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WEAK APPROXIMATION OF THE ψ–ω EQUATIONS WITH EXPLICIT VISCOUS DIFFUSION

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This paper describes a variational formulation for solving the 2-D time-dependent incompressible Navier–Stokes equations expressed in the stream function and vorticity. The difference between the proposed approach and the standard one is that the vorticity equation is interpreted as an evolution equation for the stream function while the Poisson equation is used as an expression for evaluating the distribution of vorticity in the domain and on the boundary. A time discretization is adopted with the viscous diffusion made explicit, which leads to split the incompressibility from the viscosity. In some sense, the present method generalizes to the variational framework a well-known idea which is used in finite differences approximations and that is based on a Taylor series expansion of the stream function near the boundary. Some conditional stability results and error estimates are given.

1. Introduction

A classical finite element procedure for solving the 2-D Stokes equations formulated in terms of the vorticity and stream function is the uncoupled solution method for the biharmonic problem introduced by Glowinski and Pironneau,[5] see also Fortin and Thomasset[3] or Quartapelle[9] for a review. Such an approach can compute the solution of the Stokes problem by a direct and uncoupled method. In the case of time-dependent equations, this method assumes necessarily an implicit treatment of the viscous diffusion. In the solution of the nonlinear Navier–Stokes equations, the Glowinski–Pironneau strategy can be pursued in one of the following manners. First, the nonlinear advection term can be made explicit so that the vorticity boundary

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value is evaluated by means of an influence matrix which is one and the same for all the time levels. Alternatively, a semi-implicit treatment of the nonlinearity can be adopted which gives a different influence matrix at each time level. Finally, the fully nonlinear problem at each time step can be tackled by an iterative technique which relies on a fixed influence matrix, as proposed for the solution of the steady equations by Ruas.[10] While all these techniques guarantee good stability properties, they are not so easy to implement; hence one may be tempted to make a trade-off between stability and simplicity. The objective of this paper is to propose one possible alternative technique for approximating the evolutionary Navier–Stokes equations in two dimensions.

In the perspective of trading stability for simplicity, Achdou and Pironneau[1] have recently proposed a new integral formulation of the equation governing time-dependent, advection dominated flows which is characterized by the introduction of an asymptotic approximation of the vorticity boundary value at high Reynolds numbers. In the context of finite differences, which are known for their simplicity, one classical approach is to assume an explicit treatment of the viscosity to derive vorticity boundary formulas. In unsteady calculations, it has been shown that the Neumann boundary condition for the stream function can be used as the last piece of the time-stepping algorithm. More precisely, the derivative boundary condition can be recast as a relationship specifying the boundary distribution of the new vorticity after the time advancement of the (internal) vorticity has been completed and after the new stream function has been determined, for details see e.g. Peyret and Taylor,[8] E and Liu[2] or Napolitano *et al.*[7] To the authors' knowledge, the possible implications of this technique within a variational framework have not been explored so far. The aim of this paper is to develop the variational counterpart of this idea. It is shown that the weak form of the vorticity equation can be interpreted as an evolutionary equation for the stream function and that the Poisson equation can be used as an expression for the distribution of vorticity in the domain and on the boundary. By discretizing the equations in time with the viscous diffusion term made explicit, one observes an exchange of roles between the vorticity and the Poisson equations: the former provides the time advancement of the stream function, whereas the latter becomes an expression for evaluating the new vorticity.

The paper is organized as follows. Section 2 is devoted to preliminary definitions; we also introduce the mathematical statement of the problem and the discrete setting for the approximation in space. In Sec. 3 we introduce a first-order accurate time discretization, and we make the error analysis of the proposed scheme. We extend the scheme to second-order accuracy in time in Sec. 4, and we prove a bound uniform in time for the velocity in the L^2 norm. The last section is devoted to concluding remarks. Some numerical results of an implementation of the proposed technique by means of linear elements have been presented in Guermond and Quartapelle.[6]

