A Fully Discrete NonLinear Galerkin Method for the 3D Navier–Stokes Equations

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The purpose of this paper is twofold: (i) We show that the Fourier-based Nonlinear Galerkin Method (NLGM) constructs suitable weak solutions to the periodic Navier–Stokes equations in three space dimensions provided the large scale/small scale cutoff is appropriately chosen. (ii) If smoothness is assumed, NLGM always outperforms the Galerkin method by a factor equal to 1 in the convergence order of the $\mathbf{H}^1$-norm for the velocity and the L^2-norm for the pressure. This is a purely linear superconvergence effect resulting from standard elliptic regularity and holds independently of the nature of the boundary conditions (whether periodicity or no-slip BC is enforced). © 2007 Wiley Periodicals, Inc. Numer Methods Partial Differential Eq 24: 759–775, 2008

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I. INTRODUCTION

A dissipative evolution equation over an Hilbert space H is said to have an Inertial Manifold if the manifold in question contains the global attractor, is positively invariant under the flow, attracts all the orbits exponentially, and is given as the graph of a $\mathcal{C}^1$ map over a finite-dimensional subspace of H. This class of object has been proved to exist for many equations, but for the Navier–Stokes equations, even in dimension two, the question of the existence of an Inertial Manifold is still open. To fill this gap, the concept of Approximate Inertial Manifold (AIM) has been introduced [1, 2]. An AIM is a sequence of finite-dimensional manifolds in H, of increasing dimension, which are constructed so that the global attractor lies in small neighborhoods of these manifolds and the width of which rapidly shrinks as the dimension of the manifolds goes to infinity.

Approximate Inertial Manifold have been shown to exist for the Navier–Stokes equations in two space dimensions [1]. The Nonlinear Galerkin Method (NLGM) is an approximation technique that aims at constructing Approximate Inertial Manifolds (AIM) of nonlinear PDE's; see

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[1–4]. The main principle sustaining NLGM consists of expanding the solution of the dynamical system in a two-scale fashion (large and small scales) and to simplify the dynamics of the small scales in such a way that they solve a linear PDE whose source term only depends on the large scales; in other words the small scales are slaved to the large scales. This technique sparkled a lot of interest in the 1990's as the concept, accompanied with substantial mathematical results, seemed well suited for turbulence modeling. In particular, it has been shown that NLGM has better approximations properties than the Galerkin method restricted to the large scales only.

Heywood and Rannacher [5] later argued that when applied to the Navier–Stokes equations the seemingly improved performance of NLGM over the standard Galerkin method could not be attributed to turbulence modeling. The authors advanced that the observed improved accuracy was in part to be attributed to the fact that NLGM has a better ability than the Galerkin method to handle the Gibbs phenomenon induced by higher-order boundary incompatibilities induced by the no-slip boundary condition. They further argued that in periodic domain, both NLGM and the Galerkin method perform identically. The mathematical argumentation in [5] is clear and convincing, and [5] probably rightly watered down some earlier, possibly overblown, claims about NLGM.

The goal of the present article is to revisit NLGM and to offer an alternative point of view to that of [5] which, we think, should give some credit back to the method. First we show that, when using Fourier expansions and passing to the limit, the NLGM approximation converges (up to subsequences) to a weak solution which is suitable in the sense of Scheffer [6], provided the large scale/small scale cutoff is appropriately chosen. This is an improvement over the Galerkin method since this property is not known (yet?) to hold for Fourier-based Galerkin solutions, see Theorem 5.1. This difference of behavior between NLGM and Galerkin solutions stems from the particular treatment of the nonlinear terms in NLGM. Second, if arbitrary smoothness is assumed, NLGM always outperform the Galerkin method by a factor equal to 1 in the convergence order of the $\mathbf{H}^1$-norm for the velocity and the L^2-norm for the pressure, see Theorem 6.1. And this result holds independently of the nature of the boundary conditions (whether periodicity or no-slip BC is enforced). As suspected in [5], we confirm that this superconvergence property has nothing to do with turbulence modeling but is instead a very simple consequence of a seemingly not well known result by Wheeler [7] stating that for parabolic equations, the elliptic projection of the solution is always superconvergent in the $\mathbf{H}^1$-norm by one order. This is a purely linear superconvergence effect resulting from standard elliptic regularity.

The article is organized as follows. We recall in Section II the notion of suitable weak solutions of the Navier–Stokes equations. We briefly review Nonlinear Galerkin methods in Section III and we reinterpret one of its version as a means to construct suitable solutions. The proof of the main result of the paper, i.e., Theorem 5.1, is done in Sections IV and V. We prove in Section VI that provided the Navier–Stokes solution is smooth enough, the NLGM approximation is as accurate as that that would be obtained by retaining the time derivative and the nonlinearity in the momentum equation for the small scales.

II. PRELIMINARIES

A. Navier–Stokes Equations and Suitable Weak Solutions

Let $\Omega \subset \mathbb{R}^3$ be an open connected domain with smooth boundary Γ. Let $(0, T)$ be a finite time interval and set $Q_T = \Omega \times (0, T)$. The time evolution for the velocity $\mathbf{u}$ and the pressure p fields of a fluid occupying Ω is described by the (nondimensional) time-dependent Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } Q_T, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = \mathbf{u}_0, \end{cases} \tag{2.1}$$

where $\mathbf{u}_0$ is a solenoidal vector field, $\mathbf{f}$ a source term, and ν is the inverse of the Reynolds number. To account for the boundary conditions, we consider the space $\mathbf{X}$ defined as:

$$\mathbf{X} = \begin{cases} \mathbf{H}^1_0(\Omega) & \text{if homogeneous Dirichlet} \\ \left\{\mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v} \text{ periodic}, \int_\Omega \mathbf{v} = 0\right\} & \text{if periodicity} \end{cases} \tag{2.2}$$

We also introduce the spaces:

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X},\ \nabla\cdot\mathbf{v} = 0\}, \quad \mathbf{H} = \overline{\mathbf{V}}^{\mathbf{L}^2}. \tag{2.3}$$

Unless explicitly stated otherwise, the minimal regularity assumed for the data is $\mathbf{u}_0 \in \mathbf{H}$ and $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ and in the periodic situation $\mathbf{u}_0$ (resp. $\mathbf{f}(t)$ for a.e. t in $(0,T)$) is assumed to be of zero mean over Ω.

We now recall the notion of suitable weak solutions of the Navier–Stokes equations as introduced by Scheffer [6].

Definition 2.1. *A weak solution to the Navier–Stokes equations* $(\mathbf{u}, p)$ *is suitable if* $\mathbf{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{H})$, $p \in L^{3/2}((0,T);L^{3/2}(\Omega))$ *and the local energy balance*

$$\partial_t\left(\tfrac{1}{2}\mathbf{u}^2\right) + \nabla\cdot\left(\left(\tfrac{1}{2}\mathbf{u}^2 + p\right)\mathbf{u}\right) - \nu\nabla^2\left(\tfrac{1}{2}\mathbf{u}^2\right) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \leq 0 \tag{2.4}$$

is satisfied in the distributional sense.

By analogy with nonlinear conservation laws, (2.4) can be viewed as an entropy-like condition which may (hopefully?) selects the physical solutions of (2.1). An explicit form of the distribution $D(\mathbf{u})$ that is missing in the left-hand side of (2.4) to reach equality has been given by Duchon and Robert [8]. For a smooth flow, the distribution $D(\mathbf{u})$ is zero; but for nonregular flow, $D(\mathbf{u})$ may be nontrivial. Suitable solutions are those which satisfy $D(\mathbf{u}) \geq 0$, i.e., if singularities appear, only those that dissipate energy pointwise are admissible. It is expected that suitable solutions are more regular than weak solutions. In this respect, the so-called Caffarelli–Kohn–Nirenberg (CKN) Theorem states that the one-dimensional Hausdorff measure of singular points of suitable solutions is zero. Whether these solutions are indeed classical is still far from being clear.

The techniques that are commonly used to construct suitable weak solutions mainly consist of regularizing the Navier–Stokes equations by adding hyperviscosity [9] or regularizing the nonlinear term [8, 10, 11]. It is remarkable that the weak solutions constructed by Leray [12] are actually suitable. It has been shown recently that Galerkin approximations of (2.1) also converge to suitable solutions if the approximation spaces have local interpolation properties. This has been shown for periodic boundary conditions in [13] and for Dirichlet boundary conditions in [14]. Finite Elements and wavelet spaces have the interpolation properties in question but approximation spaces constructed with trigonometric polynomials and more generally spectral approximation spaces do not have this property. As a result, at the present time, it is not known whether Fourier-based Galerkin approximations converge to suitable weak solutions. It is with this open question in mind that Theorem 5.1 is of interest.

