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NONSTANDARD NONCONFORMING APPROXIMATION OF THE STOKES PROBLEM, I: PERIODIC BOUNDARY CONDITIONS

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Abstract. This paper analyzes a nonstandard form of the Stokes problem where the mass conservation equation is expressed in the form of a Poisson equation for the pressure. This problem is shown to be wellposed in the d-dimensional torus. A nonconforming approximation is proposed and, contrary to what happens when using the standard saddle-point formulation, the proposed setting is shown to yield optimal convergence for every pairs of approximation spaces.

Key Words. Stokes equations, finite elements, nonconforming approximation, incompressible flows and Poisson equation

1. Introduction

Consider the Stokes equations in a bounded domain Ω:

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \nabla{\cdot}u = 0. \tag{1.1}$$

The objective of the present work is to analyze the following nonstandard form of the Stokes equations:

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \Delta p = \nabla{\cdot}f; \quad \partial_n p|_{\partial\Omega} = (-\nabla{\times}\nabla{\times}u + f){\cdot}n|_{\partial\Omega}. \tag{1.2}$$

The Poisson equation for the pressure is obtained formally by taking the divergence of the momentum equation, and the Neumann boundary condition is obtained by taking the normal component of the momentum equation at the boundary of the domain and substituting $-\Delta u$ by $\nabla{\times}\nabla{\times}u$ since $\nabla{\cdot}u$ is expected to be zero (recall that $-\Delta u = -\nabla\nabla{\cdot}u + \nabla{\times}\nabla{\times}u$). This way of solving the Stokes (or Navier–Stokes) equations seems to be standard in the literature dedicated to the analysis of turbulence in the d-torus. It currently seems also to attract a growing interest in the literature dealing with the approximation of the time-dependent Stokes (and Navier–Stokes) equations. This form of the Stokes equations is one building block of a splitting algorithm proposed by Orszag et al. [7] and Karniadakis et al. [6]. This problem has also been shown to play an important role in a new type of splitting algorithm proposed in [5]. A recurrent claim in the literature about this strange

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form of the Stokes equations is that when discretized it does not require the velocity and the pressure spaces to satisfy the so-called Babuška–Brezzi condition, i.e., there are no spurious pressure modes. To the present time, this claim has never been proved. The main reason for the lack of proof is that, mathematically speaking, the problem (1.2) is far from being standard. Actually, this form of the problem is more prone to raise eyebrows of mathematically minded readers than to attract their interest. Since the usual setting for this problem is to assume that f is in $[H^{-1}(\Omega)]^d$, the velocity is in $[H^1(\Omega)]^d$ and the pressure is in $L^2(\Omega)$. This type of regularity is incompatible with the boundary condition $\partial_n p|_{\partial\Omega} = (\nabla\times\nabla\times u + f)\cdot n|_{\partial\Omega}$ since it is not legitimate to speak of the normal derivative of a function in $L^2(\Omega)$, nor is it legitimate to speak of the normal component of a $\mathbb{R}^d$-valued distribution in $[H^{-1}(\Omega)]^d$.

This work is an attempt at tackling the above issue. We first analyze a slightly modified version of (1.2), (See problem (2.1)) and we show that this modified version is wellposed. Although, we do not solve exactly (1.2), we think that the setting used for the analysis of the modified problem gives hints of what should be used to seriously tackle (1.2). Since the bothering issue in (1.2) is the boundary condition, in the second part of this work we analyze (1.2) in the periodic d-torus. To the best of our knowledge, the analysis of this problem using finite elements does not seem to have been done yet. In this setting we are able to conduct a full analysis. We propose a discrete formulation and we show that it is optimally convergent. The main result of the paper is Theorem 3.1. The main conclusion of our analysis is that, yes indeed, (1.2) in the d-torus yields an optimal approximation setting that does not require the approximation spaces to satisfy the Babuška–Brezzi condition.

2. The continuous problem

This section is composed of two subsections. First we consider a slightly modified version of (1.2), which we prove to be wellposed. Second we analyze (1.2) adopting periodic boundary conditions.

