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On the construction of suitable solutions to the Navier–Stokes equations and questions regarding the definition of large eddy simulation

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Abstract

This paper discusses the notion of large eddy simulation (LES) as a mathematical concept and proposes to use the concept of suitable weak solutions as a building block for the development of a mathematical theory of LES. Various techniques for constructing suitable weak solutions to the three-dimensional Navier–Stokes equations are reviewed and many features of these mathematical constructions are shown to comply with well-accepted heuristic characterizations of LES.
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1. Introduction

1.1. Introductory comments

At the present time, computer predictions of turbulence phenomena by so-called direct numerical simulation (DNS) of the Navier–Stokes equations are still considered a formidable task for Reynolds numbers larger than a few thousands. Since the times of Boussinesq and Reynolds, numerous turbulence models based on time-averaged or space-averaged quantities (Reynolds averaged Navier–Stokes models, k-ϵ models, etc.) have been developed and then used in engineering applications as a means of overcoming, though often with limited success, the lack

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of sufficient computer resources required by DNS. During the past forty years a new class of turbulence models, collectively known as large eddy simulation (LES) models [27], has emerged in the literature. These models are founded on the observation that representing the whole range of flow scales may not be important in many engineering applications as one is generally interested only in the large scale features of the flow. With this objective in mind, LES modelers have devised artifacts for representing the interactions between the unreachable small scales and the large ones. An extended variety of LES models is now available; see e.g. [7,13,21] for reviews. However no satisfactory mathematical theory for LES has yet been proposed (for preliminary attempts of formalization see [17,22,21]). More surprisingly, no mathematical definition of LES has been stated either (to the best of our knowledge), although some qualitative attempts have been made in this direction. For instance the following formal definition is proposed in [9]: "We define a large eddy simulation as any simulation of a turbulent flow in which the large-scale motions are explicitly resolved while the small-scale motions are represented approximately by a model (in engineering nomenclature) or parameterization (in the geosciences)." The objective of this paper is to go beyond qualitative statements like the one above by proposing a list of mathematical criteria that we think should be considered to establish a reasonable definition of LES.

This paper is organized as follows: in the remainder of the Introduction, we recall the definition of suitable weak solutions to the Navier–Stokes equations. In Section 2 we define suitable approximations to the Navier–Stokes equations and introduce the concept of pre-LES-models. In Section 3, we list existing models that fall into the category of the pre-LES-models. We proceed in Section 4 by providing examples of suitable approximations and for each of these examples we show how our definition of suitable approximations helps us to determine the relationship between the discretization parameter h and the large eddy scale ϵ. Concluding remarks and comments on possible definitions of LES are reported in Section 5.

1.2. Suitable weak solutions

It is generally accepted that the Navier–Stokes equations stand as a reasonable model to predict the behavior of turbulent incompressible flows of viscous fluids. Upon denoting by $\Omega \subset \mathbb{R}^3$ the open smooth connected domain occupied by the fluid, $]0, T[$ some time interval, $\mathbf{u}$ the velocity field, and p the pressure, the problem is formulated as follows:

$$\begin{cases} \partial_t \mathbf{u} + \mathbf{u}\cdot\nabla\mathbf{u} + \nabla p - \nu\nabla^2\mathbf{u} = \mathbf{f} \quad \text{in } Q_T, \\ \nabla\cdot\mathbf{u} = 0 \quad \text{in } Q_T, \\ \mathbf{u}|_\Gamma = 0 \quad \text{or } \mathbf{u} \text{ is periodic}, \\ \mathbf{u}|_{t=0} = u_0, \end{cases} \tag{1.1}$$

where $Q_T = \Omega \times (0, T)$, Γ is the boundary of Ω, $\mathbf{u}_0$ the solenoidal initial data, $\mathbf{f}$ a source term, ν the viscosity, and the density is chosen equal to unity. Henceforth, we assume that (1.1) has been nondimensionalized, i.e., ν is the inverse of the Reynolds number.

To implicitly account for boundary conditions, we introduce

$$\mathbf{X} = \begin{cases} \mathbf{H}^1_0(\Omega) & \text{If Dirichlet conditions,} \\ \{\mathbf{v} \in \mathbf{H}^1(\Omega),\ \mathbf{v} \text{ periodic}\} & \text{If periodic conditions.} \end{cases} \tag{1.2}$$

$$\mathbf{V} = \{\mathbf{v} \in \mathbf{X},\ \nabla\cdot\mathbf{v} = 0\}, \tag{1.3}$$

$$\mathbf{H} = \bar{\mathbf{V}}^{L^2} \tag{1.4}$$

In mathematical terms, the turbulence question is an elusive one. Since the bold definition of turbulence by Leray in the 1930's [28], calling *solution turbulente* any weak solution of the Navier–Stokes equations, progress has been frustratingly slow. The major obstacle in analyzing the Navier–Stokes equations has to do with the question of uniqueness of solutions in three dimensions, a question not yet solved owing to the possibility that the occurrence of so-called vorticity bursts reaching scales smaller than the Kolmogorov scale cannot be excluded.

If weak solutions are not unique, a fundamental question is then to distinguish the physically relevant solutions. A possible piece of the maze might be the notion of suitable weak solutions proposed by Scheffer [34]:

Definition 1.1 ([Scheffer]). A weak solution to the Navier–Stokes equations $(\mathbf{u}, p)$ is suitable if $\mathbf{u} \in L^2(0,T;\mathbf{X}) \cap L^\infty(0,T;\mathbf{L}^2(\Omega))$, $p \in L^{5/4}(Q_T)$ and the local energy balance

$$\partial_t \left(\frac{1}{2}\mathbf{u}^2\right) + \nabla\cdot\left(\left(\frac{1}{2}\mathbf{u}^2 + p\right)\mathbf{u}\right) - \nu\nabla^2\left(\frac{1}{2}\mathbf{u}^2\right) + \nu(\nabla\mathbf{u})^2 - \mathbf{f}\cdot\mathbf{u} \leq 0 \tag{1.5}$$

is satisfied in the distributional sense.

By analogy with nonlinear conservative laws, (1.5) can be viewed as an entropy-like condition which may (hopefully?) selects the physical solutions of (1.1). An explicit form of the distribution $D(\mathbf{u})$ that is missing in the left-hand side of (1.5) to reach equality has been given by Duchon and Robert [8]. For a smooth flow, the distribution $D(\mathbf{u})$ is zero; but for nonregular flow, $D(\mathbf{u})$ may be nontrivial. Suitable solutions are those which satisfy $D(\mathbf{u}) \geq 0$, i.e., if singularities appear, only those that dissipate energy pointwise are admissible. It is expected that suitable solutions are more regular than weak solutions. In this respect, the so-called Caffarelli-Kohn-Nirenberg (CKN) Theorem states that the one-dimensional Hausdorff measure of singular points of suitable solutions is zero, Caffarelli et al. [3], Lin [29], Scheffer [34]. Whether these solutions are indeed classical is still far from being clear. Although it has been proved recently by He [19] that the result of the CKN Theorem also hods for weak solutions it is not known whether indeed weak solutions are suitable.

