Mixed Finite Element Approximation of an MHD Problem Involving Conducting and Insulating Regions: The 3D Case

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In this paper we extend in 3D a 2D Lagrange finite element technique proposed previously in Guermond and Minev (Mod. Math. Anal. Num. 2001) for solving the Maxwell equations in the low-frequency limit in a domain composed of conducting and insulating regions. Numer Methods Partial Differential Eq 19: 709–731, 2003

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1. INTRODUCTION

The present article is the second part of a work started in [1]. The problem under consideration stems from the Dynamo effect that is at the origin of magnetic fields around stars and planets. This problem has attracted the interest of astrophysicists for a long time (see e.g., Moffatt [2] for a survey). The physical setting and the importance of the MHD problem under consideration have been discussed in our previous article [1], where we concentrated on a 2D version of this problem. It was shown in [1] that, in a 2D domain composed of insulating and conducting regions, the Maxwell equations in the low frequency limit has a saddle point structure, where the electric field in the insulating region is the Lagrange multiplier that enforces the curl-free constraint on the magnetic field. The goal of the present study is to show that, in 3D also, the problem has a saddle point structure and Lagrange finite elements offer an alternative to edge

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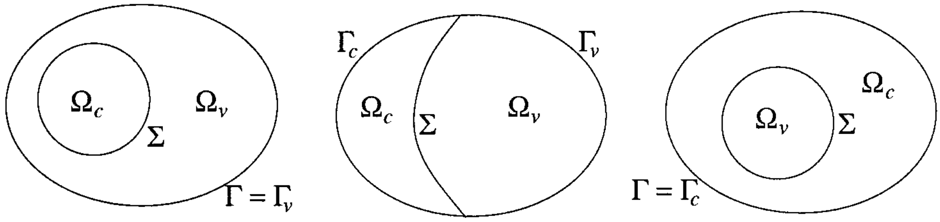

FIG. 1. Three possible settings for the domain.

finite elements, provided that the problem is formulated in an adequate mixed weak form, and the data satisfy some mild regularity criteria (see e.g., [3, 4] and the references therein for details on the use of edge finite elements). Moreover, we show that, under more stringent regularity assumptions on the data, a coercive formulation for both **H** and **E** can be derived.

We consider an incompressible conductive viscous fluid occupying a domain Ω_c in contact with a nonconducting region Ω_v so that the global 3D-domain under consideration, Ω, is nontrivially partitioned as follows:

$$\bar{\Omega} = \bar{\Omega}_c \cup \bar{\Omega}_v, \qquad \Omega_c \cap \Omega_v = \emptyset. \tag{1.1}$$

The subscripts c and v stand for conductor and vacuum, respectively. To refer to boundary conditions easily, we introduce

$$\Gamma_c = \partial\Omega \cap \partial\Omega_c, \quad \Gamma_v = \partial\Omega \cap \partial\Omega_v, \quad \Sigma = \partial\Omega_c \cap \partial\Omega_v, \quad \Gamma = \partial\Omega = \Gamma_v \cup \Gamma_c.$$

See Figure 1 for examples of three possible geometrical configurations.

The conducting fluid in Ω_c may be plasma in the stellar case, molten iron in the planetary case, or liquid gallium or sodium in experimental setups. The insulating region, Ω_v, may be either vacuum, magma, or air.

The MHD flow is governed by the incompressible Navier-Stokes equations and the Maxwell equations with displacement-currents neglected, the two sets of equations being coupled via the Lorentz force and the Ohm's law:

$$\begin{cases} \partial_t \mathbf{u} + (\mathbf{u}\cdot\nabla)\mathbf{u} - \eta\nabla^2\mathbf{u} + \nabla p = (\nabla\times\mathbf{H})\times\mu\mathbf{H} + \mathbf{f} & \text{in } \Omega_c \\ \nabla\cdot\mathbf{u} = 0 & \text{in } \Omega_c \end{cases}$$

$$\begin{cases} \partial_t(\mu\mathbf{H}) = -\nabla\times\mathbf{E} & \text{in } \Omega \\ \nabla\times\mathbf{H} = \sigma(\mathbf{E} + \mathbf{u}\times(\mu\mathbf{H})) + \mathbf{j} & \text{in } \Omega_c \\ \nabla\times\mathbf{H} = 0 & \text{in } \Omega_v \\ \nabla\cdot\mathbf{E} = 0 & \text{in } \Omega_v, \end{cases} \tag{1.2}$$

where **u**, p, **E**, and **H** are the velocity, pressure, electric field, and magnetic field, while **f** and **j** are given source terms. Note that the equations $\nabla\times\mathbf{H}|_{\Omega_v} = 0$, $\nabla\cdot\mathbf{E}|_{\Omega_v} = 0$, and $\mathbf{E}\cdot\mathbf{n}|_{\Gamma_v} = 0$ are the trace of Ampère's theorem $\varepsilon_0\partial_t\mathbf{E} = \nabla\times\mathbf{H}$, where $\varepsilon_0 = 1/\mu_0 c^2$ (μ_0 being a universal constant), and the speed of light c is assumed to be much larger than the characteristic scale of the velocity field **u**. Actually, denoting by l the characteristic lengthscale of Ω and denoting by [**u**], [**H**], and [**E**] the characteristic magnitude of **u**, **H**, and **E**, respectively, the characteristic time of interest is $T = l/[\mathbf{u}]$. Equating energy production to energy dissipation, we obtain $[\mathbf{E}] = \mu[\mathbf{u}][\mathbf{H}]$. Hence, Ampère's theorem yields

$$[\varepsilon_0 \partial_t \mathbf{E}] = \frac{1}{\mu_0 c^2}\frac{[\mathbf{E}]}{T} = \frac{1}{\mu_0 c^2}\frac{\mu_0[\mathbf{u}]^2[\mathbf{H}]}{l} = \frac{[\mathbf{u}]^2}{c^2}\frac{[\mathbf{H}]}{l} = \frac{[\mathbf{u}]^2}{c^2}[\nabla\times\mathbf{H}] \ll [\nabla\times\mathbf{H}],$$

which means that $\nabla \times \mathbf{H}$ is almost zero in the nonconducting region (see Bossavit [3, 5] for more details). The magnetic permeability $\mu = \mu(\mathbf{x})$ and conductivity $\sigma = \sigma(\mathbf{x})$ are supposed to be smooth positive functions of $\mathbf{x}$ only and to be bounded from below by positive constants μ^- and σ^-.

The initial conditions are $\mathbf{H}|_{t=0} = \mathbf{H}_0$ and $\mathbf{u}|_{t=0} = \mathbf{u}_0$, where both $\mathbf{H}_0$ and $\mathbf{u}_0$ are solenoidal vector fields. Denoting by $\mathbf{n}$ the outward normal to Γ, we shall hereafter consider the following boundary conditions:

$$\begin{cases} \mathbf{u}|_\Sigma = \mathbf{u}_\Sigma \\ \mathbf{H}\times\mathbf{n}|_\Gamma = \mathbf{H}_\Gamma \\ \mathbf{E}\cdot\mathbf{n}|_{\Gamma_v} = E_{\Gamma_v}. \end{cases} \tag{1.3}$$

Similarly to the 2D case, we shall not discuss of the treatment of the Navier-Stokes part of the MHD system since it is well studied, but we shall rather concentrate ourselves on the part of the system that controls the magnetic ($\mathbf{H}$) and electric ($\mathbf{E}$) fields. The main difference between the 2D and 3D cases is that in 2D the constraint $\nabla \cdot \mathbf{E} = 0$ in Ω_v is automatically satisfied when the electric field is a scalar while in 3D this constraint must be imposed explicitly. The equation $\nabla \times \mathbf{H} = 0$ in Ω_v is a linear constraint on the solution of the third equation of (1.2). In [1] we showed that in 2D the electric field (which is a scalar) is the Lagrange multiplier for the imposition of $\nabla \times \mathbf{H} = 0$ in Ω_v. The subsequent Galerkin formulation is a saddle point problem similar to the well studied Stokes problem, and therefore it can be analyzed in a similar fashion. As a result, classical Lagrange finite element pairs used to solve the Stokes problem are generally adequate for approximating $\mathbf{H}$ and $\mathbf{E}$. In the 3D situation, $\mathbf{E}$ is still the Lagrange multiplier for the constraint $\nabla \times \mathbf{H} = 0$ in Ω_v, but this multiplier must be chosen to satisfy the additional constraint $\nabla \cdot \mathbf{E} = 0$ in Ω_v. This creates some technical difficulties, and in this paper we suggest two different ways to overcome them. The first formulation consists of a saddle point problem where the constraint $\nabla \cdot \mathbf{E} = 0$ in Ω_v is enforced in the least-squares sense. The second formulation leads to a coercive problem for both, $\mathbf{H}$ and $\mathbf{E}$, and the error estimates for the finite element approximation of $\mathbf{E}$ are improved in comparison with those from the first approach.

The remainder of this article is organized as follows. The two proposed approaches are presented and analyzed in section 2. We present the finite element discretization in section 3. The capabilities of the suggested techniques are demonstrated on numerical examples presented in section 4.

2. WEAK FORMULATIONS

2.1. Preliminaries

We assume hereafter that Ω is a Lipschitz, open, bounded, connected domain in $\mathbb{R}^3$, which, in addition to (1.1), satisfies the following topological conditions:

(Ht1) The boundary of each connected component of Ω_v is either connected or has two connected components, provided one of them lies on Γ.

(Ht2) Ω is simply connected.
(Ht3) Ω_c is connected.

These hypotheses are *ad hoc*; they can be removed by introducing additional technicalities in the proofs. In astrophysical applications Ω_c has usually a connected boundary, $\Omega_c = \emptyset$, and the two connected components of Ω_v are Γ and Σ.

We denote by $\mathbb{L}^2(\Omega)$ [resp. $\mathbb{H}^1(\Omega)$] the space of vector-valued functions whose components are in $L^2(\Omega)$ [resp. in $H^1(\Omega)$]. The norms of the Sobolev spaces $W^{m,p}(\Omega)$ and $\mathbb{W}^{m,p}(\Omega)$ are denoted by $\|\cdot\|_{m,p,\Omega}$ with no distinction between scalar- and vector-valued functions. The norms of $H^s(\Omega)$ and $\mathbb{H}^s(\Omega)$ are denoted by $\|\cdot\|_{s,\Omega}$. For any given subdomain $\Omega_s \subset \Omega$, we denote by $(\cdot,\cdot)_{\Omega_s}$ the L^2-scalar product on Ω_s.

Partial derivatives of a function φ with respect to a variable t are denoted by $\partial_t\varphi$; in the case in which φ depends only on t, we will write $d_t\varphi$.

For every Banach space, E, we denote by $L^p(0, T; E)$, $\mathscr{C}^j(0, T; E)$, and $\mathscr{D}'(0, T; E)$ the space of measurable functions $v : (0, T) \to E$ s.t. $\|v(\cdot)\|_E$ is in $L^p(0, T)$, of class $\mathscr{C}^j$, or a distribution in $\mathscr{D}'(0, T)$, respectively. We also define

$$W(0, T; E) = \{v \in L^2(0, T; E);\ d_t v \in L^2(0, T; E')\}.$$

Throughout this article we denote by c a generic constant that does not depend on the discretization parameter h (i.e. the mesh size) but whose value may change at each occurrence.

As mentioned above, in this article we concentrate on the part of the MHD system that controls the electric and magnetic fields. We suppose hereafter that $\mathbf{u}$ is a known smooth vector field, and we restrict ourselves to the following initial-boundary-value problem:

$$\begin{cases} \partial_t(\mu\mathbf{H}) = -\nabla\times\mathbf{E} & \text{in } \Omega \\ \nabla\times\mathbf{H} = \sigma(\mathbf{E} + \mathbf{u}\times(\mu\mathbf{H})) + \mathbf{j} & \text{in } \Omega_c \\ \nabla\times\mathbf{H} = 0 & \text{in } \Omega_v \\ \nabla\cdot\mathbf{E} = 0 & \text{in } \Omega_v \\ \mathbf{H}\times\mathbf{n} = 0 & \text{on } \Gamma \\ \mathbf{E}\cdot\mathbf{n} = 0 & \text{on } \Gamma_v \\ \mathbf{H} = \mathbf{H}_0 & \text{at } t = 0. \end{cases} \tag{2.1}$$

For convenience of the reader, we recall a result that we shall use repeatedly in the rest of the article.

