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# $L^1$-minimization methods for Hamilton–Jacobi equations: the one-dimensional case

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**Abstract** A new approximation technique based on $L^1$-minimization is introduced. It is proven that the approximate solution converges to the viscosity solution in the case of one-dimensional stationary Hamilton–Jacobi equation with convex Hamiltonian.

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## 1 Introduction

The goal of the present paper is to investigate the approximation properties of a new class of $L^1$-minimization techniques for approximating stationary Hamilton–Jacobi equations in one space dimension.

Most approximation algorithms of Hamilton–Jacobi equations are based on monotonicity and Lax–Friedrichs approximate Hamiltonians, see, e.g., Kao et al. [13]. Monotonicity is very often invoked to prove convergence of low-order approximations, see, e.g., Crandall and Lions [6], Barles and Souganidis [2]. In this spirit, Abgrall [1] proved convergence for a class of first-order schemes on meshes composed of triangles using a monotonicity-based argument by Crandal and Lions [6]. For higher-order approximations, limiters must be brought into the game as monotonicity cannot be preserved. For instance, second-order MUSCL-type finite difference approximations have been shown to converge to viscosity solutions by Lions and Souganidis [17], see also Osher and Shu [19] for higher-order discretizations. We refer to Osher and Fedkiw

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[18] and Sethian [21] for reviews of the approximation literature of Hamilton–Jacobi equations.

In the present paper we take a radically different point of view by formulating the discrete problem as a minimization in $L^1(a, b)$. The motivation behind this approach is based on observations made in [8] that $L^1$-minimization is capable of selecting viscosity solutions of transport equations equipped with ill-posed boundary conditions. This fact has indeed been proved in [10] in one space dimension. This encouraged us to build a research program in this direction and the purpose of the present work is to show that indeed $L^1$-minimization is a viable technique. The idea of using $L^1$ to construct nonlinear approximations of PDE's is quite new (although, for early attempts in computational fluid dynamics we refer to [8,12,15,16]) but seems to be gaining momentum in the image processing/denoising community [3–5,20].

In this paper we describe a nonlinear finite element technique to approximate viscosity solutions of stationary Hamilton–Jacobi equations in one space dimension using continuous finite elements of arbitrary degree. The method is based on $L^1$-minimization: a functional containing the $L^1$-norm of the Hamiltonian plus discrete entropy terms is minimized over the finite element space. Under appropriate hypotheses on the Hamiltonian (say convexity and Lipschitz boundedness), it is shown that the algorithm converges to the unique viscosity solution. The main results of the paper are Theorems 5.2 and 6.2.

The paper is organized as follows. The continuous problem is introduced in Sect. 2. The discrete finite element setting along with the minimization problem is introduced in Sect. 3. The existence of minimizers to the discrete problem is proved in Sect. 4. The passage to the limit is done in Sect. 5, i.e., it is shown in this section that the limit solution solves the PDE almost everywhere. The proof that the limit solution is indeed a viscosity solution is reported in Sect. 6. The main argument consists of proving a one-sided bound on second-order finite differences of the solution.

## 2 The continuous problem

We consider the following stationary Hamilton–Jacobi equation

$$H(x, u, u') = 0, \text{ in } (a, b), \quad \text{with } u(a) = \alpha, \ u(b) = \beta, \tag{2.1}$$

where $[a, b]$ is a bounded interval. We henceforth assume that Hamiltonian $H$ satisfies the following properties:

$$|p| \le c_s\,(|H(x, v, p)| + |v| + 1), \quad \forall (x, v, p) \in [a, b]\times\mathbb{R}\times\mathbb{R}, \tag{2.2}$$

$$H(x, v, p) \text{ is uniformly Lipschitz on } [a, b]\times[-R, R]\times\overline{B(0, R)} \text{ for all } R > 0. \tag{2.3}$$

A typical example is the eikonal equation or any stationary Hamilton–Jacobi equations derived from scalar conservation laws with convex flux, see Evans [7], Kružkov [14], or Lions and Souganidis [17].

We assume that (2.1) has a unique viscosity solution $u$ such that

$$u \in W^{1,\infty}(a, b) \cap \mathcal{C}^0[a, b], \tag{2.4}$$

$$u \text{ is } q\text{-semiconcave for some } q > 1, \tag{2.5}$$

where we understand $q$-semiconcavity in the following sense:

**Definition 2.1** A function $u$ in $W^{1,\infty}(a, b)$ is said to be $q$-semiconcave if there is a concave function $v_c \in W^{1,\infty}(a, b)$ and a function $w \in W^{2,q}(a, b)$ so that $u = v_c + w$.

*Remark 2.1* Recall that a function $v$ in $W^{1,\infty}(a, b)$ is usually called uniformly semi-concave in textbooks if and only if it can be decomposed into $v(x) = v_c(x) + c_v x^2$ where $c_v$ is a nonnegative constant and $v_c$ is concave and in $W^{1,\infty}(a, b)$. Definition 2.1 is a slight generalization of semiconcavity.

