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A REINTERPRETATION OF THE NON-LINEAR GALERKIN METHOD AS A LARGE EDDY SIMULATION TECHNIQUE

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Abstract. *The purpose of this paper is to show that the Fourier-based Nonlinear Galerkin Method (NLGM) constructs suitable weak solutions to the periodic Navier–Stokes equations in three dimensions. We re-interpret NLGM as a Large-Eddy Simulation technique (LES) and we rigorously deduce a relationship between the mesh size and the large-eddy scale.*

1 Introduction

Large eddy simulation methods for approximating turbulent flows are commonly viewed as techniques in which the governing equations are derived by applying a low-pass filter to the Navier–Stokes equations. These filtered equations are similar to the original equations but for the presence of the so-called subgrid scale stresses accounting for the influence of the small scales onto the large ones. Assuming that the behavior of the small scale structures is more or less universal, the objective of LES is then to find some models for the subgrid scale stresses, the so-called closure problem, and to compute the dynamics of the large scales by using the filtered equations. Although this description of LES has been widely accepted over the years (see for example the books by Geurts [8], John [13], or Sagaut [17]), it nevertheless falls short of an unambiguous mathematical theory. Indeed, the filtering operators, which implicitly appear in the definition of the subgrid scale tensor, are often ignored while constructing the LES models. It is a common practice to work with filter length scales regardless of the actual filters being used. More importantly, it is now known that the closure problem actually yields a paradox; namely that it is possible to close exactly the LES equations, i.e., without invoking ad hoc hypotheses, by choosing a bijective filtering operator, see [7, 9]. In this case, there exists a one-to-one correspondence between the solution set of the Navier-Stokes equations and that of the filtered equations, which means that the same "number of degrees of freedom" should be used in both cases to represent any given solution. Another unjustified practice very often consists of assuming that the filtering length scale is equal to the mesh size of the approximation method that is used, regardless on the method in question.

The above observations have led us to develop a research program aiming at constructing a mathematical framework for the large eddy simulation of turbulent flows. Our first step in this direction is to introduce the concept of suitable approximation (see Section 2.2 and [10]). The definition is essentially based on two requirements. A suitable approximation is a sequences of finite-dimensional approximations converging in an appropriate sense to a weak(s) solution of the Navier-Stokes equations. Second, we require that these finite-dimensional approximations be constructed in such a way that the weak solutions(s) at the limit is (are) suitable in the sense defined by Scheffer [18] (see Section 2.1). One rationale behind this definition is that suitable solutions are expected to be more regular than weak solutions (see Duchon-Robert [4]) and the best partial regularity result as stated in the so-called Caffarelli-Kohn-Nirenberg (CKN) Theorem, see Caffarelli *et al.* [1], Lin [14], Scheffer [18] was first proved for the class of the suitable solutions. Note however that the CKN result has been recently extended to the weak solutions by He Cheng [11].

The goal of this paper we is to show that the Nonlinear Galerkin Method (NLGM) [16, 5, 6] constructs suitable approximations (see Theorem 5.1). We show also that NLGM shares many of the heuristic features that are usually assigned to LES methods. The paper is organized as follows. We recall in Section 2 the notion of suitable weak solutions of the Navier–Stokes equations and we define what we mean by sequence of suitable approximations. We briefly review Nonlinear Galerkin methods in Section 3 and we reinterpret one of its version as a means to construct suitable approximations. The proof of the main result of the paper, i.e., Theorem 5.1, is done in § 4 and § 5. We prove in § 6 that provided the Navier–Stokes solution is smooth enough, the NLGM approximation is as accurate as that that would be obtained by retaining all the nonlinearities and the time derivative in the momentum equations for the small scales. Finally, concluding remarks and comments on our interpretation of NLGM as a LES technique are reported in § 7.

2 Preliminaries

2.1 Navier–Stokes equations and suitable weak solutions

Let $\Omega \subset \mathbb{R}^3$ be an open connected domain with smooth boundary Γ. Let $(0,T)$ be a finite time interval and set $Q_T = \Omega \times (0,T)$. The time evolution for the velocity $\mathbf{u}$ and the pressure p fields of a fluid occupying Ω is described by the Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } Q_T, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{2.1}$$

where $\mathbf{u}_0$ is a solenoidal vector field, $\mathbf{f}$ a source term, and ν the viscosity. Note that the density is chosen equal to unity, that is, (2.1) is a nondimensional form of the Navier–Stokes equations and ν is the inverse of the Reynolds number. In order to account for the boundary conditions, we consider the space $\mathbf{X}$ defined as:

$$\mathbf{X} = \begin{cases} \mathbf{H}^1_0(\Omega) & \text{if homogeneous Dirichlet} \\ \{\mathbf{v} \in \mathbf{H}^1(\Omega),\ \mathbf{v} \text{ periodic}\} & \text{if periodicity is prescribed} \end{cases} \tag{2.2}$$

We also introduce the spaces:

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X},\ \nabla\cdot\mathbf{v} = 0\}, \qquad \mathbf{H} = \overline{\mathbf{V}}^{L^2}. \tag{2.3}$$

Unless explicitly stated otherwise, the minimal regularity assumed for the data is $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and in the periodic situation $\mathbf{u}_0$ (resp. $\mathbf{f}(t)$ a.e. t in $(0,T)$) is assumed to be of zero mean over Ω.

We now recall the notion of suitable weak solutions of the Navier–Stokes equations as introduced by Scheffer [18].

Definition 2.1. *A weak solution to the Navier–Stokes equations* $(\mathbf{u}, p)$ *is suitable if* $\mathbf{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$*,* $p \in L^{3/2}(0,T;L^{3/2}(\Omega))$ *and the local energy balance*

$$\partial_t(\tfrac{1}{2}\mathbf{u}^2) + \nabla\cdot((\tfrac{1}{2}\mathbf{u}^2 + p)\mathbf{u}) - \nu\nabla^2(\tfrac{1}{2}\mathbf{u}^2) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \le 0 \tag{2.4}$$

is satisfied in the distributional sense.

By analogy with nonlinear conservative laws, (2.4) can be viewed as an entropy-like condition which may (hopefully?) selects the physical solutions of (2.1). An explicit form of the distribution $D(\mathbf{u})$ that is missing in the left-hand side of (2.4) to reach equality has been given by Duchon and Robert [4]. For a smooth flow, the distribution $D(\mathbf{u})$ is zero; but for nonregular flow, $D(\mathbf{u})$ may be nontrivial. Suitable solutions are those which satisfy $D(\mathbf{u}) \ge 0$, i.e., if singularities appear, only those that dissipate energy pointwise are admissible. It is expected that suitable solutions are more regular than weak solutions. In this respect, the so-called Caffarelli-Kohn-Nirenberg (CKN) Theorem states that the one-dimensional Hausdorff measure of singular points of suitable solutions is zero. Whether these solutions are indeed classical is still far from being clear. Although it has been proved recently by He Cheng [11] that the result of the CKN Theorem also hods for weak solutions it is not known whether indeed weak solutions are suitable.