Theorem 2.1 (Lions). *Let $X \subset L$ two separable Hilbert spaces, the injection being continuous and X dense in L. Let $r(\cdot,\cdot)$ be a real bilinear form on $L \times L$, and for each $t \in [0, T]$ denote by $a(t; \cdot,\cdot)$ a real bilinear form on $X \times X$ measurable in t. Assume that*

(1) *r is symmetric and $\forall u \in L$,* $\quad r(u, u) \geq 0$.
(2) $\exists c_1 > 0$, $\forall u, v \in X$, $\forall t \in [0, T]$, $\quad |a(t; u, v)| \leq c_1\|u\|_X\|v\|_X$.
(3) $\exists \gamma \geq 0$, $c_2 > 0$, $\forall u \in X$, $\forall t \in [0, T]$, $\quad a(t; u, u) + \gamma r(u, u) \geq c_2\|u\|_X^2$.

Then, for all $f \in L^2(0, T; X')$ and all $u_0 \in L$, there is a unique $u \in W(0, T; X)$ s.t.

$$\begin{cases} r(u|_{t=0}, v) = r(u_0, v), & \forall v \in L, \\ d_t r(u, v) + a(t; u, v) = f(v) & \forall v \in X, \text{ in } L^2(0, T). \end{cases}$$

Proof. The proof for this variant of Lions' theorem [6, pp. 253–258], [7, p. 218] can be found in Showalter [8, pp. 114–117]. ∎

In order to formalize the problems discussed in the sequel of the article, we need to introduce the following functional spaces:

$$\begin{cases} \mathbf{X}_1 = \{\mathbf{b} \in \mathbb{L}^2(\Omega), \nabla \times \mathbf{b} \in \mathbb{L}^2(\Omega), \mathbf{b} \times \mathbf{n}_{|\Gamma} = 0\} \\ \mathbf{V}_1 = \{\mathbf{b} \in \mathbf{X}_1, \nabla \times \mathbf{b} = 0 \text{ in } \Omega_v\} \\ \mathbf{L}_1 = \{\mathbf{b} \in \mathbb{L}^2(\Omega), \nabla \times \mathbf{b} = 0 \text{ in } \Omega_v, \mathbf{b} \times \mathbf{n}|_{\Gamma_v} = 0\} \\ \mathbf{N}_1 = \{\mathbf{e} \in \mathbb{L}^2(\Omega_v), \nabla \cdot \mathbf{e} = 0 \text{ in } \Omega_v, \mathbf{e} \cdot \mathbf{n}|_{\Gamma_v} = 0\} \\ \mathbf{W}_1(0,T) = \{\mathbf{b}(t) \in L^2(0,T;\mathbf{X}_1), d_t\mathbf{b}(t) \in L^2(0,T;\mathbf{V}_1')\}, \end{cases} \tag{2.2}$$

$$\begin{cases} \mathbf{X}_2 = \{\mathbf{b} \in \mathbb{L}^2(\Omega), \nabla \times \mathbf{b} \in \mathbb{L}^2(\Omega), \nabla \cdot (\mu\mathbf{b}) \in \mathbb{L}^2(\Omega), \mathbf{b} \times \mathbf{n}_{|\Gamma} = 0\} \\ \mathbf{V}_2 = \{\mathbf{b} \in \mathbf{X}_2, \nabla \times \mathbf{b} = 0 \text{ in } \Omega_v\} \\ \mathbf{L}_2 = \mathbf{L}_1 \\ \mathbf{M}_2 = \{\mathbf{e} \in \mathbb{L}^2(\Omega_v), \nabla \cdot \mathbf{e} \in \mathbb{L}^2(\Omega_v), \mathbf{e} \cdot \mathbf{n}_{|\Gamma_v} = 0\} \\ \mathbf{N}_2 = \mathbf{N}_1 \\ \mathbf{W}_2(0,T) = \{\mathbf{b}(t) \in L^2(0,T;\mathbf{X}_2), d_t\mathbf{b}(t) \in L^2(0,T;\mathbf{V}_2')\}, \end{cases} \tag{2.3}$$

$$\begin{cases} \mathbf{X}_3 = \mathbf{X}_2 \\ \mathbf{M}_3 = \{\mathbf{e} \in \mathbb{L}^2(\Omega_v), \nabla \times \mathbf{e} \in \mathbb{L}^2(\Omega), \nabla \cdot \mathbf{e} \in L^2(\Omega_v), \mathbf{e} \cdot \mathbf{n}_{|\Gamma_v} = 0\} \\ \mathbf{Z}_3 = \mathbf{X}_3 \times \mathbf{M}_3 \\ \mathbf{Y}_3 = \mathbb{L}^2(\Omega) \times \mathbb{L}^2(\Omega). \end{cases} \tag{2.4}$$

The norms associated with $\mathbf{X}_1$, $\mathbf{X}_2$, $\mathbf{M}_2$, and $\mathbf{M}_3$ are defined as

$$\|\mathbf{b}\|_{\mathbf{X}_1} = \|\mathbf{b}\|_{0,\Omega} + \|\nabla \times \mathbf{b}\|_{0,\Omega}$$

$$\|\mathbf{b}\|_{\mathbf{X}_2} = \|\mathbf{b}\|_{0,\Omega} + \|\nabla \times \mathbf{b}\|_{0,\Omega} + \|\nabla \cdot (\mu\mathbf{b})\|_{0,\Omega}$$

$$\|\mathbf{e}\|_{\mathbf{M}_2} = \|\mathbf{e}\|_{0,\Omega_v} + \|\nabla \cdot \mathbf{e}\|_{0,\Omega_v}$$

$$\|\mathbf{e}\|_{\mathbf{M}_3} = \|\mathbf{e}\|_{0,\Omega_v} + \|\nabla \cdot \mathbf{e}\|_{0,\Omega_v} + \|\nabla \times \mathbf{e}\|_{0,\Omega},$$

and $\mathbf{Z}_3$, $\mathbf{Y}_3$ are equipped with a product norm.

Using the above notations, we have the following lemma which will be repeatedly used hereafter.

Lemma 2.1. *Assuming* ($Ht1$), ($Ht2$), *and* ($Ht3$) *hold, the operators* $R_1 : \mathbf{X}_1 \ni \mathbf{b} \mapsto \nabla \times \mathbf{b}|_{\Omega_v} \in \mathbf{N}_1$ *and* $R_2 : \mathbf{X}_2 \ni \mathbf{b} \mapsto \nabla \times \mathbf{b}|_{\Omega_v} \in \mathbf{N}_2$ *are surjective.*

Proof. The result follows from theorem 3.6 of [9, p. 48]. Let $\mathbf{e} \in \mathbf{N}_2$ and let us extend $\mathbf{e}$ to the entire domain Ω as follows. Consider the problem

$$\begin{cases} \text{Find } \Phi \text{ defined in } \Omega_c \text{ s.t.} \\ \nabla^2\Phi = 0 \\ \dfrac{\partial\Phi}{\partial\mathbf{n}} = \mathbf{e}\cdot\mathbf{n} \text{ on } \Sigma \quad \text{and } \dfrac{\partial\Phi}{\partial\mathbf{n}} = 0 \text{ on } \Gamma_c. \end{cases}$$

Note, that since we have assumed that Ω_c is connected [hypothesis (Ht3)], this problem is well-posed owing to

$$\int_{\Sigma} \mathbf{e} \cdot \mathbf{n} = \int_{\Sigma \cup \Gamma_v} \mathbf{e} \cdot \mathbf{n} = \int_{\partial \Omega_v} \mathbf{e} \cdot \mathbf{n} = \int_{\Omega_v} \nabla \cdot \mathbf{e} = 0,$$

which is the compatibility condition for this problem. Therefore, it has a unique solution $\Phi \in H^1(\Omega_c)/\mathbb{R}$. We then extend $\mathbf{e}$ to Ω by the definition

$$\tilde{\mathbf{e}} = \begin{cases} \nabla \Phi \text{ in } \Omega_c \\ \mathbf{e} \text{ in } \Omega_v. \end{cases}$$

In the sense of distributions, the divergence of $\tilde{\mathbf{e}}$ satisfies

$$\forall \phi \in \mathcal{D}(\Omega), \qquad \langle \nabla \cdot \tilde{\mathbf{e}}, \phi \rangle = -\int_{\Omega_v} \mathbf{e} \cdot \nabla \phi - \int_{\Omega_c} \nabla \Phi \cdot \nabla \phi.$$

After integrating by parts and taking into account the boundary condition $\partial_n \Phi = \mathbf{e} \cdot \mathbf{n}$ on Σ, we obtain that $\langle \nabla \cdot \tilde{\mathbf{e}}, \phi \rangle = 0$, i.e. $\nabla \cdot \tilde{\mathbf{e}} = 0$. This, together with the hypotheses (Ht1)–(Ht2), allows us to apply theorem 3.6 of [9, p. 48], which yields that there exists a vector potential $\boldsymbol{\Psi}$ such that $\tilde{\mathbf{e}} = \nabla \times \boldsymbol{\Psi}$, $\nabla \cdot \boldsymbol{\Psi} = 0$ in Ω and $\boldsymbol{\Psi} \times \mathbf{n} = 0$ on Γ. Thus, there exists $\boldsymbol{\Psi} \in \mathbf{X}_2$ s.t. $R_2(\boldsymbol{\Psi}) = \mathbf{e}$. The result extends easily to R_1 since $\mathbf{X}_2 \subset \mathbf{X}_1$ and $\mathbf{N}_1 = \mathbf{N}_2$. ∎

We give now equivalent forms of lemma 2.1 that are more useful in practice.

Lemma 2.2. *The following propositions are equivalent.*

(*i*) *the operator* $R_2 : \mathbf{X}_2 \ni \mathbf{b} \mapsto \nabla \times \mathbf{b}|_{\Omega_v} \in \mathbf{N}_2$ *is surjective.*
(*ii*) *There* $\beta > 0$ *s.t.*

$$\forall\ \mathbf{e} \in \mathbf{N}_2, \exists\ \mathbf{b} \in \mathbf{X}_2, \qquad \beta \|\mathbf{e}\|_{0,\Omega} \leq \|R_2^t \mathbf{e}\|_{\mathbf{X}_2'}. \tag{2.5}$$

(*iii*) *There exists* $\beta > 0$ *s.t.*

$$\forall\ \mathbf{e} \in \mathbf{N}_2, \qquad \sup_{\mathbf{b} \in \mathbf{X}_2} \frac{(\nabla \times \mathbf{b}, \mathbf{e})_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_2}} \geq \beta \|\mathbf{e}\|_{0,\Omega_v}. \tag{2.6}$$

(*iv*) *There exists* $\beta > 0$ *and* $\gamma \geq 0$ *s.t.*

$$\forall\ \mathbf{e} \in \mathbf{M}_2, \qquad \sup_{\mathbf{b} \in \mathbf{X}_2} \frac{(\nabla \times \mathbf{b}, \mathbf{e})_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_2}} \geq \beta \|\mathbf{e}\|_{0,\Omega_v} - \gamma \|\nabla \cdot \mathbf{e}\|_{0,\Omega_v}. \tag{2.7}$$

Moreover the constant β *appearing in* (*ii*), (*iii*), *and* (*iv*) *is the same.*

Proof. The equivalence of (i), (ii), and (iii) is standard. It is evident that (iv) implies (iii). Let us prove (iii) ⇒ (iv). Let $\mathbf{e}$ be in $\mathbf{M}_2$. Let p in $H^1(\Omega_v)$ s.t.

$$\begin{cases} \nabla^2 p = \nabla \cdot \mathbf{e} & \text{in } \Omega_v \\ \partial_n p = 0 & \text{on } \Gamma_v \\ p = 0 & \text{on } \Sigma. \end{cases}$$

Let us set $\mathbf{v} = \mathbf{e} - \nabla p$. It is clear that $\mathbf{v}$ is in $\mathbf{N}_2$ and

$$\|\mathbf{v}\|_{0,\Omega_v} \geq \|\mathbf{e}\|_{0,\Omega_v} - \|\nabla p\|_{0,\Omega_v} \geq \|\mathbf{e}\|_{0,\Omega_v} - c\|\nabla \cdot \mathbf{e}\|_{0,\Omega_v}.$$

Note also that, given the definition of the boundary conditions on p, we have for all $\mathbf{b}$ in $\mathbf{X}_2$

$$(\nabla \times \mathbf{b}, \nabla p)_{\Omega_v} = \int_{\Gamma_v \cup \Sigma} p(\nabla \times \mathbf{b} \cdot \mathbf{n}) = 0,$$

the surface integral being understood in the duality sense (note that on Γ_v the integral vanishes because of the boundary condition satisfied by $\mathbf{b}$). Hence,

$$\sup_{\mathbf{b}\in\mathbf{X}_2} \frac{(\nabla \times \mathbf{b}, \mathbf{e})_{0,\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_2}} \geq \sup_{\mathbf{b}\in\mathbf{X}_2} \frac{(\nabla \times \mathbf{b}, \mathbf{v} + \nabla p)_{0,\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_2}} \geq \sup_{\mathbf{b}\in\mathbf{X}_2} \frac{(\nabla \times \mathbf{b}, \mathbf{v})_{0,\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_2}}$$
$$\geq \beta\|\mathbf{v}\|_{0,\Omega_v} \geq \beta\|\mathbf{e}\|_{0,\Omega_v} - \gamma\|\nabla \cdot \mathbf{e}\|_{0,\Omega_v}.$$

The proof is complete. ■

It is the *inf-sup* condition (2.7) that will have to be satisfied also at the discrete level.