2. LES and suitable approximations

2.1. LES

At the present time LES is a rather fuzzy concept: our impression is that there exist almost as many interpretations of LES as researchers working on this topic. As originally introduced in [27], LES is often viewed as a technique to derive equations for the large scales by applying a low-pass filter to the Navier–Stokes equations. The filtered equations then include the so-called subgrid scale stresses accounting for the influence of the small scales onto the large ones. Assuming that the behavior of the small scales is more or less universal, the objective is then to find models for the subgrid scale stresses, the so-called closure problem. In consequence, some authors think of LES as solutions to the filtered equations, whereas others think of LES in terms of finite-dimensional approximations. Another commonly shared expectation is that LES should reproduce the statistics of the large scales instead of approximating individual solutions. These observations naturally suggest the following questions: How are defined the large scales in question? Can some sort of universality of the small scales be proven? What does LES solutions actually approximate? How should the numerics be accounted for? What should be the relationship between the scale of the large eddies and the mesh size when an approximation is constructed? Giving a mathematical meaning to the above questions is far from being trivial; as a consequence, a consensual and indisputable mathematical definition for LES seems out of reach for the near future. However, as a first attempt, we suggest in the next section some criteria that seem reasonable to take into consideration for a definition of LES.

2.2. Suitable approximations

One point which we would like to focus our attention on is the gap that exists between the current so-called LES modeling theories and their various numerical implementations. The existence of this gap has been repeatedly acknowledged in the literature without being seriously addressed; see e.g. Ferziger [10]: "In general, there is a close connection between the numerical methods and the modeling approach used in simulation; this connection has not been sufficiently appreciated by many authors." We thus believe that a reasonable definition of LES should at least be founded on the following criteria: (1) A LES approximation should be finite-dimensional, i.e., it should be computable; (2) A LES approximation should solve a problem which is consistent with the Navier–Stokes equations; (3) Finally, a sequence of LES approximations should select a physical solution of the Navier–Stokes equations under the appropriate limiting process, i.e., one which is suitable.

A general definition for LES is out of the scope of the present paper; however, the above discussion leads us to define the notion of suitable approximations to the Navier–Stokes equations.

Definition 2.1. A sequence $(\mathbf{u}_\gamma, p_\gamma)_{\gamma>0}$ with $\mathbf{u}_\gamma \in L^\infty(0,T;\mathbf{L}^2(\Omega)) \cap L^2(0,T;\mathbf{X})$ and $p_\gamma \in \mathcal{D}'(]0,T[, L^2(\Omega))$ is said to be a suitable approximation to (1.1) if

(i) There are two finite-dimensional vectors spaces $\mathbf{X}_\gamma \subset \mathbf{X}$ and $M_\gamma \subset L^2(\Omega)$ such that $\mathbf{u}_\gamma \in \mathcal{C}^0([0,T];\mathbf{X}_\gamma)$ and $p_\gamma \in L^2(]0,T[; M_\gamma)$ for all $T > 0$.
(ii) The sequence converges (up to subsequences) to a weak solution of (1.1), say $\mathbf{u}_\gamma \rightharpoonup \mathbf{u}$ weakly in $L^2(0,T;\mathbf{X})$ and $p_\gamma \to p$ in $\mathcal{D}'(]0,T[, L^2(\Omega))$.
(iii) The weak solution $(\mathbf{u}, p)$ is suitable.

At this point, we want to emphasize that two parameters are actually hidden in the above definition. Since $\mathbf{X}_\gamma$ and M_γ are finite-dimensional, there is a discretization parameter h associated with the size of the smallest scale that can be represented in $\mathbf{X}_\gamma$, roughly $\dim(\mathbf{X}_\gamma) = \mathcal{O}((L/h)^3)$ where $L = \mathrm{diam}(\Omega)$. The definition also implicitly involves a regularization parameter ε associated with some filtering of the Navier–Stokes equations. This parameter is the lengthscale of the smallest eddies that are allowed to be nonlinearly active in the flow. In the above definition the parameter γ is yet an unspecified combination of the two parameters h and ε.

2.3. *Practical construction of suitable approximations*

In practice, the construction of a suitable approximations can be decomposed into the following three steps:

(1) Construction of what we hereafter call the pre-LES-model. This step consists of regularizing the Navier–Stokes equations by introducing the parameter ε. The purpose of the regularization technique is to yield a well-posed problem for all times. Moreover, the limit solution of the pre-LES-model must be a weak solution to the Navier–Stokes equations as $\varepsilon \to 0$ and should be suitable. The pre-LES model can be thought of as a filtered version of the Navier–Stokes equations where the subgrid scale stresses have been modeled in such a way that the resulting PDE is well-posed and yields a unique weak solution that converges (up to subsequences) to a suitable weak solution to the Navier–Stokes equations.
(2) Discretization of the pre-LES-model. This step introduces the meshsize parameter h and the finite-dimensional spaces $\mathbf{X}_\gamma$, M_γ for the approximate velocity and the approximate pressure, respectively.
(3) Determination of a (possibly maximal) relationship between ε and h. The large eddy scale ε and the mesh size h must be selected in such a way that the sequence of discrete solutions is ensured to converge to a suitable solution of the Navier–Stokes equations when $\varepsilon \to 0$ and $h \to 0$.

The novelty in the proposed definition is that enforcing the limit solution $(\mathbf{u}, p)$ to be suitable yields a constraint on the limiting processes $\lim_{\varepsilon\to 0}$ and $\lim_{h\to 0}$. Because of this constraint, we prefer to use the neutral parameter γ than either ε or h, and whenever we write $\gamma \to 0$, it will be understood that $\varepsilon \to 0$ and $h \to 0$ in some manner yet to be specified.

3. Review of existing pre-LES-models

The purpose of this section is to show that our definition of suitable approximations shares many heuristic features of techniques that are identified in the literature as LES models. In particular, we show that regularization techniques recognized in the literature as LES models are indeed pre-LES-models in the sense of our definition, i.e., they all select suitable solutions as the large eddy scale ε tends to zero. In other words these models comply with items (ii) and (iii) of Definition 2.1.