2.2 Definition of suitable approximations

The following definition has been introduced in [10]

Definition 2.2. *A sequence* $(\mathbf{u}_\gamma, p_\gamma)_{\gamma>0}$ *with* $\mathbf{u}_\gamma \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$ *and* $p_\gamma \in \mathcal{D}'((0,T),L^2(\Omega))$ *is said to be a suitable approximation to (2.1) if*

- (i) *There are two finite-dimensional vectors spaces* $\mathbf{X}_\gamma \subset \mathbf{X}$ *and* $M_\gamma \subset L^2(\Omega)$ *such that* $\mathbf{u}_\gamma \in \mathcal{C}^0([0,T];\mathbf{X}_\gamma)$, $\partial_t \mathbf{u}_\gamma \in L^2(0,T;\mathbf{X}_\gamma)$, *and* $p_\gamma \in L^2(0,T;M_\gamma)$ *for all* $T>0$.
- (ii) *The sequence converges to a weak solution of (2.1), say* $\mathbf{u}_\gamma \rightharpoonup \mathbf{u}$ *weakly in* $L^2(0,T;\mathbf{X})$, *weakly-$*$ in* $L^\infty(0,T;\mathbf{L}^2(\Omega))$, *and* $p_\gamma \to p$ *in* $\mathcal{D}'((0,T),L^2(\Omega))$.
- (iii) *The weak solution* $(\mathbf{u},p)$ *is suitable.*

In practice, suitable approximations in the sense defined above are constructed in three steps.
(1) We first construct what we hereafter call the pre-LES-model. This step consists of regularizing the Navier-Stokes equations by introducing a parameter ε representing the large eddy scale beyond which the nonlinear effects are dampened. The purpose of the regularization technique is to yield a well-posed problem for all times, and when passing to the limit in ε the limit solution must be a suitable weak solution to the Navier–Stokes equations.
(2) Second, we discretize the pre-LES-model. Since $\mathbf{X}_\gamma$ and M_γ are finite-dimensional, there is a discretization parameter h associated with the size of the smallest scale that can be represented in $\mathbf{X}_\gamma$, roughly $\dim(\mathbf{X}_\gamma) = \mathcal{O}((L/h)^3)$ where $L = \mathrm{diam}(\Omega)$.
(3) The third step consists of choosing the relative size of the large eddy scale ε and the mesh size h in such a way that the discrete solution converges to a suitable solution of the Navier-Stokes equations when $\varepsilon \to 0$ and $h \to 0$.

The item (i) in our definition is meant to shortcut an issue that is almost never addressed in the LES literature, namely, that of the discretization of the so-called LES models. Actually, the LES literature concentrates essentially on what we herein refer to as pre-LES-model, that is on regularized Navier–Stokes equations involving a large eddy scale ε. When approximating these equations the ad hoc choice $\varepsilon = h$ is very often made without any justification. The item (ii) is simply a consistency hypothesis, that is, the couple $(\mathbf{u}_\gamma, p_\gamma)$ must solve something that is a perturbation of the Navier–Stokes equations. The item (iii) is the condition that enables us to fix a reasonable (i.e., suitability of the limit) relation between ε and h, i.e., it is a condition which says which distinguished limit $(\lim_{h\to 0,\varepsilon\to 0})$ should be chosen.

2.3 Notations and conventions

For the sake of simplicity, we limit ourselves in this paper to periodic boundary conditions and Fourier approximations techniques. The domain Ω is henceforth assumed to be the three-dimensional torus $(0,2\pi)^3$.

We use the convention that $\mathbb{R}^3$-valued variables are represented by boldfaced characters or symbols. For all $\mathbf{z} \in \mathbb{C}^3$, we denote by $\overline{\mathbf{z}}$ the conjugate of $\mathbf{z}$, by $|\mathbf{z}|$ the Euclidean norm, and by $|\mathbf{z}|_\infty$ the maximum norm.

The Sobolev spaces $H^s(\Omega)$, $s \geq 0$ is defined in terms of Fourier series as follows

$$H^s(\Omega) = \left\{u(\mathbf{x}) = \textstyle\sum_{\mathbf{k}\in\mathbb{Z}^3} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}},\ u_{\mathbf{k}} = \overline{u}_{-\mathbf{k}},\ \sum_{\mathbf{k}\in\mathbb{Z}^3} (1+|\mathbf{k}|^2)^s |u_{\mathbf{k}}|^2 < +\infty\right\}.$$

In other words, the set of trigonometric polynomials $\exp(i\mathbf{k}\cdot\mathbf{x})$, $\mathbf{k} \in \mathbb{Z}^3$, is complete and orthogonal in $H^s(\Omega)$ for all $s \geq 0$. The scalar product in $L^2(\Omega)$ is denoted by $(u,v) = (2\pi)^{-3} \int_\Omega u\overline{v}$ and the dual of $H^s(\Omega)$ by $H^{-s}(\Omega)$. We introduce the closed subspace $\dot{H}^s(\Omega) \subset H^s(\Omega)$ composed of those functions in $H^s(\Omega)$ that are of zero mean value.

Let N be a positive integer and set

$$h = \frac{1}{N}. \tag{2.5}$$

For approximating the velocity and the pressure fields we will consider the set of trigonometric polynomials of partial degree less than or equal to N:

$$\mathbb{P}_N = \left\{p(\mathbf{x}) = \textstyle\sum_{|\mathbf{k}|_\infty \leq N} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, c_{\mathbf{k}} = \overline{c}_{-\mathbf{k}}\right\}.$$

Since in the torus the mean value of the velocity and that of the pressure are irrelevant, we introduce $\dot{\mathbb{P}}_N$ the subspace of $\mathbb{P}_N$ composed of the trigonometric polynomials of zero mean value. We finally introduce the truncation operator $P_h : H^s(\Omega) \longrightarrow \mathbb{P}_N$ such that

$$v = \sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \longmapsto P_h v = \sum_{|\mathbf{k}|_\infty \leq N} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Let us recall that

Lemma 2.1. *P_h satisfies the following properties:*

(i) *P_h is the restriction on $H^s(\Omega)$ of the L^2 projection onto $\mathbb{P}_N$.*

(ii) $\forall s \geq 0$, $\|P_h\|_{\mathcal{L}(H^s(\Omega);H^s(\Omega))} = 1$.

(iii) *P_h commutes with differentiation operators.*

(iv) $\exists c > 0,\ \forall v \in H^s(\Omega),\ \forall \mu,\ 0 \leq \mu \leq s,\ \|v - P_h v\|_{H^\mu} \leq c\, N^{\mu - s} \|v\|_{H^s}$.

(v) $\exists c > 0,\ \forall v \in \mathbb{P}_N,\ \forall \mu, s,\ s \leq \mu,\ \|P_h v\|_{H^\mu} \leq c\, N^{\mu - s} \|v\|_{H^s}$.

The symbol c is henceforth a generic constant that may depend on the data $\boldsymbol{f}$, $\mathbf{u}_0$, ν, T, or Ω, and which value may change from one occurrence to an other.

3 The Nonlinear Galerkin Method

3.1 Brief review

The Nonlinear Galerkin Method (NLGM) is an approximation technique that aims at constructing Approximate Inertial Manifolds (AIM) of nonlinear PDE's; see [2, 5, 6, 16].

A dissipative evolution equation over H is said to have an Inertial Manifold if the manifold in question contains the global attractor, is positively invariant under the flow, attracts all the orbits exponentially, and is given as the graph of a $\mathcal{C}^1$ mapping over a finite-dimensional subspace of H. This class of object has been proved to exist for many equations, but for the Navier–Stokes equations, even in dimension two, the question of the existence of an Inertial Manifold is still open. Then, the concept of Approximate Inertial Manifold has been introduced to remedy this [5, 6]. In this case, a sequence of manifolds of increasing dimension in H is explicitly constructed and the global attractor is shown to lie in small neighborhoods of these manifolds, the width of the neighborhoods rapidly shrinking as the dimension of the manifolds goes to infinity.

NLGM consists of expanding the solution of the dynamical system in a two-scale fashion (large and small scales) and to simplify the dynamics of the small scales in such a way that they solve a linear PDE whose source term only depends on the large scales; in other words the small scales are slaved to the large scales. Then, the small scales are inserted into the Galerkin equations approximating the large scales. This technique is expected to have better approximations properties than the Galerkin method restricted to the large scales only. This technique was shown to construct an Approximate Inertial Manifold for the Navier–Stokes equations in two dimensions [5], and for some time, sparkled a lot of interest as the concept, accompanied with substantial mathematical results, seemed well suited for turbulence modeling.