B. Notations and Conventions

For the sake of simplicity, we limit ourselves in the rest of the article to periodic boundary conditions and Fourier approximations techniques. The domain Ω is henceforth assumed to be the three-dimensional torus $(0, 2\pi)^3$.

We use the convention that $\mathbb{R}^3$-valued variables are represented by boldfaced characters or symbols. For all $\mathbf{z} \in \mathbb{C}^3$, we denote by $\bar{\mathbf{z}}$ the conjugate of $\mathbf{z}$, by $|\mathbf{z}|$ the Euclidean norm, and by $|\mathbf{z}|_\infty$ the maximum norm.

The Sobolev spaces $H^s(\Omega)$, $s \geq 0$ is defined in terms of Fourier series as follows

$$H^s(\Omega) = \left\{u(\mathbf{x}) = \textstyle\sum_{\mathbf{k}\in\mathbb{Z}^3} u_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, \quad u_{\mathbf{k}} = \bar{u}_{-\mathbf{k}},\ \sum_{\mathbf{k}\in\mathbb{Z}^3} (1+|\mathbf{k}|^2)^s |u_{\mathbf{k}}|^2 < +\infty\right\}.$$

In other words, the set of trigonometric polynomials $\exp(i\mathbf{k}\cdot\mathbf{x})$, $\mathbf{k} \in \mathbb{Z}^3$, is complete and orthogonal in $H^s(\Omega)$ for all $s \in \mathbb{R}$. The scalar product in $L^2(\Omega)$ is denoted by $(u, v) = (2\pi)^{-3} \int_\Omega u\bar{v}$. The dual of $H^s(\Omega)$ is identified with $H^{-s}(\Omega)$. We introduce the closed subspace $\dot{H}^s(\Omega) \subset H^s(\Omega)$ composed of those functions in $H^s(\Omega)$ that are of zero mean value.

Let N be a positive integer. For approximating the velocity and the pressure fields we will consider the set of trigonometric polynomials of partial degree less than or equal to N:

$$\mathbb{P}_N = \left\{p(\mathbf{x}) = \textstyle\sum_{|\mathbf{k}|_\infty \leq N} c_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}, c_{\mathbf{k}} = \bar{c}_{-\mathbf{k}}\right\}.$$

Since the mean value of the velocity and that of the pressure are irrelevant in the torus, we introduce $\dot{\mathbb{P}}_N$ the subspace of $\mathbb{P}_N$ composed of the trigonometric polynomials of zero mean value.

Upon introducing the notation

$$h = \frac{1}{N}. \tag{2.5}$$

we define the truncation operator $P_h : H^s(\Omega) \longrightarrow \mathbb{P}_N$ so that

$$v = \sum_{\mathbf{k}\in\mathbb{Z}^3} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} \longmapsto P_h v = \sum_{|\mathbf{k}|_\infty \leq N} v_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}}.$$

Let us recall that

Lemma 2.1. *P_h satisfies the following properties:*

(i) *P_h is the restriction on $H^s(\Omega)$ of the L^2-projection onto $\mathbb{P}_N$.*
(ii) $\forall s \geq 0$, $\|P_h\|_{\mathcal{L}(H^s(\Omega);H^s(\Omega))} = 1$.
(iii) *P_h commutes with differentiation operators.*
(iv) $\exists c > 0,\ \forall v \in H^s(\Omega),\ \forall \mu,\ 0 \leq \mu \leq s,\ \|v - P_h v\|_{H^\mu} \leq c\, N^{\mu - s} \|v\|_{H^s}$.
(v) $\exists c > 0,\ \forall v \in \mathbb{P}_N,\ \forall \mu, s,\ s \leq \mu,\ \|P_h v\|_{H^\mu} \leq c\, N^{\mu - s} \|v\|_{H^s}$.

The symbol c is henceforth represents a generic constant that may depend on the data $\boldsymbol{f}$, $\mathbf{u}_0$, ν, T, or Ω, and which value may change from one occurrence to an other.

In the rest of the paper T is a fixed (possibly arbitrarily large) time. Being given a Banach space E, the notation $L^r(E)$ is short for the space $L^r((0, T); E)$ which is composed of the functions mapping $(0, T)$ to E and whose norm in E is Bochner L^r-integrable.

III. THE NONLINEAR GALERKIN METHOD

The goal of this section is to recall the NonLinear Galerkin Method as originally introduced in the literature and to discuss some of its features. We also introduce a variant of the method that we think handles the nonlinear energy cascade a little better than the original one.

A. The Semi-Discretized NLGM

We introduce in this section the Nonlinear Galerkin Method in an infinite-dimensional setting.

Let ε be a positive number that from now on we mentally associate with the smallest scale of the flow down to which we really want to represent well the nonlinear interactions (i.e., the large eddy scale). Without loss of generality we assume that ε is the inverse of an integer. Then we introduce the integer N_ε so that

$$\varepsilon = \frac{1}{N_\varepsilon}. \tag{3.1}$$

We now introduce the following finite-dimensional vector spaces:

$$\mathbf{X}_\varepsilon = \dot{\mathbf{P}}_{N_\varepsilon}, \quad \text{and} \quad M_\varepsilon = \dot{\mathbb{P}}_{N_\varepsilon}. \tag{3.2}$$

where the dot above $\mathbf{P}$ and $\mathbb{P}$ means that we restrict ourselves to trigonometric polynomials of zero mean over Ω. We are going to use $\mathbf{X}_\varepsilon$ to represent the large scales of the velocity and $\mathbf{M}_\varepsilon$ to represent the large scales of the pressure. We abuse notation by using P_ε to denote the L^2-projection onto $\mathbf{X}_\varepsilon$. (A more cumbersome but correct notation would be $\dot{\mathbf{P}}_\varepsilon$.) We define the projection $Q_\varepsilon = I - P_\varepsilon$, where I is the identity. From this definition it is clear that any field in $\dot{\mathbf{L}}^2(\Omega)$, say $\mathbf{v}$, can be decomposed as follows: $\mathbf{v} = P_\varepsilon \mathbf{v} + Q_\varepsilon \mathbf{v}$. The component $P_\varepsilon \mathbf{v}$ living in $\mathbf{X}_\varepsilon$ is referred to as the large scale component of $\mathbf{v}$ and the remainder $Q_\varepsilon \mathbf{v}$ is called the small scale component.

The nonlinear Galerkin method as introduced in [1,2,4] can be recast into the following form: Seek $\mathbf{u}_\varepsilon$ and p_ε in the Leray class such that for all $\mathbf{v} \in \dot{\mathbf{H}}^1(\Omega)$, $q \in \dot{L}^2(\Omega)$,

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{u}_\varepsilon, \mathbf{v}) + \nu(\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{v}) + \mathrm{NL}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) - (p_\varepsilon, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_\varepsilon, q) = 0, \\ (\mathbf{u}_\varepsilon, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{3.3}$$

where the nonlinear term is decomposed as follows:

$$\mathrm{NL}(\mathbf{w}, \mathbf{w}, \mathbf{v}) = (P_\varepsilon \mathbf{w} \cdot \nabla (P_\varepsilon \mathbf{w}), \mathbf{v}) + (P_\varepsilon \mathbf{w} \cdot \nabla (Q_\varepsilon \mathbf{w}), P_\varepsilon \mathbf{v}) + (Q_\varepsilon \mathbf{w} \cdot \nabla (P_\varepsilon \mathbf{w}), P_\varepsilon \mathbf{v}). \tag{3.4}$$

This approximation of the nonlinear advection is deduced from the usual trilinear form $(\mathbf{w} \cdot \nabla \mathbf{w}, \mathbf{v})$ by substituting $\mathbf{w}$ and $\mathbf{v}$ by their two-scale decomposition and by removing the terms that are formally of second- and third-order, i.e., those involving $Q_\varepsilon \mathbf{w}$ or $Q_\varepsilon \mathbf{v}$ two or three times (there are three second-order terms and one third-order term).

For reasons that we do not fully understand yet (see also Section V. B), this form of the nonlinear term does not seem to lend itself easily to analysis. In particular we have not been able to show that the sequence $(\mathbf{u}_\varepsilon, p_\varepsilon)_{\varepsilon>0}$ solving (3.3) using (3.4) converges (up to subsequences) to a weak solution that is suitable.

