

UNCONDITIONALLY STABLE FINITE-ELEMENT METHOD FOR THE UNSTEADY NAVIER-STOKES EQUATIONS

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Abstract

This paper describes a fractional-step projection method to compute incompressible viscous flows under quite general boundary conditions and using unstructured triangular meshes. The distinctive features of the method are the replacement of the continuity equation by a Poisson equation for the pressure increment, the elimination of the end-of-step velocity from the numerical algorithm and the use of an unconditionally stable semi-implicit approximation of the nonlinear term to eliminate any stability restriction on the time step. Numerical results for two-dimensional problems are presented to demonstrate the performance of the proposed method.

1 Introduction

The fractional-step projection method of Chorin [2, 3] and Temam [16] is the most frequently employed technique for the numerical solution of the primitive variable Navier-Stokes equations. Several implementations of this approach using finite-element-based spatial discretizations have been proposed, see *e.g.* [15, 5, 8]. In particular, the fractional-step method can be formulated introducing a Poisson equation for the pressure as an equivalent substitute for the continuity equation in the step enforcing the incompressibility and determining the pressure field, *cf.* [12].

An important aspect in formulating a finite element fractional-step projection method is the definition of the appropriate functional framework capable of accommodating the different mathematical character of the equations of the two steps, namely the viscous equations of the advection-diffusion step and the inviscid equations of the incompressible projection step. Such a functional analytic setting has been provided in

[9] where different possible realizations of projection methods for solving the unsteady incompressible Navier–Stokes equations have been proposed. The aim of this communication is to outline a finite element projection method derived from that analysis, which employs a Poisson equation for pressure and can be operated using unstructured triangular meshes. Numerical results are given to demonstrate the flexibility of the method to deal with arbitrary geometries and boundary conditions. A detailed description of the method is given in [11] which contains additional numerical results.

2 The unsteady Navier–Stokes problem

Let Ω be an open connected bounded domain of \mathbb{R}^d ($d \leq 3$) with a smooth boundary $\partial\Omega$. To have a fractional-step projection method capable of handling problems with boundary conditions of different kind, we consider the following Navier–Stokes problem: for a given body force \mathbf{f} (possibly dependent on time) and a prescribed divergence-free initial velocity field \mathbf{u}_0 , find a velocity field \mathbf{u} and a pressure field p so that at $t = 0$, $\mathbf{u} = \mathbf{u}_0$, and at all subsequent times

$$\begin{cases} \frac{\partial \mathbf{u}}{\partial t} - \nu \nabla^2 \mathbf{u} + (\mathbf{u} \cdot \nabla) \mathbf{u} + \nabla p = \mathbf{f}, \\ \nabla \cdot \mathbf{u} = 0, \end{cases} \quad (1)$$

the velocity and the pressure being subject to the following boundary conditions:

$$\begin{aligned} \mathbf{u}|_{\partial\Omega_1} &= \mathbf{b}_1; \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_2} &= \mathbf{b}_2 \cdot \mathbf{n}, \quad (\nabla \times \mathbf{u}) \times \mathbf{n}|_{\partial\Omega_2} = 0; \\ \mathbf{u} \times \mathbf{n}|_{\partial\Omega_3} &= \mathbf{b}_3 \times \mathbf{n}, \quad p|_{\partial\Omega_3} = P; \end{aligned} \quad (2)$$

where $\partial\Omega_1, \partial\Omega_2, \partial\Omega_3$ is a partition of $\partial\Omega$. The functions \mathbf{b}_i , $i = 1, 2, 3$, and P are defined on the appropriate parts of $\partial\Omega$ and are assumed to be conveniently smooth. Moreover, the initial and boundary data are assumed to satisfy the compatibility condition $\mathbf{u}_0 \cdot \mathbf{n}|_{\partial\Omega_{1,2}} = \mathbf{b}_{1,2} \cdot \mathbf{n}|_{t=0}$. [10].

