}(M_F(v) + \mathcal{D}v, w)_{L,F} \\ &+ \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} \frac{1}{2}([\![\mathcal{D}v]\!],[\![w]\!])_{L,F} + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (S_F([\![v]\!]),[\![w]\!])_{L,F}. \end{aligned} \tag{4.26}$$

Let R_1 to R_4 be the four terms in the right-hand side. Using the Cauchy–Schwarz inequality yields

$$|R_1| \le c \sum_{K\in\mathcal{T}_h} \|v\|_{L,K}(\|w\|_{L,K} + \|Aw\|_{L,K}) \le c\|v\|_{h,\frac{1}{2}}\|w\|_{h,A}.$$

Use (DG4) together with the Cauchy–Schwarz inequality to infer

$$|R_2| \le c \sum_{F\in\mathcal{F}_h^\partial} \|v\|_{L,F}|w|_{M,F} \le c\|v\|_{h,\frac{1}{2}}\|w\|_{h,A}.$$

For the third and fourth term, use (DG6) and (DG7), together with the fact that $[\![\mathcal{D}v]\!] = 2\mathcal{D}_{\partial K_1(F)}\{v\}$, to obtain

$$|R_3| + |R_4| \le c \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} (\|\{v\}\|_{L,F} + \|[\![v]\!]\|_{L,F})|w|_{J,F} \le c\|v\|_{h,\frac{1}{2}}\|w\|_{h,A}.$$

The result follows easily. ∎

LEMMA 4.5 (consistency). *Let u solve* (2.23) *and let u_h solve* (4.13). *If $u \in [H^1(\Omega)]^m$, then,*

$$\forall v_h \in W_h, \quad a_h(u - u_h, v_h) = 0. \tag{4.27}$$

Proof. Since $u \in [H^1(\Omega)]^m$ solves (2.23), $\mathcal{M}u = \mathcal{D}u$ a.e. on $\partial\Omega$ and $Tu = f$ in L. Assumption (DG2) yields $M_F(u|_F) = \mathcal{D}u|_F$ for all $F \in \mathcal{F}_h^\partial$. Moreover, $u \in [H^1(\Omega)]^m$ implies that $\{\mathcal{D}u\} = 0$ and $[\![u]\!] = 0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$. As a result,

$$\forall v_h \in W_h, \quad a_h(u, v_h) = (Tu, v_h)_L = (f, v_h)_L = a_h(u_h, v_h).$$

The conclusion follows readily. ∎

THEOREM 4.6 (convergence). *Let u solve* (2.23) *and let u_h solve* (4.13). *Assume that $u \in [H^1(\Omega)]^m$. Then, there is c, independent of h, such that*

$$\|u - u_h\|_{h,A} \le c \inf_{v_h\in W_h} \|u - v_h\|_{h,\frac{1}{2}}. \tag{4.28}$$

Proof. Simple application of Strang's Second Lemma; see, e.g., [15, p. 94]. Let $v_h \in W_h$. Owing to Lemmas 4.3, 4.4, and 4.5,

$$\begin{aligned}\|v_h - u_h\|_{h,A} &\le c \sup_{w_h\in W_h\setminus\{0\}} \frac{a_h(v_h - u_h, w_h)}{\|w_h\|_{h,A}} \\ &\le c \sup_{w_h\in W_h\setminus\{0\}} \frac{a_h(v_h - u, w_h)}{\|w_h\|_{h,A}} \le c\,\|u - v_h\|_{h,\frac{1}{2}}.\end{aligned}$$

Conclude using the triangle inequality. ∎

Owing to the definition of W_h, and the regularity of the mesh family $\{\mathcal{T}_h\}_{h>0}$, the following interpolation property holds. There is c, independent of h, such that for all $v \in [H^{p+1}(\Omega)]^m$, there is $v_h \in W_h$ satisfying

$$\|v - v_h\|_{h,\frac{1}{2}} \le ch^{p+\frac{1}{2}}\|v\|_{[H^{p+1}(\Omega)]^m}. \tag{4.29}$$

COROLLARY 4.7. *If u is in $[H^{p+1}(\Omega)]^m$, there is c, independent of h, such that*

$$\|u - u_h\|_{h,A} \le c\,h^{p+\frac{1}{2}}\|u\|_{[H^{p+1}(\Omega)]^m}. \tag{4.30}$$