But as hinted in [3, 4] many other admissible forms of the nonlinear term are possible, and instead of (3.4) we propose to use

$$\mathrm{NL}(\mathbf{u}_\varepsilon, \mathbf{u}_\varepsilon, \mathbf{v}) = (P_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon, \mathbf{v}). \tag{3.5}$$

Then, the semi-discrete NLGM that we henceforth consider is the following:

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{u}_\varepsilon, \mathbf{v}) + \nu(\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{v}) + (P_\varepsilon \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon, \mathbf{v}) - (p_\varepsilon, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_\varepsilon, q) = 0, \\ (\mathbf{u}_\varepsilon, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{3.6}$$

It is then possible to prove that (3.6) has a unique solution and that this solution converges up to subsequences to a suitable weak solution of the Navier–Stokes equations as $\varepsilon \to 0$. We omit the details since the essential arguments will be repeated in the analysis of the fully discrete problem in Sections IV and V. Considering that the nonlinearity is inactive at wavenumbers larger than N_ε (i.e., scales smaller than ε), we also conjecture that (3.6) has an Inertial Manifold.

Note that (3.6) bears some resemblance to the regularization originally proposed by Leray [12] and which consists of solving

$$\begin{cases} (\partial_t \mathbf{u}_\varepsilon, \mathbf{v}) + \nu(\nabla \mathbf{u}_\varepsilon, \nabla \mathbf{v}) + (\phi_\varepsilon * \mathbf{u}_\varepsilon \cdot \nabla \mathbf{u}_\varepsilon, \mathbf{v}) - (p_\varepsilon, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_\varepsilon, q) = 0, \\ (\mathbf{u}_\varepsilon, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}), \end{cases} \tag{3.7}$$

where $\phi_\varepsilon(x) = \varepsilon^{-3}\phi(\frac{x}{\varepsilon})$ is a mollifying sequence. In (3.6) the nonlinearities at scales smaller than ε are deactivated by cutting off the Fourier tail of the velocity whereas in (3.7) this is done by mollification. The major difference between the two formulations is that the time derivative of the small scales is deactivated in (3.6) whereas it remains active in (3.7). This feature allows for the elimination (static condensation) of the small scale modes in the spirit of the AIM theory.

Following [1, 2, 4] we now show that we are indeed on our way to construct an AIM. To remove the incompressibility constraint and the pressure from the above formulation, we set $\mathbf{Z}_\varepsilon = P_\varepsilon(\mathbf{V})$ and $\mathbf{Y} = Q_\varepsilon(\mathbf{V})$. Clearly $\mathbf{V} = \mathbf{Z}_\varepsilon \oplus \mathbf{Y}$ and the decomposition is orthogonal with respect to the $\mathbf{L}^2$- and the $\mathbf{H}^1$-scalar product. Let $\mathbf{u} = \mathbf{z}_\varepsilon + \mathbf{y}$ be the corresponding two-scale decomposition of $\mathbf{u}(t)$ in $\mathbf{V}$. To simplify the argumentation, we assume (in this section only) that the spectrum of $\mathbf{f}$ is restricted to low wavenumbers, i.e., there exists ε_0 so that for all $\varepsilon \le \varepsilon_0$, $Q_\varepsilon \mathbf{f} = 0$. Then (3.6) reduces to

$$\begin{cases} \mathbf{z}_\varepsilon|_{t=0} = P_\varepsilon \mathbf{u}_0 \\ (\partial_t \mathbf{z}_\varepsilon, \boldsymbol{\phi}) + \nu(\nabla \mathbf{z}_\varepsilon, \nabla \boldsymbol{\phi}) + (\mathbf{z}_\varepsilon \cdot \nabla(\mathbf{z}_\varepsilon + \mathbf{y}), \boldsymbol{\phi}) = (\mathbf{f}, \boldsymbol{\phi}), \quad \forall \boldsymbol{\phi} \in \mathbf{Z}_\varepsilon \\ \qquad \nu(\nabla \mathbf{y}_\varepsilon, \nabla \boldsymbol{\psi}) + (\mathbf{z}_\varepsilon \cdot \nabla(\mathbf{z}_\varepsilon + \mathbf{y}), \boldsymbol{\psi}) = 0, \quad \forall \boldsymbol{\psi} \in \mathbf{Y}. \end{cases} \tag{3.8}$$

Let $\boldsymbol{\Psi} : \mathbf{Z}_\varepsilon \longrightarrow \mathbf{Y}$ be the map so that

$$\nu(\nabla \boldsymbol{\Psi}(\mathbf{z}_\varepsilon), \nabla \boldsymbol{\psi}) + (\mathbf{z}_\varepsilon \cdot \nabla \boldsymbol{\Psi}(\mathbf{z}_\varepsilon), \boldsymbol{\psi}) = -(\mathbf{z}_\varepsilon \cdot \nabla \mathbf{z}_\varepsilon, \boldsymbol{\psi}), \quad \forall \boldsymbol{\psi} \in \mathbf{Y}.$$

Then, clearly

$$\mathbf{u}(t) = \mathbf{z}_\varepsilon(t) + \boldsymbol{\Psi}(\mathbf{z}_\varepsilon(t)), \quad \text{a.e. } t \text{ in } (0, T). \tag{3.9}$$

In other words $\mathbf{Z}_\varepsilon \oplus \boldsymbol{\Psi}(\mathbf{Z}_\varepsilon)$ is a good candidate to be an AIM. We conjecture that we have indeed constructed an AIM for the 2D Navier–Stokes, but we leave the matter at this point since it is the 3D situation which is of interest to us in this article.

B. The Fully Discrete NLGM

We now want to construct a finite-dimensional approximation of the solution to (3.6). To this end we introduce an integer N that we suppose to be larger than N_ε. We set

$$h = \frac{1}{N} \tag{3.10}$$

and we define

$$\mathbf{X}_h = \dot{\mathbf{P}}_N, \quad \text{and} \quad M_h = \dot{\mathbb{P}}_N. \tag{3.11}$$

$\mathbf{X}_h$ will be used to approximate the velocity and $\mathbf{M}_\varepsilon$ to approximate the pressure. We abuse notation again by using P_h to denote the L^2-projection onto $\mathbf{X}_h$. (A more cumbersome but correct notation would be $\dot{\mathbf{P}}_h$.)

To be able to control the separation between the large eddy scale ε and the discretization scale h, we introduce a parameter $\theta \in (0, 1)$ and we assume that ε and h are related by the following identity

$$\varepsilon = h^\theta. \tag{3.12}$$

This can be equivalently rewritten as: $N_\varepsilon = N^\theta$.

Then, (3.6) is approximated as follows: Seek $\mathbf{u}_h \in \mathcal{C}^0([0, T]; \mathbf{X}_h)$, and $p_h \in L^2(0, T; M_h)$ such that $\forall t \in (0, T])$, $\forall \mathbf{v} \in \mathbf{X}_h$, and $\forall q \in M_h$

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{u}_h, \mathbf{v}) + \nu(\nabla \mathbf{u}_h, \nabla \mathbf{v}) + (P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, \mathbf{v}) - (p_h, \nabla \cdot \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_h, q) = 0, \\ \mathbf{u}_h|_{t=0} = P_\varepsilon \mathbf{u}_0. \end{cases} \tag{3.13}$$

We show next that the discrete problem (3.13) yields a suitable weak solution at the limit $h \to 0$, up to subsequences, provided the ratio between ε and h, i.e., the parameter θ, is well chosen.

IV. A PRIORI ESTIMATES AND CONVERGENCE TO A WEAK SOLUTION

We start with standard a priori estimates, then we prove that the solution to (3.13) converges, up to subsequences, to a weak solution of (2.1).

Lemma 4.1. *Let $\mathbf{f} \in L^2(0, T; \mathbf{H}^{-1}(\Omega))$ and $\mathbf{u}_0 \in \mathbf{H}$, then the solution to (3.13) satisfies*

$$\max_{0 \le t \le T} \|P_\varepsilon \mathbf{u}_h(t)\|^2_{\mathbf{L}^2} + \nu \int_0^T \|\nabla P_\varepsilon \mathbf{u}_h\|^2_{\mathbf{L}^2} + \|\nabla Q_\varepsilon \mathbf{u}_h\|^2_{\mathbf{L}^2} \le c. \tag{4.1}$$

Proof. These are the basic energy estimates. ■

Corollary 4.1. *Under the assumptions of Lemma* 4.1

$$\|P_\varepsilon \mathbf{u}_h\|_{L^r(\mathbf{H}^{2/r})} + \|P_\varepsilon \mathbf{u}_h\|_{L^r(\mathbf{L}^q)} \le c, \quad \text{with} \quad \tfrac{3}{q} + \tfrac{2}{r} = \tfrac{3}{2}, \quad 2 \le r,\ 2 \le q \le 6.$$