The boundary conditions involving the tangential components of \mathbf{u} or $\nabla \times \mathbf{u}$ couple the vector components of the velocity field when the boundaries $\partial\Omega_2$ and $\partial\Omega_3$ are curved or are flat but oblique with respect to the Cartesian axes. The equations for the vector components of velocity uncouple with boundary conditions different from the purely Dirichlet ones on \mathbf{u} provided that these boundaries are parallel to the Cartesian planes.

3 The variational formulation

To recast the unsteady problem (1)-(2) in a variational form, we define the following Hilbert spaces (using standard notation for Sobolev spaces):

$$\mathbf{X}_0 = \{\mathbf{v} \in \mathbf{H}^1(\Omega), \mathbf{v}|_{\partial\Omega_1} = 0, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega_2} = 0, \mathbf{v} \times \mathbf{n}|_{\partial\Omega_3} = 0\} \quad (3)$$

$$M = L^2(\Omega) \quad (4)$$

$$\mathbf{V}_0 = \{\mathbf{v} \in \mathbf{X}_0, \nabla \cdot \mathbf{v} = 0\} \quad (5)$$

$$\mathbf{H}_0 = \{\mathbf{v} \in \mathbf{L}^2(\Omega), \nabla \cdot \mathbf{v} = 0, \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega_1 \cup \partial\Omega_2} = 0\} \quad (6)$$

Denote by $H_{0,\partial\Omega_3}^1(\Omega)$ the space of scalar functions of $H^1(\Omega)$ the trace of which is zero on $\partial\Omega_3$. The importance of \mathbf{H}_0 and $H_{0,\partial\Omega_3}^1(\Omega)$ is due to the following orthogonal decomposition of $L^2(\Omega)$

$$L^2(\Omega) = \mathbf{H}_0 \oplus \nabla(H_{0,\partial\Omega_3}^1(\Omega)), \quad (7)$$

which follows from the application of the divergence theorem. The discrete counterpart of this decomposition plays a key role in the projection technique outlined in section 4.

A variational version of problem (1)-(2) reads: For $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $\mathbf{u}_0 \in \mathbf{H}^1(\Omega)$ with $\nabla \cdot \mathbf{u}_0 = 0$, find a pair (\mathbf{u}, p) with

$$\begin{aligned} \mathbf{u} \in \mathbf{H}^1(\Omega), \quad & \begin{cases} \mathbf{u}|_{\partial\Omega_1} = \mathbf{b}_1, \\ \mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_2} = \mathbf{b}_2 \cdot \mathbf{n}, \\ \mathbf{u} \times \mathbf{n}|_{\partial\Omega_3} = \mathbf{b}_3 \times \mathbf{n}, \end{cases} \\ p \in M, \end{aligned} \quad (8)$$

and such that

$$\left\{ \begin{array}{l} \left(\frac{\partial \mathbf{u}}{\partial t}, \mathbf{v} \right) + a(\mathbf{u}, \mathbf{v}) + d(\mathbf{u}, \mathbf{u}, \mathbf{v}) - (p, \nabla \cdot \mathbf{v}) \\ \quad = (\mathbf{f}, \mathbf{v}) - \int_{\partial\Omega_3} P \mathbf{v} \cdot \mathbf{n}, \quad \forall \mathbf{v} \in \mathbf{X}_0, \\ (\nabla \cdot \mathbf{u}, q) = 0, \quad \forall q \in M. \end{array} \right. \quad (9)$$

Here, to simplify the momentum equation, we have introduced the following notations for bilinear and trilinear forms

$$\begin{aligned} a(\mathbf{u}, \mathbf{v}) &= \nu (\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + \nu (\nabla \times \mathbf{u}, \nabla \times \mathbf{v}), \\ d(\mathbf{u}, \mathbf{v}, \mathbf{w}) &= ((\mathbf{u} \cdot \nabla) \mathbf{v}, \mathbf{w}) + \frac{1}{2} (\nabla \cdot \mathbf{u}, \mathbf{v} \cdot \mathbf{w}). \end{aligned} \quad (10)$$

The conditions on the trace, the normal trace and the tangential trace of the velocity on $\partial\Omega_1$, $\partial\Omega_2$ and $\partial\Omega_3$, respectively, are all essential boundary conditions. The condition involving the tangential components of $\nabla \times \mathbf{u}$ on $\partial\Omega_2$ is a natural boundary condition. The pressure boundary condition on $\partial\Omega_3$ is natural as well. Actually, the weak formulation naturally enforces the boundary condition $(p - \nabla \cdot \mathbf{u})|_{\partial\Omega_3} = P$.