2.2. The Nonstabilized Weak Formulation

To obtain a weak form of the system (2.1), we multiply the first equation by test functions from $\mathbf{X}_1$ and integrate over Ω. The resulting equation is

$$(\mu\partial_t \mathbf{H}, \mathbf{b})_\Omega + (\nabla \times \mathbf{E}, \mathbf{b})_\Omega = 0, \qquad \forall\, \mathbf{b} \in \mathbf{X}_1.$$

Integrating by parts the second term in this equation and taking into account the essential boundary conditions satisfied by the functions belonging to $\mathbf{X}_1$, we obtain

$$(\mu\partial_t \mathbf{H}, \mathbf{b})_\Omega + (\mathbf{E}, \nabla \times \mathbf{b})_\Omega = 0, \qquad \forall\, \mathbf{b} \in \mathbf{X}_1.$$

The second integral in this equation can be represented as a sum of the integrals over Ω_c and Ω_v and substituting $\mathbf{E}_{|\Omega_c}$ from the second equation of (2.1) we obtain

$$(\mu\partial_t \mathbf{H}, \mathbf{b})_\Omega + \left(\frac{1}{\sigma}\nabla \times \mathbf{H}, \nabla \times \mathbf{b}\right)_{\Omega_c} - (\mathbf{u} \times \mu\mathbf{H}, \nabla \times \mathbf{b})_{\Omega_c} + (\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v} = \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c},$$
$$\forall \mathbf{b} \in \mathbf{X}_1.$$

In addition, we also need the weak form of the constraint $\nabla \times \mathbf{H} = 0$ in Ω_v, which reads

$$(\nabla \times \mathbf{H}, \mathbf{e})_{\Omega_v} = 0 \qquad \forall\, \mathbf{e} \in \mathbf{N}_1.$$

Now, it is convenient to introduce the bilinear form

$$a_1(\mathbf{b}, \mathbf{b}') = \left(\frac{1}{\sigma} \nabla \times \mathbf{b}, \nabla \times \mathbf{b}'\right)_{\Omega_c} - (\mathbf{u} \times \mu\mathbf{b}, \nabla \times \mathbf{b}')_{\Omega_c}, \qquad \forall\, \mathbf{b}, \mathbf{b}' \in \mathbf{X}_1,$$

which satisfies the following Gårding inequality.

Lemma 2.3. *Under the assumptions above on σ and μ, there are $\gamma \geq 0$ and $\alpha > 0$ s.t.*

$$\forall\, \mathbf{H} \in \mathbf{V}_1, \qquad a_1(\mathbf{H}, \mathbf{H}) + \gamma\|\mathbf{H}\|^2_{0,\Omega} \geq \alpha\|\mathbf{H}\|^2_{\mathbf{X}_1}.$$

With this definition in hand, the final weak formulation of the problem is as follows: For $\mathbf{j} \in L^2(0, T; \mathbf{L}^2(\Omega))$ and $\mathbf{H}_0 \in \mathbf{L}_1$,

$$\begin{cases} \text{Find } \mathbf{H} \in \mathbf{W}_1(0, T) \text{ and } \mathbf{E} \in \mathcal{D}'(0, T; \mathbf{N}_1) \text{ s.t. } \forall\, \mathbf{b} \in \mathbf{X}_1, \qquad \forall\, \mathbf{e} \in \mathbf{N}_1 \\ (\mu\partial_t\mathbf{H}, \mathbf{b})_\Omega + a_1(\mathbf{H}, \mathbf{b}) + (\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v} = \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c} \qquad \text{in } \mathcal{D}'(0, T) \\ (\nabla \times \mathbf{H}, \mathbf{e})_{\Omega_v} = 0 \qquad \text{in } L^2(0, T). \end{cases} \tag{2.8}$$

Theorem 2.2. *Under the hypotheses of Lemma* 2.1, *problem* (2.8) *is well posed.*

Proof. See the proof of Theorem 2.3 below; the two proofs being very similar, we do not repeat the arguments. See also [1]. The main arguments are the Gårding inequality stated in Lemma 2.3, Lions' theorem 2.1, and Lemma 2.1. ■

Note that the electric field $\mathbf{E}$ comes into play in the nonconducting medium only and appears to be the Lagrange multiplier for enforcing the linear constraint $\nabla \times \mathbf{H} = 0$ in Ω_v. The most important difficulty here is that the Lagrange multiplier must be solenoidal. One standard approach to approximate this problem consists in using edge finite elements and eliminating the electric field $\mathbf{E}$ completely by working in the space $\mathbf{V}_1$, hence enforcing the constraint $\nabla \times \mathbf{H} = 0$ in an essential way. This can be done by using the gradient of continuous nodal Lagrange finite elements with supporting nodes in $\bar{\Omega}_v$ as basis functions (see also Bossavit [3, 5] for details on this technique). The two alternative approaches to be described below retain $\mathbf{E}$ as unknown and enforce the constraint $\nabla \times \mathbf{H} = 0$ only weakly.

2.3. A Partially Stabilized Weak Formulation

In the previous article [1] we suggested a stabilization technique for approximating the Faraday's law that is also applicable in the 3D case, and therefore we shall adopt it here. The essentially new element in 3D is that the equation for $\mathbf{E}$ can also be regularized as shown below.

Let $\tilde{\sigma}$ be a smooth extension of σ to the entire domain Ω so that for all $x \in \Omega$, $\inf_{y\in\Omega_c}\sigma(y) \leq \tilde{\sigma}(x) \leq \sup_{y\in\Omega_c}\sigma(y)$. We introduce the following bilinear forms a_2 and a_4 such that for all $\mathbf{b}$, $\mathbf{b}'$ in $\mathbf{X}_2$, and for all $\mathbf{e}$, $\mathbf{e}'$ in $\mathbf{M}$ we have

$$a_2(\mathbf{b}, \mathbf{b}') = \left(\frac{1}{\tilde{\sigma}} \nabla \times \mathbf{b}, \nabla \times \mathbf{b}'\right)_\Omega + (\nabla \cdot (\mu \mathbf{b}), \nabla \cdot (\mu \mathbf{b}'))_\Omega - (\mu \mathbf{u} \times \mathbf{b}, \nabla \times \mathbf{b}')_{\Omega_c}$$

$$a_4(\mathbf{e}, \mathbf{e}') = (\nabla \cdot \mathbf{e}, \nabla \cdot \mathbf{e}')_{\Omega_v}.$$

Lemma 2.4. *There are $\gamma \geq 0$ and $\alpha > 0$ s.t.*

$$\forall\ \mathbf{b} \in \mathbf{X}_2, \qquad a_2(\mathbf{b}, \mathbf{b}) + \gamma \|\mathbf{b}\|_{0,\Omega} \geq \alpha \|\mathbf{b}\|^2_{\mathbf{X}_2}. \tag{2.9}$$

Proof. For all $\mathbf{H} \in \mathbf{X}_2$ we have the following:

$$\begin{aligned} a_2(\mathbf{H}, \mathbf{H}) &\geq \frac{1}{\sigma^-} \|\nabla \times \mathbf{H}\|^2_{0,\Omega} + \|\nabla \cdot (\mu \mathbf{H})\|^2_{0,\Omega} - \mu \|\mathbf{u}\|_{0,\infty,\Omega_c} \|\mathbf{H}\|_{0,\Omega_c} \|\nabla \times \mathbf{H}\|_{0,\Omega_c} \\ &\geq \frac{1}{\sigma^-} \|\nabla \times \mathbf{H}\|^2_{0,\Omega} + \|\nabla \cdot (\mu \mathbf{H})\|^2_{0,\Omega} - \frac{1}{2\sigma^-} \|\nabla \times \mathbf{H}\|^2_{0,\Omega_c} - \frac{\sigma^- \mu^2}{2} \|\mathbf{u}\|^2_{0,\infty,\Omega_c} \|\mathbf{H}\|^2_{0,\Omega_c}. \end{aligned}$$

The conclusion follows readily. ■

Remark 2.1. If, in addition to the topological hypothesis (Ht2), we assume that Γ is connected, then (see e.g., [9, p. 52]) there exists a constant $c > 0$, s.t.

$$\forall\ \mathbf{H} \in \mathbf{X}_2, \qquad \|\nabla \times \mathbf{H}\|^2_{0,\Omega} + \|\nabla \cdot \mathbf{H}\|^2_{0,\Omega} \geq c \|\mathbf{H}\|^2_{\mathbf{X}_2}.$$

Moreover, if μ is constant, we have

$$a_2(\mathbf{H}, \mathbf{H}) \geq \left(c \min\left(\frac{1}{2\sigma^-}, \mu^2\right) - \frac{\sigma^- \mu^2}{2} \|\mathbf{u}\|^2_{0,\infty,\Omega_c}\right) \|\mathbf{H}\|^2_{\mathbf{X}_2},$$

which show that if $\|\mathbf{u}\|_{0,\infty,\Omega_c}$ is small enough, the bilinear form a_2 is fully coercive in $\mathbf{X}_2$. Finally, if Ω is a convex polyhedron, then the norm of $\mathbf{X}_2$ is equivalent to that of $\mathbb{H}^1(\Omega)$ (see [9, p. 44]).

Remark 2.2. In astrophysical problems, Γ can easily be chosen to satisfy the hypotheses of the remark above, since it is generally an artificial boundary.

The (partially regularized) Galerkin formulation of (2.1) reads (see also [1]): For $\mathbf{j} \in L^2(0, T; \mathbf{L}^2(\Omega))$ and an initial condition $\mathbf{H}_0 \in \mathbf{L}_2$ s.t. $\nabla \cdot (\mu \mathbf{H}_0) = 0$,

$$\begin{cases} \text{Find } \mathbf{H} \in \mathbf{W}_2(0, T) \text{ and } \mathbf{E} \in \mathcal{D}'(0, T; \mathbf{M}_2) \text{ s.t. } \forall\ \mathbf{b} \in \mathbf{X}_2, \forall\ \mathbf{e} \in \mathbf{M}_2 \\ \mathbf{H}|_{t=0} = \mathbf{H}_0, \\ (\mu \partial_t \mathbf{H}, \mathbf{b})_\Omega + a_2(\mathbf{H}, \mathbf{b}) + (\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v} = \left(\frac{1}{\sigma} \mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c} \text{ in } \mathcal{D}'(0, T), \\ (\nabla \times \mathbf{H}, \mathbf{e})_{\Omega_v} - \delta_2 a_4(\mathbf{E}, \mathbf{e}) = 0 \text{ in } L^2(0, T), \end{cases} \tag{2.10}$$

where δ_2 is a positive constant that will be chosen after we discretize the problem.