An immediate consequence of Definition 2.1 is that if a function $u$ is $q$-semiconcave, then there is $c > 0$ such that for all $\delta > 0$ and all $\omega \subset (a, b)$ so that $\omega \pm \delta \subset \Omega$, the following hold

$$u(x + \delta) - 2u(x) + u(x - \delta) \le c\, \delta^{2-\frac{1}{q}}, \quad \forall x \in \omega \tag{2.6}$$

$$\|(u(\cdot + \delta) - 2u(\cdot) + u(\cdot - \delta))_+\|_{L^q(\omega)} \le c\, \delta^2, \tag{2.7}$$

where $(t)_+ := \frac{1}{2}(t + |t|)$ denotes the positive part of $t$ for all $t \in \mathbb{R}$. Henceforth $c$ denotes a generic constant which may vary at each occurrence but does not depend on $\delta$ (nor on the mesh parameter $h$, see Sect. 3).

*Remark 2.2* Note that (2.4)–(2.5) implies that $u' \in \mathrm{BV}(a, b)$. Actually, Definition 2.1 implies $u = v_c + w$ where $v_c \in W^{1,\infty}(a, b)$ is concave and $w \in W^{2,q}(a, b) \subset W^{2,1}(a, b)$. Hence,

$$|u'|_{\mathrm{BV}(a,b)} \le 2\|v_c'\|_{L^\infty(a,b)} + \|w'\|_{\mathrm{BV}(a,b)}.$$

To be able to collectively refer to (2.4)-(2.5), we define

$$X = \left\{ v \in W^{1,\infty}(\Omega) \cap \mathcal{C}^0[a, b];\ v \text{ is } q\text{-semiconcave} \right\}. \tag{2.8}$$

The goal of this paper is to construct a sequence of approximate solution to (2.1) using continuous finite elements and by minimizing the residual in $L^1(a, b)$. We show that upon introducing an appropriate entropy, the sequence of approximate solutions converges to the unique viscosity solution to (2.1). The fact that the residual is minimized in $L^1(a, b)$ is a key.

## 3 The discrete problem

### 3.1 The meshes

Let $\{\mathcal{T}_h\}_{h>0}$ be an indexed family of finite element meshes. We assume that for any given $h > 0$, $\mathcal{T}_h$ is a partition of the interval $[a, b]$. Namely, for all index $h > 0$, there is an integer $n > 0$ such that $\mathcal{T}_h = \cup_{i=0}^{n}[x_i, x_{i+1}]$ with $x_0 = a$, $x_{n+1} = b$, and $h_i = x_{i+1} - x_i$. The quantity $h = \max_{0\le i\le n} h_i$ is called the meshsize of the partition $\mathcal{T}_h$. We denote by $\mathcal{V}_h = \{x_i;\ 0 \le i \le n\}$ and $\mathcal{V}_h^i = \mathcal{V}_h \cap (a, b) = \{x_i\}_{i=1}^{n}$ the collection of the mesh vertices and interior mesh vertices, respectively. Let $N_h(x, \delta)$ denote the number of interior vertices in the interval $(x - \delta, x + \delta)$, i.e., $N_h(x, \delta) = \operatorname{card}(\mathcal{V}_h^i \cap (x - \delta, x + \delta))$.

**Definition 3.1** We say that the mesh family $\{\mathcal{T}_h\}_{h>0}$ is *almost uniform* if there are $c_1, c_2 > 0$ such that for all $h > 0$

$$x_1 - a \ge c_1 h, \quad b - x_n \ge c_1 h, \tag{3.1}$$

$$N_h(x, \delta) \le c_2 \left(\frac{\delta}{h} + 1\right), \quad \forall x, \delta,\ \text{s.t. } x - \delta, x + \delta \in [a, b]. \tag{3.2}$$

In this paper, we will only consider almost uniform mesh families.

Let $k \ge 1$ be an integer and denote by $\mathbb{P}_k$ the set of real-valued polynomials in $[a, b]$ of total degree at most $k$. We introduce

$$X_h = \left\{v_h \in \mathcal{C}^0[a, b];\ v_{h|K} \in \mathbb{P}_k,\ \forall K \in \mathcal{T}_h;\ v_h(a) = \alpha,\ v_h(b) = \beta\right\}, \tag{3.3}$$

$$X_{(h)} = X + X_h. \tag{3.4}$$

For every function $v$ in $X_{(h)}$ we denote by $\{-\partial_n v\}_+ : \mathcal{V}_h^i \longrightarrow \mathbb{R}^+$ the map such that for all $\{x\} = K_1 \cap K_2 \in \mathcal{V}_h^i$,

$$\{-\partial_n v\}_+(x) = \left(-\tfrac{1}{2}(v'_{|K_1}(x)\cdot n_1 + v'_{|K_2}(x)\cdot n_2)\right)_+,$$

where $n_1$ and $n_2$ are the unit outward normals to the mesh cells $K_1$ and $K_2$ at $x$ respectively. Note that, if $K_1 = [x_{i-1}, x_i]$ and $K_2 = [x_i, x_{i+1}]$, then $\{-\partial_n v\}(x_i)$ is the jump of $v'_h$ at $x_i$, i.e., $\{-\partial_n v\}(x_i) = v'_h(x_i + 0) - v'_h(x_i - 0)$. That gives $\{-\partial_n v\}_+(x_i) = (v'_h(x_i + 0) - v'_h(x_i - 0))_+$.

