L^1-APPROXIMATION OF STATIONARY HAMILTON–JACOBI EQUATIONS*

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Abstract. We describe a nonlinear finite element technique to approximate the solutions of stationary Hamilton–Jacobi equations in two space dimensions using continuous finite elements of arbitrary degree. The method consists of minimizing a functional containing the L^1-norm of the Hamiltonian plus a discrete entropy. It is shown that the approximate sequence converges to the unique viscosity solution under appropriate hypotheses on the Hamiltonian and the mesh family.

Key words. finite elements, best L^1-approximation, viscosity solution, Hamilton–Jacobi equations

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1. Introduction.

1.1. Formulation of the problem. Let Ω be an open, bounded, Lipschitz, and connected domain in $\mathbb{R}^2$. We consider the stationary Hamilton–Jacobi equation

$$(1.1) \qquad H(x, u, Du) = 0, \quad \text{a.e. } x \text{ in } \Omega, \qquad u_{|\Gamma} = 0,$$

where Du denotes the gradient of u. We restrict ourselves to homogeneous boundary conditions to simplify the analysis. Nonhomogeneous boundary conditions can be accounted for by introducing appropriate continuous liftings provided the boundary data are compatible with our solution class; see (1.4)–(1.6).

The problem (1.1) has been extensively studied and is known to be particularly challenging in regard to the question of uniqueness. It turns out that adding a vanishing viscosity to the equation and passing to the limit usually leads to unique solutions under appropriate assumptions on the structure of the Hamiltonian H. We refer to Evans [10] for an introduction to this topic. Crandall and Lions [8] thoroughly characterized limit solutions by using the maximum principle and introducing the notions of subsolution and supersolution. They showed that the solution obtained by the vanishing viscosity limit is a subsolution and a supersolution. We refer to Barles [4] for additional details on this technique. When H is convex with respect to the gradient, Kružkov [16] characterized the limit solution by proving that second finite differences satisfy a one-sided bound. This criteria has been significantly weakened by Lions and Souganidis [18]. It is the one-sided bound characterization of Lions and Souganidis that will be used in the present paper; see hypothesis (5.5).

The literature on the approximation of Hamilton–Jacobi equations is abundant; we refer to Sethian [19] for a thorough review. Most successful algorithms are based

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on monotonicity and Lax–Friedrichs approximate Hamiltonians; see, e.g., Kao, Osher, and Tsai [15]. Monotonicity is at the core of most convergence proofs for low-order approximations; see, e.g., Crandall and Lions [9], Barles and Souganidis [5], and Abgrall [1, 2]. For higher-order approximations, limiters are typically used and monotonicity cannot be preserved. Convergence results are difficult to obtain. For instance, it is shown in [18] that MUSCL-like finite difference approximations converge to viscosity solutions. In the present paper we take a radically different point of view by formulating the discrete problem as a minimization in $L^1(\Omega)$. The motivation behind this approach is based on observations made in [11] that L^1-minimization is capable of selecting viscosity solutions of transport equations equipped with ill-posed boundary conditions. This fact has indeed been proved in [13] in one space dimension. Numerical computations in [12] confirm that this is also the case for stationary one-dimensional Hamilton–Jacobi equations. Moreover, results from Lin and Tadmor [17] show that the L^1-metric is appropriate for deriving error estimates for time-dependent Hamilton–Jacobi equations. This encouraged us to develop a research program in this direction, and the purpose of this paper is to report that indeed L^1-minimization is a viable technique.

In the present paper we describe a nonlinear finite element technique for approximating viscosity solutions to (1.1) in two space dimensions using continuous finite elements of arbitrary degree. The method is based on the minimization over the finite element space of a functional containing the L^1-norm of the Hamiltonian plus a discrete entropy. Under appropriate hypotheses on the Hamiltonian, it is shown that the algorithm converges to the unique viscosity solution. The main results of this paper are Theorems 4.5, 5.3, and 6.3.

The paper is organized as follows. In the rest of this section we introduce notation and structural hypotheses for (1.1). The discrete finite element setting, along with the minimization problem, is introduced in section 2. The existence of minimizers for the discrete problem is proved in section 3. The passage to the limit is done in section 4; i.e., it is shown in this section that the limit solution solves (1.1). The proof that the limit solution is indeed a viscosity solution is reported in section 5. Since the proof reported in section 5 is based on a hypothesis which we do not know how to verify on arbitrary grids (see (3.2)), we give an alternative proof in section 6 using a vertex-based entropy on Cartesian grids. The main argument in sections 5 and 6 consists of proving a one-sided bound.

1.2. Structure hypotheses. We make the following assumptions on the Hamiltonian:

$$(1.2) \qquad \|p\| \leq c_s \left(|H(x, v, p)| + |v| + 1 \right) \quad \forall (x, v, p) \in \Omega \times \mathbb{R} \times \mathbb{R}^2,$$

$$(1.3) \qquad H(x, \cdot, \cdot) \in \mathcal{C}^{0,1}(\overline{B_{\mathbb{R}}(0, R)} \times \overline{B_{\mathbb{R}^2}(0, R)}; \mathbb{R}) \ \forall R > 0 \text{ uniformly in } x \in \Omega,$$

where $\|\cdot\|$ denotes the Euclidean norm in $\mathbb{R}^2$. A typical example is the eikonal equation, $H(x, v, p) = |p| - 1$, or modified versions of this equation, say, $H(x, v, p) = v + F(|p|) - f(x)$, where F is a convex and f a bounded positive function.

DEFINITION 1.1. *A function u in $W^{1,\infty}(\Omega)$ is said to be q-semiconcave if there is a concave function $v_c \in W^{1,\infty}(\Omega)$ and a function $w \in W^{2,q}(\Omega)$ so that $u = v_c + w$.*

We assume that (1.1) has a unique viscosity solution u such that

$$(1.4) \qquad\qquad u \in W^{1,\infty}(\overline{\Omega}),$$

$$(1.5) \qquad\qquad u \text{ is } q\text{-semiconcave for some } q > 2,$$

$$(1.6) \qquad\qquad Du \in \mathrm{BV}(\Omega),$$

where we have set $W^{1,\infty}(\overline{\Omega}) := W^{1,\infty}(\Omega) \cap \mathcal{C}^0(\overline{\Omega})$ and D is the gradient operator.

Remark 1.1. The class of problems we are working on is not empty. In particular, it is known that the unique viscosity solution to (1.1) satisfies the above hypotheses when the Hamiltonian is convex; see [16, 18].

Remark 1.2. Hypothesis (1.2) seems nonstandard. Typical hypotheses in the literature (see [4, p. 189]) consist of assuming H to be convex with respect to p and $H(x, v, p) \to +\infty$ when $|p| \to +\infty$. It can be shown that these two conditions (convexity plus growth at infinity) imply (1.2).

Remark 1.3. Recall that a function v in $W^{1,\infty}(\Omega)$ is usually called uniformly semiconcave in textbooks if and only if it can be decomposed into $v(x) = v_c(x) + c_v x^2$, where c_v is a nonnegative constant and v_c is concave and in $W^{1,\infty}(\Omega)$; see [10, p. 130]. Definition 1.1 is a slight generalization of semiconcavity.

Remark 1.4. When Ω can be finitely covered by open convex subsets, it can be shown that (1.4) and (1.5) imply (1.6). Actually, Definition 1.1 implies $u = v_c + w$, where $v_c \in W^{1,\infty}(\Omega)$, $w \in W^{2,q}(\Omega) \subset W^{2,1}(\Omega)$. Clearly $Dw \in W^{1,1}(\Omega) \subset \mathrm{BV}(\Omega)$. It is also known that convex functions in $W^{1,\infty}(\Omega)$ have gradients in $\mathrm{BV}(\Omega)$; see [3, Prop. 5.1, 7.11].

To be able to collectively refer to (1.4)–(1.6), we define

$$(1.7) \qquad\qquad X = \{v \in W^{1,\infty}(\Omega); \ Dv \in \mathrm{BV}(\Omega); \ v \text{ is } q\text{-semiconcave}\}$$

with the norm

$$(1.8) \qquad\qquad \|v\|_X := \|v\|_{W^{1,\infty}(\Omega)} + \|Dv\|_{\mathrm{BV}(\Omega)} + \inf_{v = v_c + w} \|w\|_{W^{2,q}(\Omega)}.$$

In the remainder of the paper c is a generic constant that does not depend on the mesh size and whose value may change at each occurrence. For any real number $r \geq 1$, we denote by r' the conjugate of r, i.e., $\frac{1}{r} + \frac{1}{r'} = 1$.

2. The discrete problem.

2.1. The meshes. Let $\{\mathcal{T}_h\}_{h>0}$ be a family of shape regular finite element meshes. For the sake of simplicity we assume that the mesh elements are triangles and the mesh family is quasi uniform. For each mesh $\mathcal{T}_h$, the subscript h refers to the maximum mesh size in the mesh. We denote by $\mathcal{F}_h^i$ the set of mesh interfaces: F is a member of $\mathcal{F}_h^i$ if and only if there are two elements $K_1(F)$, $K_2(F)$ in $\mathcal{T}_h$ such that $F = K_1(F) \cap K_2(F)$. The intersection of two cells is either empty, a vertex, or an entire edge. For every function $v \in \mathcal{C}^0(K_1(F)) \cup \mathcal{C}^0(K_2(F))$, we denote

$$(2.1) \qquad \forall x \in F, \qquad \{v\}(x) = \frac{1}{2}(v_{|K_1(F)}(x) + v_{|K_2(F)}(x)).$$

DEFINITION 2.1 (see Figure 1). (1) *We call a chain a numbered collection of triangles* $\Lambda = \{K_j\}_{1 \leq J}$ *such that K_1 has one edge on Γ, K_j shares one edge with K_{j-1} and one edge with K_{j+1} for $1 < j < J$, and $K_i \neq K_j$ if $i \neq j$.*
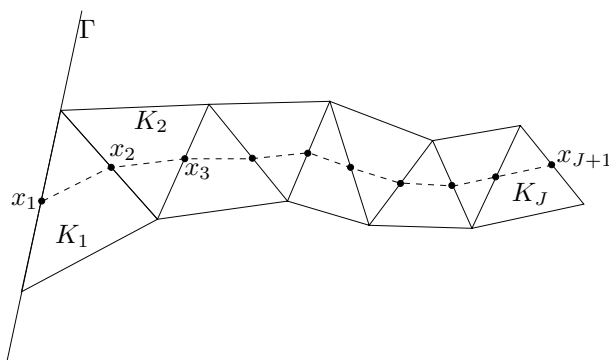
FIG. 1. *Illustration of the notation for the chain and path in Definition* 2.1.