Theorem 2.3. *Under the hypotheses of Lemma* 2.1, *problem* (2.10) *is well posed. Provided we set* $\mathbf{E}|_{\Omega_c} = (1/\sigma)(\nabla \times \mathbf{H} - \mathbf{j}) - \mathbf{u} \times (\mu\mathbf{H})$, *its solution is a solution to* (2.1) *in the distribution sense.*

Proof. (a) Existence of $\mathbf{H}$. Let us consider the following problem:

$$\begin{cases} \text{Find } \mathbf{H} \in W(0, T; \mathbf{V}_2) \text{ such that} \\ (\mu\partial_t\mathbf{H}, \mathbf{b})_\Omega + a_2(\mathbf{H}, \mathbf{b}) = \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c}, \forall\, \mathbf{b} \in \mathbf{V}_2, \text{ in } L^2(0, T) \\ \mathbf{H}|_{t=0} = \mathbf{H}_0. \end{cases} \tag{2.11}$$

Owing to the coercivity property (2.9), it is easy to show that all the hypotheses of Theorem 2.1 hold. Hence, problem (2.11) has a unique solution $\mathbf{H} \in W(0, T; \mathbf{V}_2) \subset \mathbf{W}_2(0, T)$, which satisfies the initial condition due to the fact that $W(0, T; \mathbf{V}_2) \subset \mathscr{C}^0(0, T; \mathbf{L}_2)$.

(b) Existence of $\mathbf{E}$. Let us consider the following problem:

$$\begin{cases} \text{Find } \mathbf{E} \in \mathscr{D}'(0, T; \mathbf{M}_2) \text{ such that } \forall\, \mathbf{b} \in \mathbf{X}_2 \\ (\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v} = -(\mu\partial_t\mathbf{H}, \mathbf{b})_\Omega - a_2(\mathbf{H}, \mathbf{b}) + \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c}. \end{cases}$$

Let us introduce the linear form $\phi \in \mathbf{X}_2'$ so that for all $\mathbf{b}$ in $\mathbf{X}_2$ we have

$$\langle \phi(\mathbf{H}), \mathbf{b}\rangle = -(\mu\partial_t\mathbf{H}, \mathbf{b})_\Omega - a_2(\mathbf{H}, \mathbf{b}) + \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c}.$$

It is clear that due to the definition of $\mathbf{H}$, the restriction of $\phi(\mathbf{H})$ to $\mathbf{V}_2$ is zero. That is, $\phi(\mathbf{H})$ is in the polar set of $\mathbf{V}_2$, i.e., $\phi(\mathbf{H}) \in \mathbf{V}_2^\perp$. Let us define the linear operator $R_2 : \mathbf{X}_2 \ni \mathbf{b} \mapsto \nabla \times \mathbf{b}|_{\Omega_v} \in \mathbf{N}_2$. It is clear that $\mathbf{V}_2$ is the null space of R_2, i.e., $\mathbf{V}_2 = \mathcal{N}(R_2)$. As a result we have $\phi(\mathbf{H}) \in \mathbf{V}_2^\perp = \mathcal{N}(R)^\perp$, that is $\phi(\mathbf{H})$ is in $\overline{\mathcal{R}(R_2^t)}$, i.e. the closure of the range of the adjoint of R_2. Owing to Lemma 2.1 and Banach's closed range theorem, we infer that $\phi(\mathbf{H}) \in \overline{\mathcal{R}(R_2^t)} = \mathcal{R}(R_2^t)$. That is, there is some $\mathbf{E}$ in $\mathscr{D}'(0, T; \mathbf{N}_2) \subset \mathscr{D}'(0, T; \mathbf{M}_2)$ so that $R_2^t(\mathbf{E}) = -\phi(\mathbf{H})$, the equality holding in $\mathscr{D}'(0, T)$. In other words, there is $\mathbf{E} \in \mathscr{D}'(0, T; \mathbf{M}_2)$ so that for all $\mathbf{b}$ in $\mathbf{X}_2$,

$$\langle \phi(\mathbf{H}), \mathbf{b}\rangle = -\langle R_2^t(\mathbf{E}), \mathbf{b}\rangle = -(\mathbf{E}, R_2(\mathbf{b}))_{\Omega_v} = -(\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v}, \qquad \text{in } \mathscr{D}'(0, T).$$

Note, furthermore, that $\mathbf{H}$ being in $L^2(0, T; \mathbf{V}_2)$ and $\mathbf{E}$ being in $\mathscr{D}'(0, T; \mathbf{N}_2)$, the last equation in (2.10) is satisfied.

(c) Uniqueness. Assume $\mathbf{j} = 0$ and $\mathbf{H}_0 = 0$. From the *a priori* estimate

$$\frac{1}{2}\|\mu^{1/2}\mathbf{H}\|_{0,\Omega}^2 + \delta_2 \int_0^T \|\nabla \cdot \mathbf{E}\|_{0,\Omega_v}^2 \leq 0,$$

we infer that $\mathbf{H} = 0$, hence uniqueness for $\mathbf{H}$, and $\mathbf{E}$ belongs to $\mathbf{N}_2$. From the *inf sup* inequality (2.6), together with the relation $(\mathbf{E}, \nabla \times \mathbf{b})_{\Omega_v} = 0$ for all $\mathbf{b}$ in $\mathbf{X}_2$, we infer that $\mathbf{E} = 0$.

(d) Consistency with (2.1). In the sense of distributions $\mathbf{H}$, $\mathbf{E}$ satisfy the following PDEs:

$$\begin{cases} \mu\partial_t \mathbf{H} + \nabla\times\left(\dfrac{1}{\tilde{\sigma}}\nabla\times\mathbf{H}\right) - \mu\nabla\nabla\cdot(\mu\mathbf{H}) - \nabla\times(1_{\Omega_c}\mathbf{u}\times\mu\mathbf{H}) \\ \quad + \nabla\times(1_{\Omega_v}\mathbf{E}) = \nabla\times\left(\dfrac{1_{\Omega_c}}{\tilde{\sigma}}\mathbf{j}\right) & \text{in } \Omega \\ \nabla\times\mathbf{H} + \delta_2\nabla\nabla\cdot\mathbf{E} = 0 & \text{in } \Omega_v \\ \nabla\cdot(\mu\mathbf{H}) = 0 & \text{on } \Gamma \\ \nabla\cdot\mathbf{E} = 0 & \text{on } \partial\Omega_v \\ \mathbf{H}\times\mathbf{n} = 0 & \text{on } \Gamma \\ \mathbf{H} = \mathbf{H}_0 & \text{at } t = 0. \end{cases}$$

Taking the divergence of the second equation and defining $\phi = \nabla\cdot\mathbf{E}$ we obtain the following problem for ϕ:

$$\begin{cases} -\nabla^2\phi = 0 & \text{in } \Omega_v \\ \phi = 0 & \text{on } \partial\Omega_v \end{cases}$$

Hence, $\nabla\cdot\mathbf{E} = 0$ in Ω_v and therefore $\nabla\times\mathbf{H} = 0$ in Ω_v. Using the fact that $\nabla\cdot(\mu\mathbf{H}_0) = 0$, we deduce that $\nabla\cdot(\mu\mathbf{H}) = 0$ for all $t \geq 0$. Now setting $\mathbf{E} = \frac{1}{\sigma}(\nabla\times\mathbf{H} - \mathbf{j}) - \mathbf{u}\times(\mu\mathbf{H})$ in Ω_c, it is clear that (2.1) holds. ■

Remark 2.3. Note that the coercivity of a_2 is not mandatory for (2.10) to be well posed, a Gårding type inequality is enough to guaranty wellposedness.

2.4. A Fully Regularized Weak Formulation

Under stronger regularity assumptions on $\mathbf{u}$ and $\mathbf{j}$ we can avoid the restriction of the *inf-sup* condition (2.6). Let us make the following regularity hypotheses on the data:

$$\begin{gathered} \mathbf{j} \in H^1(0,T;\mathbb{L}^2(\Omega)) \cap L^2(0,T;\mathbb{H}^1(\Omega)), \\ \mathbf{u} \in W^{1,\infty}(0,T;\mathbb{L}^\infty(\Omega_c)) \cap L^\infty(0,T;\mathbb{W}^{1,\infty}(\Omega_c)), \\ \mathbf{H}_0 \in \mathbf{X}_3, \\ \nabla\cdot(\mu\mathbf{H}_0) = 0. \end{gathered} \tag{2.12}$$

In order to derive the fully regularized weak formulation, we first divide the Faraday's law in (2.1) by μ, multiply the equation by $\nabla\times\mathbf{e}$, $\mathbf{e}\in\mathbf{M}_3$, and integrate over Ω. After integrating the left-hand side by parts and taking into account the boundary condition for $\mathbf{H}$, we obtain

$$\partial_t(\nabla\times\mathbf{H},\mathbf{e})_\Omega + \left(\frac{1}{\mu}\nabla\times\mathbf{E},\nabla\times\mathbf{e}\right)_\Omega = 0.$$

Substituting $\nabla\times\mathbf{H}$ from the second and third equations of (2.1), we obtain

$$\partial_t[(\sigma\mathbf{E},\mathbf{e})_{\Omega_c} + (\sigma\mu\mathbf{u}\times\mathbf{H},\mathbf{e})_{\Omega_c} + (\mathbf{j},\mathbf{e})_{\Omega_c}] + \left(\frac{1}{\mu}\nabla\times\mathbf{E},\nabla\times\mathbf{e}\right)_\Omega = 0.$$

Using once more the equation $\partial_t(\mu\mathbf{H}) = -\nabla \times \mathbf{E}$, we end up with the following residual equation:

$$(\sigma\partial_t\mathbf{E}, \mathbf{e})_{\Omega_c} + \left(\frac{1}{\mu}\nabla\times\mathbf{E}, \nabla\times\mathbf{e}\right)_{\Omega} + (\sigma(\mu\partial_t\mathbf{u}\times\mathbf{H} - \mathbf{u}\times\nabla\times\mathbf{E}), \mathbf{e})_{\Omega_c} = -(\partial_t\mathbf{j}, \mathbf{e})_{\Omega_c}. \quad (2.13)$$

This equation yields a control on $\mathbf{E}$ and $\nabla \times \mathbf{E}$ in $\mathbb{L}^2(\Omega_c)$ and $\mathbb{L}^2(\Omega)$, respectively.

Likewise, to obtain a control on the divergence of $\mathbf{E}$ in the nonconducting medium, we enforce

$$(\nabla\cdot\mathbf{E}, \nabla\cdot\mathbf{e})_{\Omega_v} = 0 \quad (2.14)$$

Hence, by weighting (2.13) and (2.14) with some coefficients $\delta_3 > 0$, $\delta_3' > 0$, and adding the result to $-(\nabla \times \mathbf{H}, \mathbf{e})_{\Omega_v} = 0$, we obtain

$$-(\nabla\times\mathbf{H}, \mathbf{e})_{\Omega_v} + \delta_3'(\nabla\cdot\mathbf{E}, \nabla\cdot\mathbf{e})_{\Omega_v} + \delta_3\Bigg[(\sigma\partial_t\mathbf{E}, \mathbf{e})_{\Omega_c} + \left(\frac{1}{\mu}\nabla\times\mathbf{E}, \nabla\times\mathbf{e}\right)_{\Omega}$$
$$+ (\sigma(\mu\partial_t\mathbf{u}\times\mathbf{H} - \mathbf{u}\times\nabla\times\mathbf{E}), \mathbf{e})_{\Omega_c}\Bigg] = -\delta_3(\partial_t\mathbf{j}, \mathbf{e})_{\Omega_c}. \quad (2.15)$$

At this point it is convenient to define the bilinear forms a_3 and r_3 such that

$$a_3((\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e})) = \left(\frac{1}{\tilde{\sigma}}\nabla\times\mathbf{H}, \nabla\times\mathbf{b}\right)_{\Omega} + (\nabla\cdot(\mu\mathbf{H}), \nabla\cdot(\mu\mathbf{b}))_{\Omega} - (\mu\mathbf{u}\times\mathbf{H}, \nabla\times\mathbf{b})_{\Omega_c}$$
$$+ (\nabla\times\mathbf{b}, \mathbf{E})_{\Omega_v} - (\nabla\times\mathbf{H}, \mathbf{e})_{\Omega_v} + \delta_3'(\nabla\cdot\mathbf{E}, \nabla\cdot\mathbf{e})_{\Omega_v} + \delta_3\left(\frac{1}{\mu}\nabla\times\mathbf{E}, \nabla\times\mathbf{e}\right)_{\Omega}$$
$$+ \delta_3(\sigma(\mu\partial_t\mathbf{u}\times\mathbf{H} - \mathbf{u}\times\nabla\times\mathbf{E}), \mathbf{e})_{\Omega_c}], \qquad \forall\, (\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e}) \in \mathbf{Z}_3$$
$$r_3((\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e})) = (\mu\mathbf{H}, \mathbf{b})_{\Omega} + \delta_3(\sigma\mathbf{E}, \mathbf{e})_{\Omega_c}, \qquad \forall\, (\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e}) \in \mathbf{Y}_3.$$

Furthermore, we adopt the following simplified topological hypotheses on Ω

(Ht4) $\quad \Omega_v$ is simply connected, $\Gamma_c = \emptyset$, and Σ is connected.