2.1. First formulation. The problem that we consider can be written formally in the following form

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \Delta\nabla\cdot u = 0; \quad \partial_n \nabla\cdot u|_{\partial\Omega} = 0. \tag{2.1}$$

To give sense to the above problem we introduce the spaces

$$X = [H^1_0(\Omega)]^d; \quad M = L^2_{\int=0}(\Omega); \quad Z = \{\phi \in H^1_{\int=0}; \Delta\phi \in L^2(\Omega); \partial_n\phi|_{\partial\Omega} = 0\}, \tag{2.2}$$

where $L^2_{\int=0}(\Omega)$ is composed of those functions in $L^2(\Omega)$ whose mean-value is zero. We equip X, M and Z with the following norms $\|u\|_X = \|u\|_{1,\Omega}$, $\|p\|_M = \|p\|_{0,\Omega}$, $\|q\|_Z = \|q\|_{1,\Omega} + \|\Delta q\|_{0,\Omega}$, where $\|\cdot\|_{s,\Omega}$ denotes the norm in $H^s(\Omega)$. No notational distinction is made between the norm of scalar-valued and vector-valued functions. The product spaces $X\times M$ and $X\times Z$ are equipped with the norms $\|(u,p)\|_{X\times M} = \|u\|_X + \|p\|_M$ and $\|(u,p)\|_{X\times Z} = \|u\|_X + \|p\|_Z$. All the above normed vector spaces are clearly Hilbert spaces. We define the bilinear form

(2.3)
$$a : (X\times M)\times(X\times Z) \ni ((u,p),(v,q)) \longmapsto (\nabla u,\nabla v) - (p,\nabla\cdot v) + (\nabla\cdot u,\Delta q) \in \mathbb{R}.$$

This bilinear form is clearly continuous.

The formal problem (2.1) can be reformulated as follows: For $f \in [H^{-1}(\Omega)]^d$, seek $(u,p) \in X\times M$ such that

$$a((u,p),(v,q)) = \langle f, v\rangle_{-1,1,\Omega}, \quad \forall (v,q) \in X\times Z, \tag{2.4}$$

where $\langle\cdot,\cdot\rangle_{-1,1,\Omega}$ denotes the duality pairing between $[H^{-1}(\Omega)]^d$ and $[H^1_0(\Omega)]^d$.

Lemma 2.1. *There is $\alpha > 0$ such that*

(i)
$$\inf_{\substack{(u,p)\in X\times M\\ u\neq 0, p\neq 0}} \sup_{\substack{(v,q)\in X\times Z\\ v\neq 0, q\neq 0}} \frac{a((u,p),(v,q))}{\|(u,p)\|_{X\times M}\|(v,q)\|_{X\times M}} \geq \alpha.$$

(ii) $\forall (v,q) \in X\times Z, \quad \big(\forall (u,p) \in X\times M,\ a((u,p),(v,q)) = 0\big) \Rightarrow \big((v,q) = (0,0)\big).$

Proof. (1) Let $(u,p) \in X\times M$, then

$$a((u,p),(u,0)) = \|\nabla u\|^2_{0,\Omega} - (p,\nabla\cdot u) \geq \|\nabla u\|^2_{0,\Omega} - \gamma_1\|p\|^2_{0,\Omega} - c_{\gamma_1}\|\nabla\cdot u\|^2_{0,\Omega},$$

where $\gamma_1 > 0$ can be chosen as small as we want, and c_{γ_1} is a constant that only depends on γ_1. Define $q \in Z$ solving $\Delta q = \nabla\cdot u$. Clearly, $\|q\|_Z \leq c\|\nabla\cdot u\|_{0,\Omega}$. Then we observe that

$$a((u,p),(0,q)) = \|\nabla\cdot u\|^2_{0,\Omega}.$$

Combining these two bounds we obtain

$$a((u,p),(u,c_{\gamma_1}q)) \geq \|\nabla u\|^2_{0,\Omega} - \gamma_1\|p\|^2_{0,\Omega}.$$

Now, using the fact that the linear mapping $\nabla\cdot : X \longrightarrow L^2_{\int=0}(\Omega)$ is continuous and surjective, we deduce form the Open Mapping Theorem that there is $\beta > 0$ such that for all $p \in L^2_{\int=0}(\Omega)$ there is $v \in X$ verifying $\nabla\cdot v = -p$ and $\beta\|\nabla v\|_{0,\Omega} \leq \|p\|_{0,\Omega}$. Then

$$\begin{aligned} a((u,p),(v,0)) &\geq -\|\nabla u\|_{0,\Omega}\|\nabla v\|_{0,\Omega} + \|p\|^2_{0,\Omega} \\ &\geq -c_\beta\|\nabla u\|^2_{0,\Omega} - \tfrac{\beta}{2}\|\nabla v\|^2_{0,\Omega} + \|p\|^2_{0,\Omega} \\ &\geq -c_\beta\|\nabla u\|^2_{0,\Omega} + \tfrac{1}{2}\|p\|^2_{0,\Omega}. \end{aligned}$$