Heywood and Rannacher [12] later argued that the seemingly improved performance of NLGM over the standard Galerkin method could not be attributed to turbulence modeling. The authors advanced that the observed improved accuracy was in part to be attributed to the fact that NLGM has a better ability than the Galerkin method to handle the Gibb's phenomenon induced by higher-order boundary incompatibilities induced by the no-slip boundary condition. They further argued that in periodic domain, both NLGM and the Galerkin method perform identically. The mathematical argumentation in [12] is clear and convincing, and [12] probably rightly watered down some earlier, possibly overblown, claims about NLGM. Nevertheless, we want to offer in the present paper an alternative point of view of NLGM that, we think, should give some credit back to the method.

First we show that, when using Fourier expansions and passing to the limit, the NLGM solution converge (up to subsequences) to a suitable weak solution, whereas it is not known whether this property holds for pure Galerkin solutions, see Theorem 5.1. This difference of behavior is due to the particular treatment of the nonlinear terms in NLGM (turbulence modeling?). Second, if arbitrary smoothness is assumed, NLGM always outperform the

Galerkin method by a factor equal to 1 in the convergence order of the H^1-norm for the velocity and the L^2-norm for the pressure, see Theorem 6.1. And this result holds independently of the nature of the boundary conditions (whether periodicity or no-slip BC is enforced). As suspected in [12], we confirm that this superconvergence property has nothing ado with turbulence modeling but is instead a very simple consequence of a seemingly not wellknown result by Wheeler [19] stating that for parabolic equations, the elliptic projection of the solution is always superconvergent in the H^1-norm by one order. This is a purely linear superconvergence effect resulting from standard elliptic regularity. However, this result somewhat contradicts the claim in [12] stating that both NLGM and the Galerkin method should perform identically in periodic domains.

3.2 NLGM as a pre-LES model

We introduce in this section the Nonlinear Galerkin Method in an infinite-dimensional setting. We show that in this case the infinite-dimensional NLGM is a pre-LES model in the sense we introduced in § 2.2.

Let ε be a positive number that from now on we mentally associate with the smallest scale of the flow that we really want to represent well (i.e., the Large Eddy Scale). Let us set $N_\varepsilon = \frac{1}{\varepsilon}$ (or the integer the closest to $\frac{1}{\varepsilon}$). We now introduce the following finite-dimensional vector spaces:

$$\mathbf{X}_\varepsilon = \dot{\mathbf{P}}_{N_\varepsilon}, \quad \text{and} \quad M_\varepsilon = \dot{\mathbb{P}}_{N_\varepsilon}, \tag{3.1}$$

and we introduce the projection $Q_\varepsilon = I - P_\varepsilon$, where I is the identity. From this definition it is clear that any field in $\dot{\mathbf{L}}^2(\Omega)$, say $\mathbf{v}$, can be decomposed as follows: $\mathbf{v} = P_\varepsilon\mathbf{v} + Q_\varepsilon\mathbf{v}$. The component $P_\varepsilon\mathbf{v}$ living in $\mathbf{X}_\varepsilon$ is referred to as the large scale component of $\mathbf{v}$ and the remainder $Q_\varepsilon\mathbf{v}$ is called the small scale component.

The nonlinear Galerkin method can be recast into the following form: Seek $\mathbf{u}_\varepsilon$ and p_ε in the Leray class such that for all $\mathbf{v} \in \dot{\mathbf{H}}^1(\Omega)$, $q \in \dot{L}^2(\Omega)$, that

$$\begin{cases} (\partial_t P_\varepsilon\mathbf{u}_\varepsilon, \mathbf{v}) + \nu(\nabla\mathbf{u}_\varepsilon, \nabla\mathbf{v}) + \mathrm{NL}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) - (p_\varepsilon, \nabla\cdot\mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla\cdot\mathbf{u}_\varepsilon, q) = 0, \\ (\mathbf{u}_\varepsilon, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{3.2}$$

where the nonlinear term is decomposed as follows:

$$\begin{aligned} \mathrm{NL}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, v) = {} & (P_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla(P_\varepsilon\mathbf{u}_\varepsilon), \mathbf{v}) + (P_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla(Q_\varepsilon\mathbf{u}_\varepsilon), P_\varepsilon\mathbf{v}) \\ & + (Q_\varepsilon\mathbf{u}_\varepsilon\cdot\nabla(P_\varepsilon\mathbf{u}_\varepsilon), P_\varepsilon\mathbf{v}). \end{aligned}$$

For reasons we do not yet fully understand, this form of the nonlinear does not seem to lend itself easily to analysis. In particular we have not been able to show that, at the limit $\varepsilon \to 0$, the weak solution is suitable. Thus this form of NLGM seems to be failing to comply with item (iii) in our definition. But as hinted in [2, 16] many other admissible forms of the nonlinear term are possible. We then propose to set $\mathrm{NL}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) = (P_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon, v)$ so that the pre-LES model we henceforth consider is the following:

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{u}_\varepsilon, \mathbf{v}) + \nu(\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{v}) + (P_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon, v) - (p_\varepsilon, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_\varepsilon, q) = 0, \\ (\mathbf{u}_\varepsilon, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{3.3}$$

It is then possible to prove that (3.3) has a unique solution and that this solution converges up to subsequences to a suitable weak solution of the Navier–Stokes equations. We omit the details since the essential arguments will be repeated for the analysis of the fully discrete problem in §4 and §5. Considering that no nonlinearity operates at wavenumbers larger that ε, we also conjecture that (3.3) has an Inertial Manifold.

3.3 The NLGM-based LES approximation

We now want to construct a finite-dimensional approximation to the solution to (3.3). To this end we introduce an integer N that we suppose to be larger than ε. We set

$$h = \frac{1}{N} \tag{3.4}$$

and we define

$$\mathbf{X}_h = \dot{\mathbb{P}}_N, \quad \text{and} \quad M_h = \dot{\mathbb{P}}_N, \tag{3.5}$$

To be able to control the separation between the large eddy scale ε and the discretization scale h, we introduce a parameter θ, $0 < \theta < 1$, and we assume that ε and h are related by the following relation

$$\varepsilon = h^\theta. \tag{3.6}$$

This can be equivalently be rewritten as: $\varepsilon = N^{-\theta}$.

Then, (3.3) is approximated as follows: Seek $\mathbf{u}_h \in \mathcal{C}^0([0,T]; \mathbf{X}_h)$, and $p_h \in L^2(0,T;M_h)$ such that $\forall t \in (0,T]$, $\forall \mathbf{v} \in \mathbf{X}_h$, and $\forall q \in M_h$

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{u}_h, \mathbf{v}) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}) + (P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_h, q) = 0, \\ \mathbf{u}_h|_{t=0} = P_\varepsilon \mathbf{u}_0. \end{cases} \tag{3.7}$$

Following [5, 6, 16] we now show that we are on our way to construct an AIM. To remove the incompressibility constraint and the pressure from the above formulation we define $\mathbf{V}_h = \mathbf{X}_h \cap \mathbf{V}$ and we set $\mathbf{Z}_\varepsilon = P_\varepsilon(\mathbf{V}_h)$ and $\mathbf{Y}_h = Q_\varepsilon(\mathbf{V}_h)$. Clearly $\mathbf{V}_h = \mathbf{Z}_\varepsilon \oplus \mathbf{Y}_h$ and the decomposition is orthogonal with respect to the L^2- and the H^1-scalar product. Let $\mathbf{u}_h = \mathbf{z}_\varepsilon + \mathbf{y}_h$ be the corresponding decomposition of $\mathbf{u}_h(t)$ in $\mathbf{V}_h$. Let us assume moreover that the spectrum of $\mathbf{f}$ is restricted to low wavenumbers, i.e., when N is large enough (or h small enough) $Q_\varepsilon \mathbf{f} = 0$. Then (3.7) reduces to

$$\begin{cases} \mathbf{z}_\varepsilon|_{t=0} = P_\varepsilon \mathbf{u}_0 \\ (\partial_t \mathbf{z}_\varepsilon, \boldsymbol{\phi}) + \nu(\nabla \mathbf{z}_\varepsilon, \nabla \boldsymbol{\phi}) + (\mathbf{z}_\varepsilon \cdot \nabla(\mathbf{z}_\varepsilon + \mathbf{y}_h), \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in \mathbf{Z}_\varepsilon \\ \qquad \nu(\nabla \mathbf{y}_\varepsilon, \nabla \boldsymbol{\psi}) + (\mathbf{z}_\varepsilon \cdot \nabla(\mathbf{z}_\varepsilon + \mathbf{y}_h), \boldsymbol{\psi}) = 0, \qquad \forall \boldsymbol{\psi} \in \mathbf{Y}_h. \end{cases} \tag{3.8}$$