3.1. Hyperviscosity

Lions [30,31] proposed the following hyperviscosity perturbation of the Navier–Stokes equations:

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nu\nabla^2\mathbf{u}_\varepsilon + \varepsilon^{2\alpha}(-\nabla^2)^\alpha\mathbf{u}_\varepsilon = \mathbf{f} \quad \text{in } Q_T, \\ \nabla\cdot\mathbf{u}_\varepsilon = 0 \quad \text{in } Q_T, \\ \mathbf{u}_\varepsilon|_\Gamma,\ \partial_n\mathbf{u}_\varepsilon|_\Gamma, \dots,\ \partial_n^{\alpha-1}\mathbf{u}_\varepsilon|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon \text{ is periodic} \\ u|_{t=0} = u_0, \end{cases} \tag{3.1}$$

where $\varepsilon > 0$ and α is an integer. The appealing aspect of this perturbation is that it yields a well-posed problem in the classical sense when $\alpha \geq \frac{5}{4}$ in three space dimensions. More precisely, upon denoting by $d \geq 2$ the space dimension, the following result (see [18,30,31]) holds

Theorem 3.1. *Assume $f \in L^2(0,T;\mathbf{L}^2(\Omega))$ and $u_0 \in \mathbf{H}^\alpha(\Omega)\cap\mathbf{X}$. Problem (3.5) has a unique solution $\mathbf{u}_\varepsilon$ in $L^\infty(0,T;\mathbf{H}^\alpha(\Omega)\cap\mathbf{X})$ for all times $T>0$ if $\alpha \geq \frac{d+2}{4}$. Up to subsequences, $\mathbf{u}_\varepsilon$ converges to a weak solution $\mathbf{u}$ of (1.1), weakly in $L^2(0,T;\mathbf{X})$. Moreover, if periodic boundary conditions are enforced, the limit solution $(\mathbf{u}, p)$ is suitable.*

It is remarkable that hyperviscosity models are frequently used in so-called LES simulations of oceanic and atmospheric flows [1,2,26] or to control the Navier–Stokes equations [36].

3.2. Leray mollification

A simple construction yielding a suitable solution has indeed been proposed by Leray [28] before this very notion was introduced in the literature. Leray proved the existence of weak solutions by using a, now very popular, mollification technique.

Assume that Ω is the three-dimensional torus $(0,2\pi)^3$. Denoting by $B(0,\varepsilon) \subset \mathbb{R}^3$ the ball of radius ε centered at 0, consider a sequence of non-negative mollifying functions $(\phi_\varepsilon)_{\varepsilon>0}$ satisfying:

$$\phi_\varepsilon \in \mathcal{C}_0^\infty(\mathbb{R}^3;\mathbb{R}_+), \quad \operatorname{supp}(\phi_\varepsilon) \subset B(0,\varepsilon), \quad \int_{\mathbb{R}^3} \phi_\varepsilon(\mathbf{x})\,\mathrm{d}\mathbf{x} = 1. \tag{3.2}$$

Defining the convolution product $\phi_\varepsilon * \mathbf{v}(\mathbf{x}) = \int_{\mathbb{R}^3} \mathbf{v}(\mathbf{y})\phi_\varepsilon(\mathbf{x}-\mathbf{y})\,\mathrm{d}\mathbf{y}$, Leray suggested to regularize the Navier–Stokes equations as follows:

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + (\phi_\varepsilon * \mathbf{u}_\varepsilon)\cdot\nabla\mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nu\nabla^2\mathbf{u}_\varepsilon = \phi_\varepsilon * \mathbf{f}, \\ \nabla\cdot\mathbf{u}_\varepsilon = 0, \\ \mathbf{u}_\varepsilon \text{ is periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \phi_\varepsilon * \mathbf{u}_0. \end{cases} \tag{3.3}$$

The following holds (see [28] and [8]):

Theorem 3.2. *For all* $\mathbf{u}_0 \in \mathbf{H}$, $\mathbf{f} \in \mathbf{H}$, *and* $\varepsilon > 0$, (3.3) *has a unique* $\mathcal{C}^\infty$ *solution. The velocity is bounded uniformly in* $L^\infty(0,T;\mathbf{H}) \cap L^2(0,T;\mathbf{V})$ *and one subsequence converges weakly in* $L^2(0,T;\mathbf{V})$. *The limit solution as* $\varepsilon \to 0$ *is a suitable weak solution of the Navier–Stokes equations.*

Hence the above construction complies with items (ii) and (iii) of Definition 2.1. The mollification technique can be extended to account for the homogeneous Dirichlet boundary condition as done in [3]. The limit solution is suitable in this case as well.

Roughly speaking, the convolution process removes scales that are smaller than ε. Hence, by using $\phi_\varepsilon * \mathbf{u}_\varepsilon$ as the advection velocity, scales smaller than ε are not allowed to be nonlinearly active. This feature is a characteristic of most LES models.

3.3. NS-α and Leray-α models

Introduce the so-called Helmholtz filter $\overline{(\cdot)} : v \mapsto \overline{v}$ such that

$$\overline{\mathbf{v}} := (I - \varepsilon^2\nabla^2)^{-1}\mathbf{v}, \tag{3.4}$$

where either homogeneous Dirichlet boundary conditions or periodic boundary conditions are enforced depending on the setting considered. The so-called Navier–Stokes–alpha model introduced in Chen et al. [5] and Foias *et al.* [11,12] consists of the following:

$$\begin{cases} \partial_t \mathbf{u}_\varepsilon + \overline{\mathbf{u}_\varepsilon}\cdot\nabla\mathbf{u}_\varepsilon + (\nabla\overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon - \nu\nabla^2\mathbf{u}_\varepsilon + \nabla\pi_\varepsilon = \mathbf{f}, \\ \nabla\cdot\overline{\mathbf{u}_\varepsilon} = 0, \\ \mathbf{u}_\varepsilon|_\Gamma = 0,\ \overline{\mathbf{u}_\varepsilon}|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon, \text{ and } \overline{\mathbf{u}_\varepsilon} \text{ are periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0, \end{cases} \tag{3.5}$$

Once again, regularization yields uniqueness as stated in the following

Theorem 3.3 (Foias, Holm, and Titi [11,12]). *Assume* $\mathbf{f} \in \mathbf{H}$, $\mathbf{u}_0 \in \mathbf{V}$. *Problem* (3.5) *with the Helmholtz filter* (3.4) *has a unique solution* $\mathbf{u}_\varepsilon$ *in* $\mathcal{C}^0([0,T);\mathbf{V})$ *with* $\partial_t\mathbf{u}_\varepsilon \in L^2(]0,T[;\mathbf{H})$. *The solution* $\overline{\mathbf{u}_\varepsilon}$ *is uniformly bounded in* $L^\infty(0,+\infty;\mathbf{H}) \cap L^2(0,+\infty;\mathbf{V})$ *and one subsequence converges weakly in* $L^2_{\mathrm{loc}}(0,+\infty;\mathbf{V})$ *to a weak Navier–Stokes solution as* $\varepsilon \to 0$.