It should be remarked that the use of the bilinear form $(\nabla \cdot \mathbf{u}, \nabla \cdot \mathbf{v}) + (\nabla \times \mathbf{u}, \nabla \times \mathbf{v})$ is mandatory when the boundaries $\partial\Omega_2$ and $\partial\Omega_3$, where the prescribed boundary conditions are different from a purely Dirichlet condition for \mathbf{u} alone, are curved. On the contrary, the more common bilinear form $(\nabla \mathbf{u}, \nabla \mathbf{v})$ can be used when Dirichlet conditions are specified or when the boundaries $\partial\Omega_2$ and $\partial\Omega_3$ are flat and parallel to the Cartesian axes. Note also that the coupling between the velocity components, engendered by the presence of mixed boundary conditions, appears in the definition of the test functions of \mathbf{X}_0 .

The conservative form of the nonlinear term defined in (10) is frequently considered along with the hypothesis $\text{meas}(\partial\Omega_3) = 0$. This form can be used also for channel or exterior flows with an ‘‘outflow’’ boundary, provided the boundary in question is located downstream, far enough of any recirculatory zone, so that the condition $\mathbf{u} \cdot \mathbf{n}|_{\partial\Omega_3} \geq 0$ is assured. In practice this treatment of the nonlinear term can guarantee some ‘‘unconditional’’ stability to the numerical scheme [11].

4 The fractional-step algorithm

The weak formulation (8)-(9) is suitable for approximating the Navier-Stokes equations to obtain a coupled solution method. On the contrary, to build a fractional-step projection method additional tools are required since this kind of method characterizes itself by a separate treatment of the viscous and incompressible parts of the problem. As a consequence, the appropriate functional setting for a fractional-step method must accommodate another functional space for representing the velocity field calculated in the incompressible inviscid step of the method [9], defined as follows

$$\mathbf{H}_{0,\partial\Omega_{1,2}}^{\text{div}}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega), \nabla \cdot \mathbf{v} \in L^2(\Omega), \mathbf{v} \cdot \mathbf{n}|_{\partial\Omega_{1,2}} = 0\}. \quad (11)$$

Then, one is led to introduce finite dimensional counterparts $\mathbf{X}_{0,h}$ and $\mathbf{Y}_{0,h}$ of \mathbf{X}_0 and $\mathbf{H}_{0,\partial\Omega_{1,2}}^{\text{div}}(\Omega)$, respectively, to have a discrete representation of the velocity fields $\tilde{\mathbf{u}}$ and \mathbf{u} calculated in the viscous and incompressible phases. One can choose to approximate the pressure introducing a finite dimensional space $M_h \subset H^1(\Omega)$, and also choose $\mathbf{Y}_{0,h} = \mathbf{X}_{0,h} + \nabla M_h$. In this framework, the viscous step of the incremental fractional-step method, assuming a semi-implicit approximation of the nonlinear term, amounts to looking for $\tilde{\mathbf{u}}_h^{k+1}$ in $\mathbf{X}_{b^{k+1},h}$ and such that, for all $\mathbf{v}_h \in \mathbf{X}_{0,h}$, we have

$$\begin{aligned} \frac{(\tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) - (\mathbf{u}_h^k, \mathbf{v}_h)}{\delta t} + a(\tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) + d(\tilde{\mathbf{u}}_h^k, \tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) \\ = (\mathbf{f}^{k+1}, \mathbf{v}_h) + (p_h^k, \nabla \cdot \mathbf{v}_h) - \int_{\partial\Omega_3} P^k \mathbf{v}_h \cdot \mathbf{n}. \end{aligned} \quad (12)$$