(2) *The path associated with a chain* $\Lambda = \{K_j\}_{1 \leq J}$ *is the broken line traversing the chain in such a way that* (i) *it crosses* K_j, $1 < j < J$, *by connecting the two midpoints of the two interfaces that connect* K_j *to the chain;* (ii) *it connects the midpoint of the interface connecting* K_1 *to the chain and the midpoint of the face of* K_1 *that lies on* Γ; *and* (iii) *it connects the midpoint of the interface connecting* K_J *to the chain and the midpoint of another arbitrary face of* K_J.

Since the mesh family $\{\mathcal{T}_h\}_{h>0}$ is quasi uniform, the following property holds: For each mesh $\mathcal{T}_h$, there exists a collection of chains $\{\Lambda_l\}_{1 \leq l < L_h}$ such that every triangle in $\mathcal{T}_h$ belongs to at least one chain (there are $L_h \in \mathbb{N}$ of those chains). Moreover, there is c independent of h such that

$$(2.2) \qquad\qquad \operatorname{card} \Lambda_l \leq ch^{-1}, \qquad 1 \leq l \leq L_h,$$

$$(2.3) \qquad\qquad L_h \leq ch^{-1}.$$

Let $k \geq 1$ be an integer and denote by $\mathbb{P}_k$ the set of real-valued polynomials in $\mathbb{R}^2$ of total degree at most k. We introduce

$$(2.4) \qquad\qquad X_h = \{v_h \in \mathcal{C}^0(\overline{\Omega}); \ v_{h|K} \in \mathbb{P}_k \ \forall K \in \mathcal{T}_h; \ v_{h|\Gamma} = 0\},$$

$$(2.5) \qquad\qquad X_{(h)} = X + X_h.$$

For every function v in $X_{(h)}$ we denote by $\lambda_+(v) : \mathcal{T}_h \longrightarrow \mathbb{R}^+$ the mapping such that for every $x \in K \in \mathcal{T}_h$, $\lambda_+(v)(x)$ is the largest positive eigenvalue of the Hessian of v at x. Observe that Remark 1.3 implies that λ_+ is well defined on X; that is, λ_+ is well defined on the space $X_{(h)}$.

Similarly, for every function v in $X_{(h)}$ we denote by $\{-\partial_n v\}_+ : \mathcal{F}_h^{\mathrm{i}} \longrightarrow \mathbb{R}^+$ the mapping such that for all $x \in F = K_1 \cap K_2 \in \mathcal{F}_h^{\mathrm{i}}$,

$$\{-\partial_n v\}_+(x) = \left(-\frac{1}{2} (Dv_{|K_1}(x) \cdot n_1 + Dv_{|K_2}(x) \cdot n_2) \right)_+,$$

where n_1 and n_2 are the unit outward normals to K_1 and K_2 at x, respectively, and $(t)_+ := \frac{1}{2}(t + |t|)$ denotes the positive part of t for all $t \in \mathbb{R}$.

2.2. The discrete minimization problem. Let p_1 and p_2 be two fixed real numbers such that

$$(2.6) \qquad\qquad 1 \leq p_1 \leq q \quad \text{and} \quad 1 \leq p_2 \leq q.$$

We now define the following functional $J_h : X_{(h)} \ni v \longmapsto J_h(v) \in \mathbb{R}^+$ by

$$(2.7) \quad J_h(v) = \int_\Omega |H(x, v, Dv)| dx$$

$$+ h \sum_{K \in \mathcal{T}_h} \int_K [\lambda_+(v)]^{p_1} dx + h^{2-p_2} \sum_{F \in \mathcal{F}_h^i} \int_F \{-\partial_n v\}_+^{p_2} d\sigma.$$

For every function v in $X_{(h)}$ we refer to $H(x, v, Dv)$ as the residual. The two extra terms in the right-hand side above are referred to as the volume entropy and the interface entropy, respectively.

Remark 2.1. Whenever $v \in W^{1,\infty}(\Omega)$ is concave, the two entropy terms are zero; i.e., these two terms do not add extra viscosity. They act to prevent the occurrence of large positive second derivatives.

The discrete problem on which we shall henceforth focus our attention consists of the following minimization problem: Seek u_h in X_h such that

$$(2.8) \qquad\qquad J_h(u_h) = \inf_{v_h \in X_h} J_h(v_h).$$

The goal of the rest of the paper is to show that minimizers exist for each mesh and every sequence of minimizers converges to the unique viscosity solution to (1.1).

3. Existence of minimizers. The goal of this section is to show the existence of (at least) one minimizer to problem (2.8). This is done by deriving a priori bounds and using a simple compactness argument.

3.1. Consistency. We start by deriving a consistency property. Since the mesh is quasi uniform, one can always construct a family of linear approximation operators on piecewise linear polynomials $\mathcal{I}_h : X \longrightarrow X_h$ that are stable on $W^{1,\infty}(\Omega)$ and such that the following property holds:

$$(3.1) \qquad\qquad \|v - \mathcal{I}_h v\|_{W^{1,1}(\Omega)} \leq ch \|Dv\|_{\mathrm{BV}(\Omega)} \qquad \forall v \in X.$$

This is a standard approximation property; for instance, the linear Clément operator [7, 6] satisfies this property (see also section 6.2 for a precise definition of the Clément approximation). It is also clear that for $p_2 = 1$ the Clément approximation (or any other reasonable approximation operator) satisfies

$$(3.2) \qquad\qquad h^{2-p_2} \sum_{F \in \mathcal{F}_h^i} \int_F \{-\partial_n \mathcal{I}_h(u)\}_+^{p_2} d\sigma \leq c(\|u\|_X) \, h.$$

When $p_2 > 1$, (3.2) becomes a nontrivial property. In our previous paper [14], which deals with the one-dimensional case, we showed that the piecewise linear Lagrange interpolant of the exact solution satisfies (3.2) with $p_2 > 1$. Unfortunately, this argument does not hold in two space dimensions. To see this, assume that the gradient of u is discontinuous across a line L and the mesh family is such that $\mathcal{O}(h^{-1})$ cell interfaces cross L. Then the left-hand side of (3.2) is bounded from below and above by $c\,h^{2-p_2}$ which is larger than $c\,h$ unless $p_2 = 1$. In other words, (3.2) is hard to verify when $p_2 > 1$. Of course (3.2) can be shown to hold with $p_2 > 1$ if we are allowed to optimize or control the mesh. For instance, the Lagrange interpolant of u satisfies (3.2) with $p_2 > 1$ if the mesh family is such that no cell interface crosses the

lines across which the normal derivative of u is discontinuous; i.e., the mesh is aligned with the discontinuity lines of the gradient of u.

Henceforth we make the following assumption:

$$(3.3) \qquad \begin{cases} \text{If } p_2 > 1, \text{ there exists an approximation operator } \mathcal{I}_h \text{ such} \\ \text{that } (3.1) \text{ and } (3.2) \text{ hold simultaneously.} \end{cases}$$

The existence of such an operator (for $p_2 > 1$) is an open question when the mesh is not aligned with the discontinuities of the gradient of u. Note that the assumption is empty when $p_2 = 1$.

The following lemma is the first key step of the theory.

LEMMA 3.1. *Let u solve (1.1) and assume (3.3); then there is $c(u)$ independent of h such that*

$$(3.4) \qquad\qquad J_h(\mathcal{I}_h u) \leq c(u)\, h.$$

Proof. (1) Since $\mathcal{I}_h u$ is piecewise linear, the restriction of the Hessian of $\mathcal{I}_h u$ to every mesh element is zero, i.e., $\lambda_+(\mathcal{I}_h u)_{|K} = 0$ for all $K \in \mathcal{T}_h$.

(2) Since $\mathcal{I}_h$ is uniformly stable in $W^{1,\infty}(\Omega)$, there is $c \geq 0$, independent of h, such that $\|\mathcal{I}_h u\|_{W^{1,\infty}} \leq c\,\|u\|_{W^{1,\infty}}$. Let us set $R = c\,\|u\|_{W^{1,\infty}}$; then owing to (1.3), there is $c_R \geq 0$ such that for all $x \in \Omega$,

$$|H(x, \mathcal{I}_h u, D(\mathcal{I}_h u))| = |H(x, \mathcal{I}_h u, D(\mathcal{I}_h u)) - H(x, u, Du)|$$

$$\leq c_R(|\mathcal{I}_h u - u| + \|D(\mathcal{I}_h u - u)\|).$$

Then together with (3.1), this implies

$$\int_\Omega |H(x, \mathcal{I}_h u, D(\mathcal{I}_h u))| \leq c_R \|\mathcal{I}_h u - u\|_{W^{1,1}} \leq c\,c_R h \|Du\|_{\mathrm{BV}(\Omega)}.$$

(3) We now conclude by using (3.2) and collecting the above results:

$$J_h(\mathcal{I}_h u) \leq c(\|u\|_X)\, h \leq c' h.$$

This concludes the proof. $\square$

Remark 3.1. Note that it was critical to use the L^1-norm of the residual to obtain (3.4). This is compatible with the fact that Du is in $\mathrm{BV}(\Omega)$ only. Using any other L^p-norm would yield a suboptimal exponent on h.

3.2. The $W^{1,1}(\Omega) \cap L^\infty(\Omega)$ bound. Let $\alpha > 0$ be a positive real number. Define the set $S_{h,\alpha} = \{v_h \in X_h;\ \int_\Omega |H(x, v_h, Dv_h)|dx \leq \alpha\, h\}$. Using (3.4), we infer that $S_{h,c(u)}$ is not empty, i.e., $\mathcal{I}_h(u) \in S_{h,c(u)}$.