It is clear that the small scale component of $\mathbf{u}_h$ is solution to a linear equation forced by $-\mathbf{z}_\varepsilon \cdot \nabla \mathbf{z}_\varepsilon$. Let $\mathbf{\Psi} : \mathbf{Z}_\varepsilon \longrightarrow \mathbf{Y}_h$ be the mapping such that

$$\nu(\nabla \mathbf{\Psi}(\mathbf{z}_\varepsilon), \nabla \boldsymbol{\psi}) + (\mathbf{z}_\varepsilon \cdot \nabla \mathbf{\Psi}(\mathbf{z}_\varepsilon), \boldsymbol{\psi}) = -(\mathbf{z}_\varepsilon \cdot \nabla \mathbf{z}_\varepsilon, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathbf{Y}_h.$$

Then, clearly

$$\mathbf{u}_h(t) = \mathbf{z}_\varepsilon(t) + \mathbf{\Psi}(\mathbf{z}_\varepsilon(t)), \quad \text{a.e. } t \text{ in } (0,T). \tag{3.9}$$

We show next that the discrete problem (3.7) yields a suitable approximation in the sense of Definition 2.2.

4 A priori estimates and convergence

We start with standard a priori estimates, then we prove that the solution to (3.7) converges, up to subsequences, to a weak solution of (2.1).

Lemma 4.1. *Let $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}_0 \in \mathbf{H}$, then the solution to (3.7) satisfies*

$$\max_{0 \le t \le T} \|P_\varepsilon \mathbf{u}_h(t)\|_{L^2}^2 + \nu \int_0^T \|\nabla P_\varepsilon \mathbf{u}_h\|_{L^2}^2 + \|\nabla Q_\varepsilon \mathbf{u}_h\|_{L^2}^2 \le c. \tag{4.1}$$

Proof. These are the basic energy estimates. □

Corollary 4.1. *Under the assumptions of Lemma 4.1*

$$\|P_\varepsilon \mathbf{u}_h\|_{L^r(H^{2/r})} + \|P_\varepsilon \mathbf{u}_h\|_{L^r(L^q)} \le c, \quad \textit{with} \quad \tfrac{3}{q} + \tfrac{2}{r} = \tfrac{3}{2}, \quad 2 \le r, \ 2 \le q \le 6.$$

Proof. This result is standard and is a consequence of the interpolation inequality $\|v\|_{H^{2/r}} \lesssim \|v\|_{L^2}^{1-2/r} \|v\|_{H^1}^{2/r}$, when $2 \le r$, and the embedding $H^{2/r}(\Omega) \subset L^q(\Omega)$ for $1/q = 1/2 - 2/(3r)$, (see *e.g.* [3, p. 208]), □

Lemma 4.2. *Under the assumptions of Lemma 4.1, the approximate pressure and the approximate time derivative of the velocity from (3.7) satisfy*

$$\|p_h\|_{L^{4/3}(L^2)} \le c \tag{4.2}$$

$$\|\partial_t P_\varepsilon \mathbf{u}_h\|_{L^{4/3}(H^{-1})} \le c. \tag{4.3}$$

Proof. (1) We first prove the pressure estimate (4.2). We observe that $\nabla^2 : M_h \longrightarrow M_h$ is bijective, and we denote by ∇^{-2} the inverse operator. Then, observing that $\nabla\nabla^{-2}p_h \in \mathbf{X}_h$, we multiply the momentum equation in (3.7) by $\nabla\nabla^{-2}p_h$. By using several integrations by parts, we obtain

$$\begin{aligned}\|p_h\|_{L^2}^2 &= -(\nabla p_h, \nabla\nabla^{-2}p_h)\\ &= (\partial_t P_\varepsilon \mathbf{u}_h - \nu\nabla^2\mathbf{u}_h + P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h - \mathbf{f}, \nabla\nabla^{-2}p_h)\\ &= (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h - \mathbf{f}, \nabla\nabla^{-2}p_h), \quad \text{since } \mathbf{u}_h \text{ and } P_\varepsilon\mathbf{u}_h \text{ are solenoidal}\\ &= (\nabla\cdot(P_\varepsilon\mathbf{u}_h\otimes\mathbf{u}_h) - \mathbf{f}, \nabla\nabla^{-2}p_h)\\ &= (P_\varepsilon\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\nabla\nabla^{-2}p_h) - (\mathbf{f}, \nabla\nabla^{-2}p_h)\\ &\le c(\|P_\varepsilon\mathbf{u}_h\|_{L^3}\|\mathbf{u}_h\|_{L^6} + \|\mathbf{f}\|_{H^{-1}})\|p_h\|_{L^2}.\end{aligned}$$

This yields $\|p_h\|_{L^2}^{4/3} \le c(\|P_\varepsilon\mathbf{u}_h\|_{L^3}^{4/3}\|\mathbf{u}_h\|_{L^6}^{4/3} + \|\mathbf{f}\|_{H^{-1}}^{4/3})$. We proceed further by noticing that

$$\|p_h\|_{L^{4/3}(L^2)}^{4/3} \le c\left(\|P_\varepsilon\mathbf{u}_h\|_{L^4(L^3)}^{4/3}\|\mathbf{u}_h\|_{L^2(H^1)}^{4/3} + \|\mathbf{f}\|_{L^2(H^{-1})}^{4/3}\right).$$

The conclusion is a consequence of Lemma 4.1 together with Corollary 4.1 with $q = 3$ and $r = 4$.
(2) We now prove the estimate on the time derivative of $P_\varepsilon\mathbf{u}_h$. Using the H^1-stability of P_h (see Lemma 2.1(ii)), we infer

$$\begin{aligned}\|\partial_t P_\varepsilon\mathbf{u}_h\|_{H^{-1}} &= \sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, \mathbf{v})}{\|\mathbf{v}\|_{H^1}} = \sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, P_h\mathbf{v})}{\|\mathbf{v}\|_{H^1}}\\ &\le c\sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, P_h\mathbf{v})}{\|P_h\mathbf{v}\|_{H^1}} \le c\sup_{\mathbf{v}\in X_h}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{H^1}}\\ &\le c(\nu\|u_h\|_{H^1} + \|P_\varepsilon\mathbf{u}_h\|_{L^3}\|\mathbf{u}_h\|_{H^1} + \|p_h\|_{L^2} + \|\mathbf{f}\|_{H^{-1}}).\end{aligned}$$

We conclude by proceeding as in step 1. □

We are now in measure of proving

Theorem 4.1. *Under the assumptions of Lemma 4.1, $P_\varepsilon\mathbf{u}_h$ converges up to subsequences to a weak solution to (2.1) in $L^2(0,T;\mathbf{H}^1)$ weak and in any $L^r(0,T;\mathbf{L}^q)$ strong ($1 \le q < \frac{6r}{3r-4}$, $2 \le r < \infty$); each subsequence of $P_\varepsilon\mathbf{u}_h$ and $\mathbf{u}_h$ have the same limit; p_h converge up to subsequences in $L^{\frac{4}{3}}(0,T;L^2)$.*

Proof. We only outline the main steps of the proof for the arguments are quite standard.