### 3.2 The discrete minimization problem

Let $p$ be a fixed real number such that

$$1 < p \le q. \tag{3.5}$$

We define the following functional $J_h : X_{(h)} \ni v \longmapsto J_h(v) \in \mathbb{R}^+$ by:

$$J_h(v) = \int_a^b |H(x, v, v')|dx + h \sum_{K\in\mathcal{T}_h} \int_K (v''(x))_+^p dx + h^{2-p} \sum_{x_i\in\mathcal{V}_h^i} (\{-\partial_n v\}_+(x_i))^p . \tag{3.6}$$

For every function $v$ in $X_{(h)}$ we refer to $\int_a^b |H(x, v, v')|dx$ as the residual. The two extra terms in the right-hand side above are referred to as the volume entropy term and the interface entropy term.

*Remark 3.1* Whenever $v \in W^{1,\infty}(a, b)$ is a concave function, for example in the case of the eikonal equation, the two entropy term are zero, i.e., these two terms do not add extra viscosity. They are meant to prevent the occurrence of large positive second derivatives.

*Remark 3.2* If the solution to (2.1) is known to be smooth and if high-order finite elements are used (i.e., $k > 1$), the functional $J_h$ must be modified as follows

$$\begin{aligned} J_h(v) &= \int_a^b |H(x, v, v')|dx \\ &\quad + h^k \sum_{K\in\mathcal{T}_h} \int_K (v''(x))_+^p dx + h^{1+k(1-p)} \sum_{i=1}^n (\{-\partial_n v\}_+(x_i))^p , \end{aligned} \tag{3.7}$$

to benefit from the high-order interpolating capability of the finite elements. Rescaled this way, the three terms composing $J_h(v)$ are each of order $h^k$ whenever $v$ is a smooth function. Of course, better expressions for the functional can be devised by using the local mesh size $h_K = \mathrm{diam}(K)$ instead of $h$.

We henceforth focus our attention on the following minimization problem: Seek $u_h$ in $X_h$ such that

$$J_h(u_h) = \inf_{v_h\in X_h} J_h(v_h). \tag{3.8}$$

The goal of the rest of the paper is to show that (at least) one minimizer exists and every sequence of minimizers (or sequence of almost minimizers) converges to the unique viscosity solution to (2.1).

## 4 Existence of minimizers

The goal of this section is to prove the existence of (at least) one minimizer for the discrete problem (3.8). This is done by deriving a decay rate (in terms of $h$) of the discrete residual, deriving a priori bounds, and using a compactness result.

### 4.1 The $W^{1,1}(a,b)$ bound

For every positive real number $\gamma > 0$ we define the set $S_{h,\gamma} = \{v_h \in X_h;\ \int_a^b |H(x, v_h, v_h')|dx \le \gamma h\}$. The following result shows that the collection $\{S_{h,\gamma}\}_{h>0}$ is uniformly bounded.

**Lemma 4.1** *Let $\gamma > 0$ and assume that $S_{h,\gamma}$ is not empty, then there is $c_0(\gamma) > 0$, independent of $h$, and $h_0 > 0$ such that*

$$\forall h < h_0, \quad \forall v_h \in S_{h,\gamma}, \qquad \|v_h\|_{W^{1,1}} + \|v_h\|_{L^\infty} \le c_0(\gamma). \tag{4.1}$$

*Proof* Let $v_h$ be a member of $S_{h,\gamma}$. For $a \le x \le b$ we define

$$F(x) = \int_a^x |v_h'(s)|ds.$$

Owing to (2.2), we infer

$$F(x) \le \int_a^x c_s(|H(s, v_h, v_h')| + |v_h| + 1)ds.$$

The fact that $v_h$ is a member of $S_{h,\gamma}$ implies

$$F(x) \le c_s\left(\gamma h + \int_a^x |v_h(s)|ds + x - a\right).$$

Note that

$$|v_h(s)| \le |v_h(a)| + |v_h(s) - v_h(a)| \le |\alpha| + F(s). \tag{4.2}$$

Using the above, we derive

$$F(x) \le c_s\,(\gamma + (|\alpha| + 1)(b - a)) + c_s \int_a^x F(s)ds$$

for all $a \le x \le b$ and all $h \le h_0 := 1$. Applying Gronwall's lemma, we infer

$$F(x) \le c_s\,(\gamma + (|\alpha| + 1)(b - a))\, e^{c_s(x-a)} \le c, \tag{4.3}$$

where $c$ is a constant which depends on $c_s$, $\gamma$, the size of the interval $b - a$, and the boundary value $u(a) = \alpha$ but is independent of $h$ for any $h \le 1$. Using (4.2) and (4.3), we derive the following maximum principle and variation bound

$$\|v_h\|_{L^\infty} + \|v_h\|_{\mathrm{BV}} = \|v_h\|_{L^\infty} + \|v_h\|_{W^{1,1}} \le c_0(\gamma).$$

□

4.2 Consistency

We are going to derive a consistency property. Let $\mathcal{I}_h : X \longrightarrow X_h$ be the linear Lagrange interpolation operator on piecewise linear polynomials. Note that $\mathcal{I}_h$ is stable on $W^{1,\infty}(a,b)$, and convexity preserving. The following lemma gives an estimate of the discrete residual $J_h$ in terms of $h$.