LEMMA 3.2. *Let $\alpha > 0$ and assume that $S_{h,\alpha}$ is not empty; then there is $c_0(\alpha) > 0$, independent of h, and $h_0 > 0$ such that*

$$(3.5) \qquad \forall h < h_0, \quad \forall v_h \in S_{h,\alpha}, \qquad \|v_h\|_{W^{1,1}} + \|v_h\|_{L^\infty} \leq c_0(\alpha).$$

Proof. Let Λ_l be a chain in the collection $\{\Lambda_l\}_{1 \leq l \leq L_h}$ (see Figure 1). Set $N_l = \mathrm{card}(\Lambda_l)$, let v_h be a member of the nonempty set $S_{h,\alpha}$, and define

$$F_j^l = \sum_{i=1}^{j} \int_{K_i} \|Dv_h\|dx, \qquad 1 \leq j \leq N_l.$$

Owing to (1.2), we infer

$$F_j^l \leq \sum_{i=1}^{j} \int_{K_i} c_s \left(|H(x, v_h, Dv_h)| dx + |v_h| + 1 \right) dx.$$

Then using the fact that v_h is a member of $S_{h,\alpha}$ implies

$$F_j^l \leq \alpha c_s h + c_s N_l \max_{K \in \mathcal{T}_h} \operatorname{meas}(K) + \sum_{i=1}^{j} c_s \int_{K_i} |v_h| dx.$$

In other words, using (2.2) together with $\max_{K \in \mathcal{T}_h} \operatorname{meas}(K) \leq ch^2$, we infer

$$(3.6) \qquad F_j^l \leq ch + c_s \sum_{i=1}^{j} \int_{K_i} |v_h| dx.$$

Let $x_1, \ldots, x_i$ be the points of the path traversing Λ_l such that $(x_m, x_{m+1}) = \Lambda_l \cap K_m$, $1 \leq m < N_l$ (see Figure 1). Denote $\tau_m = (x_{m+1} - x_m)/\|x_{m+1} - x_m\|$. Let us now consider a cell K_i, $1 \leq i \leq j$, and let y be an arbitrary point in K_i; then the fundamental theorem of calculus implies

$$v_h(y) = v_h(x_1) + \sum_{m=1}^{i-1} \int_{x_m}^{x_{m+1}} \tau_m \cdot Dv_h(x) d\sigma + \int_{x_i}^{y} \frac{y - x_i}{\|y - x_i\|} \cdot Dv_h(x) d\sigma,$$

with the obvious convention if $i = 1$ or $y = x_i$. This in turn implies

$$|v_h(y)| \leq c \, h \sum_{m=1}^{i} \|Dv_h\|_{L^\infty(K_i)} \qquad \forall y \in K_i.$$

Since $v_{h|K_m}$ is a polynomial, the equivalence of norms on finite-dimensional normed spaces gives in two space dimensions

$$(3.7) \qquad |v_h(y)| \leq c \, h^{-1} \sum_{m=1}^{i} \|Dv_h\|_{L^1(K_i)} = c \, h^{-1} F_i^l \qquad \forall y \in K_i.$$

By integrating over K_i, we obtain

$$(3.8) \qquad \int_{K_i} |v_h| dx \leq ch F_i^l.$$

By combining (3.6) and (3.8), we infer $F_j^l \leq ch + c'h \sum_{i=1}^{j} F_i^l$, which (provided h is small enough) immediately implies

$$F_j^l \leq \frac{ch}{1 - c'h} \left(1 + \frac{c'h}{1 - c'h} \right)^j.$$

Then, owing to (2.2) we have $j \leq N_l \leq ch^{-1}$, which in turn yields

$$F_j^l \leq c \, h,$$

which owing to (3.7) implies the desired L^∞-bound on v_h.

We obtain the $W^{1,1}$-bound by using the property saying that Ω can be covered with chains, i.e.,

$$\|Dv_h\|_{L^1(\Omega)} \le \sum_{l=1}^{L_h} \sum_{K \in \Lambda_l} \|Dv_h\|_{L^1(K)} \le c \sum_{l=1}^{L_h} F_l^{N_l} \le chL_h \le c,$$

which concludes the proof. □

We are now in a position to prove the existence of a minimizer solving problem (2.8).

COROLLARY 3.3. *The discrete problem* (2.8) *has at least one minimizer* u_h, *and there is* $c > 0$ *independent of* h *such that*

$$(3.9) \qquad\qquad\qquad J_h(u_h) \le ch,$$

$$(3.10) \qquad\qquad\qquad \|u_h\|_{W^{1,1}} + \|u_h\|_{L^\infty} \le c.$$

Proof. Observe first that Lemma 3.1 implies that $S_{h,c(u)}$ is not empty, since $\mathcal{I}_h(u) \in S_{h,c(u)}$. Second, define $\mathcal{K}_h = \{v_h \in X_h; J_h(v_h) \le J_h(\mathcal{I}_h u)\}$. Clearly $\mathcal{I}_h u$ is a member of $\mathcal{K}_h$. Moreover, owing to Lemma 3.1, for every v_h in $\mathcal{K}_h$,

$$\int_\Omega |H(x, v_h, Dv_h)| dx \le J_h(v_h) \le J_h(\mathcal{I}_h u) \le c(u)\, h.$$

That is, $\mathcal{K}_h \subset S_{h,c(u)}$. Lemma 3.2 implies that there is $c'(u)$ independent of h such that for all $v_h \in \mathcal{K}_h$, $\|v_h\|_{L^\infty} + \|v_h\|_{W^{1,1}} \le c'(u)$. In other words, $\mathcal{K}_h$ is uniformly bounded in $W^{1,1}(\Omega) \cap L^\infty(\Omega)$. Finite-dimensionality then implies that $\mathcal{K}_h$ is compact. It is also clear that $J_h : \mathcal{K}_h \longrightarrow \mathbb{R}$ is continuous in every norm (possibly not uniformly with respect to h). Then, there exists $u_h \in \mathcal{K}_h$ that minimizes J_h on $\mathcal{K}_h$. Since for every function v_h in $X_h \backslash \mathcal{K}_h$, $J_h(v_h)$ is larger than $J_h(\mathcal{I}_h u)$, we conclude that

$$\inf_{v_h \in X_h} J_h(v_h) = \inf_{v_h \in \mathcal{K}_h} J_h(v_h) = \min_{v_h \in \mathcal{K}_h} J_h(v_h) = J_h(u_h),$$

which concludes the proof. □

Since in practice u_h might not be computed exactly or might be approximated to some extent by using some iterative process (the details of the process in question are irrelevant for our discussion), we now define the notion of an *almost minimizer*.

DEFINITION 3.1. *We say that a family of functions* $\{v_h \in X_h\}_{h>0}$ *is a sequence of* almost minimizers *for* (1.1) *if there is* $c > 0$ *such that for all* $h > 0$,

$$(3.11) \qquad\qquad\qquad J_h(v_h) \le c\, h.$$

It is clear that minimizers are almost minimizers, thus showing that the class of almost minimizers is not empty. Almost minimizers also satisfy the following uniform bound owing to Lemma 3.2:

$$(3.12) \qquad\qquad\qquad \|v_h\|_{W^{1,1}} + \|v_h\|_{L^\infty} \le c.$$

The rest of the paper consists of proving that sequences of almost minimizers for (1.1) converge to the viscosity solution of (1.1).

4. Passage to the limit. Henceforth $\{u_h \in X_h\}_{h>0}$ denotes a sequence of almost minimizers for (1.1) as defined above. We show in this section that sequences of almost minimizers for (1.1) converge to weak solutions of (1.1). The main result of this section is Theorem 4.5. That the limit solution is indeed the viscosity limit will be shown in sections 5 and 6 by proving a one-sided bound.

4.1. The $W^{1,\infty}(\overline{\Omega})$-bound. We prove a $W^{1,\infty}(\overline{\Omega})$-bound by using a regularization technique. Let $\rho : \mathbb{R}^2 \longrightarrow \mathbb{R}^+$ be a positive kernel, i.e., ρ is compactly supported in $B_{\mathbb{R}^2}(0,1)$ and $\int_{\mathbb{R}^2} \rho\, dx = 1$. We then define the sequence of mollifiers $\rho_\epsilon(x) = \rho(x/\epsilon)$ with

$$(4.1) \qquad\qquad\qquad\qquad \epsilon = h^{1/2}.$$

This scaling is justified by the fact that $h\|\rho_\epsilon\|_{L^\infty} \leq c$, and this property is used in the proof of Lemma 4.1; see also Remark 4.1. Let us denote by $\tilde{u}_h$ the extension of u_h to $\mathbb{R}^2$ by setting $u_{h|\mathbb{R}\backslash\Omega} = 0$. Since $u_{h|\Gamma} = 0$, u_h is continuous and piecewise polynomial in $\overline{\Omega}$, this extension is $W^{1,\infty}$-stable, i.e., $\|\tilde{u}_h\|_{W^{1,\infty}(\mathbb{R}^2)} \leq \|u_h\|_{W^{1,\infty}(\Omega)}$. We now define

$$(4.2) \qquad\qquad\qquad\qquad u_{\epsilon,h} = \rho_\epsilon * \tilde{u}_h.$$

The main result of this section is the following lemma.

LEMMA 4.1. *There is a constant c, independent of h, such that*

$$\|u_{\epsilon,h}\|_{W^{1,\infty}(\overline{\Omega})} \leq c.$$

Proof. Let x be any point in $\mathbb{R}^2$. We have

$$\|Du_{\epsilon,h}(x)\| = \left| \int_{\mathbb{R}^2} \rho_\epsilon(x-y)D\tilde{u}_h(y)dy \right|$$

$$\leq \int_\Omega \rho_\epsilon(x-y)\|D\tilde{u}_h(y)\|dy = \int_\Omega \rho_\epsilon(x-y)\|Du_h(y)\|dy$$

$$\leq c_s \int_\Omega \rho_\epsilon(x-y)(|H(y,u_h(y),Du_h(y))| + |u_h(y)| + 1)dy$$

$$\leq c_s\|\rho_\epsilon\|_{L^\infty(\Omega)} \int_\Omega |H(y,u_h(y),Du_h(y))|dy + (\|u_h\|_{L^\infty(\Omega)} + 1)\|\rho_\epsilon\|_{L^1(\Omega)}.$$

The estimates (3.11) and (3.12) imply

$$\|Du_{\epsilon,h}(x)\| \leq c\,(h\|\rho_\epsilon\|_{L^\infty(\Omega)} + \|\rho_\epsilon\|_{L^1(\Omega)}).$$

Then the estimates $\|\rho_\epsilon\|_{L^\infty(\Omega)} \leq c\epsilon^{-2}$ and $\|\rho_\epsilon\|_{L^1(\Omega)} = 1$ along with the definition of ϵ (4.1) imply the desired result. $\qquad\square$

Remark 4.1. The above result generalizes to any space dimension d if estimates (3.11) and (3.12) hold and provided we take the scaling $\epsilon = h^{1/d}$.

4.2. The BV-bound on Du_h. We prove in this section an a priori bound on the BV-norm of Du_h. We start with a technical result concerning interface averages of the gradient of functions in X_h. Let (e_1, e_2) be the canonical basis of $\mathbb{R}^2$.

LEMMA 4.2. *For all $v_h \in X_h$ and all $F \in \mathcal{F}_h^i$, the following holds:*

$$(4.3) \qquad \{(n{\cdot}e_j)\partial_i v_h\}_{|F} = \{\partial_n v_h\}_{|F}(n_1{\cdot}e_i)(n_1{\cdot}e_j) = \{\partial_n v_h\}_{|F}(n_2{\cdot}e_i)(n_2{\cdot}e_j),$$

where n_1 (resp., n_2) is the unit outer normal of $K_1(F)$ on F (resp., $K_2(F)$ on F).