The following properties hold.

Lemma 2.5. *Under the hypothesis* (*Ht*4), *there is a constant* $c > 0$ *s.t. for all* $\mathbf{E}$ *in* $\mathbf{M}_3$

$$\|\mathbf{E}\|^2_{\mathbb{H}^{-1/2}(\Sigma)} + \|\nabla\times\mathbf{E}\|^2_{0,\Omega_v} + \|\nabla\cdot\mathbf{E}\|^2_{0,\Omega_v} \geq c\|\mathbf{E}\|^2_{0,\Omega_v}.$$

Proof. We prove this by contradiction. Assume that there is a sequence $(\mathbf{E}_n)_{n\geq 0}$ s.t. $\|\mathbf{E}_n\|_{\Omega_v} = 1$, $\|\mathbf{E}_n\|_{\mathbb{H}^{-1/2}(\Sigma)} \to 0$, $\|\nabla \times \mathbf{E}_n\|_{0,\Omega_v} \to 0$, and $\|\nabla \cdot \mathbf{E}_n\|_{0,\Omega_v} \to 0$. Then, there is a subsequence $\mathbf{E}_{n_k}$ that converges weakly to some $\mathbf{E}_0$ in $\mathbb{L}^2(\Omega_v)$. For this subsequence, we have also $\nabla \times \mathbf{E}_{n_k} \to \nabla \times \mathbf{E}_0$ weakly in $\mathbb{H}^{-1}(\Omega_v)$ and $\nabla \cdot \mathbf{E}_{n_k} \to \nabla \cdot \mathbf{E}_0$ weakly in $H^{-1}(\Omega_v)$. Hence,

$\nabla \times \mathbf{E}_0 = 0$ and $\nabla \cdot \mathbf{E}_0 = 0$. Then it follows that $\mathbf{E}_{n_k} \times \mathbf{n} \to \mathbf{E}_0 \times \mathbf{n}$ weakly in $\mathbb{H}^{-1/2}(\Sigma)$ and $\mathbf{E}_{n_k} \cdot \mathbf{n} \to \mathbf{E}_0 \cdot \mathbf{n}$ weakly in $H^{-1/2}(\Gamma)$. Hence, $\mathbf{E}_0 \times \mathbf{n} = 0$ on Σ and $\mathbf{E}_0 \cdot \mathbf{n} = 0$ on Γ. Since Ω_v is simply connected and Σ is connected we deduce that $\mathbf{E}_0 = 0$, which is a contradiction. ■

Lemma 2.6. *If* (*Ht*4) *holds*, *there are* $\gamma \geq 0$ *and* $\alpha > 0$ *s.t.*

$$\forall(\mathbf{b},\mathbf{e}) \in \mathbf{Z}_3, \qquad a_3((\mathbf{b},\mathbf{e}),(\mathbf{b},\mathbf{e})) + \gamma r_3((\mathbf{b},\mathbf{e}),(\mathbf{b},\mathbf{e})) \geq \alpha \|(\mathbf{b},\mathbf{e})\|^2_{\mathbf{Z}_3}. \tag{2.16}$$

Proof. It is clear that (2.16) holds if there is a constant $c > 0$ s.t. for all $\mathbf{e}$ in $\mathbf{M}_3$,

$$\|\mathbf{e}\|^2_{0,\Omega_c} + \|\nabla \times \mathbf{e}\|^2_{0,\Omega} + \|\nabla \cdot \mathbf{e}\|^2_{0,\Omega_v} \geq c\|\mathbf{e}\|^2_{\mathbf{M}_3}. \tag{2.17}$$

Actually, owing to (Ht4) and Lemma 2.5, we have

$$\|\mathbf{e}\|^2_{0,\Omega_c} + \|\nabla \times \mathbf{e}\|^2_{0,\Omega} + \|\nabla \cdot \mathbf{e}\|^2_{0,\Omega_v} \geq c\|\mathbf{e}\|^2_{\mathbb{H}^{-1/2}(\Sigma)} + \|\nabla \times \mathbf{e}\|^2_{0,\Omega_v} + \|\nabla \cdot \mathbf{e}\|^2_{0,\Omega_v} \geq c\|\mathbf{e}\|^2_{0,\Omega_v}.$$

Then, (2.17) follows readily. ■

Remark 2.4. The hypothesis (Ht4) is a technicality that simplifies the proofs, but we think that it could be removed.

Now we are in the position to state and analyze the fully regularized weak formulation of the problem. It reads

$$\begin{cases} \text{Find } (\mathbf{H},\mathbf{E}) \in W(0,T;\mathbf{Z}_3) \textit{ such that} \\ r_3((\mathbf{H},\mathbf{E})|_{t=0},(\mathbf{b},\mathbf{e})) = r_3\left(\left(\mathbf{H}_0, \frac{1}{\sigma}\nabla \times \mathbf{H}_0\right),(\mathbf{b},\mathbf{e})\right), \qquad \forall\, (\mathbf{b},\mathbf{e}) \in \mathbf{Y}_3 \\ d_t r_3((\mathbf{H},\mathbf{E}),(\mathbf{b},\mathbf{e})) + a_3((\mathbf{H},\mathbf{E}),(\mathbf{b},\mathbf{e})) = \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}\right)_{\Omega_c} - \delta_3(\partial_t \mathbf{j}, \mathbf{e})_{\Omega_c}, \\ \qquad \forall\, (\mathbf{b},\mathbf{e}) \in \mathbf{Z}_3, \text{ in } L^2(0,T). \end{cases} \tag{2.18}$$

Theorem 2.4. *If* (2.12) *and* (*Ht*4) *hold*, *then* (2.18) *is well posed*, *and its solution is also solution to* (2.1) *in the distribution sense*.

Proof. We apply a variant of the Lions' theorem 2.1. It is clear that, owing to Lemma (2.6), the conditions (1), (2), and (3) of Theorem 2.1 hold. The rest of the proof does not pose any particular difficulty. ■

3. FINITE ELEMENT DISCRETIZATION

In this section we study finite element discretizations of the two stabilized formulations introduced above. Throughout this section we denote by $(\mathcal{T}_h)_{h>0}$ a family of regular meshes of Ω.

3.1. The Saddle-point Formulation

Let us introduce $\mathbf{X}_{2,h} \subset \mathbf{X}_2$ and $\mathbf{M}_{2,h} \subset \mathbf{M}_2$ two finite dimensional finite element spaces based on the mesh $\mathcal{T}_h$. We assume that the couple $(\mathbf{X}_{2,h}, \mathbf{M}_{2,h})$ satisfy the discrete version of (2.7) uniformly: There exist $\beta > 0$, $\gamma > 0$ s.t.

$$\forall\ \mathbf{e}_h \in \mathbf{M}_{2,h}, \qquad \sup_{\mathbf{b}_h \in \mathbf{X}_{2,h}} \frac{(\nabla \times \mathbf{b}_h, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}_h\|_{X_1}} \geq \beta \|\mathbf{e}_h\|_{0,\Omega_v} - \gamma \|\nabla \cdot \mathbf{e}_h\|_{0,\Omega_v}. \tag{3.1}$$

For the sake of simplicity, we assume also that the bilinear form a_2 is coercive in $\mathbf{X}_2$, i.e., there exists $\alpha > 0$ s.t.

$$\forall\ \mathbf{b} \in \mathbf{X}_2, \qquad a_2(\mathbf{b}, \mathbf{b}) \geq \alpha \|\mathbf{b}\|_{\mathbf{X}_2}. \tag{3.2}$$

This hypothesis holds if $\mathbf{u}$ is small enough and μ is constant, cf. Remark 2.1.

Furthermore, we assume that these spaces satisfy the following interpolation properties: there exist $c > 0$, $k \geq k' \geq 1$ such that for all $0 \leq r \leq k$, $0 \leq r' < k'$, $\forall \mathbf{b} \in \mathbf{H}^{r+1}(\Omega) \cap \mathbf{X}_2$ and $\mathbf{e} \in \mathbf{H}^{r'+1}(\Omega_v) \cap \mathbf{M}_2$:

$$\begin{cases} \inf\limits_{\mathbf{b}_h \in \mathbf{X}_{1,h}} (\|\mathbf{b} - \mathbf{b}_h\|_{0,\Omega} + h\|\mathbf{b} - \mathbf{b}_h\|_{\mathbf{X}_2}) \leq c h^{r+1} \|\mathbf{b}\|_{r+1,\Omega} \\ \inf\limits_{\mathbf{e}_h \in \mathbf{M}_h} (\|\mathbf{e} - \mathbf{e}_h\|_{0,\Omega_v} + h\|\nabla \cdot (\mathbf{e} - \mathbf{e}_h)\|_{0,\Omega_v}) \leq c h^{r'+1} \|\mathbf{e}\|_{r'+1,\Omega_v}. \end{cases} \tag{3.3}$$

The discrete problem is formulated as a system of ODEs as follows:

$$\begin{cases} \text{Find } \mathbf{H}_h \in \mathscr{C}^1(0, T; \mathbf{X}_{2,h}) \text{ and } \mathbf{E}_h \in \mathscr{C}^0(0, T; \mathbf{M}_{2,h}) \text{ s.t.} \\ \forall\ \mathbf{b}_h \in \mathbf{X}_{2,h},\ \forall\ \mathbf{e}_h \in \mathbf{M}_{2,h} \\ (\mathbf{H}|_{t=0}, \mathbf{b}_h)_\Omega = (\mathbf{H}_0, \mathbf{b}_h)_\Omega, \\ (\mu \partial_t \mathbf{H}_h, \mathbf{b}_h)_\Omega + a_2(\mathbf{H}_h, \mathbf{b}_h) + (\mathbf{E}_h, \nabla \times \mathbf{b}_h)_{\Omega_v} = \left(\frac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}_h\right)_{\Omega_c}, \\ (\nabla \times \mathbf{H}_h, \mathbf{e}_h)_{\Omega_v} - \delta_2(h) a_4(\mathbf{E}_h, \mathbf{e}_h) = 0, \end{cases} \tag{3.4}$$

where $\delta_2(h) \sim h^{k-k'}$, as this choice gives the best approximation results according to Lemma 3.1.