The incompressible part of the calculation can be formulated as a Poisson problem for the incremental pressure, which reads: find p_h^{k+1} in M_h with $p_h^{k+1}|_{\partial\Omega_3} = P^{k+1}$ so that

$$(\nabla(p_h^{k+1} - p_h^k), \nabla q_h) = -\frac{(\nabla \cdot \tilde{\mathbf{u}}_h^{k+1}, q_h)}{\delta t}, \quad \forall q_h \in M_h, q_h|_{\partial\Omega_3} = 0, \quad (13)$$

and then set

$$\mathbf{u}_h^{k+1} = \tilde{\mathbf{u}}_h^{k+1} - \delta t \nabla(p_h^{k+1} - p_h^k). \quad (14)$$

The projection step amounts to solving a (discrete) Poisson equation with a homogeneous Neumann condition on $\partial\Omega_1 \cup \partial\Omega_2$ and a Dirichlet boundary condition on $\partial\Omega_3$.

In practice it is not convenient to use equation (12) directly, since it contains the end-of-step velocity \mathbf{u}_h^k which belongs to a space different from that of the intermediate velocity $\tilde{\mathbf{u}}_h^{k+1}$. The former can however be eliminated [9] and the equation of the first step can be implemented as:

$$\begin{aligned} & \frac{(\tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) - (\tilde{\mathbf{u}}_h^k, \mathbf{v}_h)}{\delta t} + a(\tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) + d(\tilde{\mathbf{u}}_h^k, \tilde{\mathbf{u}}_h^{k+1}, \mathbf{v}_h) \\ & = (\mathbf{f}^{k+1}, \mathbf{v}_h) + (2p_h^k - p_h^{k-1}, \nabla \cdot \mathbf{v}_h) - \int_{\partial\Omega_3} (2P^k - P^{k-1}) \mathbf{v}_h \cdot \mathbf{n}. \end{aligned} \quad (15)$$

This equation is used for $k \geq 2$. For $k = 2$ one step of the incremental algorithm in the original form (12) is used, while the first time step is performed using the nonincremental form of the fractional-step method. In this way the end-of-step velocity is made to disappear from the algorithm, thus eliminating the weird velocity space $\mathbf{Y}_{0,h}$ from practical calculations.

5 Numerical examples

Spatial discretization by finite element. The unconditionally stable fractional-step method based on the Poisson equation for pressure described in the previous section has been implemented using Delaunay triangular meshes with a linear interpolation of velocity and pressure on one and the same grid, generated by means of the procedure of Rebay [13]. The large sparse linear systems of algebraic equations for both the velocity and the pressure equations are solved by direct methods after internal reordering of the unknowns by means of the algorithms of SPARSPAK library of George [6].

The driven cavity problem. The first example is the driven cavity problem introduced by Burggraf [1]. The steady solution for $\text{Re} = 1000$ has been calculated on a nonuniform mesh of ≈ 8800 elements and ≈ 4500 nodes, starting from rest with $\delta t = 0.5$ and reaching a relative difference $|\tilde{\mathbf{u}}_h^{k+1} - \tilde{\mathbf{u}}_h^k|_0 / |\tilde{\mathbf{u}}_h^{k+1}|_0 < 10^{-3}$ in 138 time steps. The contour lines of vorticity and the streamlines are given in Fig. 1. The streamlines are in perfect agreement with those of the reference solutions of [7]; the same applies to the vorticity contours except for some wiggles in the central zone of the cavity where the employed mesh is coarsest, while the benchmark solution was calculated on a uniform 129×129 grid. Comparison

for the pressure fields reported in [11] are similarly satisfactory: we emphasize that no spatial oscillation has ever been found using the present method, even on uniform meshes.

Steady solutions for $Re = 3200$ and 5000 are also shown in the same figure: the streamlines compare very well with the reference solutions [7] calculated on uniform 129×129 and 257×257 grids, respectively: all features of the secondary vortices are correctly predicted by the proposed primitive variable method.

We have calculated also the transient solution of the square cavity problem with a sudden start of the sliding top wall at $Re = 1000$, using $\delta t = 0.1$. The plots of vorticity and stream function at $t = 5$ calculated by a second-order accurate scheme are shown in the same figure. The comparison with unsteady solutions calculated by a vorticity–stream function formulation confirm the correctness of the proposed primitive variable method for the accurate simulation of time-dependent flows.