In particular,

$$\|u - u_h\|_L \le c\,h^{p+\frac{1}{2}}\|u\|_{[H^{p+1}(\Omega)]^m}, \tag{4.31}$$

and if the mesh family $\{\mathcal{T}_h\}_{h>0}$ is quasi-uniform,

$$\left(\sum_{K\in\mathcal{T}_h} \|A(u - u_h)\|_{L,K}^2\right)^{\frac{1}{2}} \le c\,h^p\|u\|_{[H^{p+1}(\Omega)]^m}. \tag{4.32}$$

The above estimates show that, provided the exact solution is smooth enough, the method yields optimal order convergence in the broken graph norm and $(p+\frac{1}{2})$-order convergence in the L-norm.

Remark 4.5. The estimates (4.30) to (4.32) are identical to those that can be obtained by other stabilization methods like GaLS [5, 19, 21] or subgrid viscosity [18] and many other methods.

Finally, when the exact solution is not smooth enough to be in $[H^1(\Omega)]^m$ but only in the graph space W, we use a density argument to infer the convergence of the DG approximation in the L-norm.

COROLLARY 4.8. *Let u solve* (2.23) *and let u_h solve* (4.13)*. Assume that $[H^1(\Omega)]^m \cap V$ is dense in V. Then,*

$$\lim_{h\to 0} \|u-u_h\|_L = 0. \tag{4.33}$$

Proof. Let $\epsilon > 0$. There is $u_\epsilon \in [H^1(\Omega)]^m \cap V$ such that $\|u-u_\epsilon\|_W \le \frac{\epsilon}{2}$. Let $u_{\epsilon h}$ be the unique solution in W_h such that $a_h(u_{\epsilon h}, v_h) = (Tu_\epsilon, v_h)_L$ for all $v_h \in W_h$. From the regularity of u_ϵ together with Theorem 4.6 and Corollary 4.7 applied with $p = 0$, it is inferred that $\lim_{h\to 0} \|u_{\epsilon h} - u_\epsilon\|_{h,A} = 0$. Furthermore, using the discrete inf-sup condition (4.19) yields

$$\begin{aligned}\|u_{\epsilon h} - u_h\|_L &\le \sup_{v_h\in W_h\setminus\{0\}} \frac{a_h(u_{\epsilon h}, v_h) - a_h(u_h, v_h)}{\|v_h\|_{h,A}} = \sup_{v_h\in W_h\setminus\{0\}} \frac{(T(u_\epsilon - u), v_h)_L}{\|v_h\|_{h,A}}\\ &\le \|T(u_\epsilon - u)\|_L \sup_{v_h\in W_h\setminus\{0\}} \frac{\|v_h\|_L}{\|v_h\|_{h,A}} \le \|u-u_\epsilon\|_W \le \frac{\epsilon}{2},\end{aligned}$$

where we have used the fact that for all $v_h \in W_h$, $a_h(u_h, v_h) = (Tu, v_h)_L$. Finally, using the triangle inequality

$$\|u-u_h\|_L \le \|u-u_\epsilon\|_L + \|u_\epsilon - u_{\epsilon h}\|_L + \|u_{\epsilon h} - u_h\|_L,$$

it is deduced that $\limsup_{h\to 0} \|u-u_h\|_L \le \epsilon$. $\square$

4.6. Localization, fluxes, and adjoint-fluxes. The purpose of this section is to discuss briefly some equivalent formulations of the discrete problem (4.13) in order to emphasize the link with other formalisms derived previously for DG methods, namely that of Lesaint and Raviart [23, 24] and Johnson et al. [21, 22] for Friedrichs' systems. To this end, we rewrite the bilinear form (4.12) in various equivalent ways and introduce the concept of element fluxes and that of element adjoint-fluxes.