Theorem 3.1. *If* $\delta_2 \sim h^{k-k'}$, *then under the hypotheses of Theorem* 2.3 *together with the hypotheses* (3.1), (3.2), (3.3), *and provided that* $\mathbf{H}$ *and* $\mathbf{E}$ *and smooth enough in time and space we have*

$$\sup_{0 \leq t \leq T} \|\mathbf{H} - \mathbf{H}_h\|_{0,\Omega} + \left[\int_0^T \|\mathbf{H} - \mathbf{H}_h\|_{\mathbf{X}_2}^2\right]^{1/2} \leq c(T, \mathbf{H}, \mathbf{E}) h^{(k+k')/2}, \left[\int_0^T \|\mathbf{E} - \mathbf{E}_h\|_{\mathbf{M}_2}^2\right]^{1/2} \leq c(T, \mathbf{H}, \mathbf{E}) h^{k'}.$$

Proof. The proof of the result is a standard exercise and is based on the existence on particular interpolates of $\mathbf{H}$ and $\mathbf{E}$ as stated in Lemma 3.1. ■

Lemma 3.1. *Under the hypotheses of Theorem* (3.1), *the solution of the problem*

$$\begin{cases} \text{Find } I_h\mathbf{H} \text{ in } \mathbf{X}_{2,h} \text{ and } J_h\mathbf{E} \text{ in } \mathbf{M}_{2,h}, \text{ s.t. } \forall\ \mathbf{b}_h \in \mathbf{X}_{1,h} \text{ and } \forall\ \mathbf{e}_h \in \mathbf{M}_{2,h} \\ a_2(I_h\mathbf{H}, \mathbf{b}_h) + (J_h\mathbf{E}, \nabla \times \mathbf{b}_h)_{\Omega_v} = a_2(\mathbf{H}, \mathbf{b}_h) + (\mathbf{E}, \nabla \times \mathbf{b}_h)_{\Omega_v}, \\ (-\nabla \times (I_h\mathbf{H}), \mathbf{e}_h)_{\Omega_v} + \delta_2(h)(\nabla \cdot (J_h\mathbf{E}), \nabla \cdot \mathbf{e}_h)_{\Omega_v} \\ \qquad = (-\nabla \times \mathbf{H}, \mathbf{e}_h)_{\Omega_v} + \delta_2(h)(\nabla \cdot \mathbf{E}, \nabla \cdot \mathbf{e}_h)_{\Omega_v}, \end{cases}$$

is such that

$$\|\mathbf{H} - I_h\mathbf{H}\|_{\mathbf{X}_2} + \delta_2(h)^{1/2}\|\mathbf{E} - J_h\mathbf{E}\|_{\mathbf{M}_2} \leq c(h^k\delta_2(h)^{-1/2} + h^{k'}\delta_2(h)^{1/2}).$$

Remark 3.1. Note that the error estimates stated in Theorem 3.1 are not optimal. It is likely that the bounds we derived are not sharp, since the numerical tests reported in §4 show that, for $\mathbb{P}_2/\mathbb{P}_1$ finite elements, numerical error estimates are optimal when choosing $\delta_2(h) = h^{2-1} = h$.

Remark 3.2. If we assume that there exists an interpolation operator $C_h \in \mathscr{L}(\mathbf{X}_2; \mathbf{X}_{2,h})$ so that for all $\mathbf{b}$ in $\mathbf{X}_2$

$$\forall\ \mathbf{e}_h \in \mathbf{M}_{2,h} \int_\Sigma ((\mathbf{b} - C_h\mathbf{b}) \times \mathbf{n}) \cdot \mathbf{e}_h = 0$$

$$\|\mathbf{b} - C_h\mathbf{b}\|_{0,\Omega} + h\|\mathbf{b} - C_h\mathbf{b}\|_{\mathbf{X}_2} \leq ch\|\mathbf{b}\|_{\mathbf{X}_2}. \tag{3.5}$$

then, by proceeding as in [1], it is possible to prove that $\mathbb{P}_1$-Bubble/$\mathbb{P}_1$ mixed finite elements satisfy the inf-sup condition (3.1). We also expect in this case that $\mathbb{P}_2/\mathbb{P}_1$ mixed finite elements satisfy the inf-sup condition, but we have not been able to prove it yet.

3.2. The Elliptic-Parabolic Formulation

Let $\mathbf{X}_{3,h} \subset \mathbf{X}_3$ and $\mathbf{M}_{3,h} \subset \mathbf{M}_3$ be finite element spaces based on the mesh $\mathscr{T}_h$ and satisfying the following interpolation properties: there exist $c > 0$ and $k > 1$ such that for all $0 \leq r, r' \leq k$, $\forall \mathbf{b} \in \mathbf{H}^{r+1}(\Omega) \cap \mathbf{X}_3$ and $\mathbf{e} \in \mathbf{H}^{r'+1}(\Omega_v) \cap \mathbf{M}_3$:

$$\begin{cases} \inf\limits_{\mathbf{b}_h \in \mathbf{X}_{1,h}} (\|\mathbf{b} - \mathbf{b}_h\|_{0,\Omega} + h\|\mathbf{b} - \mathbf{b}_h\|_{\mathbf{X}_3}) \leq ch^{r+1}\|\mathbf{b}\|_{r+1,\Omega} \\ \inf\limits_{\mathbf{e}_h \in \mathbf{M}_h} (\|\mathbf{e} - \mathbf{e}_h\|_{0,\Omega_v} + h\|\nabla \cdot (\mathbf{e} - \mathbf{e}_h)\|_{\mathbf{M}_3}) \leq ch^{r'+1}\|\mathbf{e}\|_{r'+1,\Omega_v}. \end{cases} \tag{3.6}$$

Note that standard equal order continuous Lagrange finite element spaces are suitable, for no particular *inf-sup* condition must be satisfied.

Setting $\mathbf{Z}_{3,h} = \mathbf{X}_{3,h} \times \mathbf{M}_{3,h}$, we build an approximate solution to (2.18) by solving the following problem:

$$\begin{cases} \text{Find } (\mathbf{H}_h, \mathbf{E}_h) \in L^2(0, T; \mathbf{Z}_{3,h}) \text{ such that for all } (\mathbf{b}_h, \mathbf{e}_h) \text{ in } \mathbf{Z}_{3,h}, \\ r_3((\mathbf{H}_h, \mathbf{E}_h)|_{t=0}, (\mathbf{b}_h, \mathbf{e}_h)) = r_3\left(\left(H_0, \dfrac{1}{\sigma}\nabla \times \mathbf{H}_0\right), (\mathbf{b}_h, \mathbf{e}_h)\right), \\ d_t r_3((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) + a_3((\mathbf{H}_h, \mathbf{E}_h), (\mathbf{b}_h, \mathbf{e}_h)) = \left(\dfrac{1}{\sigma}\mathbf{j}, \nabla \times \mathbf{b}_h\right)_{\Omega_c} - (\partial_t\mathbf{j}, \mathbf{e}_h)_{\Omega_c}. \end{cases} \tag{3.7}$$

Theorem 3.2. *If* $\delta_3 \sim \delta_3' \sim 1$, *then under the hypotheses of Theorem* 2.4, *the hypotheses* (3.6), *and provided that* $\mathbf{H}$ *and* $\mathbf{E}$ *are smooth enough in time and space*, *the solution to* (3.7) *satisfies the following bounds*:

$$\sup_{0\le t\le T} \|\mathbf{H}-\mathbf{H}_h\|_{0,\Omega} + \left[\int_0^T \|\mathbf{H}-\mathbf{H}_h\|^2_{\mathbf{X}_3} + \|\mathbf{E}-\mathbf{e}_h\|^2_{\mathbf{M}_3}\right]^{1/2} \le c(T,\mathbf{H},\mathbf{E})h^k.$$

Proof. When $\delta_3 \sim \delta_3' \sim 1$, the proof of the result is standard since the coercivity constant in (2.16) is uniform in h. ■

Remark 3.3. Note that the convergence estimates in the $\mathbb{L}^2$-norm can be improved to $\mathcal{O}(h^{k+1})$ if a duality argument can be applied, i.e., if the bilinear form a_3 is fully coercive in $\mathbf{Z}_3$, and the following dual problem:

$$\begin{cases}\text{For a given pair } (\mathbf{b}'', \mathbf{e}'') \in \mathbf{Y}_3, \text{ find } (\mathbf{b},\mathbf{e}) \in \mathbf{Z}_3 \text{ s.t.} \\ a_3((\mathbf{b}',\mathbf{e}'),(\mathbf{b},\mathbf{e})) = r_3((\mathbf{b}'',\mathbf{e}''),(\mathbf{b}',\mathbf{e}')),\ \forall\,(\mathbf{b}',\mathbf{e}') \in \mathbf{Z}_3,\end{cases} \tag{3.8}$$

yields enough regularity on the couple $(\mathbf{b}, \mathbf{e})$. Actually, full coercivity in $\mathbf{Z}_3$ can be guaranteed if $\mathbf{u}$ is small enough and the divergence of $\mathbf{E}$ can be controlled in the entire domain. In general, such a control should not be expected since $\mathbf{E}\cdot\mathbf{n}|_\Sigma$ can be discontinuous. But, if one knows *a priori* that $\mathbf{E}$ is smooth throughout the domain, then it is possible to enforce the necessary extra stability by replacing a_3 in (3.7) by

$$\tilde{a}_3((\mathbf{H},\mathbf{E}),(\mathbf{b},\mathbf{e})) = a_3((\mathbf{H},\mathbf{E}),(\mathbf{b},\mathbf{e})) + (\nabla\cdot(\sigma\mathbf{E} - \sigma\mu\mathbf{u}\times\mathbf{H}), \nabla\cdot(\sigma\mathbf{e}))_{\Omega_c} \tag{3.9}$$

the right-hand side in (3.7) being replaced by

$$\left(\frac{1}{\sigma}\mathbf{j}, \nabla\times\mathbf{b}\right)_{\Omega_c} - (\nabla\cdot\mathbf{j}, \nabla\cdot(\sigma\mathbf{e}))_{\Omega_c} - (\partial_t\mathbf{j}, \mathbf{e})_{\Omega_c}.$$

Numerical convergence tests reported in §4 show that these extra terms do improve the convergence on $\mathbf{E}$ in the $\mathbb{L}^2$-norm when $\mathbf{E}$ is smooth.

Let us now assume that there exists an interpolation operator $C_h \in \mathcal{L}(\mathbf{X}_3; \mathbf{X}_{3,h})$ so that for all $\mathbf{b}$ in $\mathbf{X}_3$

$$\forall\ \mathbf{e}_h \in \mathbf{M}_{3,h} \qquad \int_\Sigma ((\mathbf{b} - C_h\mathbf{b})\times\mathbf{n})\cdot\mathbf{e}_h = 0$$

$$\|\mathbf{b} - C_h\mathbf{b}\|_{0,\Omega} + h\|\mathbf{b} - C_h\mathbf{b}\|_{\mathbf{X}_3} \le ch\|\mathbf{b}\|_{\mathbf{X}_3}. \tag{3.10}$$

Lemma 3.2. *If* $1 \gg \delta_3 \gtrsim h^2$, *then under the hypotheses of Lemma* 2.1 *and* (3.10), *there exists* $\beta > 0$ *s.t. for all* $\mathbf{e}_h$ *in* $\mathbf{M}_{3,h}$:

$$\sup_{\mathbf{b}_h\in\mathbf{X}_{3,h}} \frac{(\nabla\times\mathbf{b}_h, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}_h\|_{\mathbf{X}_3}} \ge \beta\|\mathbf{e}_h\|_{0,\Omega_v} - c_1\|\nabla\cdot\mathbf{e}_h\|_{0,\Omega_v} - c_2h\|\nabla\times\mathbf{e}_h\|_{0,\Omega_v}.$$

Proof. Using the properties of the interpolate C_h defined in (3.10), we infer

$$\sup_{\mathbf{b}_h\in\mathbf{X}_{3,h}} \frac{(\nabla\times\mathbf{b}_h, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}_h\|_{\mathbf{X}_3}} \geq \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(\nabla\times C_h\mathbf{b}, \mathbf{e}_h)_{\Omega_v}}{\|C_h\mathbf{b}\|_{\mathbf{X}_3}} \geq c \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(\nabla\times C_h\mathbf{b}, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}}$$
$$\geq c \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(\nabla\times\mathbf{b}, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}} - c \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(\nabla\times(C_h\mathbf{b}-\mathbf{b}), \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}}$$
$$\geq c \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(\nabla\times\mathbf{b}, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}} - c \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(C_h\mathbf{b}-\mathbf{b}, \nabla\times\mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}}.$$