Since $P_\varepsilon\mathbf{u}_h$ is uniformly bounded in $L^2(0,T;\mathbf{H}^1)\cap L^\infty(0,T;\mathbf{L}^2)$, and $\partial_t P_\varepsilon\mathbf{u}_h$ is uniformly bounded in $L^{4/3}(0,T;\mathbf{H}^{-1}(\Omega))$, the Aubin-Lions compactness lemma (see Lions [15, p. 57]) implies that there exists a subsequence $(\mathbf{u}_{h_l})$ such that $P_{\varepsilon_l}\mathbf{u}_{h_l}$ converges weakly in $L^2(0,T;\mathbf{H}^1)$ and strongly in any $L^r(0,T;\mathbf{L}^q)$, such that $1 \le q < \frac{6r}{3r-4}$, $2 \le r < \infty$, and that

$\partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l})$ converges weakly in $L^{4/3}(0,T;\mathbf{H}^{-1})$. Moreover, since p_h is bounded uniformly in $L^{4/3}(0,T;L^2)$, there exists a subsequence (p_{h_l}) converging weakly in $L^{4/3}(0,T;L^2)$. Let $\mathbf{u}$ and p denote these limits, and let us show that the couple $(\mathbf{u},p)$ is a weak solution to (2.1).

Observing that $\|P_{\varepsilon_l}\mathbf{u}_{h_l}-\mathbf{u}_{h_l}\|_{L^2}\le c\,\varepsilon_l\|\mathbf{u}_{h_l}\|_{H^1}$ it is clear that the subsequences $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ and $(\mathbf{u}_{h_l})$ have the same limit in $L^2(\mathbf{L}^2)$. Note that this also implies that $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ and $(\mathbf{u}_{h_l})$ have the same limit in $L^2(\mathbf{H}^1)$ weak; in other words $Q_{\varepsilon_l}\mathbf{u}_{h_l}\rightharpoonup 0$ is $L^2(\mathbf{H}^1)$.

Let $s>4$ be a real number and let s^* be such that $\frac{1}{s}+\frac{1}{s^*}=\frac{1}{2}$. Let v be an arbitrary function in $L^s(0,T;\mathbf{H}^1)$ and let $(v_{h_l})_{h_l}$ be a sequence of functions in $L^s(0,T;\mathbf{X}_{h_l})$ strongly converging to $\mathbf{v}$ in $L^s(0,T;\mathbf{H}^1)\subset L^4(0,T;\mathbf{H}^1)$.

(1) $\int_{Q_T}\partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l})\cdot\mathbf{v}_{h_l}\to\int_{Q_T}\partial_t\mathbf{u}\cdot\mathbf{v}$, since $\partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l})\rightharpoonup\partial_t\mathbf{u}$ in $L^{4/3}(\mathbf{H}^{-1})$.

(2) $\int_{Q_T}\nabla\mathbf{u}_{h_l}{:}\nabla\mathbf{v}_{h_l}\to\int_{Q_T}\nabla\mathbf{u}{:}\nabla\mathbf{v}$, since $\nabla\mathbf{u}_{h_l}\rightharpoonup\nabla\mathbf{u}$ in $L^2(\mathbf{L}^2)\subset L^{4/3}(\mathbf{L}^2)$.

(3) $\int_{Q_T}p_{h_l}\nabla\cdot\mathbf{v}_{h_l}\to\int_{Q_T}p\nabla\cdot\mathbf{v}$, since $p_h\rightharpoonup p$ in $L^{4/3}(L^2)$.

(4) Since $P_{\varepsilon_l}\mathbf{u}_{h_l}\to\mathbf{u}$ in $L^{s^*}(\mathbf{L}^3)$ and $\mathbf{v}_{h_l}\to\mathbf{v}\in L^s(\mathbf{H}^1)\subset L^s(\mathbf{L}^6)$, we infer that $\mathbf{v}_{h_l}\otimes(P_{\varepsilon_l}\mathbf{u}_{h_l})\to\mathbf{v}\otimes\mathbf{u}$ in $L^2(\mathbf{L}^2\otimes\mathbf{L}^2)$. As a result, $\int_{Q_T}[\mathbf{v}_{h_l}\otimes(P_{\varepsilon_l}\mathbf{u}_{h_l})]{:}\nabla\mathbf{u}_{h_l}\to\int_{Q_T}[\mathbf{v}\otimes\mathbf{u}]{:}\nabla\mathbf{u}$ since $\nabla\mathbf{u}_{h_l}\rightharpoonup\nabla\mathbf{u}$ in $L^2(\mathbf{L}^2\otimes\mathbf{L}^2)$.

(5) Since $\nabla\cdot\mathbf{u}_{h_l}=0$ and $\mathbf{u}_{h_l}\rightharpoonup\mathbf{u}$ in $L^2(\mathbf{H}^1)$, $\nabla\cdot\mathbf{u}=0$ in $L^2(\mathbf{H}^1)$.

(6) Clearly $\int_0^T<\mathbf{f},\mathbf{v}_{h_l}>\to\int_0^T<\mathbf{f},\phi\mathbf{v}>$ since $\mathbf{v}_{h_l}\to\mathbf{u}$ in $L^s(\mathbf{H}^1)\subset L^2(\mathbf{H}^1)$ and $f\in L^2(\mathbf{H}^{-1})$.

(7) Finally since the subsequence $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ converges in $\mathcal{C}^0(0,T;\mathbf{L}^2_w)$ (space of the functions that are continuous over $[0,T]$ with value in $\mathbf{L}^2$ equipped with the weak topology) we have $\mathbf{u}_0\leftarrow P_{\varepsilon_l}\mathbf{u}_0=P_{\varepsilon_l}\mathbf{u}_{h_l}(0)\rightharpoonup\mathbf{u}(0)$ in $\mathbf{L}^2$; hence, $\mathbf{u}(0)=\mathbf{u}_0$. The theorem is proved. □

5 Convergence to a suitable solution

5.1 Formulation (3.7)

The main contribution in this section is Theorem 5.1 which establishes that the solution of (3.7) converges to a suitable solution of the Navier–Stokes equations.

Theorem 5.1. *Let $\mathbf{f}\in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}_0\in\mathbf{H}$. Let $N>0$ and $\varepsilon=h^\theta$. Provided*

$$0<\theta<\tfrac{2}{3},\tag{5.1}$$

the limit solution(s) of (3.7) is (are) suitable.

Proof. Let ϕ be a smooth nonnegative function, periodic with respect to space, and compactly supported with respect to time in $(0,T)$. Taking $P_h(\mathbf{u}_h\phi)$ to test the momentum equation in (3.7) and integrating in time, we obtain

$$\int_0^T(\partial_t P_\varepsilon\mathbf{u}_h,P_h(\mathbf{u}_h\phi))+\nu(\nabla\mathbf{u}_h,\nabla P_h(\mathbf{u}_h\phi))-(p_h,\nabla\cdot P_h(\mathbf{u}_h\phi))$$
$$+(P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h,P_h(\mathbf{u}_h\phi))=\int_0^T(\mathbf{f},P_h(\mathbf{u}_h\phi)).$$

Using the fact that P_ε and P_h commute with differentiation operators and after integrating by parts in time and space, we obtain

$$\int_0^T -\tfrac{1}{2}((|P_\varepsilon \mathbf{u}_h|^2, \partial_t\phi) + \nu(|\nabla \mathbf{u}_h|^2, \phi) - \tfrac{1}{2}\nu(|\mathbf{u}_h|^2, \nabla^2\phi) - (p_h, \nabla\cdot(\mathbf{u}_h\phi)) \\ + (P_\varepsilon \mathbf{u}_h\cdot\nabla\mathbf{u}_h, P_h(\mathbf{u}_h\phi)) = \int_0^T (\mathbf{f}, P_h(\mathbf{u}_h\phi)).$$