Here again, it is a simple matter to show that when periodic boundary conditions are enforced $\mathbf{u}_\varepsilon$ converges, up to subsequences, to a suitable solution. Hence, the above model complies with items (ii) and (iii) of Definition 2.1.

A variant of the above regularization technique consists of replacing the term $(\nabla\overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon$ in (3.5) by $\nabla\frac{1}{2}\mathbf{u}_\varepsilon^2$. The resulting model then falls in the class of the Leray regularization in the sense that the momentum equation is the same as that in (3.3) but for the advection velocity $\phi_\varepsilon * \mathbf{u}_\varepsilon$ which is replaced by $\overline{\mathbf{u}_\varepsilon}$. This model has been analyzed in [6] and is called the Leray-α model. It has been reported in [14] to be a good candidate for a LES model: "The Leray model was found to predict the momentum thickness properly while exhibiting both forward and backward transfer of energy. Further analysis shows reliable levels of turbulence intensities and correct behavior of kinetic energy. [...] The regularized dynamics shows an appealing robustness at high Re."

3.4. Nonlinear viscosity models

Recalling that the Navier–Stokes equations are based on Newton's linear hypothesis, Ladyženskaja and Kaniel proposed to modify the incompressible Navier–Stokes equations to take into account possible large velocity gradients, [23–25].

Ladyženskaja introduced a nonlinear viscous tensor $\mathbf{T}_{ij}(\nabla\mathbf{u})$, $1 \le i, j \le 3$ satisfying the following conditions:

L1. **T** is continuous and there exists $\mu \ge \frac{1}{4}$ such that

$$\forall\boldsymbol{\xi} \in \mathbb{R}^{3\times3}, \quad |\mathbf{T}(\boldsymbol{\xi})| \le c(1 + |\boldsymbol{\xi}|^{2\mu})|\boldsymbol{\xi}|. \tag{3.6}$$

L2. **T** satisfies the coercivity property:

$$\forall\boldsymbol{\xi} \in \mathbb{R}^{3\times3}, \quad \mathbf{T}(\boldsymbol{\xi}) : \boldsymbol{\xi} \ge c|\boldsymbol{\xi}|^2(1 + c'|\boldsymbol{\xi}|^{2\mu}). \tag{3.7}$$

L3. **T** possesses the following monotonicity property: There exists a constant $c > 0$ such that for all solenoidal fields $\boldsymbol{\xi}, \boldsymbol{\eta}$ in $\mathbf{W}^{1,2+2\mu}(\Omega)$ either coinciding on the boundary Γ or being periodic,

$$\int_\Omega (\mathbf{T}(\nabla\boldsymbol{\xi}) - T(\nabla\boldsymbol{\eta})) : (\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}) \ge c\int_\Omega |\nabla\boldsymbol{\xi} - \nabla\boldsymbol{\eta}|^2. \tag{3.8}$$

The three above conditions are satisfied if

$$\mathbf{T}(\boldsymbol{\xi}) = \beta(|\boldsymbol{\xi}|^2)\boldsymbol{\xi}, \tag{3.9}$$

provided the viscosity function $\beta(\tau)$ is a positive monotonically-increasing function of $\tau \ge 0$ and for large values of τ the following inequality holds

$$c\tau^\mu \le \beta(\tau) \le c'\tau^\mu,$$

with $\mu \ge \frac{1}{4}$ and c, c' are some strictly positive constants.

After introducing the large eddy scale $\varepsilon > 0$, the modified Navier–Stokes equations take the form

$$\begin{cases} \partial_t\mathbf{u}_\varepsilon + \mathbf{u}_\varepsilon\cdot\nabla\mathbf{u}_\varepsilon + \nabla p_\varepsilon - \nabla\cdot(\nu\nabla\mathbf{u}_\varepsilon + \varepsilon^{2\mu+1}\mathbf{T}(\nabla\mathbf{u}_\varepsilon)) = \mathbf{f}, \\ \nabla\cdot\mathbf{u}_\varepsilon = 0 \\ \mathbf{u}_\varepsilon|_\Gamma = 0, \quad \text{or } \mathbf{u}_\varepsilon \text{ is periodic}, \\ \mathbf{u}_\varepsilon|_{t=0} = \mathbf{u}_0. \end{cases} \tag{3.10}$$

The main result from [25,24] (see [23] for a similar result where monotonicity is also assumed) is the following theorem

Theorem 3.4. *Assume* $\mathbf{u}_0 \in \mathbf{H}$ *and* $\mathbf{f} \in L^2(]0, +\infty[; \mathbf{L}^2(\Omega))$. *Provided conditions* L1, L2, *and* L3 *hold, then* (3.10) *has a unique weak solution, for all* $R > 0$, *in* $L^{2+2\mu}(]0, T[; \mathbf{W}^{1,2+2\mu}(\Omega) \cap \mathbf{V}) \cap C^0([0, T]; \mathbf{H})$.

Moreover, for periodic boundary conditions, $(\mathbf{u}_\varepsilon, p_\varepsilon)$ converges, up to subsequences, to a suitable solution, of (1.1) as $\varepsilon \to 0$, i.e., items (ii) and (iii) of Definition 2.1 hold.

Possibly one of the most popular LES models is that proposed by Smagorinsky [35], which corresponds to setting $\mathbf{T}(\nabla\mathbf{u}) = |\mathbf{D}|\mathbf{D}$. (i.e., $\beta(\tau) = \tau^\mu$ with $\mu = \frac{1}{2}$). Hence in addition to other possible appealing features LES modelers may see in Smagorinsky-like LES models, the one of interest to us is that they guarantee well-posedness and ensure the limit solutions be suitable as required by items (ii) and (iii) of Definition 2.1.

4. Discretization

The purpose of this section is to introduce discrete versions of some of the pre-LES-models described above, i.e., we want to comply with item (i) of Definition 2.1. (Recall once again that what we herein call a pre-LES-model is usually referred to in the literature as a LES model.) In each case we show that requiring the approximate solutions to be suitable approximations determines the relationship between the mesh size h and the large eddy scale ε, thus solving a question very often left open or solved heuristically in the LES literature.

For the sake of simplicity, we restrict ourselves to periodic boundary conditions and spectral approximation techniques.

4.1. The discrete hyperviscosity model

We turn our attention to the hyperviscosity model introduced in Section 3.1 and we construct a Galerkin-Fourier approximation.