Let $K \in \mathcal{T}_h$. Define the operator $M^{\mathrm{L}}_{\partial K} \in \mathcal{L}([L^2(\partial K)]^m; [L^2(\partial K)]^m)$ as follows. For $v \in [L^2(\partial K)]^m$ and a face $F \subset \partial K$, set

$$M^{\mathrm{L}}_{\partial K}(v)|_F = \begin{cases} M_F(v|_F) & \text{if } F \in \mathcal{F}^\partial_h,\\ 2S_F(v|_F) & \text{if } F \in \mathcal{F}^{\mathrm{i}}_h.\end{cases} \tag{4.34}$$

Furthermore, for $v \in W(h)$ and $x \in \partial K$, set

$$v^i(x) = \lim_{\substack{y\to x\\ y\in K}} v(y), \qquad v^e(x) = \lim_{\substack{y\to x\\ y\notin K}} v(y), \tag{4.35}$$

$$[\![v]\!]_{\partial K}(x) = v^i(x) - v^e(x), \qquad \{v\}_{\partial K}(x) = \frac{1}{2}(v^i(x) + v^e(x)), \tag{4.36}$$

with $v^e(x) = 0$ if $x \in \partial\Omega$. Then, a straightforward calculation shows that the bilinear form a_h defined in (4.12) can be rewritten as follows:

$$a_h(u,v) = \sum_{K\in\mathcal{T}_h} (Tu,v)_{L,K} + \sum_{K\in\mathcal{T}_h} \frac{1}{2}(M^{\mathrm{L}}_{\partial K}([\![u]\!]_{\partial K}) - \mathcal{D}_{\partial K}[\![u]\!]_{\partial K}, v^i)_{L,\partial K} \tag{4.37}$$

$$= \sum_{K\in\mathcal{T}_h} (u,\tilde{T}v)_{L,K} + \sum_{K\in\mathcal{T}_h} \frac{1}{2}(M^{\mathrm{L}}_{\partial K}([\![u]\!]_{\partial K}) + 2\mathcal{D}_{\partial K}\{u\}_{\partial K}, v^i)_{L,\partial K}. \tag{4.38}$$

The bilinear form (4.37) is that analyzed by Lesaint and Raviart [24, 23] and further investigated by Johnson et al. [21] in the particular case where the operator $M^{\mathrm{L}}_{\partial K}$ is defined pointwise using a matrix-valued field on ∂K; see section 5.1 for further discussion.

DEFINITION 4.9. *Let $K\in\mathcal{T}_h$ and let $v\in W(h)$. The* element flux *of v on ∂K, say $\phi_{\partial K}(v)\in[L^2(\partial K)]^m$, is defined on a face $F\subset\partial K$ by*

$$\phi_{\partial K}(v)|_F = \begin{cases} \frac{1}{2}M_F(v|_F) + \frac{1}{2}\mathcal{D}_{\partial\Omega}v & \text{if } F\subset\partial K^{\partial}, \\ S_F([\![v]\!]_{\partial K}|_F) + \mathcal{D}_{\partial K}\{v\}_{\partial K} & \text{if } F\subset\partial K^{\mathrm{i}}, \end{cases} \tag{4.39}$$

where ∂K^{i} denotes that part of ∂K that lies in Ω and ∂K^{∂} denotes that part of ∂K that lies on $\partial\Omega$. Likewise, the element adjoint-flux *of v on ∂K, say, $\tilde{\phi}_{\partial K}(v)\in[L^2(\partial K)]^m$, is defined on a face $F\subset\partial K$ by*

$$\tilde{\phi}_{\partial K}(v)|_F = \begin{cases} \frac{1}{2}M_F(v|_F) - \frac{1}{2}\mathcal{D}_{\partial\Omega}v & \text{if } F\subset\partial K^{\partial}, \\ S_F([\![v]\!]_{\partial K}|_F) - \frac{1}{2}\mathcal{D}_{\partial K}[\![v]\!]_{\partial K} & \text{if } F\subset\partial K^{\mathrm{i}}. \end{cases} \tag{4.40}$$

The relevance of the notion of flux and adjoint-flux is clarified by the following proposition.