Proof. Let $\tau_1 := -\tau_2$ be one of the two unit vectors that are parallel to F (see Figure 2). Upon observing that $e_i = (n_1{\cdot}e_i)n_1 + (\tau_1{\cdot}e_i)\tau_1 = (n_2{\cdot}e_i)n_2 + (\tau_2{\cdot}e_i)\tau_2$,
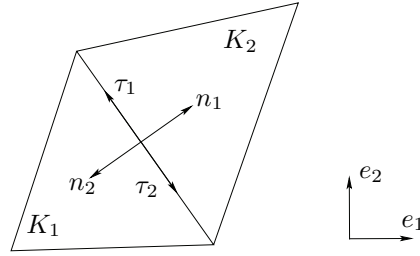
Fig. 2. *Illustration of the notation for Lemma 4.2.*

$(n_1 \cdot e_i)(n_1 \cdot e_j) = (n_2 \cdot e_i)(n_2 \cdot e_j)$, and $(n_1 \cdot e_i)(\tau_1 \cdot e_j) = (n_2 \cdot e_i)(\tau_2 \cdot e_j)$, we infer

$$\{(n \cdot e_j)\partial_i v_h\} = \frac{1}{2}((n_1 \cdot e_j)\partial_i v_{h|K_1} + (n_2 \cdot e_i)\partial_i v_{h|K_2})$$

$$= \frac{1}{2}((n_1 \cdot e_j)(Dv_{h|K_1}) \cdot ((n_1 \cdot e_i)n_1 + (\tau_1 \cdot e_i)\tau_1)$$

$$+ (n_2 \cdot e_j)(Dv_{h|K_2}) \cdot ((n_2 \cdot e_i)n_2 + (\tau_2 \cdot e_i)\tau_2))$$

$$= \{\partial_n v_h\}(n_1 \cdot e_i)(n_1 \cdot e_j) + \{\partial_\tau v_h\}(n_1 \cdot e_i)(\tau_1 \cdot e_j).$$

Then, we conclude by observing that functions in X_h are continuous across interfaces, which implies $\{\partial_\tau v_h\} = 0$.   ☐

Another preliminary result consists of bounding the normal derivative of u_h at the boundary of the domain. This is achieved by means of the following lemma.

LEMMA 4.3. *There is c, independent of h, such that*

$$(4.4) \qquad\qquad\qquad \int_\Gamma |\partial_n u_h| dx \le c.$$

Proof. Let us denote by $\mathcal{L}_h$ the layer of triangles that have at least one edge on Γ. Using an inverse inequality and (1.2), we deduce

$$\int_\Gamma |\partial_n u_h| dx \le c\,h^{-1} \sum_{K \in \mathcal{L}_h} \int_K \|Du_h\| dx \le c\,h^{-1} \sum_{K \in \mathcal{L}_h} \int_K (|H(x, u_h, Du_h)| + |u_h| + 1)\,dx$$

$$\le c\,h^{-1} \int_\Omega |H(x, u_h, Du_h)| dx + c\,h^{-1} \sum_{K \in \mathcal{L}_h} \int_K (|u_h| + 1)\,dx.$$

Then we conclude using the estimates (3.11)–(3.12) together with the fact that $\sum_{K \in \mathcal{L}_h} \text{meas}(K) \le ch$.   ☐

LEMMA 4.4. *There is c, independent of h, such that*

$$(4.5) \qquad\qquad\qquad \|Du_h\|_{BV(\Omega)} \le c.$$

Proof. Using (4.3) and the definition of the BV-seminorm implies

$$|Du_h|_{\mathrm{BV}(\Omega)} = \sum_{i,j=1}^{2} \sup_{\substack{\phi \in \mathcal{C}_0^\infty(\Omega) \\ \|\phi\|_{L^\infty} \leq 1}} \int_\Omega \partial_i u_h \partial_j \phi dx$$

$$= \sum_{i,j=1}^{2} \left(\sup_{\substack{\phi \in \mathcal{C}_0^\infty(\Omega) \\ \|\phi\|_{L^\infty} \leq 1}} \sum_{K \in \mathcal{T}_h} \int_K -\partial_{ij} u_h \phi dx + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F 2\{\partial_i u_h (n \cdot e_j)\} \phi ds \right)$$

$$\leq \sum_{i,j=1}^{2} \left(\sum_{K \in \mathcal{T}_h} \int_K |\partial_{ij} u_h| dx + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F 2|\{-\partial_n u_h\}| ds \right)$$

$$\leq \sum_{K \in \mathcal{T}_h} \int_K (|\partial_{11} u_h| + |\partial_{22} u_h| + 2|\partial_{12} u_h|) \, dx + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F 8|\{-\partial_n u_h\}| ds.$$

Now we use the relation $|x| = 2x_+ - x$ as follows:

$$|Du_h|_{\mathrm{BV}(\Omega)} \leq \sum_{K \in \mathcal{T}_h} \int_K (2((\partial_{11} u_h)_+ + (\partial_{22} u_h)_+) - \Delta u_h + 2|\partial_{12} u_h|) \, dx$$

$$+ \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F (16\{-\partial_n u_h\}_+ - 8\{-\partial_n u_h\}) \, ds.$$

Moreover, the definition of λ_+ implies that for all $x \in K$ and all $K \in \mathcal{T}_h$,

$$\max((\partial_{11} u_h)_+(x), (\partial_{22} u_h)_+(x)) \leq \lambda_+(x),$$

$$|\partial_{12} u_h(x)| \leq \lambda_+(x) - \frac{1}{2} \Delta u_h(x).$$

Then,

$$|Du_h|_{\mathrm{BV}(\Omega)} \leq \sum_{K \in \mathcal{T}_h} \int_K (6\lambda_+ - 2\Delta u_h) \, dx + \sum_{K \in \mathcal{T}_h} \int_K 4\Delta u_h dx$$

$$+ \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F (16\{-\partial_n u_h\}_+) \, ds - \sum_{F \in \mathcal{F}_h^{\partial}} \int_F 4\partial_n u_h ds.$$

Now we use $\Delta u_h \leq 2\lambda_+$ to derive

$$|Du_h|_{\mathrm{BV}(\Omega)} \leq \sum_{K \in \mathcal{T}_h} \int_K 10\lambda_+ dx + \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F (16\{-\partial_n u_h\}_+) \, ds + \int_\Gamma 4|\partial_n u_h| ds.$$

Let R_1, R_2, and R_3 be the three terms in the right-hand side of the above inequality. We bound $R_1 + R_2$ as follows:

$$R_1 + R_2 \leq c_1 \left(\sum_{K \in \mathcal{T}_h} \mathrm{meas}(K) \right)^{1/p_1'} \left(\sum_{K \in \mathcal{T}_h} \int_K \lambda_+^{p_1} \right)^{1/p_1}$$

$$+ c_2 \left(h^{\frac{p_2-1}{p_2} p_2'} \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \mathrm{meas}(F) \right)^{1/p_2'} \left(h^{1-p_2} \sum_{F \in \mathcal{F}_h^{\mathrm{i}}} \int_F \{-\partial_n u_h\}_+^{p_2} ds \right)^{1/p_2}.$$

Then using the estimate on $J_h(u_h)$ in (3.11), we derive

$$R_1 + R_2 \leq c.$$

To conclude that R_3 is also bounded, we use the estimate (4.4). We conclude that $\|Du_h\|_{\mathrm{BV}(\Omega)}$ is uniformly bounded by using the fact that Du_h is also uniformly bounded in $L^1(\Omega)$. $\square$

4.3. Convergence to a weak solution. We say that $v \in W^{1,\infty}(\overline{\Omega})$ is a weak solution to (1.1) if v solves (1.1) almost everywhere.

THEOREM 4.5. *Assume that* (1.1) *has a solution* u *in* X *and that the mesh family satisfies* (3.3). *Then the sequence of almost minimizers* $\{u_h\}_{h>0}$ *converges, up to subsequences, to a weak solution to* (1.1).

Proof. Owing to Corollary 3.3 and Lemma 4.4, the sequence $\{u_h\}_{h>0}$ is precompact in $W^{1,1}(\Omega)$. Let u be the limit, up to subsequences, of $\{u_h\}_{h>0}$ in $W^{1,1}(\Omega)$. We need to show that u is also in $W^{1,\infty}(\overline{\Omega})$. To see this, we observe that, up to subsequences again, $\{u_h\}_{h>0}$ and $\{u_{\epsilon,h}\}_{h>0}$ have the same limit in $W^{1,1}(\Omega)$ since

$$\|u_h - u_{\epsilon,h}\|_{W^{1,1}(\Omega)} \leq \|u_h - u\|_{W^{1,1}(\Omega)} + \|u - \rho_\epsilon * u\|_{W^{1,1}(\Omega)} + \|\rho_\epsilon * (u - u_h)\|_{W^{1,1}(\Omega)},$$

and the right-hand side goes to zero as $h \to 0$, owing to well-known properties of mollifiers, recalling that $\epsilon = h^{\frac{1}{2}}$. Moreover, the sequence $\{u_{\epsilon,h}\}_{h>0}$, being uniformly bounded in $W^{1,\infty}(\overline{\Omega}) \subset W^{1,\infty}(\Omega)$, converges in $W^{1,\infty}(\Omega)$ in the weak-$*$ topology, up to subsequences. The uniqueness of limits implies that u is in $W^{1,\infty}(\Omega)$. The sequence $\{u_{\epsilon,h}\}_{h>0}$ being uniformly bounded in $W^{1,\infty}(\overline{\Omega})$ means that it is equicontinuous on $\overline{\Omega}$; as a result, the limit is continuous, i.e., $u \in \mathcal{C}^0(\overline{\Omega})$. Combining the two above results implies $u \in W^{1,\infty}(\overline{\Omega}) = \mathcal{C}^0(\overline{\Omega}) \cap W^{1,\infty}(\Omega)$.

We now prove that u is a weak solution to (1.1) by showing that $\|H(\cdot, u, Du)\|_{L^1(\Omega)} = 0$. Using that $u_h \to u$ in $W^{1,1}(\Omega)$, we conclude that, up to subsequences, $u_h \to u$ and $Du_h \to Du$ a.e. in Ω. Then, we can apply Egorov's theorem. Given $\epsilon' > 0$, there exists a set E with $\mathrm{meas}(E) < \epsilon'$, such that the convergence of $u_h \to u$ on $\Omega \backslash E$ is uniform. Therefore, for every ϵ'', $1 \geq \epsilon'' > 0$, we can find $h(\epsilon'') > 0$ such that for every $h < h(\epsilon'')$,

$$|u_h(x) - u(x)| < \epsilon'' \quad \text{and} \quad \|Du_h(x) - Du(x)\| < \epsilon'' \quad \forall x \in \Omega \backslash E.$$

Note also that for every $x \in \Omega \backslash E$ and every $h < h(1)$, we have

$$\max(|u_h(x)|, |u(x)|, \|Du_h(x)\|, \|Du(x)\|) \leq R,$$

where $R := \|u\|_{W^{1,\infty}(\Omega)} + 1$. Hence, we can use the Lipschitz continuity of H to derive that there exists a value of $\epsilon'' > 0$ such that

$$(4.6) \qquad\qquad |H(x, u, Du) - H(x, u_h, Du_h)| < \epsilon'$$

for every $x \in \Omega \backslash E$ and every $h < h(\epsilon'')$. Note that at this point the value of ϵ'' solely depends on ϵ'. We now split $\|H(\cdot, u, Du)\|_{L^1(\Omega)}$ in the following way:

$$(4.7) \qquad \|H(\cdot, u, Du)\|_{L^1(\Omega)} = \|H(\cdot, u, Du)\|_{L^1(\Omega \backslash E)} + \|H(\cdot, u, Du)\|_{L^1(E)}.$$