Set $\gamma_1 = \frac{1}{8c_\beta}$, $w = u + \frac{1}{2c_\beta}v$, and $r = \gamma_1 q$, then

$$a((u,p),(w,r)) \geq \tfrac{1}{2}\|\nabla u\|^2_{0,\Omega} + \tfrac{1}{8c_\beta}\|p\|^2_{0,\Omega}.$$

Using the bounds $\|q\|_Z \leq c\|\nabla\cdot u\|_{0,\Omega}$ and $\beta\|\nabla v\|_{0,\Omega} \leq \|p\|_{0,\Omega}$ we infer

$$\sup_{(v,q)} \frac{a((u,p),(v,q))}{\|(v,q)\|_{X\times M}} \geq \frac{a((u,p),(w,r))}{\|(w,r)\|_{X\times Z}} \geq c(\|\nabla u\|_{0,\Omega} + \|p\|_{0,\Omega}) = c\|(u,p)\|_{X\times M}.$$

(2) Assume now that $(v,q) \in X\times Z$ is such that $a((u,p),(v,q)) = 0$ for all (u,p) in $X\times M$. Using $u = 0$ yields $(p,\nabla\cdot v) = 0$ for all p in M; moreover, since $\nabla\cdot v \in M$, we infer $\nabla\cdot v = 0$. This in turn implies $0 = a((v,0),(v,q)) = \|\nabla v\|^2_{0,\Omega}$, meaning that v is zero. Moreover, $(\Delta q,\nabla\cdot u) = 0$ for all $u \in X$. Since $\nabla\cdot : X \longrightarrow M$ is surjective and Δq is in M, we infer that $\Delta q = 0$, meaning that q is zero since $\partial_n q|_{\partial\Omega} = 0$. □

As consequence of the above lemma, we infer the following

Corollary 2.1. *The problem* (2.4) *is wellposed and the couple* (u,p) *solves* (1.1) *and* (2.1).

Proof. Apply the Banach–Nečas–Babuška (BNB) Theorem, see e.g., [1, Th. 3.6] or [3, Th. 2.6]. ☐

We do not go any further in the analysis of (2.1) since it is not really the problem we started with; however, we now show how (2.1) relates to (1.2).

2.2. Second formulation with periodic boundary conditions. Using the fact that u solves $-\Delta u+\nabla p = f$, another way to reformulate (2.1) consists of observing that

$$0 = \Delta\nabla{\cdot}u = \nabla{\cdot}\Delta u = \nabla{\cdot}(\nabla p - f) = \Delta p - \nabla{\cdot}f, \tag{2.5}$$

and using the equality $\nabla\nabla{\cdot}u = \nabla{\times}\nabla{\times}u + \nabla p - f$, we infer

$$0 = \partial_n \nabla{\cdot}u = n{\cdot}\nabla\nabla{\cdot}u = n{\cdot}(\nabla{\times}\nabla{\times}u + \nabla p - f). \tag{2.6}$$

In other words, (2.1) is formally equivalent to

$$-\Delta u + \nabla p = f; \quad u|_{\partial\Omega} = 0; \quad \Delta p = \nabla{\cdot}f; \quad \partial_n p|_{\partial\Omega} = (-\nabla{\times}\nabla{\times}u + f){\cdot}n|_{\partial\Omega}. \tag{2.7}$$

Then, we introduce the bilinear form

$$b((u,p),(v,q)) = (\nabla u, \nabla v) - (p, \nabla{\cdot}v) + (\nabla p + \nabla{\times}\nabla{\times}u, \nabla q). \tag{2.8}$$