PROPOSITION 4.10. *The discrete problem* (4.13) *is equivalent to each of the following two local formulations.*

$$\begin{cases} \text{Seek } u_h\in W_h \text{ such that } \forall K\in\mathcal{T}_h \text{ and } \forall v_h\in[\mathbb{P}_p(K)]^m, \\ (u_h,\tilde{T}v_h)_{L,K} + (\phi_{\partial K}(u_h),v_h)_{L,\partial K} = (f,v_h)_{L,K}. \end{cases} \tag{4.41}$$

$$\begin{cases} \text{Seek } u_h\in W_h \text{ such that } \forall K\in\mathcal{T}_h \text{ and } \forall v_h\in[\mathbb{P}_p(K)]^m, \\ (Tu_h,v_h)_{L,K} + (\tilde{\phi}_{\partial K}(u_h),v_h)_{L,\partial K} = (f,v_h)_{L,K}. \end{cases} \tag{4.42}$$

Proof. Localize the test functions in (4.13) to the mesh elements and use the fact that $\phi_{\partial K}(v)|_F = \frac{1}{2}M^{\mathrm{L}}_{\partial K}([\![v]\!]_{\partial K}) + \mathcal{D}_{\partial K}\{v\}_{\partial K}$ and $\tilde{\phi}_{\partial K}(v)|_F = \frac{1}{2}M^{\mathrm{L}}_{\partial K}([\![v]\!]_{\partial K}) - \frac{1}{2}\mathcal{D}_{\partial K}[\![v]\!]_{\partial K}$. □

Let v be a function in $W(h)$. We define the *interface fluxes* (resp., *interface adjoint-fluxes*) of v, say, $\phi^{\mathrm{i}}(v)$, (resp., say, $\tilde{\phi}^{\mathrm{i}}(v)$), to be the two-valued function defined on $\mathcal{F}^{\mathrm{i}}_h$ that collects all the element fluxes (resp., adjoint-fluxes) of v on the interior faces. Likewise we define the *boundary fluxes* (resp., *boundary adjoint-fluxes*) of v, say, $\phi^{\partial}(v)$, (resp., say, $\tilde{\phi}^{\partial}(v)$), to be the single-valued function defined on $\mathcal{F}^{\partial}_h$ that collects all the element fluxes (resp., adjoint-fluxes) of v on the boundary faces.

Remark 4.6.

(i) The link between DG methods and the concept of element fluxes has been explored recently by Arnold et al. [1] for the Poisson equation (in [1], the terminology "numerical fluxes" is employed instead).

(ii) In engineering practice, approximation schemes such as (4.41) are often designed by a priori specifying the element fluxes. The above analysis then provides a practical means to assess the stability and convergence properties of the scheme. Indeed, once the element fluxes are given, the boundary operators M_F and the interface operators S_F can be directly retrieved from (4.39). Then, properties (DG1)–(DG8) provide sufficient conditions to analyze the scheme.

(iii) The interface fluxes are such that $\left\{\phi^{\mathrm{i}}(v)\right\} = 0$ a.e. on $\mathcal{F}_h^{\mathrm{i}}$. Approximation schemes where the interface fluxes satisfy this property are often termed *conservative*. Note that the concept of conservativity as such does not play any role in the present analysis of the method, although it can play a role when deriving improved L^2-error estimates by using the Aubin–Nitsche lemma; see, e.g., Arnold et al. [1] and the second part of this work [16].

(iv) The following relation links the element fluxes and the element adjoint-fluxes

$$\phi_{\partial K}(v) - \tilde{\phi}_{\partial K}(v) = \mathcal{D}_{\partial K} v^i. \tag{4.43}$$

In particular, the element adjoint-fluxes are not conservative.

(v) Both the element fluxes and the element adjoint-fluxes are associated with the operator T, i.e., they are derived from a DG discretization of (2.23). It is also possible to design a DG discretization of the adjoint problem (2.24) involving the operator $\tilde{T}$ and the bilinear form a^*. This would lead to two new families of fluxes, the element fluxes for $\tilde{T}$ and the element adjoint-fluxes for $\tilde{T}$. It should be noted that the element adjoint-fluxes for T are not the element fluxes for $\tilde{T}$. In particular, the former are not conservative whereas the latter are conservative.

5. Applications. This section shows how the conditions (DG1)–(DG8) can be used to design DG approximations of the model problems introduced in section 3.

5.1. Pointwise boundary and interface operators. For ease of presentation, the boundary and interface operators discussed in this section are constructed from matrix-valued fields defined on all the mesh faces. This simpler construction is sufficient to recover several DG methods considered in the literature. Examples where a more general form for the boundary and interface operators is needed will be presented in a forthcoming work [16].