We now pass to the limit in each term of the above equation separately, and to avoid cumbersome notations we still denote by $(\mathbf{u}_h)$, (p_h) the subsequences that are extracted instead of $(\mathbf{u}_{hl})$, (p_{hl}).
(1) $\int_0^T -\frac{1}{2}((|P_\varepsilon\mathbf{u}_h|^2, \partial_t\phi) \rightarrow \int_0^T -\frac{1}{2}((|\mathbf{u}|^2, \partial_t\phi)$ since $|P_\varepsilon\mathbf{u}_h|^2 \rightarrow |\mathbf{u}|^2$ in $L^r(L^1)$ for any $1 \leq r < \infty$.
(2) For the term $\nu\int_0^T(|\nabla\mathbf{u}_h|^2, \phi)$ we proceed as follows:

$$\int_0^T (|\nabla\mathbf{u}_h|^2, \phi) = \int_0^T (|\nabla(\mathbf{u}_h - \mathbf{u})|^2 + 2\nabla(\mathbf{u}_h - \mathbf{u}):\nabla\mathbf{u} + |\nabla\mathbf{u}|^2, \phi).$$

The second term in the right-hand side goes to zero since $\mathbf{u}_h \rightharpoonup \mathbf{u}$ in $L^2(\mathbf{H}^1)$. As a result

$$\liminf_{N\rightarrow+\infty} \int_0^T (|\nabla\mathbf{u}_h|^2, \phi) \geq \int_0^T (|\nabla\mathbf{u}|^2, \phi).$$

(3) $\frac{1}{2}\nu\int_0^T(|\mathbf{u}_h|^2, \nabla^2\phi) \rightarrow \frac{1}{2}\nu\int_0^T(|\mathbf{u}|^2, \nabla^2\phi)$ since $|\mathbf{u}_h|^2 \rightarrow |\mathbf{u}|^2$ in $L^2(L^1)$. To be convince of the last result observe that $\int_0^T \|\mathbf{u}_h - \mathbf{u}\|_{L^2}^2 \leq \int_0^T 2\|P_\varepsilon\mathbf{u}_h - \mathbf{u}\|_{L^2}^2 + 2\|Q_\varepsilon\mathbf{u}_h\|_{L^2}^2$. The using $\|Q_\varepsilon\mathbf{u}_h\|_{L^2} \leq c\,\varepsilon\|\mathbf{u}_h\|_{H^1}$ together with the fact that $\int_0^T 2\|P_\varepsilon\mathbf{u}_h - \mathbf{u}\|_{L^2}^2 \rightarrow 0$, we conclude $\mathbf{u}_h \rightarrow \mathbf{u}$ in $L^2(\mathbf{L}^2)$.
(4) Since $\mathbf{u}_h$ is solenoidal, the pressure term simplifies as follows $\int_0^T(p_h, \nabla\cdot(\mathbf{u}_h\phi)) = \int_0^T(p_h\mathbf{u}_h, \nabla\phi)$. As a result, $\int_0^T(p_h, \nabla\cdot(\mathbf{u}_h\phi)) \rightarrow \int_0^T(p\mathbf{u}, \nabla\phi)$ since $p_h \rightharpoonup p$ in $L^{4/3}(L^2)$ and $\mathbf{u}_h\cdot\nabla\phi \rightarrow \mathbf{u}\cdot\nabla\phi$ in $L^4(L^2)$.
(5) We treat the trouble-making nonlinear term as follows

$$\begin{aligned}(P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h, P_h(\mathbf{u}_h\phi)) &= (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \mathbf{u}_h\phi) + R_1 \\ &= -(\tfrac{1}{2}|\mathbf{u}_h|^2 P_\varepsilon\mathbf{u}_h, \nabla\phi) + R_1, \\ &= -(\tfrac{1}{2}|P_\varepsilon\mathbf{u}_h|^2 P_\varepsilon\mathbf{u}_h, \nabla\phi) + R_1 + R_2,\end{aligned}$$

where

$$\begin{aligned}R_1 &= (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h, P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi), \\ R_2 &= -\tfrac{1}{2}((P_\varepsilon\mathbf{u}_h + \mathbf{u}_h)\cdot Q_\varepsilon\mathbf{u}_h P_\varepsilon\mathbf{u}_h, \nabla\phi).\end{aligned}$$

Using the approximation property of P_h (see Lemma 2.1(iv)) and the fact that $\|\mathbf{u}_h\phi\|_{H^1} \leq c\|\mathbf{u}_h\|_{H^1}\|\phi\|_{W^{1,\infty}}$, we can bound the first residual as follows:

$$\begin{aligned}|R_1| &\leq \|P_\varepsilon \mathbf{u}_h\|_{L^\infty}\|\nabla \mathbf{u}_h\|_{L^2}\|P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi\|_{L^2},\\ &\leq c\,\varepsilon^{-\frac{3}{2}}N^{-1}\|P_\varepsilon \mathbf{u}_h\|_{L^2}\|\nabla \mathbf{u}_h\|_{L^2}\|\mathbf{u}_h\phi\|_{H^1},\\ &\leq c\,N^{\frac{3}{2}\theta-1}\|P_\varepsilon \mathbf{u}_h\|_{L^2}\|\mathbf{u}_h\|^2_{H^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

Then, it is clear that $\int_0^T |R_1| \to 0$ as $N \to \infty$ owing to (5.1). For the second residual, we use the embedding $H^1(\Omega) \subset L^6(\Omega)$, to show that:

$$\begin{aligned}|R_2| &\leq c\,\|Q_\varepsilon \mathbf{u}_h\|_{L^2}\|P_\varepsilon \mathbf{u}_h\|_{L^3}\|P_\varepsilon \mathbf{u}_h + \mathbf{u}_h\|_{L^6}\|\phi\|_{W^{1,\infty}}\\ &\leq c\,\varepsilon\|Q_\varepsilon \mathbf{u}_h\|_{H^1}\varepsilon^{-\frac{1}{2}}\|P_\varepsilon \mathbf{u}_h\|_{L^2}\|P_\varepsilon \mathbf{u}_h + \mathbf{u}_h\|_{H^1}\|\phi\|_{W^{1,\infty}}\\ &\leq c\,N^{-\frac{1}{2}\theta}\|P_\varepsilon \mathbf{u}_h\|_{L^2}\|\mathbf{u}_h\|^2_{H^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

Then, for $\theta > 0$, $\int_0^T |R_2| \to 0$ as $N \to \infty$.
(6) Passing to the limit in the source term does not pose any difficulty since

$$< \mathbf{f}, P_h(\phi\mathbf{u}_h) > = < \mathbf{f}, \phi\mathbf{u}_h > + R,$$

where $R = < \mathbf{f}, P_h(\phi\mathbf{u}_h) - \phi\mathbf{u}_h >$. Clearly $\int_0^T < \mathbf{f}, \phi\mathbf{u}_h > \to \int_0^T < \mathbf{f}, \phi\mathbf{u} >$ since $\mathbf{u}_h \rightharpoonup \mathbf{u}$ in $L^2(\mathbf{H}^1)$ and $f \in L^2(\mathbf{H}^{-1})$. Moreover,

$$\begin{aligned}\int_0^T |R| &\leq \|\mathbf{f}\|_{L^2(H^{-1})}\|P_h(\phi u_h) - \phi\mathbf{u}_h\|_{L^2(H^1)}\\ &\leq c\,N^{-1}\|\mathbf{f}\|_{L^2(H^{-1})}\|\mathbf{u}_h\|_{L^2(H^1)}.\end{aligned}$$

Then $\int_0^T |R| \to 0$ as $N \to +\infty$. □

Remark 5.1. The above theorem shows that if the sizes of the large eddy scales and the mesh size are such that $\varepsilon \gg h^{2/3}$, then the pair $(\mathbf{u}_h, p_h)$ is a suitable approximation in the sense of Definition 2.2.

5.2 Other NLGM formulations

We now briefly explain why we are not successful to prove that the discrete versions of the NLGM formulation (3.2) converge to a suitable solution.