Formally,(2.5)–(2.6) is equivalent to

$$b((u,p),(v,q)) = (f,v) + (f,\nabla q), \qquad \forall v,q. \tag{2.9}$$

To make this rigorous, we need to state the domain of b and the regularity we expect for f. It is clear that if u and v are picked in X, $(\nabla u, \nabla v)$ is well defined. The term that poses difficulties is $(\nabla p + \nabla{\times}\nabla{\times}u, \nabla q)$. Actually $(\nabla p, \nabla q)$ can be rewritten $-(p, \Delta q)$ if p is picked in M and q is picked in Z; then this term is well defined. The term that is really troublesome at this point is $(\nabla{\times}\nabla{\times}u, \nabla q)$. Formally integrating by parts, it can be rewritten $-\int_{\partial\Omega}(\nabla{\times}u{\times}n){\cdot}\nabla q$, but as far as regularity is concerned this does not really help. At this point, it seems mandatory to assume u be in $H^2(\Omega)^d$ or at least $\nabla{\times}\nabla{\times}u$ be square integrable. Note however that this difficulty does not arise if periodic boundary conditions are enforced since in this case $(\nabla{\times}\nabla{\times}u, \nabla q)$ is zero.

We henceforth assume that Ω is the d-torus in $\mathbb{R}^d$. We set

$$X_{\#} = \{v \in [H^1(\Omega)]^d;\ v \text{ periodic};\ \textstyle\int_\Omega v = 0\}, \tag{2.10}$$

$$M_{\#} = L^2_{\int=0}(\Omega), \tag{2.11}$$

$$Z_{\#} = \{\phi \in H^2(\Omega);\ \phi|_{\partial\Omega} \text{ and } \partial_n\phi|_{\partial\Omega} \text{ periodic};\ \int_\Omega \phi = 0\}. \tag{2.12}$$

Then we define

$$c((u,p),(v,q)) = (\nabla u, \nabla v) - (p, \nabla{\cdot}v) + \langle \nabla p, \nabla q\rangle_{-1,1,\Omega}. \tag{2.13}$$

Clearly c is bilinear and bounded on $(X_\#\times M_\#)\times(X_\#\times Z_\#)$. Furthermore, consider the following problem: For $f\in[H^{-1}(\Omega)]^d$, seek $(u,p)\in X_\#\times M_\#$ such that

$$c((u,p),(v,q))=(f,v)+\langle f,\nabla q\rangle_{-1,1,\Omega},\qquad \forall(v,q)\in X_\#\times Z_\#. \tag{2.14}$$

Lemma 2.2. *The is $\alpha>0$ such that*

(i) $$\inf_{\substack{(u,p)\in X_\#\times M_\#\\ u\neq0,p\neq0}}\ \sup_{\substack{(v,q)\in X_\#\times Z_\#\\ v\neq0,q\neq0}} \frac{c((u,p),(v,q))}{\|(u,p)\|_{X\times M}\|(v,q)\|_{X\times Z}}\geq\alpha.$$

(ii) $\forall(v,q)\in X_\#\times Z_\#,\ \big(\forall(u,p)\in X_\#\times M_\#,\ c((u,p),(v,q))=0\big)\Rightarrow\big((v,q)=(0,0)\big).$

Proof. (1) Let (u,p) be a nonzero member of $X_\#\times M_\#$. Let $q\in Z_\#$ solve $\Delta q=-\nabla\cdot u$. Then

$$c((u,p),(u,q))=\|\nabla u\|_{0,\Omega}^2-(p,\nabla\cdot u)-(p,\Delta q)=\|\nabla u\|_{0,\Omega}^2.$$

Since the linear mapping $\nabla\cdot:X_\#\longrightarrow M_\#$ is continuous and surjective, there is $\beta>0$ such that for all $p\in M_\#$ there is $v\in X_\#$ verifying $\nabla\cdot v=-p$ and $\beta\|\nabla v\|_{0,\Omega}\leq\|p\|_{0,\Omega}$. Then

$$\begin{aligned}c((u,p),(v,0))&\geq-\|\nabla u\|_{0,\Omega}\|\nabla v\|_{0,\Omega}+\|p\|_{0,\Omega}^2\\&\geq-c_\beta\|\nabla u\|_{0,\Omega}^2-\tfrac{\beta}{2}\|\nabla v\|_{0,\Omega}+\|p\|_{0,\Omega}^2\\&\geq-c_\beta\|\nabla u\|_{0,\Omega}^2+\tfrac12\|p\|_{0,\Omega}^2.\end{aligned}$$

Set $w=u+\frac{1}{2c_\beta}v$, and $r=q$, then

$$c((u,p),(w,r))\geq\tfrac12\|\nabla u\|_{0,\Omega}^2+\tfrac{1}{4c_\beta}\|p\|_{0,\Omega}^2.$$

The rest of the proof is similar to that of Lemma 2.1. □

As a direct consequence of the BNB Theorem, the above lemma implies the following

Corollary 2.2. *The problem* (2.14) *is wellposed and the couple* (u,p) *solves* (1.1) *and* (2.1).