For all $F \in \mathcal{F}_h^{\partial}$, let $\mathcal{M}_F$ be a matrix-valued field such that for all $\xi, \zeta \in \mathbb{R}^m$,

$$\mathcal{M}_F \text{ is positive,} \tag{DG1a}$$

$$\mathrm{Ker}(\mathcal{M} - \mathcal{D}_{\partial\Omega}) \subset \mathrm{Ker}(\mathcal{M}_F - \mathcal{D}_{\partial\Omega}), \tag{DG2a}$$

$$|\zeta^t(\mathcal{M}_F - \mathcal{D}_{\partial\Omega})\xi| \leq c(\xi^t \mathcal{M}_F \xi)^{\frac{1}{2}} \|\zeta\|_{\mathbb{R}^m}, \tag{DG3a}$$

$$|\zeta^t(\mathcal{M}_F + \mathcal{D}_{\partial\Omega})\xi| \leq c(\zeta^t \mathcal{M}_F \zeta)^{\frac{1}{2}} \|\xi\|_{\mathbb{R}^m}, \tag{DG4a}$$

where $\|\cdot\|_{\mathbb{R}^m}$ denotes the Euclidean norm in $\mathbb{R}^m$. Similarly, for all $F \in \mathcal{F}_h^{\mathrm{i}}$, let $\mathcal{S}_F$ be a matrix-valued field such that for all $\xi, \zeta \in \mathbb{R}^m$,

$$\mathcal{S}_F \text{ is positive,} \tag{DG5a}$$

$$\mathcal{S}_F \text{ is uniformly bounded,} \tag{DG6a}$$

$$|\zeta^t \mathcal{S}_F \xi| \leq c(\xi^t \mathcal{S}_F \xi)^{\frac{1}{2}} (\zeta^t \mathcal{S}_F \zeta)^{\frac{1}{2}}, \tag{DG7a}$$

$$|\zeta^t \mathcal{D} \xi| \leq c(\xi^t \mathcal{S}_F \xi)^{\frac{1}{2}} \|\zeta\|_{\mathbb{R}^m}. \tag{DG8a}$$

A straightforward verification yields the following proposition.

PROPOSITION 5.1. *For all $F \in \mathcal{F}_h^\partial$, define $M_F : [L^2(F)]^m \ni v \mapsto \mathcal{M}_F|_F v \in [L^2(F)]^m$, and for all $F \in \mathcal{F}_h^{\mathrm{i}}$, define $S_F : [L^2(F)]^m \ni v \mapsto \mathcal{S}_F|_F v \in [L^2(F)]^m$. Then,* (DG1)–(DG8) *hold.*

Remark 5.1.

(i) Whenever the matrix-valued field $\mathcal{M}$ defined in (4.7) satisfies (DG3a)–(DG4a), one simply sets $\mathcal{M}_F = \mathcal{M}$; otherwise, it is necessary to strengthen $\mathcal{M}$. This last situation occurs, for instance, when approximating advection-diffusion-reaction problems and the Maxwell equations in the elliptic regime; see sections 5.3 and 5.4.

(ii) One possible way of constructing $\mathcal{S}_F$ follows. Since $\mathcal{D}$ is symmetric, $\mathcal{D}$ is diagonalizable; hence, the absolute value of $\mathcal{D}$, say, $|\mathcal{D}|$, can be defined. Moreover, observing that $|\mathcal{D}|$ is single-valued on $\mathcal{F}_h^{\mathrm{i}}$, one can set $\mathcal{S}_F = |\mathcal{D}|$.

5.2. Advection-reaction. Consider the advection-reaction problem introduced in section 3.1. Assume that all the faces in $\mathcal{F}_h^\partial$ are in $\partial\Omega^-$, in $\partial\Omega^+$, or in $\partial\Omega \setminus (\partial\Omega^- \cup \partial\Omega^+)$. The integral representation (4.7) holds with

$$\mathcal{D}_{\partial\Omega} = \beta{\cdot}n \qquad \text{and} \qquad \mathcal{M} = |\beta{\cdot}n|. \tag{5.1}$$

Let $\alpha > 0$ (this design parameter can vary from face to face) and for all $F \in \mathcal{F}_h$, set

$$\mathcal{M}_F = \mathcal{M} = |\beta{\cdot}n| \qquad \text{and} \qquad \mathcal{S}_F = \alpha|\beta{\cdot}n_F|, \tag{5.2}$$

where n_F is a unit normal vector to F (its orientation is clearly irrelevant). It is straightforward to verify the following proposition.