For any $z \in \mathbb{C}^\ell$, $1 \le \ell \le 3$, we denote by $|z|$ the Euclidean norm of z and by $|z|_\infty$ the maximum norm. We denote by $\bar{z}$ the conjugate of z. Recall that Sobolev spaces $H^s(\Omega)$, $s \ge 0$, can be equivalently defined in terms of Fourier series as follows

$$H^s(\Omega) = \left\{ u = \textstyle\sum_{k\in\mathbb{Z}^3} u_k e^{ik\cdot x}, u_k = \bar{u}_{-k}, \displaystyle\sum_{k\in\mathbb{Z}^3} (1+|k|^2)^s |u_k|^2 < +\infty \right\}.$$

In other words, the set of trigonometric polynomials $\exp(ik\cdot x)$, $k \in \mathbb{Z}^3$, is complete and orthogonal in $H^s(\Omega)$ for all $s \ge 0$. The scalar product in $L^2(\Omega)$ is denoted by $(u, v) = (2\pi)^{-3}\int_\Omega u\bar{v}$ and the dual of $H^s(\Omega)$ by $H^{-s}(\Omega)$. We introduce the closed subspace $\dot{H}^s(\Omega)$ of $H^s(\Omega)$ composed of the functions of zero mean value.

Let N be a positive integer, henceforth referred to as the cutoff wave number. We introduce the set of trigonometric polynomials of partial degree less than or equal to N:

$$\mathbb{P}_N = \left\{ p(x) = \sum_{|k|_\infty \le N} c_k e^{ik\cdot x}, c_k = \bar{c}_{-k} \right\},$$

and denote by $\dot{\mathbb{P}}_N$ the subspace of $\mathbb{P}_N$ composed of the trigonometric polynomials of zero mean value.

Finally, to approximate the velocity and the pressure fields we introduce the following finite-dimensional vector spaces:

$$\mathbf{X}_N = \dot{\mathbb{P}}_N^3, \quad \text{and} \quad M_N = \dot{\mathbb{P}}_N. \tag{4.1}$$

The mesh size that can naturally be associated with the above setting is (up to a 2π factor)

$$h_N = \frac{1}{N} \tag{4.2}$$

We now define a large eddy scale ε_N associated with the hyperviscosity model. Let θ be a real number with $0 < \theta < 1$. Then we set

$$\varepsilon_N = h_N^\theta, \quad N_i = \frac{1}{\varepsilon_N} = N^\theta. \tag{4.3}$$

To avoid unnecessary dampening on the low wavenumbers we choose to construct an hyperviscosity operator that acts only on the high wavenumbers of the velocity field, namely for wavenumbers k such that $N_i \le |k|_\infty \le N$, This idea is similar to the spectral viscosity technique that Tadmor [37,32,4] developed for nonlinear scalar conservation laws. We introduce the hyperviscosity kernel $Q(x)$ such that

$$Q(x) = (2\pi)^{-3} \sum_{N_i \le |k|_\infty \le N} |k|^{2\alpha} e^{ik\cdot x} \tag{4.4}$$

where

$$\alpha > \frac{5}{4} \tag{4.5}$$

is the exponent of the hyperviscosity. This kernel is such that for all $v_N \in X_N$

$$Q{*}v_N(x) = \int_\Omega v_N(y)Q(x-y)\mathrm{d}y = \sum_{N_i \le |k|_\infty \le N} |k|^{2\alpha} v_k e^{ik\cdot x} \tag{4.6}$$

When α is an integer, $Q{*}(\cdot)$ is the α-th power of the Laplace operator restricted to the space spanned by the Fourier modes associated with the length scales comprised between ε_N and h_N. Note that when θ increases, ε_N decreases and the range of wavenumbers on which the kernel $Q(x)$ is active shrinks.

The spectral hyperviscosity model consists of the following:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^0([0,T];\mathbf{X}_N) \text{ and } p_N \in L^2([0,T];M_N) \text{ such that} \\ (\partial_t \mathbf{u}_N, v) + (\mathbf{u}_N \cdot \nabla \mathbf{u}_N, v) - (p_N, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) + \varepsilon_N^{2\alpha}(Q{*}\mathbf{u}_N, \mathbf{v}) \\ \quad = (\mathbf{f}, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N, \forall t \in (0,T], \\ (\nabla \cdot \mathbf{u}_N, q) = 0, \quad \forall q \in M_N, \forall t \in (0,T], \\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}), \quad \forall \mathbf{v} \in \mathbf{X}_N. \end{cases} \tag{4.7}$$

One interesting feature of the above technique is the following

Proposition 4.1. *The hyperviscosity perturbation is spectrally small, i.e.,*

$$\varepsilon_N^{2\alpha} \| Q{*}v_N \|_{L^2} \lesssim N^{-\theta s} \|v_N\|_{H^s}, \quad \forall v_N \in H^s(\Omega),\ \forall s \ge 2\alpha. \tag{4.8}$$

The interpretation of the above result is that the consistency error induced by the hyperviscosity is arbitrarily small if the exact solution to the Navier–Stokes equation is smooth.

The main result of this section is the following:

Theorem 4.1. *Let* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}^\alpha(\Omega) \cap \mathbf{V}$. *Assume that (4.5) and (4.3) hold. Moreover, assume that*

$$0 < \theta < \begin{cases} \dfrac{4\alpha - 5}{4\alpha} & \text{If } \alpha \le \frac{3}{2}, \\ \dfrac{2(\alpha-1)}{2\alpha+3} & \text{Otherwise.} \end{cases} \tag{4.9}$$

then, up to subsequences, the solution u_N *to (4.7) converges weakly in* $L^2(0,T;\mathbf{V})$ *and strongly in any* $L^r(0,T;\mathbf{L}^q(\Omega))$, *with* $1 \le q < \frac{6r}{3r-4} < +\infty$, $2 \le r < \infty$, *to a suitable solution of (1.1) as N goes to infinity.*

The above result, proved in [18], can be interpreted as follows. Since the hyperviscosity term is meant to be a perturbation of the Navier–Stokes equations, one would want this term to be as small as possible. In fact, one would like θ to be as large as possible; the choice $\theta = 1$ is that which minimizes the impact of the hyperviscosity perturbation. But, if θ is too close to one, the hyperviscosity term cannot play the role it is assigned, i.e., the limit solution cannot be guaranteed to be suitable (see item (iii)). It is shown in [18] that a sufficient condition for the limit solution to be suitable is that the bound from above in (4.9) holds. In this sense we claim that Definition 2.1 is constructive.