The rest of the paper is devoted to the analysis of nonconforming approximations of (2.14).

3. Nonconforming approximation to (2.14)

3.1. The discrete problem. Let $\{\mathcal{T}_h\}_{h>0}$ be a regular family of affine meshes. For the sake of simplicity, we assume that the mesh family is quasi-uniform. We denote by $\mathcal{F}_h$ the set of interfaces of the mesh (including those that are on the periodic boundary of Ω). We define $X_h\subset X_\#$ to be a finite element space for approximating the velocity, and we define $M_h\subset M_\#$ to be a finite element space for approximating the pressure.

We now construct a nonconforming approximation of $Z_\#$ by using M_h to approximate function in $Z_\#$; that is, we set $Z_h=M_h$. Let q_h be a function in M_h and let F be an interface in $\mathcal{F}_h$. Let K_1 and K_2 be the two elements in $\mathcal{T}_h$ such that F is the interface between K_1 and K_2. We denote by q_{h1}, q_{h2} the restriction of q_h to K_1 and K_2, respectively. The unit outward normal to K_1, K_2 is denoted by n_1, n_2, respectively. We define the jump of q_h across F to be

$$[\![q_h]\!] = q_{h1}n_1 + q_{h2}n_2.$$

We introduce the following norm

$$\|q_h\|_{1,h}^2 = \|q_h\|_{0,\Omega}^2 + \sum_{K\in\mathcal{T}_h} \|\nabla q_h\|_{0,K}^2 + \sum_{F\in\mathcal{F}_h^{\mathrm{i}}} h_F^{-1}\|[\![q_h]\!]\|_{0,F}^2 \tag{3.1}$$

Owing to the quasi-uniformity property, the following inverse inequality holds: There is $c_i > 0$ independent of h such that

$$\forall q_h \in M_h, \qquad \|q_h\|_{1,h} \le c_i h^{-1}\|q_h\|_{0,\Omega}. \tag{3.2}$$

To approximate the bilinear form $(-\Delta q, p)$, we introduce a bilinear mapping $l_h : (M_h + Z_\#)\times M_h \longrightarrow \mathbb{R}$ that we assume to be coercive in the following sense: There is $c_l > 0$ independent of h such that

$$\forall q_h \in M_h, \qquad l_h(q_h, q_h) \ge c_l\|q_h\|_{1,h}^2. \tag{3.3}$$

We assume also that the following boundedness property holds: There is $c_L > 0$ such that

$$\forall \phi_h, \psi_h \in M_h, \qquad l_h(\phi_h, \psi_h) \le c_L\|\phi_h\|_{1,h}\|\psi_h\|_{1,h}. \tag{3.4}$$

We define

$$\forall q_h \in M_h, \qquad \|q_h\|_{2,h} = \sup_{\substack{\psi_h\in M_h\\ \psi_h\neq 0}} \frac{l_h(q_h, \psi_h)}{\|\psi_h\|_{0,\Omega}}. \tag{3.5}$$

Since $\|q_h\|_{2,h} \ge c_l\|q_h\|_{1,h}$, the mapping $\|\cdot\|_{2,h} : M_h \longrightarrow \mathbb{R}_+$ is clearly a norm. We now define product norms as follows

$$\|(u_h, p_h)\|_{X_h\times M_h} = \|\nabla u_h\|_{0,\Omega} + \|p_h\|_{0,\Omega} = \|(u_h, p_h)\|_{X\times M} \tag{3.6}$$

$$\|(v_h, q_h)\|_{X_h\times Z_h} = \|\nabla v_h\|_{0,\Omega} + \|q_h\|_{2,h}. \tag{3.7}$$

We define the bilinear form

$$c_h((u_h, p_h), (v_h, q_h)) = (\nabla u_h, \nabla v_h) - (p_h, \nabla\cdot v_h) + l_h(q_h, p_h). \tag{3.8}$$