Proof. This result is standard and is a consequence of the interpolation inequality $\|v\|_{\mathbf{H}^{2/r}} \lesssim \|v\|_{\mathbf{L}^2}^{1-2/r}\|\mathbf{v}\|_{\mathbf{H}^1}^{2/r}$, when $2 \le r$, and the embedding $\mathbf{H}^{2/r}(\Omega) \subset \mathbf{L}^q(\Omega)$ for $1/q = 1/2 - 2/(3r)$, (see *e.g.* 15 p. 208]), ∎

Lemma 4.2. *Under the assumptions of Lemma* 4.1*, the approximate pressure and the approximate time derivative of the velocity from (3.13) satisfy*

$$\|p_h\|_{L^{4/3}(L^2)} \le c \tag{4.2}$$

$$\|\partial_t P_\varepsilon \mathbf{u}_h\|_{L^{4/3}(\mathbf{H}^{-1})} \le c. \tag{4.3}$$

Proof. (1) We first prove the pressure estimate (4.2). We observe that $\nabla^2 : M_h \longrightarrow M_h$ is bijective, and we denote by ∇^{-2} the inverse operator. Then, observing that $\nabla\nabla^{-2}p_h \in \mathbf{X}_h$, we multiply the momentum equation in (3.13) by $\nabla\nabla^{-2}p_h$. By using several integrations by parts, we obtain

$$\begin{aligned}
\|p_h\|_{L^2}^2 &= -(\nabla p_h, \nabla\nabla^{-2}p_h)\\
&= (\partial_t P_\varepsilon \mathbf{u}_h - \nu\nabla^2\mathbf{u}_h + P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h - \mathbf{f}, \nabla\nabla^{-2}p_h)\\
&= (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h - \mathbf{f}, \nabla\nabla^{-2}p_h), \quad \text{since } \mathbf{u}_h \text{ and } P_\varepsilon\mathbf{u}_h \text{ are solenoidal}\\
&= (\nabla\cdot(P_\varepsilon\mathbf{u}_h\otimes\mathbf{u}_h) - \mathbf{f}, \nabla\nabla^{-2}p_h)\\
&= (P_\varepsilon\mathbf{u}_h\otimes\mathbf{u}_h, \nabla\nabla\nabla^{-2}p_h) - (\mathbf{f}, \nabla\nabla^{-2}p_h)\\
&\le c(\|P_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^3}\|\mathbf{u}_h\|_{\mathbf{L}^6} + \|\mathbf{f}\|_{\mathbf{H}^{-1}})\|p_h\|_{L^2}.
\end{aligned}$$

This yields $\|p_h\|_{L^2}^{4/3} \le c(\|P_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^3}^{4/3}\|\mathbf{u}_h\|_{\mathbf{L}^6}^{4/3} + \|\mathbf{f}\|_{\mathbf{H}^{-1}}^{4/3})$. We proceed further by noticing that

$$\|p_h\|_{L^{4/3}(L^2)}^{4/3} \le c\big(\|P_\varepsilon\mathbf{u}_h\|_{L^4(\mathbf{L}^3)}^{4/3}\|\mathbf{u}_h\|_{L^2(\mathbf{H}^1)}^{4/3} + \|\mathbf{f}\|_{L^2(\mathbf{H}^{-1})}^{4/3}\big).$$

The conclusion is a consequence of Lemma 4.1 together with Corollary 4.1 with $q = 3$ and $r = 4$.

(2) We now prove the estimate on the time derivative of $P_\varepsilon\mathbf{u}_h$. Using the $\mathbf{H}^1$-stability of P_h (see Lemma 2.1(ii)), we infer

$$\begin{aligned}
\|\partial_t P_\varepsilon\mathbf{u}_h\|_{\mathbf{H}^{-1}} &= \sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, \mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}^1}} = \sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, P_h\mathbf{v})}{\|\mathbf{v}\|_{\mathbf{H}^1}}\\
&\le c\sup_{\mathbf{v}\in\mathbf{H}^1}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, P_h\mathbf{v})}{\|P_h\mathbf{v}\|_{\mathbf{H}^1}} \le c\sup_{\mathbf{v}\in X_h}\frac{(\partial_t P_\varepsilon\mathbf{u}_h, \mathbf{v}_h)}{\|\mathbf{v}_h\|_{\mathbf{H}^1}}\\
&\le c(\nu\|\mathbf{u}_h\|_{\mathbf{H}^1} + \|P_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^3}\|\mathbf{u}_h\|_{\mathbf{H}^1} + \|p_h\|_{L^2} + \|\mathbf{f}\|_{\mathbf{H}^{-1}}).
\end{aligned}$$

We conclude by proceeding as in step 1. ∎

We are now in position to prove

Theorem 4.1. *Under the assumptions of Lemma* 4.1*, $P_\varepsilon \mathbf{u}_h$ converges up to subsequences to a weak solution to (2.1) in $L^2(0,T;\mathbf{H}^1)$ weak and in any $L^r(0,T;\mathbf{L}^q)$ strong ($1 \le q < \frac{6r}{3r-4}$, $2 \le r < \infty$); each subsequence of $P_\varepsilon \mathbf{u}_h$ and $\mathbf{u}_h$ have the same limit; p_h converge up to subsequences in $L^{\frac{4}{3}}(0,T;L^2)$ weak.*

Proof. We only outline the main steps of the proof for the arguments are quite standard.

Since $P_\varepsilon \mathbf{u}_h$ is uniformly bounded in $L^2(0,T;\mathbf{H}^1) \cap L^\infty(0,T;\mathbf{L}^2)$, and $\partial_t P_\varepsilon \mathbf{u}_h$ is uniformly bounded in $L^{4/3}(0,T;\mathbf{H}^{-1}(\Omega))$, the Aubin-Lions compactness lemma (see Lions [15, p.57]) implies that there exists a subsequence $(\mathbf{u}_{h_l})$ such that $P_{\varepsilon_l}\mathbf{u}_{h_l}$ converges weakly in $L^2(0,T;\mathbf{H}^1)$ and strongly in any $L^r(0,T;\mathbf{L}^q)$, such that $1 \le q < \frac{6r}{3r-4}$, $2 \le r < \infty$, and that $\partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l})$ converges weakly in $L^{4/3}(0,T;\mathbf{H}^{-1})$. Moreover, since p_h is bounded uniformly in $L^{4/3}(0,T;L^2)$, there exists a subsequence (p_{h_l}) converging weakly in $L^{4/3}(0,T;L^2)$. Let $\mathbf{u}$ and p denote these limits, and let us show that the pair $(\mathbf{u},p)$ is a weak solution to (2.1).

Observing that $\|P_{\varepsilon_l}\mathbf{u}_{h_l} - \mathbf{u}_{h_l}\|_{\mathbf{L}^2} \le c\,\varepsilon_l\|\mathbf{u}_{h_l}\|_{\mathbf{H}^1}$ it is clear that the subsequences $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ and $(\mathbf{u}_{h_l})$ have the same limit in $L^2(\mathbf{L}^2)$. Note that this also implies that $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ and $(\mathbf{u}_{h_l})$ have the same limit in $L^2(\mathbf{H}^1)$ weak; in other words $Q_{\varepsilon_l}\mathbf{u}_{h_l} \rightharpoonup 0$ is $L^2(\mathbf{H}^1)$.

Let $s > 4$ be a real number and let s^* be such that $\frac{1}{s} + \frac{1}{s^*} = \frac{1}{2}$. Let v be an arbitrary function in $L^s(0,T;\mathbf{H}^1)$ and let $(v_{h_l})_{h_l}$ be a sequence of functions in $L^s(0,T;\mathbf{X}_{h_l})$ strongly converging to $\mathbf{v}$ in $L^s(0,T;\mathbf{H}^1) \subset L^4(0,T;\mathbf{H}^1)$.