For the other formulation (3.2) we would need to show that terms like $(\mathbf{z}\cdot\nabla\mathbf{y}, P_h(\mathbf{u}_h\phi))$ converge to zero as $N \to \infty$, where recall $\mathbf{z} = P_\varepsilon \mathbf{u}_h$, and $\mathbf{y} = Q_\varepsilon \mathbf{u}_h$. For example, we have:

$$\begin{aligned}(\mathbf{z}\cdot\nabla\mathbf{y}, P_h(\mathbf{u}_h\phi)) &= (\mathbf{z}\cdot\nabla\mathbf{y}, \mathbf{u}_h\phi) + (\mathbf{z}\cdot\nabla\mathbf{y}, P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi)\\ &= (\mathbf{z}\cdot\nabla\mathbf{y}, \mathbf{y}\phi) + (\mathbf{z}\cdot\nabla\mathbf{y}, \mathbf{z}\phi) + (\mathbf{z}\cdot\nabla\mathbf{y}, P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi)\\ &= -(\tfrac{1}{2}|\mathbf{y}|^2\mathbf{z}, \nabla\phi) + (\mathbf{z}\cdot\nabla\mathbf{y}, \mathbf{z}\phi) + (\mathbf{z}\cdot\nabla\mathbf{y}, P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi)\end{aligned}$$

It is clear that the first term in the right hand side goes to zero as $N \to \infty$, but unfortunately we have not been able to show that the last two terms vanish as $N \to \infty$. The difficulty stems from that we cannot see any way to integrate by parts the two other terms so that the derivative acts only on the test functionϕ. Hence is seems that the Fourier version of (3.2) does not yield a suitable approximation.

6 Convergence analysis assuming regularity

We (re)prove in this section that provided the solution to (2.1) is smooth enough (wishful thinking?) the velocity field from (3.7) is as accurate in the H^1-norm as that of the un-truncated Galerkin solution on $\mathbf{X}_h \times M_h$. This feature is a well-known characteristics of nonlinear Galerkin methods. However, as noted in [12] the presence or absence of nonlinearities has nothing ado with this remarkable property. The single key argument at stake here is that the elliptic projection is super-convergent in the H^1-norm. It seems to us that this property, found by Wheeler in [19], has not been emphasized enough in the literature dedicated to NLGM. The goal of this section is make this point clearer. The main result of this section is Theorem 6.1.

Denote by $(\mathbf{R}_h(\mathbf{u}), S_h(p)) \in \mathbf{X}_h \times M_h$ the elliptic projection of the couple $(\mathbf{u}, p)$, i.e., for a.e. $t \in [0, T]$, for all $\mathbf{v}_h \in \mathbf{X}_h$, and for all $q_h \in M_h$

$$\begin{cases} (\nabla \mathbf{R}_h(\mathbf{u}), \nabla \mathbf{v}_h) - (S_h(p), \nabla\cdot \mathbf{v}_h) = (\mathbf{u}, \nabla \mathbf{v}_h) - (p, \nabla\cdot \mathbf{v}_h) \\ (q_h, \nabla\cdot \mathbf{u}_h) = 0. \end{cases} \tag{6.1}$$

Define $K_1 = \|\mathbf{u}_t\|_{L^2(H^{\sigma+1})}$, $K_2 = \|\mathbf{u}\|_{L^\infty(H^s)}$, with $s > \frac{3}{2}$, $K_3 = \|\mathbf{u}\|_{L^2(H^{\sigma+1})}$, and $K_4 = \|\mathbf{u}_0\|_{H^{\sigma+1}}$. Let us set $K = K_1 + \ldots + K_4$. Throughout this section we assume that $\mathbf{f}$ and $\mathbf{u}_0$ are smooth enough so that there exist $\sigma > 0$ and $s > \frac{3}{2}$ for which K is bounded.

Lemma 6.1. *Provided the quantity K is bounded,*

$$\|\mathbf{u} - \mathbf{R}_h(\mathbf{u})\|_{L^2(L^2)} + \|\mathbf{u}_t - \mathbf{R}_h(\mathbf{u}_t)\|_{L^2(L^2)} \leq c(K_1, K_2)\varepsilon^{-(\sigma+1)}. \tag{6.2}$$

$$\|\mathbf{R}_h(\mathbf{u})\|_{L^\infty(L^\infty)} \leq c(K_2) \tag{6.3}$$

$$\|\mathbf{u}_0 - \mathbf{R}_h(\mathbf{u}_0)\|_{L^2} \leq c(K_4)\varepsilon^{-(\sigma+1)} \tag{6.4}$$

The following Lemma clarifies what we meant above when stating that the elliptic projection is super-convergent in the H^1-norm.

Lemma 6.2. *Under the regularity assumptions of Lemma 6.1, the velocity field from (3.7) satisfies the following error estimate*

$$\|\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})\|_{L^2(H^1)} \leq c(\nu, T, K)\varepsilon^{-(\sigma+1)}. \tag{6.5}$$

Proof. Let us set $\mathbf{e}_h = \mathbf{R}_h(\mathbf{u}) - \mathbf{u}_h$ and $\delta_h = S_h(p) - p_h$. Then the equations controlling these two quantities are

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{e}_h, \mathbf{v}) + \nu(\nabla \mathbf{e}_h, \nabla \mathbf{v}) - (\delta_h, \nabla\cdot \mathbf{v}) = (\mathbf{F}(\mathbf{u}_h, \mathbf{u}), \mathbf{v}) + (\mathbf{R}_0, \mathbf{v}), \\ (\nabla\cdot \mathbf{e}_h, q) = 0, \\ \mathbf{e}_h|_{t=0} = P_\varepsilon(\mathbf{R}_h(\mathbf{u}_0) - \mathbf{u}_0) \end{cases}$$

where we have defined $\mathbf{F}(\mathbf{u}_h, \mathbf{u}) = (P_\varepsilon \mathbf{u}_h)\cdot\nabla \mathbf{u}_h - \mathbf{u}\cdot\nabla\mathbf{u}$, $\mathbf{R}_0 = P_\varepsilon \mathbf{R}_h(\mathbf{u}_t) - \mathbf{u}_t$, and the test functions $\mathbf{v}$ and q span $\mathbf{X}_h$ and M_h respectively. The error estimate (6.5) is obtained by using $\mathbf{e}_h$ as a test function in the above equations and by integrating over the time interval $(0, T)$.

Owing to the assumed regularity for $\mathbf{u}$ we have

$$\begin{aligned} \|\mathbf{R}_0\|_{L^2(L^2)} &\le \|P_\varepsilon(\mathbf{R}_h(\mathbf{u}_t) - \mathbf{u}_t)\|_{L^2(L^2)} + \|P_\varepsilon \mathbf{u}_t - \mathbf{u}_t\|_{L^2(L^2)} \\ &\le c\,\varepsilon^{-(\sigma+1)} \|\mathbf{u}_t\|_{L^2(H^{\sigma+1})}. \end{aligned}$$

This immediately yields

$$\textstyle\int_0^T |(\mathbf{R}_0, \mathbf{e}_h)| \le \gamma \|\nabla \mathbf{e}_h\|^2_{L^2(L^2)} + c(\gamma, K_1)\, \varepsilon^{-2(\sigma+1)},$$

where $\gamma > 0$ is a positive real that can be chosen as small as needed.