Numerically speaking, the above result is somewhat vague. A possible algorithm for implementing this hyperviscosity model is to pick α, to take $\theta < \frac{4\alpha-5}{4\alpha}$ or $\theta < \frac{2(\alpha-1)}{2\alpha+3}$, depending on the value of α, and to finally set

Table 1
Admissible values of the parameters α and θ

α	$\frac{3}{2}$	2	3	4	5
θ	$< \frac{1}{6}$	$< \frac{2}{7}$	$< \frac{4}{9}$	$< \frac{6}{11}$	$< \frac{8}{13}$

$N_i = c_1 N^\theta$ and $\varepsilon_N = c_2 N_i^{-1}$. In this case, α is a free parameter and the constants c_1, c_2 are coefficients that can be played with, provided they are of order one. Some admissible values of the parameters α and θ are shown in Table 1.

Remark 4.1. Observe that the condition (4.9) means that asymptotically $h_N \ll \varepsilon_N$. In other words, the scales filtered by the hyperviscosity are significantly larger than the grid size.

4.2. The discrete Leray model

For the sake of simplicity, we restrict the analysis to the periodic case, i.e., Ω is assumed to be the three-dimensional torus, and we use again the Fourier setting introduced in Section 4.1. Let N be an integer and set

$$\mathbf{X}_N = \dot{\mathbb{P}}_N^3, \quad \text{and} \quad M_N = \dot{\mathbb{P}}_N. \tag{4.10}$$

Up to a 2π factor, the mesh size naturally associated with the above setting is

$$h_N = \frac{1}{N}. \tag{4.11}$$

We now define a large eddy scale ε_N to be associated with the filtering of the advection velocity. For this purpose we introduce a real number θ with $0 < \theta < 1$, and we set

$$\varepsilon_N = h_N^\theta, \quad N_i = \frac{1}{\varepsilon_N} = N^\theta. \tag{4.12}$$

Then we consider the truncation operator $P_{\varepsilon_N} : \mathbf{H}^s(\Omega) \longrightarrow \mathbb{P}_{N_i}^3$ such that

$$P_{\varepsilon_N} : \mathbf{H}^s(\Omega) \ni \sum_{k\in\mathbb{Z}^3} \mathbf{v}_k e^{ik\cdot x} = \mathbf{v} \longmapsto \sum_{|k|_\infty \le N_i} \mathbf{v}_k e^{ik\cdot x} \in \mathbb{P}_{N_i}^3$$

and we let Q_{ε_N} denote the operator $Q_{\varepsilon_N} = I - P_{\varepsilon_N}$. Upon defining $\phi_{\varepsilon_N} = \sum_{|k|_\infty \le N_i} e^{ik\cdot x}$, it is clear that for all $\mathbf{v}_N \in \mathbf{X}_N$

$$\phi_{\varepsilon_N} * \mathbf{v}_N = P_{\varepsilon_N} \mathbf{v}_N.$$

Thus, the discrete Leray model takes the following form:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^0([0,T];\mathbf{X}_N) \text{ and } p_N \in L^2([0,T];M_N) \text{ such that} \\ \text{for all } t \in (0,T], \text{ for all } \mathbf{v} \in \mathbf{X}_N, \text{ and for all } q \in M_N, \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (P_{\varepsilon_N}\mathbf{u}_N \cdot \nabla \mathbf{u}_N, \mathbf{v}) - (p_N, \nabla\cdot\mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\nabla\cdot\mathbf{u}_N, q) = 0, \\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}). \end{cases} \tag{4.13}$$

Theorem 4.2. *Let* $\mathbf{f} \in L^2(0, T; \mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}$. *Assume that* (4.12) *hold. If*

$$0 < \theta < \frac{2}{3}, \tag{4.14}$$

the solution $\mathbf{u}_N$ *to* (4.13) *converges weakly, up to subsequences, in* $L^2(0, T; \mathbf{V})$ *and strongly in any* $L^r(0, T; \mathbf{L}^q(\Omega))$, *with* $1 \le q < \frac{6r}{3r-4} < +\infty$, $2 \le r < \infty$, *to a suitable solution of* (1.1) *as N goes to infinity*.

Proof. We just sketch the proof since the details are similar to those given in the proof of Theorem 5.1 in [18]. The main difficulty revolves around the handling of the nonlinear term when proving that the limit solution is suitable. As usual, the basic a priori estimates are

$$\|\mathbf{u}_N\|_{L^2}^2 + \nu \int_0^T \|\nabla P_{\varepsilon_N}\mathbf{u}_N\|_{L^2}^2 + \|\nabla Q_{\varepsilon_N}\mathbf{u}_N\|_{L^2}^2 \le c. \tag{4.15}$$

Let $\phi \in \mathcal{D}(Q_T)$ and use $P_N(\phi\mathbf{u}_N)$ to test the momentum equation in (4.13). The nonlinear term gives

$$(P_{\varepsilon_N}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi)) = (P_{\varepsilon_N}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, \mathbf{u}_N\phi) + R_1 = -\left(\frac{1}{2}|\mathbf{u}_N|^2 P_{\varepsilon_N}\mathbf{u}_N, \nabla\phi\right) + R_1,$$
$$= -\left(\frac{1}{2}|\mathbf{u}_N|^2\mathbf{u}_N, \nabla\phi\right) + R_1 + R_2,$$

where

$$R_1 = (P_{\varepsilon_N}\mathbf{u}_N\cdot\nabla\mathbf{u}_N, P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi), \quad R_2 = \left(\frac{1}{2}|\mathbf{u}_N|^2 Q_{\varepsilon_N}\mathbf{u}_N, \nabla\phi\right).$$

The first residual is handled as follows:

$$\begin{aligned} |R_1| &\le \|P_{\varepsilon_N}\mathbf{u}_N\|_{L^\infty}\|\nabla\mathbf{u}_N\|_{L^2}\|P_N(\mathbf{u}_N\phi) - \mathbf{u}_N\phi\|_{L^2} \\ &\lesssim N_i^{(3/2)}N^{-1}\|P_{\varepsilon_N}\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}\|\mathbf{u}_N\phi\|_{H^1} \\ &\lesssim N^{(3/2)\theta-1}\|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}^2\|\phi\|_{W^{1,\infty}}, \end{aligned}$$

where $\lesssim$ means that the inequality holds up to a constant independent of N. Then, it is clear that $\int_0^T |R_1| \to 0$ as $N \to \infty$ owing to (4.14). For the second residual we use the embedding $H^{\frac{1}{2}}(\Omega) \subset L^3(\Omega)$ as follows

$$\begin{aligned} |R_2| &\lesssim \|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{L^6}\|Q_{\varepsilon_N}\mathbf{u}_N\|_{L^3}\|\phi\|_{W^{1,\infty}} \\ &\lesssim \|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}\|Q_{\varepsilon_N}\mathbf{u}_N\|_{H^{\frac{1}{2}}}\|\phi\|_{W^{1,\infty}} \\ &\lesssim N_i^{-\frac{1}{2}}\|\mathbf{u}_N\|_{L^2}\|\mathbf{u}_N\|_{H^1}^2\|\phi\|_{W^{1,\infty}}. \end{aligned}$$

As a result, $\int_0^T |R_2| \to 0$ as $N \to \infty$ since $\theta > 0$. The rest of the proof follows as that of Theorem 5.1 in [18]. □

Remark 4.2. In essence, Theorem 4.2 shows that if $\varepsilon_N^{3/2} \gg h_N$, then the pair $(\mathbf{u}_N, p_N)$ is a suitable approximation in the sense of Definition 2.1.