(1) $\int_{Q_T} \partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l}) \cdot \mathbf{v}_{h_l} \to \int_{Q_T} \partial_t \mathbf{u}\cdot\mathbf{v}$, since $\partial_t(P_{\varepsilon_l}\mathbf{u}_{h_l}) \rightharpoonup \partial_t\mathbf{u}$ in $L^{4/3}(\mathbf{H}^{-1})$.
(2) $\int_{Q_T} \nabla\mathbf{u}_{h_l}:\nabla\mathbf{v}_{h_l} \to \int_{Q_T} \nabla\mathbf{u}:\nabla\mathbf{v}$, since $\nabla\mathbf{u}_{h_l} \rightharpoonup \nabla\mathbf{u}$ in $L^2(\mathbf{L}^2) \subset L^{4/3}(\mathbf{L}^2)$.
(3) $\int_{Q_T} p_{h_l}\nabla\cdot\mathbf{v}_{h_l} \to \int_{Q_T} p\nabla\cdot\mathbf{v}$, since $p_h \rightharpoonup p$ in $L^{4/3}(L^2)$.
(4) Since $P_{\varepsilon_l}\mathbf{u}_{h_l} \to \mathbf{u}$ in $L^{s^*}(\mathbf{L}^3)$ and $\mathbf{v}_{h_l} \to \mathbf{v} \in L^s(\mathbf{H}^1) \subset L^s(\mathbf{L}^6)$, we infer that $\mathbf{v}_{h_l}\otimes(P_{\varepsilon_l}\mathbf{u}_{h_l}) \to \mathbf{v}\otimes\mathbf{u}$ in $L^2(\mathbf{L}^2\otimes\mathbf{L}^2)$. As a result, $\int_{Q_T}[\mathbf{v}_{h_l}\otimes(P_{\varepsilon_l}\mathbf{u}_{h_l})]:\nabla\mathbf{u}_{h_l} \to \int_{Q_T}[\mathbf{v}\otimes\mathbf{u}]:\nabla\mathbf{u}$ since $\nabla\mathbf{u}_{h_l} \rightharpoonup \nabla\mathbf{u}$ in $L^2(\mathbf{L}^2\otimes\mathbf{L}^2)$.
(5) Since $\nabla\cdot\mathbf{u}_{h_l} = 0$ and $\mathbf{u}_{h_l} \rightharpoonup \mathbf{u}$ in $L^2(\mathbf{H}^1)$, $\nabla\cdot\mathbf{u} = 0$ in $L^2(\mathbf{H}^1)$.
(6) Clearly $\int_0^T < \mathbf{f}, \mathbf{v}_{h_l} > \to \int_0^T < \mathbf{f}, \mathbf{v} >$ since $\mathbf{v}_{h_l} \to \mathbf{v}$ in $L^s(\mathbf{H}^1) \subset L^2(\mathbf{H}^1)$ and $f \in L^2(\mathbf{H}^{-1})$.
(7) Finally since the subsequence $(P_{\varepsilon_l}\mathbf{u}_{h_l})$ converges in $\mathcal{C}^0(0,T;\mathbf{L}^2_w)$ (space of the functions that are continuous over $[0,T]$ with value in $\mathbf{L}^2$ equipped with the weak topology) we have $\mathbf{u}_0 \leftarrow P_{\varepsilon_l}\mathbf{u}_0 = P_{\varepsilon_l}\mathbf{u}_{h_l}(0) \rightharpoonup \mathbf{u}(0)$ in $\mathbf{L}^2$; hence, $\mathbf{u}(0) = \mathbf{u}_0$. The theorem is proved. ∎

V. CONVERGENCE TO A SUITABLE SOLUTION

In this section we analyze NLGM with the two variants (3.4) and (3.5) of the nonlinear terms. We show that (3.5) yields a suitable solution at the limit whereas we are not able to conclude with (3.4).

A. Analysis with Formulation (3.5)

The main contribution in this section is Theorem 5.1 which establishes that the solution to (3.13) converges to a suitable weak solution of the Navier–Stokes equations.

Theorem 5.1. *Let* $\mathbf{f} \in L^2(0,T;\mathbf{H}^{-1}(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}$. *Let* $N = \frac{1}{h} > 0$ *and* $\varepsilon = h^\theta$. *Provided*

$$0 < \theta < \tfrac{2}{3}, \tag{5.1}$$

the limit solution(s) of (3.13) is (are) suitable.

Proof. Let ϕ be a smooth nonnegative function, periodic with respect to space, and compactly supported with respect to time in $(0,T)$. Taking $P_h(\mathbf{u}_h\phi)$ to test the momentum equation in (3.13) and integrating in time, we obtain

$$\int_0^T (\partial_t P_\varepsilon \mathbf{u}_h, P_h(\mathbf{u}_h\phi)) + \nu(\nabla\mathbf{u}_h, \nabla P_h(\mathbf{u}_h\phi)) - (p_h, \nabla\cdot P_h(\mathbf{u}_h\phi))$$
$$+ (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h, P_h(\mathbf{u}_h\phi)) = \int_0^T (\mathbf{f}, P_h(\mathbf{u}_h\phi)).$$

Using the fact that P_ε and P_h commute with differentiation operators and after integrating by parts in time and space, we obtain

$$\int_0^T -\tfrac{1}{2}((|P_\varepsilon\mathbf{u}_h|^2, \partial_t\phi) + \nu(|\nabla\mathbf{u}_h|^2, \phi) - \tfrac{1}{2}\nu(|\mathbf{u}_h|^2, \nabla^2\phi) - (p_h, \nabla\cdot(\mathbf{u}_h\phi))$$
$$+ (P_\varepsilon\mathbf{u}_h\cdot\nabla\mathbf{u}_h, P_h(\mathbf{u}_h\phi)) = \int_0^T (\mathbf{f}, P_h(\mathbf{u}_h\phi)).$$

We now pass to the limit in each term of the above equation separately, and to avoid cumbersome notations we still denote by $(\mathbf{u}_h)$, (p_h) the subsequences that are extracted instead of $(\mathbf{u}_{hl})$, (p_{hl}).

(1) $\int_0^T -\frac{1}{2}((|P_\varepsilon\mathbf{u}_h|^2, \partial_t\phi) \to \int_0^T -\frac{1}{2}((|\mathbf{u}|^2, \partial_t\phi)$ since $|P_\varepsilon\mathbf{u}_h|^2 \to |\mathbf{u}|^2$ in $L^r(L^1)$ for any $1 \le r < \infty$.

(2) For the term $\nu\int_0^T (|\nabla\mathbf{u}_h|^2, \phi)$ we proceed as follows:

$$\int_0^T (|\nabla\mathbf{u}_h|^2, \phi) = \int_0^T (|\nabla(\mathbf{u}_h - \mathbf{u})|^2 + 2\nabla(\mathbf{u}_h - \mathbf{u}):\nabla\mathbf{u} + |\nabla\mathbf{u}|^2, \phi).$$

The second term in the right-hand side goes to zero since $\mathbf{u}_h \rightharpoonup \mathbf{u}$ in $L^2(\mathbf{H}^1)$. As a result

$$\liminf_{N\to+\infty} \int_0^T (|\nabla\mathbf{u}_h|^2, \phi) \ge \int_0^T (|\nabla\mathbf{u}|^2, \phi).$$

(3) $\frac{1}{2}\nu\int_0^T (|\mathbf{u}_h|^2, \nabla^2\phi) \to \frac{1}{2}\nu\int_0^T (|\mathbf{u}|^2, \nabla^2\phi)$ since $|\mathbf{u}_h|^2 \to |\mathbf{u}|^2$ in $L^2(L^1)$. To be convinced of this result observe that $\int_0^T \|\mathbf{u}_h - \mathbf{u}\|^2_{\mathbf{L}^2} \le \int_0^T 2\|P_\varepsilon\mathbf{u}_h - \mathbf{u}\|^2_{\mathbf{L}^2} + 2\|Q_\varepsilon\mathbf{u}_h\|^2_{\mathbf{L}^2}$. Then using $\|Q_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^2} \le c\,\varepsilon\|\mathbf{u}_h\|_{\mathbf{H}^1}$ together with the fact that $\int_0^T 2\|P_\varepsilon\mathbf{u}_h - \mathbf{u}\|^2_{\mathbf{L}^2} \to 0$, we conclude $\mathbf{u}_h \to \mathbf{u}$ in $L^2(\mathbf{L}^2)$.

(4) Since $\mathbf{u}_h$ is solenoidal, the pressure term simplifies as follows $\int_0^T (p_h, \nabla\cdot(\mathbf{u}_h\phi)) = \int_0^T (p_h\mathbf{u}_h, \nabla\phi)$. As a result, $\int_0^T (p_h, \nabla\cdot(\mathbf{u}_h\phi)) \to \int_0^T (p\mathbf{u}, \nabla\phi)$ since $p_h \rightharpoonup p$ in $L^{4/3}(L^2)$ and $\mathbf{u}_h\cdot\nabla\phi \to \mathbf{u}\cdot\nabla\phi$ in $L^4(L^2)$.