We now consider discrete counterparts of (2.14). We introduce the linear form $f_h : Z_h \longrightarrow \mathbb{R}$ that we assume to be uniformly continuous with respect to the $\|\cdot\|_{2,h}$-norm and consistent with f in the sense that there exists a constant c independent of h such that for all $p \in M_\#$

$$\sup_{q_h\in M_h} \frac{|f_h(q_h) - l_h(p, q_h)|}{\|q_h\|_{2,h}} \le c \inf_{\psi_h\in M_h} \|p - \psi_h\|_{0,\Omega}. \tag{3.9}$$

Then, we consider the following problem: Seek the couple $(u_h, p_h) \in X_h\times M_h$ such that for all (v_h, q_h) in $X_h\times Z_h$

$$c_h((u_h, p_h), (v_h, q_h)) = \langle f, v_h\rangle_{-1,1,\Omega} + f_h(q_h). \tag{3.10}$$

Example 3.1.

(i) If f is only in $[H^{-1}(\Omega)]^d$, we denote by $\pi_h : [H^{-1}(\Omega)]^d \longrightarrow X_h$ the projection such that $(\pi_h f, v_h) = \langle f, v_h\rangle_{-1,1,\Omega}$. Then $f_h(q_h)$ can be defined to be equal to $-(\nabla\cdot(\pi_h f), q_h)$.

(ii) If f is in $[L^2(\Omega)]^d$ and M_h is H^1-conforming, then one can set $f_h(q_h) = (f, \nabla q_h)$.

(iii) If M_h is H^1-conforming, l_h can be simply defined by $l_h(q_h, p_h) = (\nabla q_h, \nabla p_h)$. If f is in $[L^2(\Omega)]^d$ and $f_h(q_h) = (f, \nabla q_h)$, then $u \in [H^2(\Omega)]^d$ and $p \in H^1(\Omega)$ and

$$f_h(q_h) - l_h(p, q_h) = (-\Delta u + \nabla p, \nabla q_h) - (\nabla p, \nabla q_h) = 0.$$

3.2. Error analysis. The error analysis is performed using the second Strang Lemma. For this purpose we go through a series of Lemma establishing stability, continuity, and consistency.

Lemma 3.1 (Stability). *There is $\alpha > 0$ independent of h such that*

$$\inf_{\substack{(u_h,p_h)\in X_h\times M_h\\ u_h\neq 0, p_h\neq 0}} \sup_{\substack{(v_h,q_h)\in X_h\times M_h\\ v_h\neq 0, q_h\neq 0}} \frac{c_h((u_h,p_h),(v_h,q_h))}{\|(u_h,p_h)\|_{X_h\times M_h}\|(v_h,q_h)\|_{X_h\times Z_h}} \geq \alpha. \tag{3.11}$$

Proof. We proceed similarly to the proof of Lemma 2.2.

(1) Let (u_h, p_h) be a nonzero member of $X_h\times M_h$. Let $q_h \in Z_h$ solve $l_h(q_h, \psi_h) = (\nabla\cdot u_h, \psi_h)$ for all ψ_h in M_h; note that a solution to this problem exists since l_h is coercive. Then

$$c_h((u_h,p_h),(u_h,q_h)) = \|\nabla u_h\|^2_{0,\Omega} - (p_h, \nabla\cdot u_h) + l_h(q_h,p_h) = \|\nabla u_h\|^2_{0,\Omega}.$$

We now use the fact that there is $\beta > 0$ such that for all $p_h \in M_h \subset M_\#$ there is $v \in X_\#$ verifying $\nabla\cdot v = -p_h$ and $\beta\|\nabla v\|_{0,\Omega} \leq \|p_h\|_{0,\Omega}$. Let $C_h v$ be the Clément interpolant of v or any other H^1-stable interpolant with the following local interpolation properties [2]:

$$\|C_h v - v\|_{0,K} \leq c h_K \|\nabla v\|_{0,\Delta_K}, \qquad \|C_h v - v\|_{0,F} \leq c h_K^{\frac{1}{2}} \|\nabla v\|_{0,\Delta_F},$$

where Δ_K is the collection of all the element having a nonzero intersection with K and Δ_F is that of those elements that have a nonzero intersection with F. Since $C_h v$ and v are continuous across interfaces, we have