**Lemma 4.2** *Let $u$ solve (2.1), then there is $c(u)$ independent of $h$ such that*

$$J_h(\mathcal{I}_h u) \leq c(u)\, h. \tag{4.4}$$

*Proof* Observe that

(1) Since $\mathcal{I}_h u$ is piecewise linear, the second derivative of $\mathcal{I}_h u$ is zero inside each interval $K_i = [x_i, x_{i+1}]$, i.e., $(\mathcal{I}_h u)''_{|K_i} = 0$ for all $0 \leq i \leq n$;

(2) Owing to Definition 2.1 and the linearity of $\mathcal{I}_h$, we have $(\mathcal{I}_h u)' = (\mathcal{I}_h v_c)' + (\mathcal{I}_h(w))'$. Since $\mathcal{I}_h$ is convexity preserving and $\mathcal{I}_h v_c$ is concave, we derive that for all $1 \leq i \leq n$ the following holds

$$\begin{aligned}\{-\partial_n \mathcal{I}_h u\}_+(x_i) &\leq \{-\partial_n \mathcal{I}_h v_c\}_+(x_i) + \{-\partial_n \mathcal{I}_h(w)\}_+(x_i)\\ &= \{-\partial_n \mathcal{I}_h(w)\}_+(x_i) \leq |\{-\partial_n \mathcal{I}_h(w)\}(x_i)|.\end{aligned}$$

Now we represent $\{-\partial_n \mathcal{I}_h(w)\}(x_i)$ as follows:

$$\begin{aligned}\{-\partial_n \mathcal{I}_h(w)\}(x_i) &= \frac{1}{h_i}\int_{x_i}^{x_{i+1}} w'(s)ds - \frac{1}{h_{i-1}}\int_{x_{i-1}}^{x_i} w'(s)ds\\ &= \int_0^1 \int_{x_i+(t-1)h_{i-1}}^{x_i+th_i} w''(s)dsdt.\end{aligned}$$

Then Hölder's inequality yields

$$h^{2-p}\sum_{x_i \in \mathcal{V}_h^i} |\{-\partial_n \mathcal{I}_h(w)\}(x_i)|^p \leq ch\|u\|^p_{W^{2,p}} \leq c'h\|u\|^p_{W^{2,q}} = c''h.$$

(3) Since $\mathcal{I}_h$ is uniformly stable in $W^{1,\infty}(a,b)$, there is $c \geq 0$, independent of $h$, such that $\|\mathcal{I}_h u\|_{W^{1,\infty}} \leq c\,\|u\|_{W^{1,\infty}}$. Let us set $R = c\,\|u\|_{W^{1,\infty}}$, then owing to (2.3), there is $c_R \geq 0$ such that for all $x \in [a,b]$

$$\begin{aligned}|H(x, \mathcal{I}_h u, (\mathcal{I}_h u)')| &= |H(x, \mathcal{I}_h u, (\mathcal{I}_h u)') - H(x, u, u')|\\ &\leq c_R(|\mathcal{I}_h u - u| + |(\mathcal{I}_h u)' - u'|).\end{aligned}$$

This immediately implies

$$\int_a^b |H(x, \mathcal{I}_h u, (\mathcal{I}_h u)')| dx \leq c_R \|\mathcal{I}_h u - u\|_{W^{1,1}(a,b)} \leq c\, c_R h \|u'\|_{\mathrm{BV(a,b)}},$$

where $\|u'\|_{\mathrm{BV(a,b)}}$ is bounded, see Remark 2.2.
(4) Now, we use the three items above to estimate

$$J_h(\mathcal{I}_h u) \leq c\, h(c_u + c_R \|u'\|_{\mathrm{BV(a,b)}}) \leq c' h.$$

□

*Remark 4.1* Note that it has been critical to use the $L^1$-norm of the residual to obtain (4.4). This is compatible with the fact that $u'$ is in $\mathrm{BV}(a, b)$ only. Using any other $L^r$-norm would yield a suboptimal exponent on $h$.

## 4.3 Existence of a minimizer

We now use the above a priori and consistency estimates to prove the existence of a minimizer to problem (3.8).

**Corollary 4.3** *The discrete problem* (3.8) *has at least one minimizer* $u_h$ *and there is* $c > 0$ *independent of h such that*

$$\|u_h\|_{W^{1,1}} + \|u_h\|_{L^\infty} \leq c, \tag{4.5}$$

$$J_h(u_h) \leq ch. \tag{4.6}$$

*Proof* Let $K_h = \{v_h \in X_h;\, J_h(v_h) \leq J_h(\mathcal{I}_h u)\}$. Clearly $\mathcal{I}_h u$ is a member of $K_h$. Owing to Lemma 4.2, for every $v_h$ in $K_h$

$$\int_a^b |H(x, v_h, v_h')| dx \leq J_h(v_h) \leq J_h(\mathcal{I}_h u) \leq c(u)\, h.$$