Flow past an airfoil. The second example is the unsteady flow past a NACA0012 airfoil with an angle of incidence of 34° and at $Re = 1000$. Figure 2 shows a part of the finite element mesh near the airfoil; the mesh consists of ≈ 7000 triangles and ≈ 3600 nodes. No attempt was made to refine the mesh according to the computed solution. The streamlines and pressure fields at $t_1 = 1.6$ and $t_2 = 3.6$ are shown in Fig. 2 ($\delta t = 0.04$). The comparison with the solutions calculated by a nonprimitive variable method using domain decomposition [14] and by a different primitive variable method [4] is fully satisfactory.

6 Conclusions

In this communication we have presented the implementation of a new finite element projection method for unstructured triangular grids using an unconditionally stable integration scheme, a Poisson equation for the pressure increment and equal-order interpolations for velocity and pressure. The comparison of the numerical results provided by this method with reference solutions is quite satisfactory and the method is found to be capable of predicting internal and external incompressible laminar flows up to $Re \approx 5000$ accurately, without requiring any tuning of the algorithm. In particular, the unconditionally stable semi-implicit treatment of the nonlinear term combined with adaptive mesh generation

techniques can deal with boundary layers without requiring prohibitively small time steps.

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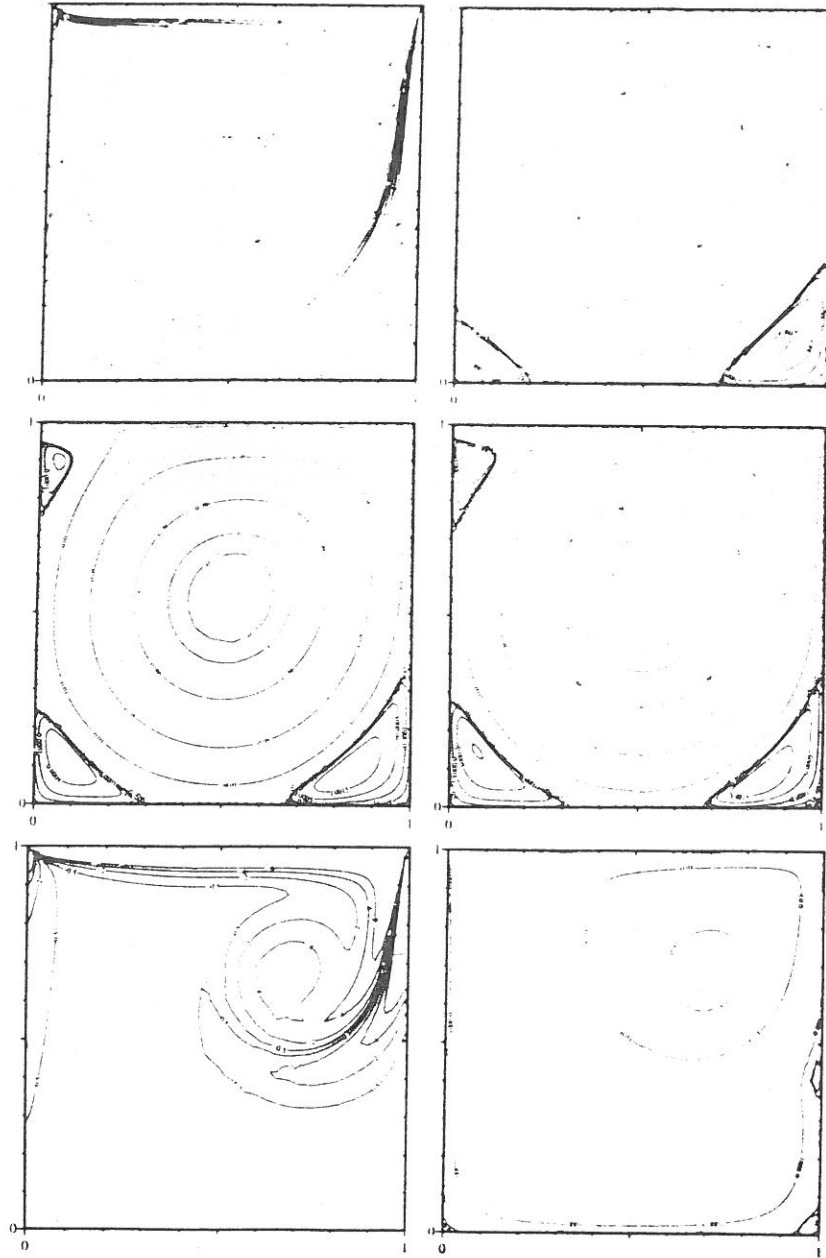