To control the nonlinear term we set

$$\begin{aligned} \mathbf{F}(\mathbf{u}_h, \mathbf{u}) = &\, P_\varepsilon \mathbf{u}_h\cdot\nabla(\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})) + P_\varepsilon(\mathbf{u}_h - \mathbf{R}_h(\mathbf{u}))\cdot\nabla \mathbf{R}_h(\mathbf{u}) \\ &+ (P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u})\cdot\nabla\mathbf{R}_h(\mathbf{u}) + \mathbf{u}\cdot\nabla(\mathbf{R}_h(\mathbf{u}) - \mathbf{u}). \end{aligned}$$

Let $\mathbf{R}_1$ to $\mathbf{R}_4$ be the four residuals in the right-hand side above. Clearly

$$\textstyle\int_0^T (\mathbf{R}_1, \mathbf{e}_h) = 0.$$

Then, integrating by parts

$$|(\mathbf{R}_2, \mathbf{e}_h)| \le \|P_\varepsilon \mathbf{e}_h\|_{L^2} \|\nabla \mathbf{e}_h\|_{L^2} \|\mathbf{R}_h(\mathbf{u})\|_{L^\infty} \le c(K_2)\, \|\mathbf{e}_h\|_{L^2} \|\nabla \mathbf{e}_h\|_{L^2}.$$

Hence

$$\textstyle\int_0^T |(\mathbf{R}_2, \mathbf{e}_h)| \le \gamma \|\nabla \mathbf{e}_h\|^2_{L^2} + c(K_2) \|\mathbf{e}_h\|^2_{L^2}.$$

For the third residual we have

$$\begin{aligned} |(\mathbf{R}_3, \mathbf{e}_h)| &\le \|P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{L^2} \|\nabla \mathbf{e}_h\|_{L^2} \|\mathbf{R}_h(\mathbf{u})\|_{L^\infty} \\ &\le c(K_2)\, \varepsilon^{-(\sigma+1)} \|\mathbf{u}\|_{H^{\sigma+1}} \|\nabla \mathbf{e}_h\|_{L^2}. \end{aligned}$$

This yields

$$\textstyle\int_0^T |(\mathbf{R}_3, \mathbf{e}_h)| \le \gamma \|\nabla \mathbf{e}_h\|^2_{L^2} + c(\gamma, K_2, K_3)\, \varepsilon^{-2(\sigma+1)}.$$

For the last residual we proceed similarly

$$
\begin{aligned}
|(\mathbf{R}_4, \mathbf{e}_h)| &\le \|\mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{L^2} \|\nabla \mathbf{e}_h\|_{L^2} \|\mathbf{u}\|_{L^\infty} \\
&\le c(K_2)\, \varepsilon^{-(\sigma+1)} \|\mathbf{u}\|_{H^{\sigma+1}} \|\nabla \mathbf{e}_h\|_{L^2}.
\end{aligned}
$$

This yields

$$
\int_0^T |(\mathbf{R}_4, \mathbf{e}_h)| \le \gamma \|\nabla \mathbf{e}_h\|_{L^2}^2 + c(\gamma, K_2, K_3)\, \varepsilon^{-2(\sigma+1)}.
$$

We obtain the desired estimate by setting $\gamma = \nu/8$ and using the Gronwall Lemma. □

Lemma 6.3. *Under the regularity assumptions of Lemma 6.1, the pressure field from (3.7) satisfies the following error estimate*

$$
\|p_h - S_h(p)\|_{L^2(L^2)} \le c(\nu, T, K)\varepsilon^{-(\sigma+1)}. \tag{6.6}
$$

Proof. The omit the details since the argument is the same as that for proving the estimate (4.2) repeating the arguments of the proof of Lemma 6.2. □

Theorem 6.1. *Under the regularity assumptions of Lemma 6.1, the velocity field and the pressure field from (3.7) satisfies the following error estimate*

$$
\|\mathbf{u} - \mathbf{u}_h\|_{L^2(H^1)} + \|p - p_h\|_{L^2(L^2)} \le c(\nu, T, K)(N^{-\sigma} + N^{-\theta(\sigma+1)}). \tag{6.7}
$$

Proof. This is a simple consequence of the triangle inequalities

$$
\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{L^2(H^1)} &\le \|\mathbf{u} - \mathbf{R}_h(\mathbf{u})\|_{L^2(H^1)} + \|\mathbf{R}_h(\mathbf{u}) - \mathbf{u}_h\|_{L^2(H^1)} \\
\|p - p_h\|_{L^2(L^2)} &\le \|p - S_h(p)\|_{L^2(L^2)} + \|S_h(p) - p_h\|_{L^2(L^2)}.
\end{aligned}
$$

together with Lemma 6.2, Lemma 6.3, and the definition of ε. □

Remark 6.1. As an immediate consequence of the above Theorem one deduces that the couple $(\mathbf{u}_h, p_h)$ is as accurate as the un-truncated Galerkin solution on $(\mathbf{X}_h, M_h)$ in the $\mathbf{H}^1 \times L^2$-norm provided the expected regularity index σ and the real θ are such that

$$
\theta \ge \frac{\sigma}{\sigma + 1}. \tag{6.8}
$$

7 Concluding remarks

Although there is no mathematical definition of LES available in the literature at the present time, it seems to us that, in addition to constructing a suitable approximation, the NLGM described in the present paper satisfies many of the heuristic criteria usually assigned to LES. More specifically: (1) This technique represents correctly all the non-linear interactions between the scales larger than the large eddy scale ε; (2) The scales below ε interact only with the large scales, i.e., there is no nonlinear interaction among the small scales; (3) For any assumed regularity index, $\sigma \ge 0$, the method is as accurate as the un-truncated technique. In other words, when smoothness is assumed the method performs as well as if the dynamics of the small scales had fully been accounted for.

REFERENCES

[1] L. Caffarelli, R. Kohn, and L. Nirenberg. Partial regularity of suitable weak solutions of the Navier-Stokes equations. *Comm. Pure Appl. Math.*, 35(6):771–831, 1982.

[2] C. Devulder, M. Marion, and E. S. Titi. On the rate of convergence of the nonlinear Galerkin methods. *Math. Comp.*, 60(202):495–514, 1993.

[3] C. Doering and J. Gibbon. *Applied analysis of the Navier–Stokes equations*. Cambridge texts in applied mathematics. Cambridge University Press, 1995.

[4] J. Duchon and R. Robert. Inertial energy dissipation for weak solutions of incompressible Euler and Navier-Stokes equations. *Nonlinearity*, 13(1):249–255, 2000.

[5] C. Foias, O. Manley, and R. Temam. Modelling of the interaction of small and large eddies in two-dimensional turbulent flows. *RAIRO M^2AN*, 22:93–114, 1988.

[6] C. Foias, G. R. Sell, and E. S. Titi. Exponential tracking and approximation of inertial manifolds for dissipative nonlinear equations. *J. Dynamics and Diff. Eq.*, 1:199–243, 1989.

[7] M. Germano. Differential filters for the large eddy numerical simulation of turbulent flows. *Phys. Fluids*, 29(6):1755–1757, 1986.

[8] B. J. Geurts. *Elements of Direct and Large-Eddy Simulation*. Springer-Verlag, R.T. Edwards, Inc., Philadelphia, PA, 2004.

[9] J.-L. Guermond, J. Oden, and S. Prudhomme. Mathematical perspectives on the Large Eddy Simulation models for turbulent flows. *J. Math. Fluid Mech.*, 2003. In press.

[10] J.-L. Guermond and S. Prudhomme. A definition for large-eddy-simulation approximations of the Navier-Stokes equations. *Physica D*, 2004. Submitted.

[11] C. He. On partial regularity for weak solutions to the Navier-Stokes equations. *J. Funct. Anal.*, 211(1):153–162, 2004.

[12] J. G. Heywood and R. Rannacher. On the question of turbulence modeling by approximate inertial manifolds and the nonlinear Galerkin method. *SIAM J. Numer. Anal.*, 30(6):1603–1621, 1993.

[13] V. John. *Large eddy simulation of turbulent incompressible flows*, volume 34 of *Lecture Notes in Computational Science and Engineering*. Springer-Verlag, Berlin, 2004.

[14] F. Lin. A new proof of the Caffarelli-Kohn-Nirenberg theorem. *Comm. Pure Appl. Math.*, 51(3):241–257, 1998.

[15] J.-L. Lions. *Quelques méthodes de résolution des problèmes aux limites non linéaires*, volume 1. Dunod, Paris, France, 1969.

[16] M. Marion and R. Temam. Nonlinear Galerkin methods. *SIAM J. Numer. Anal.*, 26(5):1139–1157, 1989.

[17] P. Sagaut. *Large eddy simulation for incompressible flows.* Scientific Computation. Springer-Verlag, Berlin, 2001.

[18] V. Scheffer. Hausdorff measure and the Navier-Stokes equations. *Comm. Math. Phys.*, 55(2):97–112, 1977.

[19] M. Wheeler. A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations. *SIAM J. Numer. Anal.*, 10:723–759, 1973.