That is, the hypothesis of Lemma 4.1 is satisfied, i.e., $S_{h,c(u)}$ is not empty. This implies that there is $c'(u)$ independent of $h$ such that for all $v_h \in K_h$, $\|v_h\|_{L^\infty} + \|v_h\|_{W^{1,1}} \leq c'(u)$. In other words, $K_h$ is uniformly bounded in $W^{1,1}(a, b) \cap L^\infty(a, b)$. Finite-dimensionality implies that $K_h$ is compact. It is also clear that $J_h : K_h \longrightarrow \mathbb{R}$ is continuous in every norm (possibly not uniformly with respect to $h$). Since for every function $v_h$ in $X_h \backslash K_h$, $J_h(v_h)$ is larger than $J_h(\mathcal{I}_h u)$, and the functional $J_h$ is continuous on a compact set, we infer

$$\inf_{v_h \in X_h} J_h(v_h) = \inf_{v_h \in K_h} J_h(v_h) = \min_{v_h \in K_h} J_h(v_h),$$

which concludes the proof. □

Since in practice $u_h$ might not be computed exactly or might be approximated to some extend through some iterative process (the details of the process in question are irrelevant for our discussion), we now define the notion of *almost minimizer*. We say that a family of functions $\{v_h \in X_h\}_{h>0}$ is a sequence of *almost minimizers* if there is $c > 0$ such that for all $h > 0$,

$$J_h(v_h) \leq c\,h. \tag{4.7}$$

It is clear that minimizers are almost minimizers, thus showing that the class of almost minimizers is not empty. The rest of the paper consists of proving that sequences of almost minimizers converge to the viscosity solution of (2.1).

## 5 Convergence to a weak solution

In this section it is proved that any sequence of almost minimizers as defined above converges to a weak solution to (2.1).

### 5.1 $W^{1,\infty}$ bound via BV gradient bound

We proceed by showing that $v_h'$ has bounded variation which in turn implies a $W^{1,\infty}$ bound on $v_h$ and convergence of the sequence to a weak solution.

**Lemma 5.1** *Let $\{v_h \in X_h\}_{h>0}$ be a sequence of almost minimizers, then*

$$\|v_h'\|_{L^\infty(a,b)} + \|v_h'\|_{BV(a,b)} \leq c. \tag{5.1}$$

*Proof* Since the mesh family is almost uniform, the number of mesh elements, $n+1$, is proportional to $\frac{b-a}{h}$, and the first and last mesh intervals are of size $O(h)$, see (3.1). We compute the variation of $v_h'$ on [a,b] as follows

$$|v_h'|_{\mathrm{BV}} = \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} |v_h''(s)|\,ds + \sum_{i=1}^{n} |\{-\partial_n v\}(x_i)|\,.$$

Using that $|t| = 2(t)_+ - t$, we obtain

$$\begin{aligned} |v_h'|_{\mathrm{BV}} &= 2\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+\,ds + 2\sum_{i=1}^{n} \{-\partial_n v\}_+(x_i) \\ &\quad - \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} v_h''(s) - \sum_{i=1}^{n} \{-\partial_n v\}(x_i). \end{aligned}$$

Recalling that $\{-\partial_n v\}(x_i) = v_h(x_i+0) - v_h(x_i-0)$, we obtain

$$|v_h'|_{\mathrm{BV}} = 2\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+ \, ds + 2\sum_{i=1}^{n} \{-\partial_n v\}_+(x_i) - (v_h'(b) - v_h'(a)). \tag{5.2}$$

Recall that $x_1 - a \geq c_1 h$ and $b - x_n \geq c_1 h$, and $v_h$ is a polynomial of fixed degree (at most $k$ on $[a, x_1)$). Therefore,

$$|v_h'(a)| \leq \|v_h'\|_{L^\infty(a,x_1)} \leq \frac{c}{x_1 - a}\|v_h'\|_{L^1(a,x_1)} \leq \frac{c}{c_1 h}\|v_h'\|_{L^1(a,x_1)},$$

where the constant $c$ depends only on $k$. Owing to (2.2) and (4.7), we infer

$$\|v_h'\|_{L^1(a,x_1)} \leq c_s \int_a^{x_1} |H(s, v_h, v_h')| ds + c_s(\|v_h\|_{L^\infty(a,b)} + 1)(x_1 - a) \leq c\, h.$$

Then, we conclude that

$$|v_h'(a)| \leq \frac{c}{c_1 h}\|v_h'\|_{L^1(a,x_1)} \leq c. \tag{5.3}$$

Similarly, we infer

$$|v_h'(b)| \leq c,$$

where the constant $c$ is independent of $h$. Using the above bounds for $|v_h'(a)|$ and $|v_h'(b)|$ in (5.2), we derive

$$|v_h'|_{\mathrm{BV}} \leq 2\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+ \, ds + 2\sum_{i=1}^{n} \{-\partial_n v\}_+(x_i) + c. \tag{5.4}$$

We now bound the two terms in the right-hand side of (5.4) using the estimate (4.7). Using Hölder's inequality, we infer

$$\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+ \, ds \leq \left( \sum_{i=0}^{n} \left( \int_{x_i}^{x_{i+1}} (v_h''(s))_+ \, ds \right)^p \right)^{\frac{1}{p}} (n+1)^{1-\frac{1}{p}}. \tag{5.5}$$

We apply Hölder's again in (5.5) and use that $x_{i+1} - x_i \leq h \leq \frac{c}{n}$ to derive

$$\sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+ \, ds \leq c \left( \sum_{i=0}^{n} \int_{x_i}^{x_{i+1}} (v_h''(s))_+^p \, ds \right)^{\frac{1}{p}}. \tag{5.6}$$

Similarly, we estimate the second term

$$\sum_{i=1}^{n}\{-\partial_n v\}_+(x_i) \le ch^{-1+\frac{1}{p}}\left(\sum_{i=1}^{n}\{-\partial_n v\}_+^p(x_i)\right)^{\frac{1}{p}}. \tag{5.7}$$

Using the estimate (4.7) in (5.6) and (5.7), we obtain

$$\sum_{i=0}^{n}\int_{x_i}^{x_{i+1}}(v_h''(s))_+\,ds + \sum_{i=1}^{n}\{-\partial_n v\}_+(x_i) \le c,$$

which implies, see (5.4),

$$|v_h'|_{\mathrm{BV}} \le c. \tag{5.8}$$

The $W^{1,\infty}$ bound for $v_h$ immediately follows from (5.8) and (5.3)

$$\|v_h'\|_{L^\infty(a,b)} \le |v_h(a)| + |v_h'|_{\mathrm{BV}} \le c, \tag{5.9}$$

which concludes the proof. □

### 5.2 Convergence to a weak solution

We say that $v \in W^{1,\infty}(a,b)\cap \mathcal{C}^0[a,b]$ is a weak solution to (2.1), if $v$ solves (2.1) almost everywhere.