PROPOSITION 5.2. *Properties* (DG1a)–(DG8a) *hold.*

Remark 5.2. The specific value $\alpha = \frac{1}{2}$ has received considerable attention in the literature. When working with the local formulation (4.42), the interface and boundary fluxes are given by

$$\begin{aligned}\tilde{\phi}^{\mathrm{i}}(u_h)|_{\partial K} &= \left(\alpha|\beta{\cdot}n_K| - \frac{1}{2}\beta{\cdot}n_K\right) [\![u_h]\!]_{\partial K},\\ \tilde{\phi}^{\partial}(u_h) &= -|\beta{\cdot}n|u_h 1_{\partial\Omega^-},\end{aligned}$$

where $1_{\partial\Omega^-}$ denotes the characteristic function of $\partial\Omega^-$. Setting $\alpha = \frac{1}{2}$, one obtains the DG method analyzed by Lesaint and Raviart [24, 23]; in this case the interface adjoint-flux $\tilde{\phi}^{\mathrm{i}}$ is nonzero only on that part of the boundary ∂K where $\beta{\cdot}n_K < 0$. Similarly, when working with the local formulation (4.41), the interface and boundary fluxes are given by

$$\begin{aligned}\phi^{\mathrm{i}}(u_h)|_{\partial K} &= (\beta{\cdot}n_K)\,\{u_h\} + \alpha|\beta{\cdot}n_K|[\![u_h]\!]_{\partial K},\\ \phi^{\partial}(u_h) &= |\beta{\cdot}n|u_h 1_{\partial\Omega^+},\end{aligned}$$

where $1_{\partial\Omega^+}$ denotes the characteristic function of $\partial\Omega^+$. Setting $\alpha = \frac{1}{2}$ leads to $\phi^{\mathrm{i}}(u_h)|_{\partial K} = (\beta{\cdot}n_K)u_h^\uparrow$, where $u_h^\uparrow = u_h^i$ if $\beta{\cdot}n_K > 0$ and $u_h^\uparrow = u_h^e$ otherwise, i.e., the well-known upwind flux is recovered as a particular case of the above analysis which is valid for any $\alpha > 0$. This coincidence has led many authors to believe that DG methods are methods of choice to solve hyperbolic problems. Actually DG methods, as presented herein, are tailored to solve symmetric positive systems of first-order PDEs, and as pointed out by Friedrichs, the notion of symmetric systems goes beyond the hyperbolic/elliptic traditional classification of PDEs. Moreover, the presence of the user-dependent interface operator S_F (see (DG5)–(DG8)) points to the fact that DG methods are merely stabilization techniques. This fact is even clearer when one realizes that the error estimates (4.30)–(4.32) are identical to those that can be obtained by using other stabilization techniques like GaLS (also sometimes called streamline diffusion) [5, 19, 21] or subgrid viscosity [18] methods.

5.3. Advection-diffusion-reaction. Consider the advection-diffusion-reaction problem introduced in section 3.2. The integral representation (4.7) for D holds with

$$\mathcal{D}_{\partial\Omega} = \left[\begin{array}{c:c} 0 & n \\ \hdashline n^t & \beta\cdot n \end{array}\right]. \tag{5.3}$$

To simplify, we assume that the parameters β and μ are of order 1, i.e., we hide the dependency on these coefficients in the constants. Special cases such as advection-dominated problems go beyond the scope of the present work. We begin with the interface operator since its design is independent of the boundary conditions imposed. Let $\alpha > 0$, $\eta > 0$, and $\delta \in \mathbb{R}^d$. For all $F \in \mathcal{F}_h^{\mathrm{i}}$, define

$$\mathcal{S}_F = \left[\begin{array}{c:c} \alpha n_F \otimes n_F & (\delta\cdot n_F) n_F \\ \hdashline -(\delta\cdot n_F) n_F^t & \eta \end{array}\right]. \tag{5.4}$$