We use that for every $R > 0$, $H(x, \cdot, \cdot)$ is Lipschitz continuous on $\overline{B_\mathbb{R}(0, R)} \times \overline{B_{\mathbb{R}^2}(0, R)}$ uniformly with respect to x to estimate

$$\|H(\cdot, u, Du)\|_{L^1(E)} \leq c \, \mathrm{meas}(E) = c\epsilon'.$$

The other term in the right-hand side of (4.7) is estimated as follows:

$$\|H(\cdot, u, Du)\|_{L^1(\Omega \setminus E)}$$

$$\leq \|H(\cdot, u, Du) - H(\cdot, u_h, Du_h)\|_{L^1(\Omega \setminus E)} + \|H(\cdot, u_h, Du_h)\|_{L^1(\Omega \setminus E)}$$

$$\leq \epsilon' \operatorname{meas}(\Omega \setminus E) + \|H(\cdot, u_h, Du_h)\|_{L^1(\Omega)}$$

$$\leq c\epsilon' + ch,$$

where we used (4.6) and (3.11) to derive the above inequality. As a result, for every $\epsilon' > 0$ and every $h < h(\epsilon')$,

$$\|H(\cdot, u, Du)\|_{L^1(\Omega)} \leq c(\epsilon' + h),$$

which means $\|H(\cdot, u, Du)\|_{L^1(\Omega)} = 0$. ∎

Remark 4.2. Recall that when $p_2 = 1$, hypothesis (3.3) is empty, i.e., Theorem 4.5 holds without any assumption on the mesh family other than that it is quasi uniform.

Now we have to address the question of whether this weak solution is indeed the viscosity solution.

5. One-sided bound. The goal of this section is to show that the algorithm described in this paper using the functional defined in (2.7) converges to the viscosity solution to (1.1) under the assumptions

$$(5.1) \qquad\qquad p_1 > 2 \quad \text{and} \quad p_2 > 2.$$

Throughout section 5 we conjecture (3.3). That is, there exists an approximation operator $\mathcal{I}_h$ satisfying (3.1) and (3.2) simultaneously for every mesh family. We have not been able to prove this statement for arbitrary meshes (unless the discontinuity lines of the gradient of u are aligned with the mesh). An alternative proof of convergence is reported in section 6 using a vertex-based entropy assuming that the mesh family is Cartesian and $p_1 = p_2 = 1$ so that the Clément interpolant always satisfies (3.1)–(3.2); i.e., the assumption (3.3) is empty.

Observe that if a function v is q-semiconcave, then there is $c > 0$ such that for all $\delta > 0$, all $\omega \subset \Omega$ so that $\omega + \delta e \subset \Omega$, and for every unit vector $e \in \mathbb{R}^2$, the following hold:

$$(5.2) \qquad u(x + \delta e) - 2u(x) + u(x - \delta e) \leq c\,\delta^{2 - \frac{2}{q}} \quad \forall x \in \omega,$$

$$(5.3) \qquad \|(u(\cdot + \delta e) - 2u(\cdot) + u(\cdot - \delta e))_+\|_{L^q(\omega)} \leq c\,\delta^2.$$

Note that (5.2) implies that for every orthonormal basis of $\mathbb{R}^2$, say (f_1, f_2), every $\delta > 0$, every $\gamma \leq 1 - \frac{2}{q}$, and every $x \in \omega$, the following holds:

$$(5.4) \qquad \Delta_\delta u(x) := \sum_{i=1}^{2} u(x + \delta f_i) - 2u(x) + u(x - \delta f_i) \leq c\delta^{1+\gamma}.$$

To stay general in the remainder of the paper we make the following assumption:

$$(5.5) \quad \left\{ \begin{array}{l} \text{A weak solution } u \text{ to (1.1) is the unique viscosity solution if } u \in \\ W^{1,\infty}(\Omega) \text{ and there exist an orthonormal basis } (f_1, f_2) \text{ of } \mathbb{R}^2 \text{ and} \\ \gamma > 0 \text{ such that (5.4) is satisfied.} \end{array} \right.$$
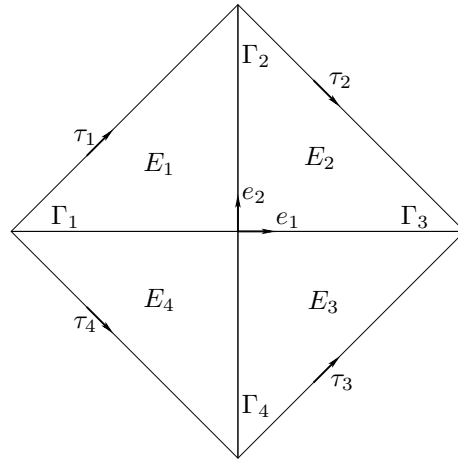
FIG. 3. *Illustration of the notation for Lemma* 5.1.

This property is known to characterize viscosity solutions to stationary Hamilton–Jacobi equations with $H(x, u, Du) = u + F(Du)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is convex as shown by Lions and Souganidis [18, Thm. 2.6].

Throughout section 5 the orthonormal basis that we use is the canonical one (e_1, e_2) and the discrete Laplacian $\Delta_\delta u(x)$ is defined using this basis.

Let $x \in \Omega$ and $\delta > 0$ such that $B_{\mathbb{R}^2}(x, \delta) \subset \Omega$ and let us consider the square whose four vertices are $x - \delta e_1$, $x + \delta e_2$, $x + \delta e_1$, and $x - \delta e_2$. This square is the union of the following four triangles:

$$
(5.6)
\begin{aligned}
E_1 &= x + \{(x_1, x_2) \in \mathbb{R}^2; \, x_1 \leq 0; \, x_2 \geq 0; \, x_1 - x_2 + \delta \geq 0\}, \\[4pt]
E_2 &= x + \{(x_1, x_2) \in \mathbb{R}^2; \, x_1 \geq 0; \, x_2 \geq 0; \, x_1 + x_2 - \delta \leq 0\}, \\[4pt]
E_3 &= x + \{(x_1, x_2) \in \mathbb{R}^2; \, x_1 \geq 0; \, x_2 \leq 0; \, x_1 - x_2 - \delta \leq 0\}, \\[4pt]
E_4 &= x + \{(x_1, x_2) \in \mathbb{R}^2; \, x_1 \leq 0; \, x_2 \leq 0; \, x_1 + x_2 + \delta \geq 0\}.
\end{aligned}
$$

The interior of E_i is henceforth denoted by $\dot{E}_i$, $i \in \{1, 2, 3, 4\}$. We also set

$$
(5.7)
\begin{aligned}
\Gamma_1 &= x + \{(x_1, 0) \in \mathbb{R}^2; \, -\delta \leq x_1 \leq 0\}, \\[4pt]
\Gamma_2 &= x + \{(0, x_2) \in \mathbb{R}^2; \, 0 \leq x_2 \leq \delta\}, \\[4pt]
\Gamma_3 &= x + \{(x_1, 0) \in \mathbb{R}^2; \, 0 \leq x_1 \leq \delta\}, \\[4pt]
\Gamma_4 &= x + \{(0, x_2) \in \mathbb{R}^2; \, -\delta \leq x_2 \leq 0\}.
\end{aligned}
$$

We now define the unit vectors $\tau_1 = 2^{-\frac{1}{2}}(e_1 + e_2)$, $\tau_2 = 2^{-\frac{1}{2}}(e_1 - e_2)$, $\tau_3 = \tau_1$, and $\tau_4 = \tau_2$. See Figure 3.

We are now in a position to derive an integral representation of $\Delta_\delta u_h(x)$ over the square $E_1 \cup E_2 \cup E_3 \cup E_4$.

LEMMA 5.1. *The following holds for all $v_h \in X_h$ and all $x \in \Omega$ and $\delta > 0$ such that $B_{\mathbb{R}^2}(x, \delta) \subset \Omega$:*

$$(5.8) \qquad \Delta_\delta v_h(x) = \sum_{l=1}^4 \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_l \neq \emptyset}} \int_{K \cap E_l} \partial_{\tau_l \tau_l} v_h + 2 \sum_{l=1}^4 \sum_{\substack{F \in \mathcal{F}_h^i \\ F \cap E_l \neq \emptyset}} \int_{F \cap E_l} \{-\partial_n v_h\}(\tau_l \cdot n)^2.$$

Proof. Consider first triangle E_1. Upon integrating by parts two times and using Lemma 4.2 and the fact that $v_h \in \mathcal{C}^0(\Omega)$, we infer the following:

$$0 = \int_{E_1} v_h \partial_{\tau_1 \tau_1}(1) dx = \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_1 \neq \emptyset}} \int_{K \cap E_1} v_h \partial_{\tau_1 \tau_1}(1) dx = \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_1 \neq \emptyset}} - \int_{K \cap E_1} \partial_{\tau_1} v_h \partial_{\tau_1}(1) dx$$

$$= \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_1 \neq \emptyset}} \int_{K \cap E_1} \partial_{\tau_1 \tau_1} v_h dx - \int_{\partial(K \cap E_1)} (\tau_1 \cdot n) \partial_{\tau_1} v_h ds$$

$$= \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_1 \neq \emptyset}} \int_{K \cap E_1} \partial_{\tau_1 \tau_1} v_h dx - \sum_{\substack{F \in \mathcal{F}_h^i \\ F \cap \dot{E}_1 \neq \emptyset}} \int_{F \cap \dot{E}_1} 2\{\partial_n v_h\}(\tau_1 \cdot n)^2 ds$$

$$+ \frac{1}{2} \int_{\Gamma_1} \partial_1 v_h ds - \frac{1}{2} \int_{\Gamma_2} \partial_2 v_h ds - \int_{\Gamma_1} \partial_n v_h (\tau_1 \cdot n)^2 ds - \int_{\Gamma_2} \partial_n v_h (\tau_1 \cdot n)^2 ds.$$

By proceeding similarly with the other triangles E_2, E_3, and E_4, and adding the four results, we obtain

$$- \int_{\Gamma_1} \partial_1 v_h ds + \int_{\Gamma_2} \partial_2 v_h ds + \int_{\Gamma_3} \partial_1 v_h ds - \int_{\Gamma_4} \partial_2 v_h ds$$

$$= \sum_{l=1}^4 \sum_{\substack{K \in \mathcal{T}_h \\ K \cap E_l \neq \emptyset}} \int_{K \cap E_l} \partial_{\tau_l \tau_l} v_h dx + 2 \sum_{l=1}^4 \sum_{\substack{F \in \mathcal{F}_h^i \\ F \cap E_l \neq \emptyset}} \int_{F \cap E_l} \{-\partial_n v_h\}(\tau_l \cdot n)^2.$$

We conclude by observing that

$$\Delta_\delta v_h(x) = - \int_{\Gamma_1} \partial_1 v_h ds + \int_{\Gamma_2} \partial_2 v_h ds + \int_{\Gamma_3} \partial_1 v_h ds - \int_{\Gamma_4} \partial_2 v_h ds.$$

This concludes the proof. $\square$

We are now in a position to prove a one-sided bound similar to that in (5.5).