**Theorem 5.2** *Let $\{v_h \in X_h\}_{h>0}$ be a family of almost minimizers, then the sequence $\{v_h\}_{h>0}$ converges, up to a subsequence, to a weak solution $v$ to* (2.1).

*Proof* Owing to Lemma 5.1, the sequence $\{v_h\}_{h>0}$ is precompact in $W^{1,\infty}(a,b)$ equipped with the weak-$\star$ topology and precompact in $W^{1,1}(a,b)$ equipped with the strong topology. Let $v \in W^{1,\infty}(a,b)$ be the limit, up to subsequences, of $\{v_h\}_{h>0}$ in $W^{1,\infty}(a,b)$. Moreover, the limit $v$ is in $\mathcal{C}^0[a,b]$, since the sequence $\{v_h\}_{h>0}$ is equi-continuous.

We now prove that $v$ is a weak solution to (2.1) by showing that $\|H(\cdot,v,v')\|_{L^1(a,b)} = 0$. Using that $v_h \to v$ in $W^{1,1}(a,b)$, we conclude that $v_h \to v$ and $v_h' \to v'$ a.e. in $(a,b)$. Then, we can apply Egorov's Theorem. Given $\epsilon > 0$, there exists a set $E$ with $\mathrm{meas}(E) < \epsilon$, such that the convergence of $v_h' \to v'$ on $(a,b)\backslash E$ is uniform. Recall that the convergence of $v_h \to v$ is uniform on $[a,b]$. Therefore, for every $\epsilon''$, $1 \ge \epsilon'' > 0$, we can find $h(\epsilon'') > 0$ such that for every $h < h(\epsilon'')$,

$$|v_h(x) - v(x)| < \epsilon'' \quad \text{and} \quad |v_h'(x) - v'(x)| < \epsilon'', \quad \forall x \in (a,b)\backslash E.$$

Note also that for every $x \in (a,b)\backslash E$ and every $h < h(1)$, we have

$$\max(|v_h(x)|, |v(x)|, |v_h'(x)|, |v'(x)|) \le R,$$

where $R := \|v\|_{W^{1,\infty}(a,b)} + 1$. Hence, we use that $H$ is uniformly Lipschitz (see (2.3)) and derive that there exists a value of $\epsilon'' > 0$ such that

$$|H(x, v, v') - H(x, v_h, v_h')| < c\epsilon'' < \epsilon \tag{5.10}$$

for every $x \in (a, b)\backslash E$ and every $h < h(\epsilon'')$. Note that at this point the value of $\epsilon''$ solely depends on $\epsilon$. We now split $\|H(\cdot, v, v')\|_{L^1(a,b)}$ in the following way

$$\|H(\cdot, v, v')\|_{L^1(a,b)} = \|H(\cdot, v, v')\|_{L^1((a,b)\backslash E)} + \|H(\cdot, v, v')\|_{L^1(E)}. \tag{5.11}$$

We use $v \in W^{1,\infty}(a, b)$ and (2.3) to estimate

$$\|H(\cdot, v, v')\|_{L^1(E)} \leq c\,\mathrm{meas}(E) = c\epsilon.$$

The other term in (5.11) is estimated as follows

$$\begin{aligned}\|H(\cdot, v, v')\|_{L^1((a,b)\backslash E)} &\leq \|H(\cdot, v, v') - H(\cdot, v_h, v_h')\|_{L^1((a,b)\backslash E)}\\ &\quad + \|H(\cdot, v_h, v_h')\|_{L^1((a,b)\backslash E)}\\ &\leq \epsilon\,\mathrm{meas}((a, b)\backslash E) + \|H(\cdot, v_h, v_h')\|_{L^1(a,b)}\\ &\leq c\epsilon + ch,\end{aligned}$$

where we used (5.10) and (4.7) to derive the above inequality. As a result for every $\epsilon > 0$ and every $h < h(\epsilon)$,

$$\|H(\cdot, v, v')\|_{L^1(a,b)} \leq c(\epsilon + h),$$

which implies $\|H(\cdot, v, v')\|_{L^1(a,b)} = 0$. □

*Remark 5.1* Theorem 5.2 still holds true if $p = 1$ since the BV bound from Lemma 5.1 holds for any $p \geq 1$. The requirement $p > 1$ (see (3.5)) is necessary for proving convergence to the viscosity solution which is dealt with in the next section.