4.3. The discrete Leray-α model

Still keeping the above Fourier framework, we now consider the following discrete version of the Leray-α model introduced at the end of Section 3.3; see also [6,16]:

$$\begin{cases} \text{Seek } \mathbf{u}_N \in \mathcal{C}^0([0,T];\mathbf{X}_N) \text{ and } p_N \in L^2([0,T];M_N) \text{ such that} \\ \text{for all } t \in (0,T], \text{ for all } \mathbf{v} \in \mathbf{X}_N, \text{ and for all } q \in M_N, \\ (\partial_t \mathbf{u}_N, \mathbf{v}) + (\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, \mathbf{v}) - (p_N, \nabla \cdot \mathbf{v}) + \nu(\nabla \mathbf{u}_N, \nabla \mathbf{v}) = (\mathbf{f}, \mathbf{v}), \\ (\bar{\mathbf{u}}_N, \mathbf{v}) + \varepsilon_N^2 (\nabla \bar{\mathbf{u}}_N, \nabla \mathbf{v}) = (\mathbf{u}_N, \mathbf{v}), \\ (\nabla \cdot \mathbf{u}_N, q) = 0, \\ (\mathbf{u}_N, \mathbf{v})|_{t=0} = (\mathbf{u}_0, \mathbf{v}), \end{cases} \tag{4.16}$$

where ε_N is the scale of the smallest eddies that we authorize to be nonlinearly active:

$$\varepsilon_N = N_i^{-1} = N^{-\theta}, \qquad 0 < \theta < 1. \tag{4.17}$$

Note that the system (4.16) is similar to (4.13) except for the advective velocity $P_{\varepsilon_N} \mathbf{u}_N$ which is now replaced by the regularized velocity $\bar{\mathbf{u}}_N$.

Theorem 4.3. *Let* $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$ *and* $\mathbf{u}_0 \in \mathbf{H}$. *Assume that* (4.17) *holds. If*

$$0 < \theta < \tfrac{2}{3}, \tag{4.18}$$

the solution $\mathbf{u}_N$ *to* (4.16) *converges weakly, up to subsequences, in* $L^2(0,T;\mathbf{V})$ *and strongly in* $L^r(0,T;\mathbf{L}^q(\Omega))$, *with* $1 \le q < \frac{6r}{3r-4} < +\infty$, $2 \le r < \infty$, *to a suitable solution of* (1.1) *as N goes to infinity.*

Proof. Using the fact that $\nabla \cdot \bar{\mathbf{u}}_N = 0$ and taking $\mathbf{v} = P_N(\mathbf{u}_N \phi)$ to test the discrete momentum equation, the nonlinear term becomes

$$(\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, P_N(\mathbf{u}_N \phi)) = (\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, \mathbf{u}_N \phi) + R_1 = -\left(\tfrac{1}{2}|\mathbf{u}_N|^2 \mathbf{u}_N, \nabla \phi\right) + R_1 + R_2,$$

where

$$R_1 = (\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, P_N(\mathbf{u}_N \phi) - \mathbf{u}_N \phi),$$
$$R_2 = \left(\tfrac{1}{2}|\mathbf{u}_N|^2 (\mathbf{u}_N - \bar{\mathbf{u}}_N), \nabla \phi\right).$$

Then

$$|R_1| = |(\bar{\mathbf{u}}_N \cdot \nabla \mathbf{u}_N, P_N(\mathbf{u}_N \phi) - \mathbf{u}_N \phi)| \lesssim \|\bar{\mathbf{u}}_N\|_{L^\infty} \|\nabla \mathbf{u}_N\|_{L^2} \|P_N(\mathbf{u}_N \phi) - \mathbf{u}_N \phi\|_{L^2}$$
$$\lesssim N^{-1} \|\bar{\mathbf{u}}_N\|_{L^\infty} \|\mathbf{u}_N\|_{H^1}^2 \|\phi\|_{W^{1,\infty}}.$$

Using $\|\bar{\mathbf{u}}_N\|_{L^\infty}^2 \lesssim \|\nabla \bar{\mathbf{u}}_N\|_{L^2} \|\Delta \bar{\mathbf{u}}_N\|_{L^2}$ and the bounds $\varepsilon_N \|\nabla \bar{\mathbf{u}}_N\|_{L^2} \lesssim \|\mathbf{u}_N\|_{L^2}$ and $\varepsilon_N^2 \|\Delta \bar{\mathbf{u}}_N\|_{L^2} \lesssim \|\mathbf{u}_N\|_{L^2}$ (from the Helmholtz problem), an estimate of the residual is

$$|R_1| \lesssim \varepsilon_N^{-3/2} N^{-1} \|\mathbf{u}_N\|_{L^2} \|\mathbf{u}_N\|_{H^1}^2 \|\phi\|_{W^{1,\infty}} \lesssim N^{3/2(\theta-1)} \|\mathbf{u}_N\|_{L^2} \|\mathbf{u}_N\|_{H^1}^2 \|\phi\|_{W^{1,\infty}}.$$

It follows that $\int_0^T |R_1| \to 0$ as $N \to \infty$ provided that $\theta < \frac{2}{3}$. For the second residual we use the fact that

$$\|\mathbf{u}_N - \bar{\mathbf{u}}_N\|_{L^2} + \varepsilon_N \|\nabla(\mathbf{u}_N - \bar{\mathbf{u}}_N)\|_{L^2} \lesssim \varepsilon_N \|\nabla \mathbf{u}_N\|_{L^2}.$$