LEMMA 5.2. *For all sequences of almost minimizers for (2.8), say $\{u_h\}_{h>0}$, there exist $c > 0$ and $\gamma := \min(\frac{p_1-2}{p_1}, \frac{p_2-2}{p_2})$ such that for all $x \in \Omega$ and $\delta > h$ such that $B_{\mathbb{R}^2}(x, \delta) \subset \Omega$, the following one-sided bound holds:*

$$(5.9) \qquad \Delta_\delta u_h(x) \le c \delta^{1+\gamma}.$$

Proof. Let us set $E := E_1 \cup E_2 \cup E_3 \cup E_4$. Using Lemma 5.1 together with the estimate (3.11), we infer

$$\Delta_\delta u_h(x) \le \sum_{K \in \mathcal{T}_h \cap E} \int_K \lambda_+(u_h) dx + 2 \sum_{F \in \mathcal{F}_h^i \cap E} \int_F \{-\partial_n u_h\}_+ ds$$

$$\le \left(\sum_{K \in \mathcal{T}_h \cap E} \operatorname{meas} K \right)^{\frac{1}{p_1'}} \left(\sum_{K \in \mathcal{T}_h} \int_K \lambda_+(u_h)^{p_1} dx \right)^{\frac{1}{p_1}}$$

$$+ 2 \left(h^{\frac{p_2-1}{p_2} p_2'} \sum_{F \in \mathcal{F}_h^i \cap E} \operatorname{meas} F \right)^{\frac{1}{p_2'}} \left(h^{1-p_2} \sum_{F \subset \mathcal{F}_h^i} \int_F \{-\partial_n u_h\}_+^{p_2} ds \right)^{\frac{1}{p_2}}$$

$$\le c \left(\delta^{\frac{2}{p_1'}} + \delta^{\frac{2}{p_2'}} \right),$$

where we used $h \le \delta$ in the last inequality. (We have bounded from above the number of cells in $\mathcal{T}_h \cap E$ and the number of interfaces in $\mathcal{F}_h^i \cap E$ by δ/h and this number cannot be less than 1.) We conclude by observing that $\frac{2}{p_i'} = 1 + \frac{p_i-2}{p_i}$ and $p_i > 2$ for $i = 1, 2$. □

THEOREM 5.3. *Let $u \in X$ be the unique solution to* (1.1). *Under the mesh assumption* (3.3), *the uniqueness assumption* (5.5), *and the restriction $p_1 > 2$, $p_2 > 2$, every sequence of almost minimizers converges to the unique viscosity solution to* (1.1).

Proof. Let $\{u_h\}_{h>0}$ be a sequence of almost minimizers. Let $\delta > 0$ and let $\Omega_\delta = \{x \in \Omega; B_{\mathbb{R}^2}(x, \delta) \subset \Omega\}$. Then owing to Lemma 5.9 the following holds for every $x \in \Omega_\delta$:

$$\Delta_\delta u_h(x) \le c\delta^{1+\gamma}.$$

Since u_h converges strongly to u in L^1, we infer that $u_h \to u$ a.e. in Ω_δ, that is,

$$\Delta_\delta u(x) \le c\delta^{1+\gamma}, \qquad \text{a.e. } x \text{ in } \Omega_\delta.$$

We then conclude that the above inequality holds for every $x \in \Omega_\delta$ since u is continuous. □

Remark 5.1. Recall that whether hypothesis (3.3) holds for every quasi-uniform mesh family is an open question. It definitely holds on aligned meshes. We remove this assumption in the next section for uniform meshes by taking $p_2 = 1$ and adding an extra term in the entropy.

6. One-sided bound on uniform meshes. The goal of this section is to prove an analogue of Theorem 5.3 in the case $p_2 = 1$, i.e., the mesh assumption (3.3) is empty. For this purpose we assume that the mesh is uniform and add a vertex-centered entropy to the functional J_h. We prove the one-sided bound (5.5) using the orthonormal basis $\frac{1}{\sqrt{2}}(e_1 + e_2, e_1 - e_2)$.

6.1. The vertex-centered entropy. We henceforth assume that the mesh is uniform in the sense that the set of vertices is

(6.1) $$\Omega_h := \{(sh, kh) \in \mathbb{R}^2; (s, k) \in I_h\} \subset \overline{\Omega},$$

where $\{I_h\}_{h>0}$ is a family of subsets of $\mathbb{N}^2$. The set of interior vertices is denoted by $\dot\Omega_h$. The mesh cells are triangles whose edges are parallel to e_1, e_2, or $e_1 + e_2$.

In order to understand how a vertex-centered entropy can be constructed, let us consider a point $x := (ih, jh) \in \Omega_h$ and $\delta > 0$ such that $x + B_h(0, \sqrt{2}\delta) \subset \Omega_h$, where $B_h(0, \mu) := \{(sh, kh) = z; (k, l) \in \mathbb{N}^2; \|z\| < \mu\}$. Assume for the time being that $\delta = nh$ with $n \geq 1$. We now define the following index sets

$$\Lambda_0 = \{(s, k); |k - j| \leq n \text{ and } |s - i| \leq n\},$$

$$\Lambda_1 = \{(s, k); 1 \leq k - j \leq n \text{ and } |s - i| \leq k - j - 1\},$$

$$\Lambda_2 = \{(s, k); 1 \leq s - i \leq n \text{ and } |k - j| \leq s - i - 1\},$$

(6.2)

$$\Lambda_3 = \{(s, k); 1 \leq j - k \leq n \text{ and } |s - i| \leq j - k - 1\},$$

$$\Lambda_4 = \{(s, k); 1 \leq i - s \leq n \text{ and } |k - j| \leq i - s - 1\},$$

$$\Lambda_5 = \{(s, k); 0 < |s - i| = |k - j| < n\}.$$

Up to a $\pi/4$ rotation and an appropriate rescaling, the sets Λ_m, $m \in \{1, 2, 3, 4\}$ correspond to the triangles E_m defined in (5.6). The set Λ_0 corresponds to the square $E_1 \cup E_2 \cup E_3 \cup E_4$. The set Λ_5 corresponds to $\Gamma_1 \cup \Gamma_2 \cup \Gamma_3 \cup \Gamma_4$ minus the center and the four corners of the square (the Γ_m's have been defined in (5.7)); see also Figure 3.

Let v be a member of X_h. To simplify notation we set $v_{s,k} := v(sh, kh)$. Now our goal is to find a discrete analogue of the integral representation (5.8) of the discrete Laplacian

(6.3) $$\Delta_{\sqrt{2}\delta} u(x) = u_{i-n,j-n} + u_{i-n,j+n} + u_{i+n,j-n} + u_{i+n,j+n} - 4u_{i,j}.$$

Let $z = (sh, kh) \in \dot\Omega_h$ be an interior vertex. We introduce the following additional notation for the second-order directional finite differences at z:

(6.4) $$D_1^2 v_{s,k} = u_{s-1,k} - 2u_{s,k} + u_{s+1,k}, \qquad D_2^2 v_{s,k} = u_{s,k-1} - 2u_{s,k} + u_{s,k+1}.$$

Then the following discrete representation of $\Delta_{\sqrt{2}\delta} v(x)$ holds:

$$\Delta_{\sqrt{2}\delta} v(x) = R_1(v, x, \delta) + D_1^2 v_{i,j} + D_2^2 v_{i,j} + \sum_{(s,k)\in\Lambda_1\cup\Lambda_3} D_1^2 v_{s,k} + \sum_{(s,k)\in\Lambda_2\cup\Lambda_4} D_2^2 v_{s,k}$$

(6.5) $$+ \sum_{(s,k)\in\Lambda_5} \frac{1}{2} D_1^2 v_{s,k} + \frac{1}{2} D_2^2 v_{s,k},$$

where the remainder $R_1(v, x, \delta)$ is defined by

$$2R_1(v, x, \delta) = (v_{i-n+1,j+n} - v_{i-n,j+n}) + (v_{i-n,j+n-1} - v_{i-n,j+n})$$

$$+ (v_{i+n-1,j+n} - v_{i+n,j+n}) + (v_{i+n,j+n-1} - v_{i+n,j+n})$$

(6.6)

$$+ (v_{i+n-1,j-n} - v_{i+n,j-n}) + (v_{i+n,j-n+1} - v_{i+n,j-n})$$

$$+ (v_{i-n+1,j-n} - v_{i-n,j-n}) + (v_{i-n,j-n+1} - v_{i-n,j-n}).$$

We now define a vertex-centered entropy as follows:

(6.7) $$E_h(v) := \sum_{(ih,jh)\in\dot\Omega_h} (D_1^2 v_{i,j})_+^{p_3} + (D_2^2 v_{i,j})_+^{p_3},$$

where

$$(6.8) \qquad\qquad\qquad p_3 > 2.$$

We modify the functional J_h by setting $p_1 = p_2 = 1$ and by adding the vertex-centered entropy

$$(6.9) \quad J_h(v) = \int_\Omega |H(x, v, Dv)|dx$$
$$+ h \sum_{K \in \mathcal{T}_h} \int_K \lambda_+(v)dx + h \sum_{F \in \mathcal{F}_h^i} \int_F \{-\partial_n v\}_+ d\sigma + h^{3-2p_3} E_h(v),$$

and we henceforth denote by $\{u_h\}_{h>0}$ a sequence of almost minimizers for (2.8) using the above modified functional.