Proposition 5.3. *Properties* (DG5a)–(DG8a) *hold.*

Proof. For $\xi \in \mathbb{R}^{d+1}$, denote by $\xi = (\xi_\sigma, \xi_u)$ its decomposition in $\mathbb{R}^d \times \mathbb{R}$ and use a similar notation for $\zeta = (\zeta_\sigma, \zeta_u) \in \mathbb{R}^{d+1}$. The field $\mathcal{S}_F$ is clearly positive and bounded, i.e., (DG5a) and (DG6a) hold. Moreover, for $\xi, \zeta \in \mathbb{R}^{d+1}$,

$$\zeta^t \mathcal{S}_F \xi = \alpha(\xi_\sigma\cdot n)(\zeta_\sigma\cdot n) + (\delta\cdot n)(\zeta_\sigma\cdot n)\xi_u - (\delta\cdot n)(\xi_\sigma\cdot n)\zeta_u + \eta\xi_u\zeta_u,$$

and $\xi^t \mathcal{S}_F \xi = \alpha(\xi_\sigma\cdot n)^2 + \eta\xi_u^2$, when (DG7a) is readily deduced. Finally, since

$$\zeta^t \mathcal{D}_{\partial K} \xi = (\xi_\sigma\cdot n_K)\zeta_u + (\zeta_\sigma\cdot n_K)\xi_u + (\beta\cdot n_K)\xi_u\zeta_u,$$

(DG8a) holds. ∎

Remark 5.3.

(i) We stress the fact that the above DG method yields $(p + \frac{1}{2})$-order estimates in the L-norm for both u_h and σ_h.

(ii) The σ- and u-component of the interface fluxes are given by

$$\phi^{\sigma,\mathrm{i}}(\sigma_h, u_h)|_{\partial K} = (\{u_h\} + \alpha n_K\cdot[\![\sigma_h]\!]_{\partial K} + (\delta\cdot n_K)[\![u_h]\!]_{\partial K}) n_K,$$
$$\phi^{u,\mathrm{i}}(\sigma_h, u_h)|_{\partial K} = n_K\cdot\{\sigma_h\} - (\delta\cdot n_K) n_K\cdot[\![\sigma_h]\!]_{\partial K} + \eta[\![u_h]\!]_{\partial K} + \beta\cdot n_K\{u_h\}.$$

Owing to the fact that $\alpha \neq 0$, the local formulation (4.41) or (4.42) cannot be used to derive a local reconstruction formula where $\sigma_h|_K$ is expressed solely in terms of u_h. To this end, the coefficient α has to be set to zero, and this requires a nontrivial modification of the analysis that will be reported in [16]. With this modification, the DG approximation does no longer yield a $(p + \frac{1}{2})$-order estimate for σ_h in the L-norm.

(iii) The design parameters α, δ, and η can vary from face to face. In particular, one can take δ to be any bounded vector-valued field on $\mathcal{F}_h^{\mathrm{i}}$; $\delta = 0$ is a suitable choice. Other particular choices lead to DG methods already reported in the literature for advection-diffusion-reaction problems. A more detailed discussion, including a comparison with methods where the unknown $\sigma_h|_K$ is eliminated locally, is postponed to [16].

5.3.1. Dirichlet boundary conditions. The integral representation (4.7) of the boundary operator M defined in (3.11) holds with

$$\mathcal{M} = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & 0 \end{array}\right]. \tag{5.5}$$

Let $\varsigma > 0$ (this design parameter can vary from face to face). For all $F \in \mathcal{F}_h^\partial$, define

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -n \\ \hdashline n^t & \varsigma \end{array}\right]. \tag{5.6}$$

It is straightforward to verify the following proposition.

PROPOSITION 5.4. *Properties* (DG1a)–(DG4a) *hold.*

Remark 5.4. Observe that setting $\mathcal{M}_F = \mathcal{M}$ is not adequate here since with this choice (DG3a) does not hold.