Then, since $\mathbf{X}_2 = \mathbf{X}_3$ and $\mathbf{M}_3 \subset \mathbf{M}_2$, from Lemma 2.2 we deduce

$$\sup_{\mathbf{b}_h\in\mathbf{X}_{3,h}} \frac{(\nabla\times\mathbf{b}_h, \mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}_h\|_{\mathbf{X}_3}} \geq \beta\|\mathbf{e}_h\|_{0,\Omega_v} - c_1\|\nabla\cdot\mathbf{e}_h\|_{0,\Omega_v} - c_2 \sup_{\mathbf{b}\in\mathbf{X}_3} \frac{(C_h\mathbf{b}-\mathbf{b}, \nabla\times\mathbf{e}_h)_{\Omega_v}}{\|\mathbf{b}\|_{\mathbf{X}_3}}.$$

The desired result follows easily from the interpolation properties of the operator C_h. Note that the possibility to integrate by parts in $(\nabla \times (C_h\mathbf{b} - \mathbf{b}), \mathbf{e}_h)_{\Omega_v}$ is a key feature of C_h. ∎

Now we introduce interpolation operators for $\mathbf{H}$ and $\mathbf{E}$ that will be useful in the sequel. Let us denote by a_3^0 the bilinear form a_3 with $\mathbf{u} = 0$ and let us define the bilinear form:

$$d((\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e})) = (\mathbf{H}, \mathbf{b})_{0,\Omega} + a_3^0((\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e})), \qquad \forall\, (\mathbf{H}, \mathbf{E}), (\mathbf{b}, \mathbf{e}) \in \mathbf{Z}_3.$$

Lemma 3.3. *Under the hypotheses of Lemma* 3.2 *and* (3.6), *if* $\delta_3' \sim 1$, *then the solution to the discrete problem*,

$$\begin{cases} \text{For } (\mathbf{H}, \mathbf{E}) \text{ in } \mathbf{Z}_3, \text{ find } (I_h\mathbf{H}, J_h\mathbf{E}) \text{ in } \mathbf{Z}_{3,h} \text{ s.t.} \\ d((I_h\mathbf{H}, J_h\mathbf{E}), (\mathbf{b}_h, \mathbf{e}_h)) = d((\mathbf{H}, \mathbf{E}), (\mathbf{b}_h, \mathbf{e}_h)) \qquad \forall\, (\mathbf{b}_h, \mathbf{e}_h) \in \mathbf{Z}_{3,h}, \end{cases}$$

satisfies the following bounds

$$\|\mathbf{H} - I_h\mathbf{H}\|_{\mathbf{X}_3} + \|\mathbf{E} - J_h\mathbf{E}\|_{0,\Omega_v} + \|\nabla\cdot(\mathbf{E} - J_h\mathbf{E})\|_{0,\Omega_v} + \delta_3^{1/2}\|\nabla\times(\mathbf{E} - J_h\mathbf{E})\|_{0,\Omega_v} \leq c(\mathbf{H}, \mathbf{E})h^k.$$

Proof. Let us pick any couple $(\mathbf{b}_h', \mathbf{e}_h')$ in $\mathbf{Z}_{3,h}$ and denote

$$\boldsymbol{\varepsilon} = I_h\mathbf{H} - \mathbf{b}_h', \qquad \boldsymbol{\epsilon} = \mathbf{H} - \mathbf{b}_h'$$
$$\boldsymbol{\varphi} = I_h\mathbf{E} - \mathbf{e}_h', \qquad \phi = \mathbf{E} - \mathbf{e}_h'.$$

We have the Galerkin orthogonality:

$$d((\boldsymbol{\varepsilon}, \boldsymbol{\varphi}), (\mathbf{b}_h, \mathbf{e}_h)) = d((\boldsymbol{\epsilon}, \phi), (\mathbf{b}_h, \mathbf{e}_h)), \qquad \forall\, (\mathbf{b}_h, \mathbf{e}_h) \in \mathbf{Z}_{3,h}.$$

Using $(\boldsymbol{\varepsilon}, \boldsymbol{\varphi})$ as a test function, we obtain

$$\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}^2 + \|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \delta_3\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}^2$$
$$\leq c(\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}\|\boldsymbol{\epsilon}\|_{\mathbf{X}_3} + \|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}\|\phi\|_{0,\Omega_v} + \|\boldsymbol{\epsilon}\|_{\mathbf{X}_3}\|\boldsymbol{\varphi}\|_{0,\Omega_v} + \|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v}\|\nabla\cdot\phi\|_{0,\Omega_v} + \delta_3\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}\|\nabla\times\phi\|_{0,\Omega_v}),$$

which gives

$$\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}^2 + \|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \delta_3\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}^2 \\ \leq c_1(\|\boldsymbol{\epsilon}\|_{\mathbf{X}_3}^2 + \|\phi\|_{0,\Omega_v}^2 + \|\nabla\cdot\phi\|_{0,\Omega_v}^2 + \delta_3\|\nabla\times\phi\|_{0,\Omega_v}^2) + c_2\|\boldsymbol{\epsilon}\|_{\mathbf{X}_3}\|\boldsymbol{\varphi}\|_{0,\Omega_v}. \quad (3.11)$$

To control the term $\|\boldsymbol{\varphi}\|_{0,\Omega_v}$ we use Lemma 3.2 as follows. By using $(\mathbf{b}_h, 0)$ as test function in the Galerkin orthogonality relation and taking the supremum, we obtain

$$\sup_{\mathbf{b}_h\in\mathbf{X}_{3,h}} \frac{(\nabla\times\mathbf{b}_h, \boldsymbol{\varphi})_{0,\Omega_v}}{\|\mathbf{b}_h\|_{\mathbf{X}_3}} \leq c(\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3} + \|\boldsymbol{\epsilon}\|_{\mathbf{X}_3} + \|\phi\|_{0,\Omega_v}).$$

As a result, we deduce the following bound:

$$\beta\|\boldsymbol{\varphi}\|_{0,\Omega_v} \leq \sup_{\mathbf{b}_h\in\mathbf{X}_{3,h}} \frac{(\nabla\times\mathbf{b}_h, \boldsymbol{\varphi})_{0,\Omega_v}}{\|\mathbf{b}_h\|_{\mathbf{X}_3}} + c(\|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v} + \delta_3^{1/2}\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}) \\ \leq c(\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3} + \|\boldsymbol{\epsilon}\|_{\mathbf{X}_3} + \|\phi\|_{0,\Omega_v} + \|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v} + \delta_3^{1/2}\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}).$$

Using this bound into (3.11), we obtain

$$\|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}^2 + \|\boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \|\nabla\cdot\boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \delta_3\|\nabla\times\boldsymbol{\varphi}\|_{0,\Omega_v}^2 \leq c(\|\boldsymbol{\epsilon}\|_{\mathbf{X}_3}^2 + \|\phi\|_{0,\Omega_v}^2 + \|\nabla\cdot\phi\|_{0,\Omega_v}^2 + \delta_3\|\nabla\times\phi\|_{0,\Omega_v}^2).$$

The desired result is obtained by taking the infimum on $(\mathbf{b}', \mathbf{e}'_h)$ in $\mathbf{Z}_{3,h}$. ∎

Remark 3.4. Note that we can take $d = a_3^0$ if a_3^0 is $\mathbf{Z}_3$-coercive.

We now state the main result of this section.

Theorem 3.3. *If* $1 \gg \delta_3 \gtrsim h^2$ *and* $\delta_3' \sim 1$, *then under the hypotheses of Theorem* 2.4, *the hypotheses* (3.6), (3.10), *and provided that* **H** *and* **E** *are smooth enough in time and space*, *the solution to* (3.7) *satisfies the following bounds*:

$$\sup_{0\leq t\leq T}\|\mathbf{H}-\mathbf{H}_h\|_{0,\Omega} + \left[\int_0^T \|\mathbf{H}-\mathbf{H}_h\|_{\mathbf{X}_3}^2 + \|\mathbf{E}-\mathbf{E}_h\|_{0,\Omega_v}^2 + \|\nabla\cdot(\mathbf{E}-\mathbf{e}_h)\|_{0,\Omega_v}^2\right]^{1/2} \leq c(T,\mathbf{H},\mathbf{E})h^k.$$

Proof. Let $I_h\mathbf{H}$ and $J_h\mathbf{E}$ be the interpolates for **H** and **E** defined in Lemma 3.3 and let us denote

$$\boldsymbol{\varepsilon} = \mathbf{H}_h - I_h\mathbf{H}, \qquad \boldsymbol{\epsilon} = \mathbf{H} - I_h\mathbf{H}$$
$$\boldsymbol{\varphi} = \mathbf{E}_h - I_h\mathbf{E}, \qquad \phi = \mathbf{E} - I_h\mathbf{E}.$$

Owing to the definition of the interpolates, we have $a_3^0((\boldsymbol{\epsilon}, \phi), (\mathbf{b}_h, \mathbf{e}_h)) = -(\boldsymbol{\epsilon}, \mathbf{b}_h)_{0,\Omega}$, where we recall that a_3^0 is the bilinear form a_3 with $\mathbf{u} = 0$. As a result, the Galerkin orthogonality takes the following form:

$$\begin{cases} r_3((\boldsymbol{\varepsilon}, \boldsymbol{\varphi})|_{t=0}, (\mathbf{b}_h, \mathbf{e}_h)) = r_3\left(\left(\boldsymbol{\epsilon}_0, \dfrac{1}{\sigma}\nabla \times \boldsymbol{\epsilon}_0\right), (\mathbf{b}_h, \mathbf{e}_h)\right), \\ d_t r_3((\boldsymbol{\varepsilon}, \boldsymbol{\varphi}), (\mathbf{b}_h, \mathbf{e}_h)) + a_3((\boldsymbol{\varepsilon}, \boldsymbol{\varphi}), (\mathbf{b}_h, \mathbf{e}_h)) \\ \qquad = d_t r_3((\boldsymbol{\epsilon}, \boldsymbol{\phi}), (\mathbf{b}_h, \mathbf{e}_h)) - (\mu \mathbf{u} \times \boldsymbol{\epsilon}, \nabla \times \mathbf{b}_h)_{\Omega_c} \\ \qquad + \delta_3[(\sigma(\mu \partial_t \mathbf{u} \times \boldsymbol{\epsilon} - \mathbf{u} \times \nabla \times \phi), \mathbf{e}_h)_{\Omega_c}] - (\boldsymbol{\epsilon}, \mathbf{b}_h)_{0,\Omega}. \end{cases}$$

Using $(\boldsymbol{\varepsilon}, \boldsymbol{\varphi})$ as test function and applying the Gronwall lemma, we obtain

$$\sup_{0 \le t \le T} \|\boldsymbol{\varepsilon}\|_{0,\Omega} + \left[\int_0^T \|\boldsymbol{\varepsilon}\|_{\mathbf{X}_3}^2 + \|\nabla \cdot \boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \delta_3 \|\nabla \times \boldsymbol{\varphi}\|_{0,\Omega_v}^2\right]^{1/2} \le ch^k.$$

Then, the desired estimates on $\mathbf{H} - \mathbf{H}_h$ are obtained by applying the triangle inequality.

To obtain the $\mathbb{L}^2$-estimate on $\mathbf{E} - \mathbf{E}_h$, we reproduce the argument above for time derivatives:

$$\sup_{0 \le t \le T} \|d_t \boldsymbol{\varepsilon}\|_{0,\Omega} \le ch^k.$$

Then, applying Lemma 3.2 to the equation

$$(\nabla \times \mathbf{b}_h, \boldsymbol{\varphi})_{0,\Omega_v} = -(d_t \boldsymbol{\varepsilon}, \mathbf{b}_h)_{0,\Omega} - a_2(\boldsymbol{\varepsilon}, \mathbf{b}_h) + (d_t \boldsymbol{\epsilon}, \mathbf{b}_h)_{0,\Omega} - (\mu \mathbf{u} \times \boldsymbol{\epsilon}, \nabla \times \mathbf{b}_h)_{\Omega_c} - (\boldsymbol{\epsilon}, \mathbf{b}_h)_{0,\Omega}$$

and using the estimates on $\|\nabla \cdot \boldsymbol{\varphi}\|_{0,\Omega_v}^2 + \delta_3 \|\nabla \times \boldsymbol{\varphi}\|_{0,\Omega_v}^2$ already obtained, we infer

$$\int_0^T \|\boldsymbol{\varphi}\|_{0,\Omega_v}^2 \le ch^k.$$

Applying the triangular inequality yields the desired result. ■

Remark 3.5. Numerical tests reported in §4 show that the method works also properly if $\delta_3 \sim \delta_3' \sim h$. Though, this has yet to be proven.