6.2. Consistency. The goal of this section is to show that, with the above choice of entropy, it is possible to construct an approximation of u, say $\mathcal{I}_h u$, that satisfies the estimate

$$(6.10) \qquad\qquad\qquad J_h(\mathcal{I}_h u) \leq ch.$$

We make use of the Clément interpolation operator [7, 6] to this purpose. Let v be an arbitrary function in $L^1(\Omega)$. We define $\mathcal{I}_h(v)$ to be a piecewise linear function on the mesh $\mathcal{T}_h$ as follows. Let a be any vertex of $\mathcal{T}_h$. If a is on Γ, we set $\mathcal{I}_h(v)(a) = 0$. If a is an interior vertex, we define Δ_a to be the set of all those triangles that have a as a vertex. We define $r(v) \in \mathbb{P}_1$ to the linear polynomial such that

$$(6.11) \qquad\qquad \int_{\Delta_a} (r(v)(x) - v(x))q(x)dx = 0 \qquad \forall q \in \mathbb{P}_1.$$

Then we set $\mathcal{I}_h(v)(a) = r(v)(a)$. The interpolant $\mathcal{I}_h$ thus defined is $W^{1,\infty}$-stable and there is $c > 0$ such that for all $m \in \{0, 1\}$, $k \in \{0, 1\}$, any number $p \geq 1$, and all $v \in W^{k+1,p}(\Omega) \cap W_0^{1,p}(\Omega)$ the following holds (see [6, 7]):

$$(6.12) \qquad\qquad \|\mathcal{I}_h(v) - v\|_{W^{m,p}(\Omega)} \leq c\, h^{k+1-m} \|v\|_{W^{k+1,p}(\Omega)},$$

$$(6.13) \qquad \left(\sum_{F \in \mathcal{F}_h^i} \|\mathcal{I}_h v - v\|_{W^{m,p}(F)}^p \right)^{1/p} \leq c\, h^{k+1-m-\frac{1}{p}} \|v\|_{W^{k+1,p}(\Omega)}.$$

LEMMA 6.1. *Under the above hypotheses and with the definition* (6.9) *of the functional J_h, there is c, uniform in h, such that the linear Clément interpolant of u, say $\mathcal{I}_h(u)$, satisfies*

$$(6.14) \qquad\qquad\qquad J_h(\mathcal{I}_h(u)) \leq c\, h.$$

Proof. The $W^{1,\infty}$-stability and the approximation property (6.12) with $m = p = k = 1$ of the Clément interpolant together with the assumption (1.3) yields $\|H(\cdot, \mathcal{I}_h(u), D\mathcal{I}_h(u))\|_{L^1(\Omega)} \leq c\, h$. Since $\mathcal{I}_h(u)$ is piecewise linear, the volume entropy in J_h involving $\lambda_+(\mathcal{I}_h(u))$ is zero. The q-semiconcavity of u implies $\{-\partial_n \mathcal{I}_h u\}_+ = \{-\partial_n(\mathcal{I}_h u - u)\}_+$ across every $F \in \mathcal{F}_h^i$, since the $W^{2,q}$-component of u has a continuous gradient by embedding (recall that $q > 2$). This together with (6.13) (using $k = 1$,

$m = 0$, $p = 1$) yields that the interface entropy involving $\{-\partial_n \mathcal{I}_h u\}_+$ is bounded from above by $c\,h$.

Now we have to make sure that the vertex-centered entropy is appropriately controlled. Let $a \in \dot\Omega_h$ be an interior mesh vertex. Let us evaluate $r(u)$ at a as defined in (6.11). To do so, we expand $r(u)$ in the following manner:

$$r(u)(x) = \alpha + \beta \cdot (x - a),$$

where $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^2$ are yet to be determined. Since the mesh is structured, the following holds:

$$\int_{\Delta_a} (x - a) dx = 0.$$

This immediately implies

$$0 = \int_{\Delta_a} (r(u) - u) dx = |\Delta_a| \left(\alpha - \Delta_a^{-1} \int_{\Delta_a} u \, dx \right),$$

i.e., $r(u)(a) = \alpha = \frac{1}{\Delta_a} \int_{\Delta_a} u \, dx$.

Let i be an index in $\{1, 2\}$. Let us set $\Delta_a^+ = \Delta_a + h e_i$ and $\Delta_a^- = \Delta_a - h e_i$ Then, with obvious notation,

$$|\Delta_a| D_i^2 r(u)(a) = \int_{\Delta_a^+} u(x) dx - 2 \int_{\Delta_a} u(x) dx + \int_{\Delta_a^-} u(x) dx$$

$$= \int_{\Delta_a} (u(x + h e_i) dx - 2u(x) + u(x - h e_i)) \, dx$$

$$= \int_{\Delta_a} D_i^2 u(x) dx.$$

Now, using the q-semiconcavity hypothesis (1.5), we have $u = v_c + w$, where $v_c \in W^{1,\infty}(\Omega)$ is concave and $w \in W^{2,q}(\Omega)$. Then, using the concavity of v_c and the $W^{2,q}$-regularity of w, infer

$$E(\mathcal{I}_h(u)) = E(\mathcal{I}_h(w)) \leq ch^{2p_3 - 2}.$$

This implies $h^{3 - 2p_3} E(\mathcal{I}_h(u)) \leq c\,h$. This completes the proof. $\quad\square$

Lemma 6.1 means that the set of almost minimizers is not empty when using definition (6.9). No extra assumptions need to be made.

Remark 6.1. Note that the fact that the mesh is structured is a key argument in the proof of Lemma 6.1.

6.3. Convergence to the viscosity solution. Let $\delta \geq h$ be a real number that we assume for the time being to be a multiple of h, i.e., $\delta = nh$ with $n \geq 1$. Consider a point $x := (ih, jh) \in \Omega_h$ such that $x + B_h(0, \sqrt{2}\delta) \subset \Omega_h$.

LEMMA 6.2. *Under the above hypotheses, for all sequences of almost minimizers of (2.8), say $\{u_h\}_{h>0}$, there is c, independent of x, h, and δ, and there is $\gamma := 1 - \frac{2}{p3} > 0$ so that*

(6.15) $$\Delta_{\sqrt{2}\delta} u_h(x) \leq c\delta^{1+\gamma} + R_1(u_h, x, \delta).$$

Proof. From (6.5) we infer

$$\Delta_{\sqrt{2}\delta}u_h(x) \leq R_1(u_h, x, \delta) + \frac{5}{2}\sum_{(s,k)\in\Lambda_0}(D_1^2(u_h)_{s,k})_+ + \frac{5}{2}\sum_{(s,k)\in\Lambda_0}(D_2^2(u_h)_{s,k})_+.$$

Using Hölder's inequality, this implies

$$\Delta_{\sqrt{2}\delta}u_h(x) \leq R_1(u_h, x, \delta) + \mathrm{card}(\Lambda_0)^{\frac{1}{p_3'}}(E_h(u_h))^{\frac{1}{p_3}},$$

where $\mathrm{card}(\Lambda_0)$ is the cardinal number of Λ_0. Clearly $\mathrm{card}(\Lambda_0) \leq c\,(\delta/h)^2$. Moreover, since $\{u_h\}_{h>0}$ is a sequence of almost minimizers, it comes that $E_h(u_h) \leq c\,h^{2(p_3-1)}$. That is to say,

$$\Delta_{\sqrt{2}\delta}u_h(x) \leq R_1(u_h, x, \delta) + c\,\delta^{2/p_3'}h^{-2/p_3'}h^{2(p_3-1)/p_3} \leq R_1(u_h, x, \delta) + c\,\delta^{1+\gamma},$$

where $\gamma = 1 - 2/p_3 > 0$ since $p_3 > 2$. ∎

We now conclude the following theorem.

THEOREM 6.3. *Let $u \in X$ be the unique solution to* (1.1). *Consider the uniform mesh family defined by* (6.1). *Under the uniqueness assumption* (5.5), *and the restriction $p_3 > 2$, every sequence of almost minimizers for the functional* (6.9) *converges to the unique viscosity solution to* (1.1).

Proof. Let $\{u_h\}_{h>0}$ be a sequence of almost minimizers. Let x be a point in Ω. There exists δ_0 such that $B_{\mathbb{R}^2}(x, \delta_0) \subset \Omega$. Let δ be a fixed number in $(0, \delta_0/2\sqrt{2}]$ and let h be an arbitrary mesh size such that $h \leq \delta$.

Since the ratio δ/h may not be an integer and/or x is almost surely not a member of Ω_h, we define $n := \lfloor \delta/h \rfloor$, $i := \lfloor (x{\cdot}e_1)/h \rfloor$, and $j := \lfloor (x{\cdot}e_2)/h \rfloor$, where $\lfloor \cdot \rfloor$ is the floor function. Then we set $\overline{\delta} = nh$ and $\overline{x} = (ih, jh)$. Note that with our choice of parameters, $\overline{z}$ is in Ω_h, where $\overline{z}$ is either $\overline{x}$ or $\overline{x} \pm \overline{\delta}e_1 \pm \overline{\delta}e_2$. These definitions imply

$$(6.16) \qquad \Delta_{\sqrt{2}\delta}u_h(x) = \Delta_{\sqrt{2}\,\overline{\delta}}u_h(\overline{x}) + R_2(u_h, x, \delta),$$

where the remainder is defined by $R_2(u_h, x, \delta) := \Delta_{\sqrt{2}\delta}u_h(x) - \Delta_{\sqrt{2}\,\overline{\delta}}u_h(\overline{x})$.

Now we use the one-sided bound (6.15) from Lemma 6.2 to obtain

$$\Delta_{\sqrt{2}\delta}u_h(x) \leq c\overline{\delta}^{1+\gamma} + R_1(u_h, \overline{x}, \overline{\delta}) + R_2(u_h, x, \delta).$$

We conclude by passing to the limit on h. Clearly $\overline{\delta} \to \delta$. Since $u_h \to u$ a.e. in Ω, we infer $\Delta_{\sqrt{2}\delta}u_h(x) \to \Delta_{\sqrt{2}\delta}u(x)$ for a.e. x in Ω. Moreover, owing to Lemma 6.4, $R_1(u_h, \overline{x}, \overline{\delta}) \to 0$ and $R_2(u_h, x, \delta) \to 0$ for a.e. x in Ω. As a result,

$$\Delta_{\sqrt{2}\delta}u(x) \leq c\delta^{1+\gamma} \text{ for a.e. } x \in \Omega,$$

and the constant c does not depend on x. Since u is continuous, this implies that the inequality holds for every x in Ω and every δ such that $B_{\mathbb{R}^2}(x, \delta_0) \subset \Omega$. ∎

Remark 6.2. Recall that the class of stationary Hamilton–Jacobi equations defined by $H(x, u, Du) = u + F(Du)$, where $F : \mathbb{R}^2 \longrightarrow \mathbb{R}$ is convex, has a unique viscosity solution characterized by (5.5); see [18, Thm. 2.6]. Moreover, it is known that the solution is ∞-semiconcave under appropriate restrictions on the domain. In other words, Theorem 6.3 holds at least for the above class of Hamilton–Jacobi equations.