$$\begin{aligned}
-(p_h, \nabla\cdot(C_h v)) &= -(p_h, \nabla\cdot(C_h v - v)) + \|p_h\|^2_{0,\Omega}\\
&= \|p_h\|^2_{0,\Omega} + \sum_{K\in\mathcal{T}_h}\int_K (C_h v - v)\cdot\nabla p_h - \sum_{F\in\mathcal{F}_h}\int_F [\![p_h]\!]\cdot(C_h v - v)\\
&\geq \|p_h\|^2_{0,\Omega} - \sum_{K\in\mathcal{T}_h}\|C_h v - v\|_{0,K}\,\|\nabla p_h\|_{0,K} - \sum_{F\in\mathcal{F}_h}\|[\![p_h]\!]\|_{0,F}\|C_h v - v\|_{0,F}\\
&\geq \|p_h\|^2_{0,\Omega} - \sum_{K\in\mathcal{T}_h} c h_K\|\nabla v\|_{0,\Delta_K}\,\|\nabla p_h\|_{0,K} - \sum_{F\in\mathcal{F}_h} h_F^{\frac{1}{2}}\|[\![p_h]\!]\|_{0,F}\|\nabla v\|_{0,\Delta_F}\\
&\geq \|p_h\|^2_{0,\Omega} - ch\|\nabla v\|_{0,\Omega}\|p_h\|_{1,h} \geq \|p_h\|^2_{0,\Omega} - c\beta h\|p_h\|_{0,\Omega}\|p_h\|_{1,h}.\\
&\geq \tfrac{1}{2}\|p_h\|^2_{0,\Omega} - c\, h^2\|p_h\|^2_{1,h}.
\end{aligned}$$

Let $\gamma > 0$ be a, yet arbitrary, positive real number. Then, using the above bound

$$\begin{aligned} c_h((u_h,p_h),(C_hv,\gamma h^2 p_h)) &\geq -\|\nabla u_h\|_{0,\Omega}\|\nabla C_h v\|_{0,\Omega} \\ &\quad - (p_h, \nabla\cdot(C_h v)) + \gamma h^2 l_h(p_h,p_h) \\ &\geq -c_1\|\nabla u_h\|_{0,\Omega}\|\nabla v\|_{0,\Omega} \\ &\quad + \tfrac{1}{2}\|p\|_{0,\Omega}^2 - c_2 h^2\|p_h\|_{1,h}^2 + c_3\gamma h^2\|p_h\|_{1,h}^2 \\ &\geq -c_\beta\|\nabla u\|_{0,\Omega}^2 - \tfrac{\beta}{4}\|\nabla v\|_{0,\Omega} + \tfrac{1}{2}\|p\|_{0,\Omega}^2 \\ &\quad + (c_3\gamma - c_2)h^2\|p_h\|_{1,h}^2 \\ &\geq -c_\beta\|\nabla u\|_{0,\Omega}^2 + \tfrac{1}{4}\|p\|_{0,\Omega}^2 + (c_3\gamma - c_2)h^2\|p_h\|_{1,h}^2. \end{aligned}$$

We now choose $\gamma = 2c_2/c_3$. We set $w_h = u_h + \frac{1}{2c_\beta}C_h v$ and $r_h = q_h + \gamma h^2 p_h$. Then,

$$c_h((u_h,p_h),(w_h,r_h)) \geq \tfrac{1}{2}\|\nabla u_h\|_{0,\Omega}^2 + \tfrac{1}{8c_\beta}\|p_h\|_{0,\Omega}^2.$$

It is clear that $\|\nabla w_h\|_{0,\Omega} \leq c\|\nabla u_h\|_{0,\Omega} + \|p_h\|_{0,\Omega}$. Furthermore, using the definition of q_h together with (3.4) and (3.2), we infer

$$\begin{aligned} \|r_h\|_{2,h} = \sup_{\psi_h\in M_h}\frac{l_h(r_h,\psi_h)}{\|\psi_h\|_{0,\Omega}} &= \sup_{\psi_h\in M_h}\frac{l_h(q_h,\psi_h)}{\|\psi_h\|_{0,\Omega}} + c\|p_h\|_{0,\Omega} \\ &\leq \|\nabla\cdot u_h\|_{0,\Omega} + c\|p_h\|_{0,\Omega}. \end{aligned}$$