5.3.2. Neumann and Robin boundary conditions. The integral representation (4.7) of the boundary operator M defined in (3.12) holds with

$$\mathcal{M} = \left[\begin{array}{c:c} 0 & n \\ \hdashline -n^t & 2\varrho + \beta{\cdot}n \end{array}\right]. \tag{5.7}$$

Consider first Neumann boundary conditions, i.e., $\varrho = 0$. Let $\lambda > 0$ (this design parameter can vary from face to face). For all $F \in \mathcal{F}_h^\partial$, define

$$\mathcal{M}_F = \left[\begin{array}{c:c} \lambda n \otimes n & n \\ \hdashline -n^t & 0 \end{array}\right]. \tag{5.8}$$

It is straightforward to verify the following proposition.

PROPOSITION 5.5. *Properties* (DG1a)–(DG4a) *hold.*

Consider next Robin boundary conditions and assume that $\varrho + \min(\beta{\cdot}n, 0) \geq 0$ (this assumption is not restrictive since Robin boundary conditions are often enforced on inflow boundaries by setting $\varrho = -\beta{\cdot}n$). Let $\lambda \in \,]0, \frac{1}{\varrho}[$ ($\lambda \in \,]0, +\infty)$ if $\varrho = 0$), $\theta = 1 - \lambda\varrho$, and $\alpha = -\lambda\varrho^2$. For all $F \in \mathcal{F}_h^\partial$, define

$$\mathcal{M}_F = \left[\begin{array}{c:c} \lambda n \otimes n & \theta n \\ \hdashline -\theta n^t & 2\varrho + \beta{\cdot}n + \alpha \end{array}\right]. \tag{5.9}$$

PROPOSITION 5.6. *Properties* (DG1a)–(DG4a) *hold.*

Proof. Since $\varrho + \beta{\cdot}n \geq 0$ by assumption and since $\varrho + \alpha > 0$ by construction, it is inferred that for all $\xi \in \mathbb{R}^{d+1}$, $\xi^t \mathcal{M}_F \xi \geq c((\xi_\sigma{\cdot}n)^2 + \xi_u^2)$ with $c > 0$. The rest of the proof is straightforward. $\square$

Remark 5.5. The bilinear forms $(u, v) \longmapsto \int_{\partial\Omega} v^t \mathcal{M}_F u$ considered above cannot be extended to $W \times W$ due to the presence of the upper-left block in $\mathcal{M}_F$. The difficulty stems from the fact that vector fields in $H(\mathrm{div};\Omega)$ may not have normal traces in $L^2(\partial\Omega)$. As a consequence, the approximate method is meaningful only if the exact solution is smooth enough; see the definition of $W(h)$ in (4.2).

5.4. Maxwell's equations in the elliptic regime. We close this series of applications with Maxwell's equations in the elliptic regime; see section 3.3. The integral representation (4.7) holds with the $\mathbb{R}^{6,6}$-valued fields

$$\mathcal{D}_{\partial\Omega} = \left[\begin{array}{c:c} 0 & \mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right] \quad \text{and} \quad \mathcal{M} = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & 0 \end{array}\right], \tag{5.10}$$

where $\mathcal{N} = \sum_{k=1}^{3} n_k \mathcal{R}^k \in \mathbb{R}^{3,3}$ and $n = (n_1, n_2, n_3)^t$ is the unit outward normal to Ω on $\partial\Omega$. Observe that $\mathcal{N}\xi = n \times \xi$ for all $\xi \in \mathbb{R}^3$.

Let $\varsigma > 0$, $\alpha_1 > 0$, and $\alpha_2 > 0$ (these design parameters can vary from face to face) and set

$$\mathcal{M}_F = \left[\begin{array}{c:c} 0 & -\mathcal{N} \\ \hdashline \mathcal{N}^t & \varsigma\mathcal{N}^t\mathcal{N} \end{array}\right] \qquad \text{and} \qquad \mathcal{S}_F = \left[\begin{array}{c:c} \alpha_1 \mathcal{N}_F^t \mathcal{N}_F & 0 \\ \hdashline 0 & \alpha_2 \mathcal{N}_F^t \mathcal{N}_F \end{array}\right], \tag{5.11}$$

where $\mathcal{N}_F$ is defined as $\mathcal{N}$ by replacing n by n_F. It is straightforward to verify the following proposition.

Proposition 5.7. *Properties* (DG1a)–(DG8a) *hold.*

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