Figure 1. Driven cavity problem. Vorticity and streamlines of the steady flow at $Re = 1000$ (top); streamlines of the steady flow at $Re = 3200$ and 5000 (middle); vorticity and streamlines at $t = 5$ for sudden start at $Re = 1000$ (bottom).

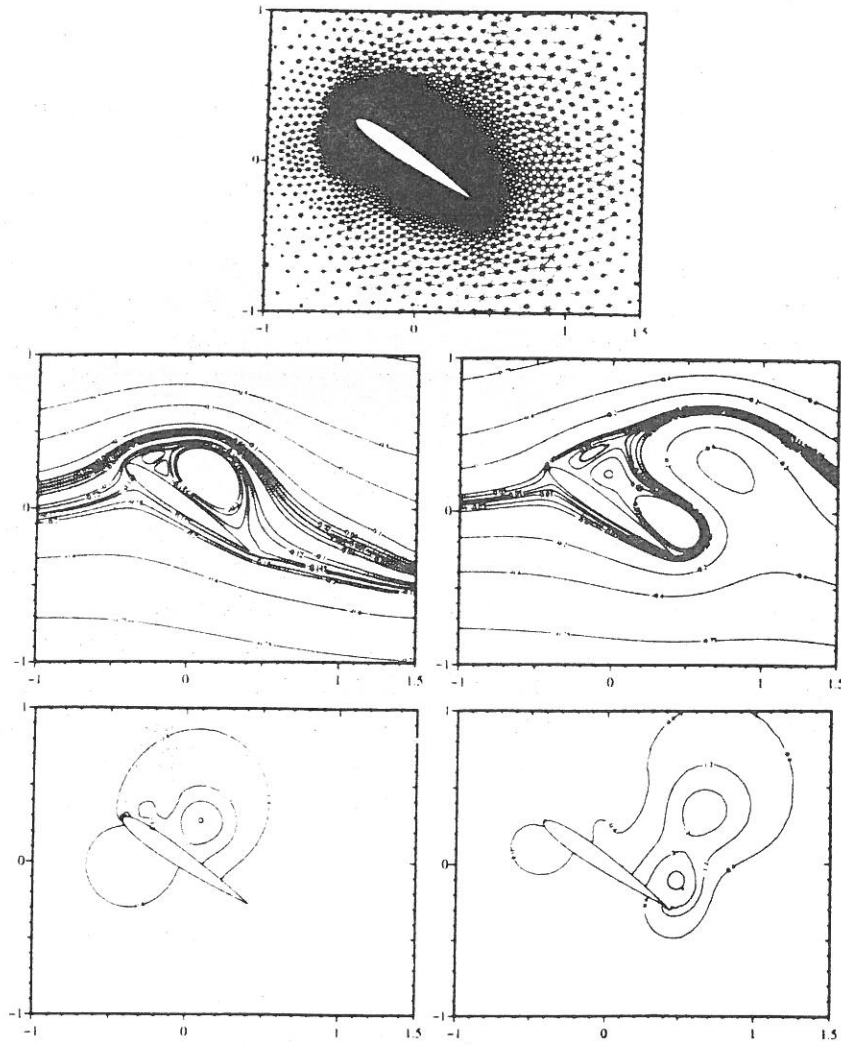


Figure 2. Transient flow past a NACA0012 airfoil with 34° of incidence at $Re = 1000$. Computational grid (top), streamlines (middle) and pressure field (bottom) at $t_1 = 1.6$ (left) and $t_2 = 3.6$ (right).