## 6 Convergence to the viscosity solution

We prove in this section that the limit solution $v$ is the viscosity solution to (2.1). We now make the following assumption:

$$\begin{cases} \text{A weak solution } u \in W^{1,\infty}(a, b) \cap \mathcal{C}^0[a, b] \text{ which} \\ \text{satisfies the one-sided bound} \\ \Delta_\delta u(x) := u(x + \delta) - 2u(x) + u(x - \delta) \leq c\,\delta^{1+\gamma} \\ \text{for all } x - \delta, x + \delta \in (a, b), \text{ with } 0 < \gamma \leq 1, \text{ is the} \\ \text{unique viscosity solution of (2.1).} \end{cases} \tag{6.1}$$

This property is known to characterize viscosity solutions to stationary (and time dependent) Hamilton–Jacobi equations with $H(x, u, u') = \lambda u + F(u')$ where $\lambda > 0$

and $F$ is convex, see Theorems 2.6–2.7 in Lions and Souganidis [17]. Note that, the property with $\gamma = 1$ is the well known semi-concavity uniqueness criterion, see, e.g., Evans [7]. If $u$ is $q$-semiconcave, then $u$ satisfies the above one-sided bound with $\gamma = 1 - \frac{1}{p}$ for any $1 < p \leq q$.

**Lemma 6.1** *Let $\{v_h \in X_h\}_{h>0}$ be a sequence of almost minimizers, then there is $c$ independent of $h$ such that*

$$\Delta_\delta v_h(x) = v_h(x+\delta) - 2v_h(x) + v_h(x-\delta) \leq c\,\delta^{2-\frac{1}{p}} \tag{6.2}$$

*for all $\delta \geq h$ and all $x-\delta, x+\delta \in (a,b)$.*

*Proof* We start with the following observation:

$$\Delta_\delta v_h(x) = \int\limits_x^{x+\delta} v_h'(s) - v_h'(s-\delta)\,ds. \tag{6.3}$$

Recall that $K_i = [x_i, x_{i+1}]$, for $0 \leq i \leq n$. Then for a.e. $s \in (x, x+\delta)$, we have

$$v_h'(s) - v_h'(s-\delta) = \sum_{K_i \cap (s-\delta,s) \neq \emptyset} \int\limits_{K_i \cap (s-\delta,s)} v_h''(\tau)\,d\tau + \sum_{x_i \in (s-\delta,s)} \{-\partial_n v\}(x_i).$$

Using that $x - \delta \leq s - \delta < s \leq x + \delta$, we derive that

$$(v_h'(s) - v_h'(s-\delta))_+ \leq \sum_{K_i \cap (x-\delta,x+\delta) \neq \emptyset} \int\limits_{K_i} (v_h''(\tau))_+\,d\tau + \sum_{x_i \in (x-\delta,x+\delta)} \{-\partial_n v\}_+(x_i).$$

Using the above bound (6.3), we infer

$$\Delta_\delta v_h(x) \leq \delta \left( \sum_{K_i \cap (x-\delta,x+\delta) \neq \emptyset} \int\limits_{K_i} (v_h''(\tau))_+\,d\tau + \sum_{x_i \in (x-\delta,x+\delta)} \{-\partial_n v\}_+(x_i) \right). \tag{6.4}$$

We now estimate each sum in (6.4). We have

$$\begin{aligned} \sum_{K_i \cap (x-\delta,x+\delta) \neq \emptyset} \int\limits_{K_i} (v_h''(\tau))_+\,d\tau &\leq c\,\delta^{1-\frac{1}{p}} \left( \sum_{K_i \subset \mathcal{T}_h} \int\limits_{K_i} (v_h''(\tau))_+^p\,d\tau \right)^{\frac{1}{p}} \\ &\leq c\,\delta^{1-\frac{1}{p}}, \end{aligned} \tag{6.5}$$

where we used Hölder's inequality and the bound (4.7). Similarly, for the second term in (6.4), we derive

$$\sum_{x_i\in(x-\delta,x+\delta)} \{-\partial_n v\}_+(x_i) \le c\,(N_h(x,\delta))^{1-\frac{1}{p}} \left(\sum_{x_i\in\mathcal{V}_h^i} \{-\partial_n v\}_+^p(x_i),\right)^{\frac{1}{p}},$$

where recall $N_h(x,\delta)$ is the number of interior vertices in $(x-\delta, x+\delta)$. Since the vertices are almost uniformly distributed, we have $N_h(x,\delta) \le c\,(\frac{\delta}{h}+1)$. The hypothesis $\delta > h$ yields $N_h(x,\delta) \le c\frac{\delta}{h}$. Using this estimate together with the bound (4.7), we obtain

$$\sum_{x_i\in(x-\delta,x+\delta)} \{-\partial_n v\}_+(x_i) \le c\left(\frac{\delta}{h}\right)^{1-\frac{1}{p}} h^{\frac{p-1}{p}} = c\,\delta^{1-\frac{1}{p}}. \tag{6.6}$$

Combining (6.4), (6.5), and (6.6), we conclude

$$\Delta_\delta v_h(x) \le c\,\delta^{2-\frac{1}{p}},$$

which proves the lemma. □

**Theorem 6.2** *Assume (6.1). Then, every sequence of almost minimizers $\{v_h \in X_h\}_{h>0}$ converges in $W^{1,1}(a,b)$ to the unique solution $u$ to (2.1).*