(5) We treat the trouble-making nonlinear term as follows

$$\begin{aligned}(P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, P_h(\mathbf{u}_h\phi)) &= (P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, \mathbf{u}_h\phi) + R_1\\ &= -(\tfrac{1}{2}|\mathbf{u}_h|^2 P_\varepsilon \mathbf{u}_h, \nabla\phi) + R_1,\\ &= -(\tfrac{1}{2}|P_\varepsilon \mathbf{u}_h|^2 P_\varepsilon \mathbf{u}_h, \nabla\phi) + R_1 + R_2,\end{aligned}$$

where we have set

$$\begin{aligned}R_1 &= (P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi),\\ R_2 &= -\tfrac{1}{2}((P_\varepsilon \mathbf{u}_h + \mathbf{u}_h)\cdot Q_\varepsilon \mathbf{u}_h P_\varepsilon \mathbf{u}_h, \nabla\phi).\end{aligned}$$

Using the approximation property of P_h (see Lemma 2.1(iv)) and the fact that $\|\mathbf{u}_h\phi\|_{\mathbf{H}^1} \le c\|\mathbf{u}_h\|_{\mathbf{H}^1}\|\phi\|_{W^{1,\infty}}$, we can bound the first residual as follows:

$$\begin{aligned}|R_1| &\le \|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^\infty}\|\nabla \mathbf{u}_h\|_{\mathbf{L}^2}\|P_h(\mathbf{u}_h\phi) - \mathbf{u}_h\phi\|_{\mathbf{L}^2},\\ &\le c\,\varepsilon^{-\frac{3}{2}}N^{-1}\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^2}\|\nabla \mathbf{u}_h\|_{\mathbf{L}^2}\|\mathbf{u}_h\phi\|_{\mathbf{H}^1},\\ &\le c\,N^{\frac{3}{2}\theta-1}\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^2}\|\mathbf{u}_h\|^2_{\mathbf{H}^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

Then, it is clear that $\int_0^T |R_1| \to 0$ as $N \to \infty$ owing to (5.1). For the second residual, we use the embedding $H^1(\Omega) \subset L^6(\Omega)$, to show that:

$$\begin{aligned}|R_2| &\le c\,\|Q_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^2}\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^3}\|P_\varepsilon \mathbf{u}_h + \mathbf{u}_h\|_{\mathbf{L}^6}\|\phi\|_{W^{1,\infty}}\\ &\le c\,\varepsilon\|\mathbf{u}_h\|_{\mathbf{H}^1}\varepsilon^{-\frac{1}{2}}\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^2}\|P_\varepsilon \mathbf{u}_h + \mathbf{u}_h\|_{\mathbf{H}^1}\|\phi\|_{W^{1,\infty}}\\ &\le c\,N^{-\frac{1}{2}\theta}\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^2}\|\mathbf{u}_h\|^2_{\mathbf{H}^1}\|\phi\|_{W^{1,\infty}}.\end{aligned}$$

Then, for $\theta > 0$, $\int_0^T |R_2| \to 0$ as $N \to \infty$. In conclusion $\int_0^T (P_\varepsilon \mathbf{u}_h \cdot \nabla \mathbf{u}_h, P_h(\mathbf{u}_h\phi)) \to -(\frac{1}{2}|\mathbf{u}|^2\mathbf{u}, \nabla\phi)$ since $|P_\varepsilon \mathbf{u}_h|^2 P_\varepsilon \mathbf{u}_h \to |\mathbf{u}|^2\mathbf{u}$ in $L^s(\mathbf{L}^1)$, $s \in [1, \frac{4}{3})$.

(6) Passing to the limit in the source term does not pose any difficulty. Observe first that $P_h\mathbf{f} \to \mathbf{f}$ in $L^2(\mathbf{H}^{-1})$ strong. As a result

$$\int_0^T <\mathbf{f}, P_h(\phi\mathbf{u}_h)> = \int_0^T <\phi P_h\mathbf{f}, \mathbf{u}_h> \to \int_0^T <\phi\mathbf{f}, \mathbf{u}>,$$

since $\phi P_h\mathbf{f} \to \phi\mathbf{f}$ in $L^2(\mathbf{H}^{-1})$ strong and $\mathbf{u}_h \rightharpoonup \mathbf{u}$ in $L^2(\mathbf{H}^1)$ weak. ∎

Remark 5.1. The above theorem shows that the size of the large eddy scales and the mesh size must be such that $\varepsilon \gg h^{2/3}$ to ascertain that the sequence $(\mathbf{u}_h, p_h)_{h>0}$ converges to a suitable solution (up to subsequences).

Remark 5.2. We emphasize again that Theorem 5.1 is not *a priori* evident. It is both the form of the nonlinear term (3.5) and the condition $\varepsilon \gg h^{2/3}$ that allows us to conclude. Whether the result still holds for $\theta \in (\frac{2}{3}, 1]$ is an open question to the best of our knowledge. Recall that the case $\theta = 1$, i.e., $h = \varepsilon$, is the Galerkin approximation. The source of the difficulty is that we are using Fourier expansions

B. Analysis with Formulation (3.4)

We now briefly outline the difficulties that we encounter when working with the NLGM solution obtained by using (3.4).

Let us denote by NL_1 the trilinear form defined in (3.4) and by NL_2 the trilinear form defined in (3.5). These two forms are related by

$$\mathrm{NL}_1(\mathbf{w},\mathbf{w},\mathbf{v}) - \mathrm{NL}_2(\mathbf{w},\mathbf{w},\mathbf{v}) = (Q_\varepsilon\mathbf{w}\cdot\nabla P_\varepsilon\mathbf{w}, P_\varepsilon\mathbf{v}) - (P_\varepsilon\mathbf{w}\cdot\nabla Q_\varepsilon\mathbf{w}, Q_\varepsilon\mathbf{v}).$$

Since we have proven in the previous section that $\int_0^T \mathrm{NL}_2(\mathbf{u}_h,\mathbf{u}_h,P_h(\phi\mathbf{u}_h))$ converges to $-\frac{1}{2}\int_0^T(|\mathbf{u}|^2\mathbf{u},\nabla\phi)$, expecting $\int_0^T \mathrm{NL}_1(\mathbf{u}_h,\mathbf{u}_h,P_h(\phi\mathbf{u}_h))$ to behave similarly would require the right-hand side in the above equation to go to zero with h. Heuristically that would not be a surprise since one term is formally a first-order residual and the other is formally a second-order residual. However, so far, we have not been able to prove that.

To better understand the problem we face, we now show where our attempts fail. Since we do not see any cancellation occurring between these two residuals, we treat them separately. Using $P_\varepsilon(\phi\mathbf{u}_h)$ as test function, we rewrite the first term as follows:

$$\begin{aligned}(Q_\varepsilon\mathbf{u}_h\cdot\nabla P_\varepsilon\mathbf{u}_h, P_\varepsilon(\phi\mathbf{u}_h)) &= (Q_\varepsilon\mathbf{u}_h\cdot\nabla P_\varepsilon\mathbf{u}_h, \phi P_\varepsilon\mathbf{u}_h) + (Q_\varepsilon\mathbf{u}_h\cdot\nabla P_\varepsilon\mathbf{u}_h, P_\varepsilon(\phi\mathbf{u}_h) - \phi P_\varepsilon\mathbf{u}_h)\\ &= -\tfrac{1}{2}(|P_\varepsilon\mathbf{u}_h|^2 Q_\varepsilon\mathbf{u}_h,\nabla\phi) + R_1 := R_2 + R_1.\end{aligned}$$

Clearly $\lim_{h\to 0}\int_0^T|R_2| = 0$. However, for R_1, the best we can see doing is the following:

$$\begin{aligned}\int_0^T|R_1| &\le \int_0^T \|Q_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^3}\|\nabla P_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^2}\|P_\varepsilon(\phi\mathbf{u}_h) - \phi P_\varepsilon\mathbf{u}_h\|_{\mathbf{L}^6}\\ &\le \|Q_\varepsilon\mathbf{u}_h\|_{L^\infty(\mathbf{L}^3)}\|\mathbf{u}_h\|_{L^2(\mathbf{H}^1)}\|P_\varepsilon(\phi\mathbf{u}_h) - \phi P_\varepsilon\mathbf{u}_h\|_{L^2(\mathbf{H}^1)}.\end{aligned}$$

That would converge to zero if $\|Q_\varepsilon\mathbf{u}_h\|_{L^\infty(\mathbf{L}^3)}$ were uniformly bounded. Unfortunately, this information is unavailable to us, although formally one would even expect this term to converge to zero since it is a first-order residual. A possible way out could be to seek a control on the commutator $[\phi, P_\varepsilon]$. For finite elements and wavelets it can be shown that $\|[\phi,P_\varepsilon]\|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^1)} \le c\,\varepsilon$, where c depends on ϕ. This estimate would be enough to conclude positively since we have the obvious bound $\|Q_\varepsilon\mathbf{u}_h\|_{L^\infty(\mathbf{L}^3)} \le c\,\varepsilon^{-\frac{1}{2}}$. Unfortunately, the above bound on $\|[\phi,P_\varepsilon]\|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^1)}$ is not true for trigonometric polynomials. Actually $\|[\phi,P_\varepsilon]\|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^1)}$ does not even converge to zero. To see this, consider $\phi = e^{i(x_1+x_2+x_2)}$ and let $\mathbf{v} = (1,0,0)e^{iN_\varepsilon(x_1+x_2+x_2)}$. Then $[\phi,P_\varepsilon]\mathbf{v} = \phi\mathbf{v}$ and there is obviously $c > 0$ so that $\|\phi\mathbf{v}\|_{\mathbf{H}^1} \ge c\,\|\mathbf{v}\|_{\mathbf{H}^1}$; as a result we obtain the defeating lower bound $\|[\phi,P_\varepsilon]\|_{\mathcal{L}(\mathbf{H}^1,\mathbf{H}^1)} \ge c$, and we cannot see why $\int_0^T|R_1|$ should go to zero without invoking additional unavailable regularity.