Then (3.11) follows readily. □

Lemma 3.2 (Continuity). *There is $c > 0$ independent of h such that*

$$\sup_{\substack{(w,r)\in(X_\# + X_h)\times(M_\# + M_h)\\ u_h\neq 0, p_h\neq 0}} \sup_{\substack{(v_h,q_h)\in X_h\times Z_h\\ v_h\neq 0, q_h\neq 0}} \frac{c_h((w,r),(v_h,q_h))}{\|(w,r)\|_{X\times M}\|(v_h,q_h)\|_{X_h\times Z_h}} \leq c. \tag{3.12}$$

Proof. Clearly

$$c_h((w,r),(v_h,q_h)) \leq \|\nabla w\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega} + c\,\|r\|_{0,\Omega}\|\nabla v_h\|_{0,\Omega} + \|q_h\|_{2,h}\|r\|_{0,\Omega}.$$

Then the conclusion follows readily. □

Lemma 3.3 (Consistency). *Let $(u,p) \in X_\#\times M_\#$ solve* (1.1)*, then*

$$\sup_{\substack{(v_h,q_h)\in X_h\times Z_h\\ v_h\neq 0, q_h\neq 0}} \frac{|\langle f, v_h\rangle_{-1,1,\Omega} + f_h(q_h) - c_h((u,p),(v_h,q_h))|}{\|(v_h,q_h)\|_{X_h\times Z_h}} \leq c \inf_{\psi_h\in M_h}\|p-\psi_h\|_{0,\Omega} \tag{3.13}$$

Proof. Clearly,

$$\langle f, v_h\rangle_{-1,1,\Omega} + f_h(q_h) - c_h((u,p),(v_h,q_h)) = f_h(q_h) - l_h(p,q_h).$$

The conclusion is a consequence of the consistency hypothesis (3.9). □

We are now in measure to state the main theorem of this paper.

Theorem 3.1. *Under the above assumptions, there is are constants c_1, c_2 independent of h such that*

$$\|u-u_h\|_{1,\Omega} + \|p-p_h\|_{0,\Omega} \leq c_1 \inf_{v_h\in X_h}\|u-v_h\|_{1,\Omega} + c_2 \inf_{\psi_h\in M_h}\|p-\psi_h\|_{0,\Omega}. \tag{3.14}$$

Proof. This is a simple consequence of Lemma 3.1, Lemma 3.2, Lemma 3.3 together with the second Strang Lemma. □

Remark 3.1.

(i) One striking property of the above approximation method is that convergence is ensured for all pairs of spaces X_h, M_h. In other words, these spaces do not need be compatible, i.e. they do not need satisfy the so-called Babuška-Brezzi condition. Note that this property is not due to the fact that we have adopted periodic boundary conditions, since, as shown in Figure 1, the usual saddle-point $\mathbb{P}_1$–$\mathbb{P}_1$ setting has spurious modes even if periodic boundary conditions are adopted.

(ii) Note that the solution method (3.10) is not a stabilized method, i.e., there is no tunable coefficient entering in the formulation, see e.g., [4, 8].

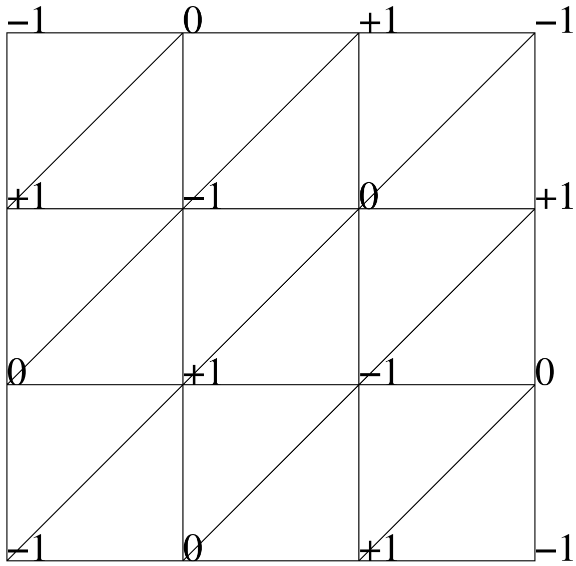

FIGURE 1. One pressure spurious mode for the $\mathbb{P}_1$–$\mathbb{P}_1$ periodic setting.

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