*Proof* Lemma 5.1, implies that $\{v_h\}_{h>0}$ is uniformly bounded in $W^{1,\infty}(a,b)$. Then (up to a subsequence), $v_h \overset{\star}{\rightharpoonup} v \in W^{1,\infty}(a,b)$, and by Arzelà–Ascoli Theorem we have that $v_h \to v$ uniformly on $[a,b]$. Hence, given $h_0 > 0$, for any $h < h_0$, $v_h$ satisfies the one-sided bound (6.2) for all $\delta > h_0$. Taking the limit as $h \to 0$, we derive that

$$\Delta_\delta v(x) = v(x+\delta) - 2v(x) + v(x-\delta) \le c\,\delta^{2-\frac{1}{p}}$$

for all $\delta > h_0$ and all $x$ such that $(x-\delta, x+\delta) \subset (a,b)$. Since $h_0 > 0$ is arbitrary, we conclude that $v$ satisfies the one-sided estimate in (6.1) for all $\delta > 0$ with $\gamma = 1 - \frac{1}{p}$ (observe that $\gamma > 0$ since $p > 1$). Therefore, Assumption (6.1) implies that $v$ is the unique viscosity solution to (2.1). □

## 7 Numerical illustration

To illustrate the method, we now perform some numerical tests. Let us consider the following stationary Hamilton–Jacobi equation on the domain $\Omega = (-a,a)$:

$$(u')^2 + 3u + \tfrac{1}{2}x^2 - |x| = 0, \qquad u(\pm a) = b, \tag{7.1}$$

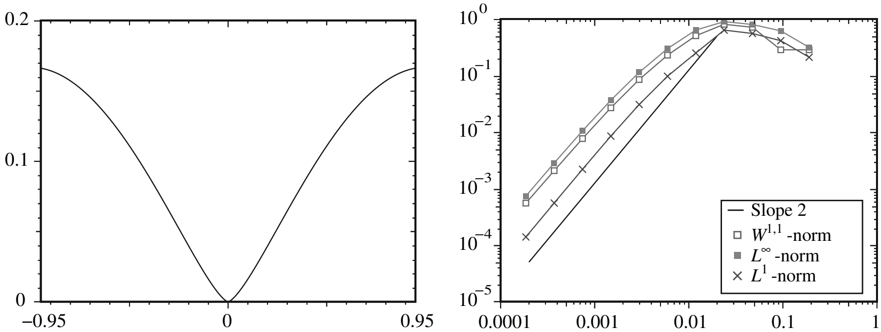


**Fig. 1** *Left* Graph of $u_{\text{visc}}$. *Right* Convergence tests

where $a = 0.95$ and the boundary data $b$ is set so that the viscosity solution $u_{\text{visc}}$ is

$$u_{\text{visc}}(x) = -\tfrac{1}{2}x^2 + \tfrac{2}{3}|x|^{\frac{3}{2}}, \tag{7.2}$$

in other words $b = u_{\text{visc}}(a)$. This solution is in $W^{1,\infty}(\Omega) \cap W^{2,q}(\Omega) \cap \mathcal{C}^0[-a, a]$ for any $q \in [1, 2)$. It also satisfies the one-sided bound (6.1) for any $\gamma \in [0, \frac{1}{2}]$. The graph of $u_{\text{visc}}$ is shown in the left panel of Fig. 1. The purpose of this example is to illustrate our introducing the $q$-semiconcavity notion, see Definition 2.1.

We discretize the integral in (3.6) by using the midpoint rule and the minimization problem (3.8) is solved approximately by means of a fast algorithm that is described in [9]. This algorithm yields an almost minimizer of (3.8) in $\mathcal{O}(n)$ operations.

We show in the right panel of Fig. 1 convergence tests on this problem using $p = 1.5$ for the entropy functional. We report the error measured in the $L^1$-, $L^\infty$-, and $W^{1,1}$-norms with respect to the mesh size. The algorithm is clearly second-order in the $L^1$-norm. The second-order slope is consistent with the $W^{2,1}$ regularity. We observe superconvergence in the $W^{1,1}$-norm and $L^\infty$-norm; the convergence rate is better than first-order.

We observe experimentally that when $p$ is close to 1, which is the lower limit in Theorem 6.2, the method yields an approximation which is far from the viscosity solution if the mesh is not fine enough, say if $h > h_0(p)$. This phenomenon is observed only in the range $p \in (1, 1.16)$. We report in Table 1 the threshold $h_0(p)$ as a function of $p$. When $p > 1.17$ no threshold phenomenon is observed. This seems to confirm that the condition $p > 1$ that we inferred in Theorem 6.2 is sharp. More thorough tests and a description of the fast algorithm mentioned above are reported in [9].

**Table 1** Threshold $h_0(p)$ in the range $p \in (1, 1.16)$

| $p$ | 1.09 | 1.1 | 1.12 | 1.13 | 1.14 | 1.15 | 1.16 |
|---|---|---|---|---|---|---|---|
| $h_0(p)$ | 4.6e-7 | 2.5e-6 | 2.7e-5 | 1.4e-4 | 7.6e-4 | 9.5e-4 | 1.1e-3 |

## 8 Conclusions

We have introduced an $L^1$-based method for solving stationary Hamilton–Jacobi equations in one space dimension assuming a $q$-semiconcavity property on the solution. A two-dimensional generalization of our main result, i.e., Theorem 6.2, is reported in [11]. We are currently investigating possible generalization of the fast 1D minimization algorithm from [9] to two space dimensions.

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