In conclusion, we cannot establish that the Fourier version of (3.4) (as originally introduced in the literature) yields a suitable approximation at the limit whereas (3.5) does. As shown in the above argument, the main source of difficulty is that we are working with a spectral basis. In other words (3.5) beats the Gibbs phenomenon, whereas it is unclear whether (3.4) does.

VI. CONVERGENCE ANALYSIS ASSUMING REGULARITY

We (re)prove in this section that provided the solution to (2.1) is smooth enough, the velocity field from (3.13) is more accurate in the $\mathbf{H}^1$-norm than the Galerkin solution on $\mathbf{X}_\varepsilon\times M_\varepsilon$. This feature is a well-known characteristics of nonlinear Galerkin methods; though, some doubts have been raised in [5] as to whether this property holds in the periodic case. We prove here that this super-convergence result is independent of the type of boundary conditions.

However, as noted in [5] the presence or absence of nonlinearities has nothing to do with this remarkable property. The single key argument at stake here is that the elliptic projection is super-convergent in the $\mathbf{H}^1$-norm. It seems to us that this property, found by Wheeler in [7], has not been emphasized enough in the literature dedicated to NLGM. The goal of this section is make this point clearer. The main result of this section is Theorem 6.1.

Unless stated otherwise, Ω is the 3D torus, i.e., periodic boundary conditions are assumed (although most of the results below can easily be extended to homogeneous Dirichlet boundary conditions).

A. Super-Convergence of the Elliptic Projection

Let us denote by $(\mathbf{R}_h(\mathbf{u}(t)), S_h(p(t))) \in \mathbf{X}_h \times M_h$ the elliptic projection of the pair $(\mathbf{u}(t), p(t))$, i.e., let $\mathbf{R}_h(\mathbf{u}(t))$ and $S_h(p(t))$ solve the following problem: For a.e. $t \in [0, T]$, for all $\mathbf{v}_h \in \mathbf{X}_h$, and for all $q_h \in M_h$

$$\begin{cases} (\nabla \mathbf{R}_h(\mathbf{u}(t)), \nabla \mathbf{v}_h) - (S_h(p(t)), \nabla\cdot \mathbf{v}_h) = (\mathbf{u}(t), \nabla \mathbf{v}_h) - (p(t), \nabla\cdot \mathbf{v}_h) \\ (q_h, \nabla\cdot \mathbf{R}_h(\mathbf{u}(t))) = 0. \end{cases} \tag{6.1}$$

In the rest of this section, we assume that $\mathbf{f}$ and $\mathbf{u}_0$ are smooth (or small) enough so that there exist $\sigma > 0$ and $s > \frac{3}{2}$ such that the following quantities are bounded: $K_1 := \|\mathbf{u}_t\|_{L^2(\mathbf{H}^{\sigma+1})}$, $K_2 := \|\mathbf{u}\|_{L^\infty(\mathbf{H}^s)}$, $K_3 := \|\mathbf{u}\|_{L^2(\mathbf{H}^{\sigma+1})}$, and $K_4 := \|\mathbf{u}_0\|_{\mathbf{H}^{\sigma+1}}$. Let us set $K = K_1 + \ldots + K_4$.

Lemma 6.1. *The following uniform bounds hold provided the quantity K is bounded,*

$$\|\mathbf{u} - \mathbf{R}_h(\mathbf{u})\|_{L^2(\mathbf{L}^2)} + \|\mathbf{u}_t - \mathbf{R}_h(\mathbf{u}_t)\|_{L^2(\mathbf{L}^2)} \leq c(K_1, K_2) h^{\sigma+1}, \tag{6.2}$$

$$\|\mathbf{R}_h(\mathbf{u})\|_{L^\infty(\mathbf{L}^\infty)} \leq c(K_2), \tag{6.3}$$

$$\|\mathbf{u}_0 - \mathbf{R}_h(\mathbf{u}_0)\|_{\mathbf{L}^2} \leq c(K_4) h^{\sigma+1}, \tag{6.4}$$

$$\|\mathbf{u} - \mathbf{R}_h(\mathbf{u})\|_{L^2(\mathbf{H}^1)} + \|p - S_h(p)\|_{L^2(L^2)} \leq c(K_1, K_2) h^{\sigma}. \tag{6.5}$$

Then, the following Lemma clarifies what we meant above when stating that the elliptic projection is super-convergent in the $\mathbf{H}^1$-norm.

Lemma 6.2. *Under the regularity assumptions of Lemma* 6.1*, the velocity and pressure fields from (3.13) satisfy the following error estimate*

$$\|\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})\|_{L^2(\mathbf{H}^1)} \leq c(\nu, T, K)\varepsilon^{\sigma+1}. \tag{6.6}$$

$$\|p_h - S_h(p)\|_{L^2(L^2)} \leq c(\nu, T, K)\varepsilon^{\sigma+1}. \tag{6.7}$$

Proof. (1) Let us set $\mathbf{e}_h = \mathbf{R}_h(\mathbf{u}) - \mathbf{u}_h$ and $\delta_h = S_h(p) - p_h$. Then the system of equations controlling these two quantities is

$$\begin{cases} (\partial_t P_\varepsilon \mathbf{e}_h, \mathbf{v}) + \nu(\nabla \mathbf{e}_h, \nabla \mathbf{v}) - (\delta_h, \nabla\cdot \mathbf{v}) = (\mathbf{F}(\mathbf{u}_h, \mathbf{u}), \mathbf{v}) + (\mathbf{G}(\mathbf{u}_h, \mathbf{u}), \mathbf{v}), \\ (\nabla\cdot \mathbf{e}_h, q) = 0, \\ \mathbf{e}_h|_{t=0} = P_\varepsilon(\mathbf{R}_h(\mathbf{u}_0) - \mathbf{u}_0) \end{cases}$$

where we have set $\mathbf{F}(\mathbf{u}_h, \mathbf{u}) = (P_\varepsilon \mathbf{u}_h)\cdot\nabla\mathbf{u}_h - \mathbf{u}\cdot\nabla\mathbf{u}$ and $\mathbf{G}(\mathbf{u}_h, \mathbf{u}) = P_\varepsilon \mathbf{R}_h(\mathbf{u}_t) - \mathbf{u}_t$, and the test functions $\mathbf{v}$ and q span $\mathbf{X}_h$ and M_h respectively.

(2) The error estimate (6.6) is obtained by using $\mathbf{e}_h$ as a test function in the above equation and by integrating over the time interval $(0, T)$. Owing to the assumed regularity for $\mathbf{u}$ we have

$$\begin{aligned}\|\mathbf{G}(\mathbf{u}_h, \mathbf{u})\|_{L^2(\mathbf{L}^2)} &\leq \|P_\varepsilon(\mathbf{R}_h(\mathbf{u}_t) - \mathbf{u}_t)\|_{L^2(\mathbf{L}^2)} + \|P_\varepsilon \mathbf{u}_t - \mathbf{u}_t\|_{L^2(\mathbf{L}^2)} \\ &\leq c\,\varepsilon^{\sigma+1}\|\mathbf{u}_t\|_{L^2(\mathbf{H}^{\sigma+1})}.\end{aligned}$$

This immediately yields

$$\int_0^T |(\mathbf{G}(\mathbf{u}_h, \mathbf{u}), \mathbf{e}_h)| \leq \gamma\|\nabla\mathbf{e}_h\|^2_{L^2(\mathbf{L}^2)} + c(\gamma, K_1)\,\varepsilon^{2(\sigma+1)},$$

where $\gamma > 0$ is a positive real that can be chosen as small as needed.