3.3. Lagrange FE vs. Edge Elements

Though the goal of the present article is to show that it is indeed possible to solve the set of Equations (2.1) by means of the standard continuous Lagrange finite elements, provided the problem is set in an adequate weak form, the reader must bear in mind that there are limitations to this approach, which may sometimes be severe. First, it is not natural to enforce the continuity of $\mathbf{H} \cdot \mathbf{n}$ across the interface Σ, for this quantity may be discontinuous if μ is discontinuous. If μ is discontinuous, the term $(\nabla \cdot (\mu \mathbf{H}), \nabla \cdot (\mu \mathbf{b}))_\Omega$ can be dropped from the bilinear forms a_2 and a_3 without modifying essentially their properties. In the same spirit, the bilinear form $\tilde{a}_3$ defined in (3.9) should not be used in general, unless $\mathbf{E}$ is known *a priori* to be smooth. Second, the most important limitation is that if $\mathbf{H}$ has no more regularity in space than that of $\mathbf{X}_1$ or $\mathbf{X}_2$, then it may happen that the sequence $(\mathbf{H}_h)_{h>0}$ does not converge to $\mathbf{H}$. The reason for this being that $\mathbf{X}_2 \cap \mathbb{H}^1(\Omega)$ is a closed subspace of $\mathbf{X}_2$ with a supplementary that is not zero if both μ and Σ are simultaneously not smooth or Ω is not convex, as shown by Costabel [10]. As a result,

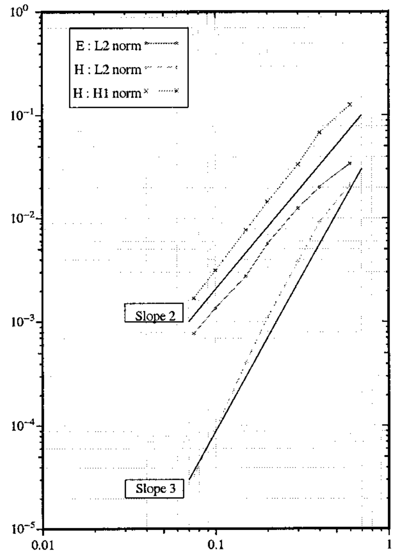

FIG. 2. Convergence analysis. First stabilized formulation with $\mathbb{P}_2/\mathbb{P}_1$ elements and $\delta_2 = h$.

$\mathbf{X}_2 \cap \mathbb{H}^1(\Omega)$ is not dense in $\mathbf{X}_2$, which means that there are elements in $\mathbf{X}_2$ that cannot be approximated by continuous Lagrange finite element functions, these functions being in $\mathbf{X}_2 \cap \mathbb{H}^1(\Omega)$. Note, however, that in the astrophysical context, either μ or Σ is smooth and Ω can easily be chosen to be convex, so that convergence is guaranteed even in the minimal regularity situation. Of course, none of the limitations mentioned above apply if edge finite element are used.

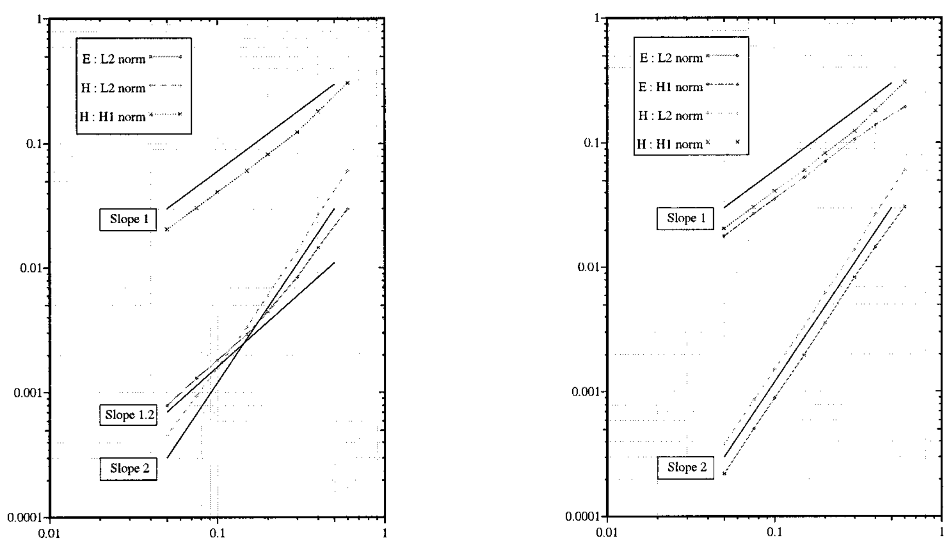

FIG. 3. Convergence analysis. Second stabilized formulation with $\mathbb{P}_1/\mathbb{P}_1$ elements and $\delta_3' = \delta_3 = 1$; left graph, formulation using a_3; right graph, formulation using $\tilde{a}_3$.

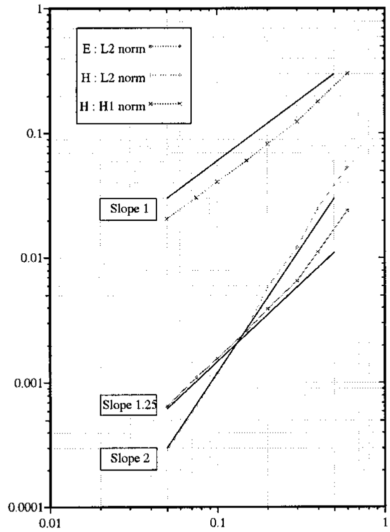

FIG. 4. Convergence analysis. Second stabilized formulation using the bilinear form $\tilde{a}_3$ with $\mathbb{P}_1$ elements and $\delta_3' = \delta_3 = h$.

4. NUMERICAL RESULTS

In this section we illustrate the numerical performances of the two stabilized methods introduced above.

4.1. Convergence Tests

We test the convergence in space by using an analytic solution to (2.1). The tests are performed in 2D, the magnetic field being normal to the plane and the electric field being in the plane. This situation retains the complexity of the 3D one for $\nabla \cdot \mathbf{E}$ is not identically zero.

We set $\Omega =]-2, +2[^2$ and $\Omega_c = \{(x, y) \in \mathbb{R}^2, \sqrt{x^2 + y^2} < 1\}$. Note that since Ω is convex, the norms of $\mathbf{X}_2$ and $\mathbb{H}^1(\Omega)$ are equivalent; hence, it is legitimate to use the $\mathbb{H}^1$-norm to measure the error. Taking $\sigma = 1$, $\mu = 1$, we choose

$$\begin{cases} \mathbf{H} = [\gamma(t) + \phi(x, y)]\mathbf{e}_z \\ E_x = \dot{\gamma}(t)(\cos(x)e^y + x + y), \\ E_y = \dot{\gamma}(t)(\sin(x)e^y - y), \end{cases} \quad \text{where} \begin{cases} \gamma(t) = t^2, \\ \phi(x, y) = 1_{\Omega_c}[1 - (x^2 + y^2)]^3. \end{cases} \tag{4.1}$$

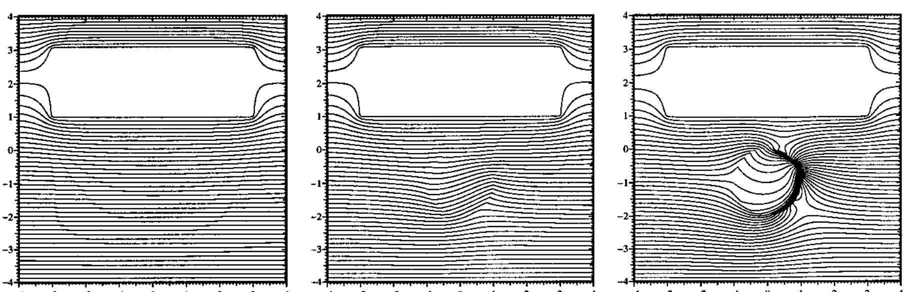

FIG. 5. Magnetic field at $t = 5$ for $\omega = 0, 1, 10$.

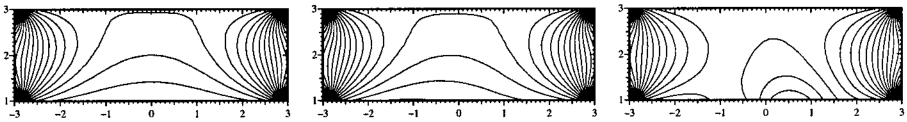

FIG. 6. Electric field in the non-conducting domain at $t = 5$ for $\omega = 0, 1, 10$.

Taking $\mathbf{u} = 0$, the corresponding source term is

$$\mathbf{j} = \begin{cases} (-6y[1 - (x^2 + y^2)]^2 - \sigma E_x, 6x[1 - (x^2 + y^2)]^2 - \sigma E_y)^T & \text{in } \Omega_c \\ 0 & \text{in } \Omega_v \end{cases}$$

For the first stabilized formulation, we use $\mathbb{P}_2/\mathbb{P}_1$ finite elements with $\delta_2(h) = h$ to approximate **H** and **E**. For the second stabilized formulation we use $\mathbb{P}_1$ finite elements for both **H** and **E**. To march in time we use the backward differencing time stepping of second order (BDF2). Owing to the special choice we made of $\gamma(t)$, the integration in time is exact; hence, the approximation error is composed of the error in space only.

The convergence results are shown in Figures 2 and 3. On the two graphs we show the maximum in time of the error on **E** in the $\mathbb{L}^2(\Omega_v)$-norm and that on **H** in the $\mathbb{L}^2(\Omega)$-norm and the $\mathbb{H}^1(\Omega)$-norm as functions of h. The errors are evaluated by performing highly accurate Gaussian integrations. The two series of tests are performed on the same grids.

The convergence results for the first method using $\mathbb{P}_2/\mathbb{P}_1$ approximation and $\delta_2 = h$ are shown in Figure 2. Both the magnetic field and the electric field converge with optimal rates.

The convergence results for the second method using $\mathbb{P}_1/\mathbb{P}_1$ interpolation with $\delta_3 = \delta_3' = 1$ are shown on the left of Figure 3. The method is optimal for the magnetic field but is suboptimal for the electric field in the $\mathbb{L}^2$-norm. Since in this test **E** is known to be smooth, we can recover optimal convergence by replacing a_3 by $\tilde{a}_3$, the right-hand side being modified accordingly, as explained in Remark 3.3. The results are shown on the right of Figure 3. The convergence on the electric field is $\mathbb{O}(h^2)$ in the $\mathbb{L}^2$-norm and $\mathbb{O}(h)$ in the $\mathbb{H}^1(\Omega)$-norm.

We finish this section by showing convergence results for the second formulation still using $\mathbb{P}_1/\mathbb{P}_1$ interpolation and the bilinear form a_3 but with $\delta_3' = \delta_3 = h$. The results are shown in Figure 4. The method seems to perform slightly better than for the choice $\delta_3' = 1 = \delta_3$.

4.2. A Numerical Illustration

We illustrate now the performance of the two methods described in this article by testing them on a model 2D problem. The domain consist of square $\Omega =]-4, +4[^2$. The nonconducting domain is the rectangle $\Omega_v =]-3, +3[\times] + 1, +3[$. The disk $\{(x,); x^2 + (y + 2)^2 \leq 1\}$ rotates with a constant angular velocity ω so that $\mathbf{u} = \omega\mathbf{k} \times (\mathbf{r} + \mathbf{e}_y)$. We show in Figure 5 the magnetic field at time $t = 5$ for $\omega = 0, 1, 10$. Note that the magnetic field is constant in the nonconducting domain. We show the x-component of the electric field in Ω_c in Figure 6. Note the deformations of the electric field as the rotation speed of the disk increases.

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