LEMMA 6.4. *Under the above hypotheses, $R_1(u_h, \overline{x}(x), \overline{\delta}) \to 0$ for a.e. $x \in \Omega$ and $R_2(u_h, x, \delta) \to 0$ for a.e. $x \in \Omega$.*

Proof. R_1 is composed of eight terms which have the generic form

$$r_h(\overline{x}, \overline{\delta}, h) := u_h(\overline{x} + s_1\overline{\delta}e_\pm + s_2he_i) - u_h(\overline{x} + s_1\overline{\delta}e_\pm),$$

where $e_\pm = e_1 \pm e_2$, $i \in \{1, 2\}$ and $s_1, s_2 \in \{-1, +1\}$. To avoid boundary issues, we extend r_h to $\mathbb{R}^2$ by replacing u_h by $\tilde{u}_h$; the extension in question is denoted by $\tilde{r}_h$. We now evaluate the L^1-norm of $\tilde{r}_h(\overline{x}(x), \overline{\delta}, h)$ as follows:

$$\|\tilde{r}_h(\overline{x}(\cdot), \overline{\delta}, h)\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\tilde{u}_h(\overline{x}(x) + s_1\overline{\delta}e_\pm + s_2he_i) - \tilde{u}_h(\overline{x}(x) + s_1\overline{\delta}e_\pm)|dx$$

$$= \int_{\mathbb{R}^2} |\tilde{u}_h(\overline{x}(x) + s_2he_i) - \tilde{u}_h(\overline{x}(x))|dx$$

$$= h^2 \sum_{k,l=-\infty}^{+\infty} |\tilde{u}_h(x_{k,l} + s_2he_i) - \tilde{u}_h(x_{k,l})|$$

$$= h^2 \sum_{k,l=-\infty}^{+\infty} \left| \int_{F_{k,l}} \partial_{e_i}\tilde{u}_h(y)dy \right|,$$

where we have denoted $x_{k,l} = (kh, lh)$ and $F_{k,l}$ is the segment $(x_{k,l}, x_{k,l} + s_2he_i)$. Note that F_{kl} is equal to $(x_{k,l}, x_{(k+s_2),l})$ if $i = 1$ and $(x_{k,l}, x_{k,(l+s_2)})$ if $i = 2$. Let us denote by $\Delta_{x_{k,l}}$ the set of all those triangles that have $x_{k,l}$ as a vertex (there are six of those). Then a trace and an inverse inequality yield

$$\left| \int_{F_{k,l}} \partial_{e_i}\tilde{u}_h(y)dy \right| \le ch^{-1} \sum_{K \in \Delta_{x_{k,l}}} \int_K \|D\tilde{u}_h\|_{L^1(K)}.$$

This then yields

$$\|\tilde{r}_h(\overline{x}(\cdot), \overline{\delta}, h)\|_{L^1(\mathbb{R}^2)} \le ch \sum_{k,l=-\infty}^{+\infty} \sum_{K \in \Delta_{x_{k,l}}} \int_K \|D\tilde{u}_h\|_{L^1(K)}$$

$$\le ch\|\tilde{u}_h\|_{W^{1,1}(\mathbb{R}^2)} \le c'h\|u_h\|_{W^{1,1}(\Omega)}.$$

This means $\tilde{r}_h(\overline{x}(\cdot), \overline{\delta}, h) \to 0$ in $L^1(\mathbb{R}^2)$, which immediately implies $r_h(\overline{x}(x), , \overline{\delta}, h) \to 0$ for a.e. x in Ω.

For R_2, we observe that R_2 is composed of five terms which have the generic form

$$r_h(x, \delta, h) := u_h(x + s\delta e_\pm) - u_h(\overline{x}(x) + s\overline{\delta}e_\pm),$$

where $s \in \{-1, 0, +1\}$. To avoid boundary issues, we again extend r_h to $\mathbb{R}^2$ by replacing u_h by $\tilde{u}_h$; the extension in question is denoted by $\tilde{r}_h$. We now evaluate the

L^1-norm of $\tilde{r}_h(x, \delta, h)$ as follows:

$$\|\tilde{r}_h(\cdot, \overline{\delta}, h)\|_{L^1(\mathbb{R}^2)} = \int_{\mathbb{R}^2} |\tilde{u}_h(x + s\delta e_\pm) - \tilde{u}_h(\overline{x}(x) + s\overline{\delta}e_\pm)|dx$$

$$\leq \int_{\mathbb{R}^2} |\tilde{u}_h(x + s\delta e_\pm) - \tilde{u}_h(x + s\overline{\delta}e_\pm)|dx$$

$$+ \int_{\mathbb{R}^2} |\tilde{u}_h(x + s\overline{\delta}e_\pm) - \tilde{u}_h(\overline{x}(x) + s\overline{\delta}e_\pm)|dx$$

$$\leq \int_{\mathbb{R}^2} |\tilde{u}_h(x) - \tilde{u}_h(x + s(\overline{\delta} - \delta)e_\pm)|dx$$

$$+ \int_{\mathbb{R}^2} |\tilde{u}_h(x) - \tilde{u}_h(\overline{x}(x))|dx.$$

Let r_1, r_2 be the two integrals in the right-hand side, respectively. For r_1 we have

$$r_1 = \int_{\mathbb{R}^2} \left| \int_0^1 D\tilde{u}_h(x + \theta s(\overline{\delta} - \delta)e_\pm)\cdot(\overline{\delta} - \delta)e_\pm d\theta \right| dx$$

$$\leq |\overline{\delta} - \delta| \int_0^1 \int_{\mathbb{R}^2} \|D\tilde{u}_h(x + \theta s(\overline{\delta} - \delta)e_\pm)\|dxd\theta \leq h\|D\tilde{u}_h\|_{L^1(\mathbb{R}^2)}$$

$$\leq c\,h\,\|u_h\|_{W^{1,1}(\Omega)}.$$

For the second residual we have

$$r_2 = \sum_{k,l=-\infty}^{+\infty} \int_{S_{k,l}} |\tilde{u}_h(x) - \tilde{u}_h(x_{k,l})|dx,$$

where $x_{k,l} = (kh, lh)$ and $S_{k,l}$ is the square $(x_{k,l}, x_{k+1,l}) \times (x_{k,l}, x_{k,l+1})$. Then a trace inequality and an inverse inequality yields

$$r_2 = \sum_{k,l=-\infty}^{+\infty} \int_{S_{k,l}} \left| \int_0^1 D\tilde{u}_h(x + \theta(x_{k,l} - x))\cdot(x_{k,l} - x)d\theta \right| dx$$

$$\leq c\,h^{-1} \sum_{k,l=-\infty}^{+\infty} \int_{S_{k,l}} \|D\tilde{u}_h\|_{L^1(S_{k,l})} \leq c\,h \sum_{k,l=-\infty}^{+\infty} \|D\tilde{u}_h\|_{L^1(S_{k,l})}$$

$$\leq c\,h\|u_h\|_{W^{1,1}(\Omega)}.$$

We then conclude as above. $\quad\square$

7. Numerical experiments. A one-dimensional theory for the L^1-approximation of stationary Hamilton–Jacobi equations is developed in [14], and efficient numerical algorithms are proposed and analyzed in [12].

The purpose of this section is to support our theory by reporting two-dimensional numerical experiments. Our goal is not to analyze or discuss the optimality of any given numerical strategy to solve (2.8) but to show that L^1-minimizers are computable
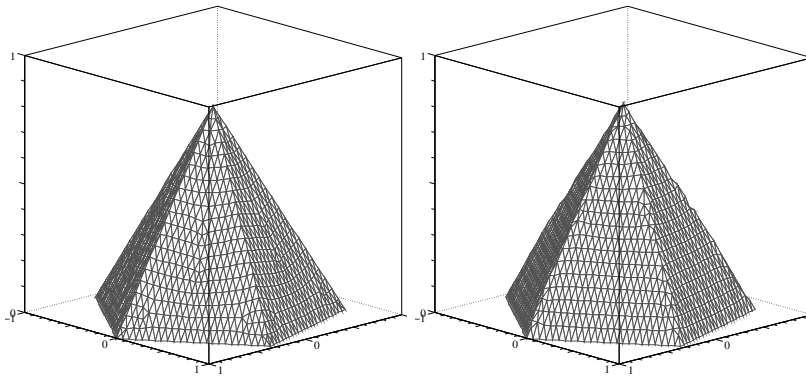
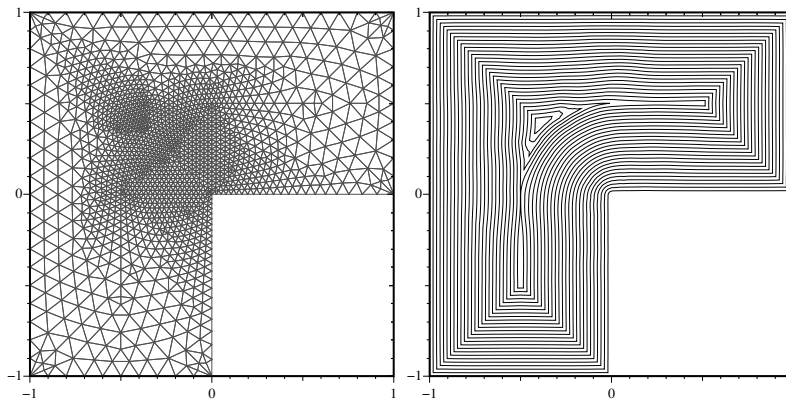Fɪɢ. 4. *Pentagon: Aligned unstructured mesh (left); nonaligned unstructured mesh (right).*

Fɪɢ. 5. *L-shaped domain: Unstructured mesh (left); iso-lines of approximate minimizer (right).*

and are very accurate nonoscillatory approximations to viscosity solutions of stationary two-dimensional Hamilton–Jacobi equations. We henceforth focus our attention on the eikonal equation, $\|Du\| = 1$, equipped with homogeneous Dirichlet boundary conditions. The computations are done using piecewise linear continuous finite elements. The entropy is defined using $p = 2$. The discrete problem (2.8) is solved by using an iterative regularization method described in [11]. In a few words, the algorithm consists of computing $\lim_{h\to 0} \lim_{\epsilon\to 0} \operatorname{argmin}_{v_h \in X_h} J_{h,\epsilon}(v_h)$. The functional $J_{h,\epsilon}(v_h)$ is a regularized version of $J_h(v_h)$, where the absolute value defining the L^1-norm and the $(\cdot)_+$ function are replaced by $x \longmapsto x^2/(|x| + \epsilon)$. The minimization problem is solved by using a Newton method. The number ϵ is used as a continuation parameter. The computation stops when $\epsilon = 1.10^{-5}$. The mesh size h is also used as a continuation parameter in the sense that the computation is done on three grids successively refined. The result on a coarse grid is used to initialize the solution on the next grid.

In the first example the domain Ω is a pentagon. The computation is done on two types of meshes. The first type is composed of meshes that are aligned with the discontinuities of the gradient and the second type consists of unstructured meshes. Typical results are reported in Figure 4. For both mesh types, we observe that the approximate L^1-minimizer is similar to the Lagrange interpolant of the exact solution

on the same mesh. This is what we should expect intuitively. The L^1-minimization process solves the equation in the region where the solution is smooth and simply ignores the PDE in the regions where the gradient of the exact solution is discontinuous. For more details we refer to [13, 14].

The second example is the eikonal equation on an L-shaped domain. The viscosity solution to this problem is in $W^{1,\infty}(\overline{\Omega})$ and is q semiconcave for every $q < 2$. This is a borderline case not covered by our theory (we a priori need $q > 2$). We nevertheless do the computations using $p = 2$ for the entropy. We show a mesh and the corresponding approximate minimizer in Figure 5. Once again, we observe that the solution is accurate. The iso-lines are not oscillating and are very sharp.

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