Using $\|\mathbf{v}\|_{L^3} \lesssim \|\mathbf{v}\|_{L^2}^{1/2} \|\mathbf{v}\|_{L^6}^{1/2}$, this implies $\|\mathbf{u}_N - \bar{\mathbf{u}}_N\|_{L^3} \lesssim \varepsilon_N^{1/2} \|\nabla \mathbf{u}_N\|_{L^2}$. Then

$$|R_2| = \left| \left(\tfrac{1}{2} |\mathbf{u}_N|^2 (\mathbf{u}_N - \bar{\mathbf{u}}_N), \nabla\phi \right) \right| \lesssim \|\mathbf{u}_N\|_{L^2} \|\mathbf{u}_N\|_{L^6} \|\mathbf{u}_N - \bar{\mathbf{u}}_N\|_{L^3} \|\phi\|_{W^{1,\infty}}$$
$$\lesssim \varepsilon_N^{1/2} \|\mathbf{u}_N\|_{L^2} \|\mathbf{u}_N\|_{H^1}^2 \|\phi\|_{W^{1,\infty}}.$$

It follows that $\int_0^T |R_1| \to 0$ as $N \to \infty$ provided that $0 < \theta$. □

Renmark 4.3. Once again, the conclusion of this section is that the Fourier-based Leray-α model (4.16) yields a suitable approximation to the Navier–Stokes equations whenever $\varepsilon_N^{3/2} \gg h_N$.

4.4. The other discrete models

We have not yet been able to show that discrete counterparts of the other pre-LES-models introduced in Section 3, namely the NS-α model and the nonlinear viscosity models, produce suitable approximations.

Regarding the nonlinear viscosity models, we are still facing technical issues to prove that the discrete solutions actually converge to suitable weak solutions of the Navier–Stokes equations. In particular, the difficulty lies in the fact that the Fourier analysis is not the proper tool to work with L^p spaces when $p \neq 2$. However, we believe that these issues, being purely technical, will eventually be overcome in the near future.

In the case of the NS-α model, the presence of the term $(\nabla\overline{\mathbf{u}_\varepsilon})^T \cdot \mathbf{u}_\varepsilon$ poses some difficulties when passing to the limit. Whether these difficulties are either technical or fundamental is not yet clear.

Surprisingly we have also observed that the Nonlinear Galerkin Method [33] can be reinterpreted in the light of Definition 2.1. Indeed, following a similar approach as above, it can be shown that the discrete solution of a slightly modified version of the Nonlinear Galerkin Method yields a suitable solution at the limit under appropriate conditions, (the modified version in question consists of replacing $\partial_t \mathbf{u}_N$ by $\partial_t P_{\varepsilon_N} \mathbf{u}_N$ in (4.13)). This interpretation of the Nonlinear Galerkin Method will be the subject of a forthcoming paper.

4.5. The case of DNS

A natural question that comes to mind is whether a sequence of direct numerical solutions (DNS) is a suitable approximation as defined above.

To clarify this issue let us consider the construction proposed by Hopf [20]. Let $\mathbf{X}_h \subset \mathbf{X}$ and $M_h \subset L^2(\Omega)$ be two finite-dimensional vector spaces and consider the following Galerkin approximation

$$\begin{cases} (\partial_t \mathbf{u}_h, \mathbf{v}) + b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) - (p_h, \nabla\cdot\mathbf{v}) + \nu(\nabla\mathbf{u}_h, \nabla\mathbf{v}) = (\mathbf{f}, \mathbf{v}) \quad \forall t \in [0,T], \forall \mathbf{v} \in \mathbf{X}_h, \\ (q, \nabla\cdot\mathbf{u}_h) = 0, \quad \forall t \in [0,T],\ \forall q \in M_h, \\ (\mathbf{u}_h|_{t=0}, v) = (\mathbf{u}_0, \mathbf{v}), \forall \mathbf{v} \in \mathbf{X}_h, \end{cases} \tag{4.19}$$

where b is a trilinear form accounting for the nonlinear advection term which is assumed to be skew-symmetric with respect to its second and third arguments. For instance $b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = (\mathbf{u}_h\cdot\nabla\mathbf{u}_h, \mathbf{v}) - \frac{1}{2}(\mathbf{u}_h \nabla\cdot\mathbf{u}_h, \mathbf{v})$ and $b(\mathbf{u}_h, \mathbf{u}_h, \mathbf{v}) = ((\nabla\times\mathbf{u}_h)\times\mathbf{u}_h, \mathbf{v})$ are admissible candidates.

The above construction is usually referred to as DNS in the literature. Owing to standard a priori estimates uniform in h, it is clear that the above defined sequence of approximate solutions complies with items (i) and (ii) of Definition 2.1; see, e.g., Lions [30,31] or Temam [38]. However, it is not yet known in general whether such a construction yields a suitable solution at the limit when $h \to 0$, i.e., surprisingly item (iii) is not guaranteed to hold. In other words, it may happen that the solution sets spanned by the limits of DNS approximations and suitable approximations are not identical.

It has however been recently proved, [15], that when low-order finite elements are used in the Galerkin construction and if periodic boundary conditions are enforced, then the sequence of Galerkin approximations yields a suitable approximation. This results underlines that the nature of the approximation technique that is used plays a key role in this matter. Low-order approximations seem to do the trick, whereas spectral methods need some regularization or extra viscosities to guarantee convergence to a suitable solution. This is related to the fact that spectral methods are prone to Gibbs phenomena. This result tends to confirm statements sometimes made in the literature that, when using low-order methods, it is preferable to let the "numerical diffusion do the job" than to perform any LES modeling.

5. Conclusions

We have defined in this paper the notion of suitable approximations to the Navier–Stokes equations. The definition introduces two parameters: a discretization scale h and a large eddy scale ε. The ratio h/ε should be chosen so that the limit solution is suitable. It is proven on three examples based on the Fourier approximation setting that h should be much smaller than ε to ensure that approximate solutions converge to a suitable solution of the Navier–Stokes equations; see Remarks 4.1, 4.2, and 4.3.

We have also shown that the notion of suitable approximations shares many heuristic features with what is often referred to in the literature as LES modeling. As a result, we think that the notion of suitable approximations is a concept that, together with other mathematical criteria yet to be clearly identified, should be seriously considered to be included in any future mathematical definition of LES.

Whether the mathematical framework proposed herein is of any help to model turbulence is far from being clear, and we do not make any claim in this respect. We identify two obstacles in the way. First, although the notion of turbulence is quite intuitive and is a daily experience, the very concept of turbulence has yet to be mathematically defined. In particular, we are not aware of any other mathematical definition of turbulence than that proposed by Leray, who identifies "solutions turbulentes" and weak solutions. Second, there is still a possibility that all the weak solutions could eventually be shown to be suitable. In this event, the notion of suitable approximations would be either irrelevant or should be adapted to the inviscid Euler equations. This last argument shows again that the question of the regularity of the Navier–Stokes equations is far from being of academic interest only.

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