To control the nonlinear term we set

$$\begin{aligned}\mathbf{F}(\mathbf{u}_h, \mathbf{u}) = {} & P_\varepsilon \mathbf{u}_h\cdot\nabla(\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})) + P_\varepsilon(\mathbf{u}_h - \mathbf{R}_h(\mathbf{u}))\cdot\nabla\mathbf{R}_h(\mathbf{u}) \\ & + (P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u})\cdot\nabla\mathbf{R}_h(\mathbf{u}) + \mathbf{u}\cdot\nabla(\mathbf{R}_h(\mathbf{u}) - \mathbf{u}).\end{aligned}$$

Let $\mathbf{R}_1$ to $\mathbf{R}_4$ be the four residuals in the right-hand side above. Clearly

$$\int_0^T (\mathbf{R}_1, \mathbf{e}_h) = 0.$$

Then, integration by parts yields

$$|(\mathbf{R}_2, \mathbf{e}_h)| \leq \|P_\varepsilon \mathbf{e}_h\|_{\mathbf{L}^2}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2}\|\mathbf{R}_h(\mathbf{u})\|_{\mathbf{L}^\infty} \leq c(K_2)\,\|P_\varepsilon \mathbf{e}_h\|_{\mathbf{L}^2}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2}.$$

Hence

$$\int_0^T |(\mathbf{R}_2, \mathbf{e}_h)| \leq \gamma\|\nabla\mathbf{e}_h\|^2_{\mathbf{L}^2} + c(K_2)\|P_\varepsilon \mathbf{e}_h\|^2_{\mathbf{L}^2}.$$

Proceeding similarly, for the third residual we have

$$\begin{aligned}|(\mathbf{R}_3, \mathbf{e}_h)| &\leq \|P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{\mathbf{L}^2}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2}\|\mathbf{R}_h(\mathbf{u})\|_{\mathbf{L}^\infty} \\ &\leq c(K_2)\,\varepsilon^{\sigma+1}\|\mathbf{u}\|_{\mathbf{H}^{\sigma+1}}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2}.\end{aligned}$$

This yields

$$\int_0^T |(\mathbf{R}_3, \mathbf{e}_h)| \leq \gamma\|\nabla\mathbf{e}_h\|^2_{\mathbf{L}^2} + c(\gamma, K_2, K_3)\,\varepsilon^{2(\sigma+1)}.$$

Integrating by parts again for the last residual we obtain

$$\begin{aligned}|(\mathbf{R}_4, \mathbf{e}_h)| &\leq \|\mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{\mathbf{L}^2}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2}\|\mathbf{u}\|_{\mathbf{L}^\infty} \\ &\leq c(K_2)\,\varepsilon^{\sigma+1}\|\mathbf{u}\|_{\mathbf{H}^{\sigma+1}}\|\nabla\mathbf{e}_h\|_{\mathbf{L}^2},\end{aligned}$$

from which we derive

$$\int_0^T |(\mathbf{R}_4, \mathbf{e}_h)| \leq \gamma\|\nabla\mathbf{e}_h\|^2_{\mathbf{L}^2} + c(\gamma, K_2, K_3)\,\varepsilon^{2(\sigma+1)}.$$

We obtain the desired estimate on $\|\mathbf{e}_h\|_{L^2(\mathbf{H}^1)}$ by setting $\gamma = \nu/8$ and using the Gronwall Lemma.

(3) The technique for proving the pressure estimate (6.7) is the same as that for proving the estimate (4.2) repeating the above arguments to control the nonlinear terms. Clearly we have

$$\begin{aligned}
\|\delta_h\|_{L^2}^2 &= -(\nabla\delta_h, \nabla\Delta^{-2}\delta_h)\\
&= -(\mathbf{F}(\mathbf{u}_h,\mathbf{u}) + \mathbf{G}(\mathbf{u}_h,\mathbf{u}) + \nu\nabla^2\mathbf{e}_h - \partial_t P_\varepsilon \mathbf{e}_h, \nabla\Delta^{-2}\delta_h)\\
&= -(\mathbf{F}(\mathbf{u}_h,\mathbf{u}), \nabla\Delta^{-2}\delta_h)\\
&= (P_\varepsilon \mathbf{u}_h \otimes (\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})) + P_\varepsilon(\mathbf{u}_h - \mathbf{R}_h(\mathbf{u})) \otimes \mathbf{R}_h(\mathbf{u})\\
&\quad + (P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u}) \otimes \mathbf{R}_h(\mathbf{u}) + \mathbf{u} \otimes (\mathbf{R}_h(\mathbf{u}) - \mathbf{u}), \nabla\nabla\Delta^{-2}\delta_h)\\
&\le c((\|P_\varepsilon \mathbf{u}_h\|_{\mathbf{L}^3} + \|\mathbf{R}_h(\mathbf{u})\|_{\mathbf{L}^3})\|\mathbf{e}_h\|_{\mathbf{H}^1}\\
&\quad + \|P_\varepsilon \mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{\mathbf{L}^2}\|\mathbf{R}_h(\mathbf{u})\|_{\mathbf{L}^\infty} + \|\mathbf{u}\|_{\mathbf{L}^\infty}\|\mathbf{R}_h(\mathbf{u}) - \mathbf{u}\|_{\mathbf{L}^2})\|\delta_h\|_{L^2}.
\end{aligned}$$

Then the conclusion follows readily. ■

Remark 6.1. The observation that the H^1-distance between the elliptic projection and the Galerkin approximation of a nonlinear parabolic PDE is super-convergent is due to Wheeler [7]. The heuristic principle that follows from this observation is that, when approximating a nonlinear parabolic PDE, if ones treats correctly the diffusion terms down to the finest scale available, say h, but ones slashes/botches terms in the time derivative or the nonlinear term at scale ε, then the H^1-distance between the elliptic projection and the Galerkin approximation is nevertheless of order $\varepsilon^{\sigma+1}$ instead of being of order ε^{σ} as intuition would suggest. (The slashing must not be too severe though; it must be such that the consistency error measured in the H^{-1}-norm is of order $\varepsilon^{\sigma+1}$.) It is this principle which is at work in NLGM. It is also this same principle which is at work in the so-called post-processing Galerkin method [17] and in the numerous two-grid solution methods for the Navier-Stokes equations that have been published lately (although, Wheeler's super-convergence mechanism [7] is rarely cited).

Remark 6.2. The statement of Lemma 6.2 is independent of the type of boundary condition. Modulo minor modifications to the proof, the reader can verify that the super-convergence argument carries through with Dirichlet boundary conditions.

B. The Super-Convergence Result

We can now conclude.

Theorem 6.1. *Under the regularity assumptions of Lemma* 6.1*, the velocity field and the pressure field from (3.13) satisfies the following error estimate*

$$\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathbf{H}^1)} + \|p - p_h\|_{L^2(L^2)} \le c(\nu, T, K)(h^\sigma + h^{\theta(\sigma+1)}). \tag{6.8}$$

Proof. This is a simple consequence of the triangle inequalities

$$\begin{aligned}
\|\mathbf{u} - \mathbf{u}_h\|_{L^2(\mathbf{H}^1)} &\le \|\mathbf{u} - \mathbf{R}_h(\mathbf{u})\|_{L^2(\mathbf{H}^1)} + \|\mathbf{R}_h(\mathbf{u}) - \mathbf{u}_h\|_{L^2(\mathbf{H}^1)}\\
\|p - p_h\|_{L^2(L^2)} &\le \|p - S_h(p)\|_{L^2(L^2)} + \|S_h(p) - p_h\|_{L^2(L^2)}.
\end{aligned}$$

together with Lemma 6.1, Lemma 6.2, and the definition of ε. ■

Remark 6.3. Contrary to what is hinted at in [5], the estimate (6.8) shows that, for any given regularity index σ, NLGM is more accurate than the Galerkin method defined on the discrete (large eddy) spaces $(\mathbf{X}_\varepsilon, M_\varepsilon)$, i.e., we obtain $(\mathcal{O}(\varepsilon^{\sigma+1}))$ estimates instead of $(\mathcal{O}(\varepsilon^{\sigma}))$ estimate. In other words, accounting for the quasi-steady linearized dynamics of the small scales (third equation in (3.8)) is just what is needed to recover near optimality in the $\mathbf{H}^1 \times L^2$-norm all the way down to scale h.

Remark 6.4. As an immediate consequence of the above Theorem, one deduces that the pair $(\mathbf{u}_h, p_h)$ is as accurate as the Galerkin solution on $(\mathbf{X}_h, M_h)$ in the $\mathbf{H}^1 \times L^2$-norm (i.e., that obtained using $\theta = 1$) provided the expected regularity index σ and the real θ are such that

$$\theta \geq \frac{\sigma}{\sigma + 1}. \tag{6.9}$$

Just to remind us finally that there is no free lunch though, observe that NLGM yields a $(\mathcal{O}(h^{\theta(\sigma+1)})$ error estimate in the $\mathbf{L}^2$-norm of the velocity, which is always $(\mathcal{O}(h^{(1-\theta)(\sigma+1)})$ suboptimal, unless $\theta = 1$.

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