2. Preliminaries

The fluid domain Ω is assumed to be an open, bounded, simply connected domain of $\mathbb{R}^2$; its boundary Γ is assumed to be as smooth as needed. As usual, $W^{s,p}(\Omega)$ denotes the real Sobolev spaces, $0 \le s < \infty$, $1 \le p \le \infty$, equipped with the norm $\|\cdot\|_{s,p}$ and semi-norm $|\cdot|_{s,p}$. The space $W_0^{s,p}(\Omega)$ is the completion of the space of smooth functions compactly supported in Ω with respect to the $\|\cdot\|_{s,p}$ norm. For $p = 2$, we denote the Hilbert spaces $W^{s,2}(\Omega)$ (resp. $W_0^{s,p}(\Omega)$) by $H^s(\Omega)$ (resp. $H_0^s(\Omega)$), and the related semi-norm and norm are denoted by $|\cdot|_s$ and $\|\cdot\|_s$, respectively. The dual space of $H_0^s(\Omega)$ is denoted by $H^{-s}(\Omega)$. The duality form between $H^{-1}(\Omega)$ and $H_0^1(\Omega)$ is denoted by $\langle\cdot\,,\cdot\rangle$. For a fixed positive real number T and a Banach space X, we denote by $L^p(X)$, $H^s(X)$, and $C(X)$ the spaces $L^p(0,T;X)$, $H^s(0,T;X)$, and $C([0,T];X)$, respectively. We embed $\mathbb{R}^2$ into $\mathbb{R}^3$ so that, $(\hat{\mathbf{x}},\hat{\mathbf{y}})$ being a unit base of $\mathbb{R}^2$, $(\hat{\mathbf{x}},\hat{\mathbf{y}},\hat{\mathbf{z}})$ is a right-handed unit base of $\mathbb{R}^3$.

In the following we consider the time-dependent Navier–Stokes equations in two dimensions formulated in terms of vorticity, ω, and stream function, ψ. The equations are supplemented by homogeneous boundary conditions to avoid technical difficulties which are not relevant to the structure of the method. For the initial solenoidal velocity field $\mathbf{u}_0$ assumed to be in $\mathbf{H}_0^1(\Omega)$ and the body force $\mathbf{f} \in L^2(0,T;\mathbf{L}^2(\Omega))$, we have the problem

$$\begin{cases} \text{Find } \psi \in L^2(0,T;H_0^2(\Omega)) \cap C(0,T;H_0^1(\Omega))\,, \quad \psi_t \in L^2(0,T;H_0^1(\Omega)) \text{ and} \\ \omega \in L^2(0,T;L^2(\Omega)) \cap C(0,T;H^{-1}(\Omega))\,, \quad \text{so that} \\ \forall\, \phi \in H_0^1(\Omega)\,, \quad ((\nabla\psi)_{|t=0}, \nabla\phi) = (\mathbf{u}_0, \boldsymbol{\nabla}\phi \times \hat{\mathbf{z}})\,, \quad \text{and for all } t > 0 \\ \forall\, \psi' \in H_0^2(\Omega)\,, \quad \left(\dfrac{\partial\nabla\psi}{\partial t}, \nabla\psi'\right) + \nu\,\langle\nabla\omega, \nabla\psi'\rangle + b(\omega,\psi,\psi') = (\mathbf{f}, \boldsymbol{\nabla}\psi' \times \hat{\mathbf{z}})\,, \\ \forall\, v \in L^2(\Omega)\,, \quad (\omega, v) + (\nabla^2\psi, v) = 0\,. \end{cases} \tag{2.1}$$

The trilinear form $b(\omega,\psi,\psi')$ associated to the advection term is defined by

$$b(\omega,\psi,\psi') = (\omega\hat{\mathbf{z}} \times (\boldsymbol{\nabla}\psi \times \hat{\mathbf{z}}),\, \boldsymbol{\nabla}\psi' \times \hat{\mathbf{z}}) = (\omega\,\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi' \times \hat{\mathbf{z}})\,. \tag{2.2}$$

Note that $b(\omega,\psi,\psi')$ is zero if $\psi = \psi'$ and this property holds in the discrete case provided the approximation of ψ is H^1-conformal. This way of writing the advection term is referred to as the *rotational form* in the sequel.

Remark 2.1. Note the unusual form of the evolutionary term which involves mixed time and space derivatives of ψ. Actually, this weak form of the dynamical equation is the most natural within the present variational setting since it stems from the original momentum equation where

(i) the velocity is replaced by $\boldsymbol{\nabla}\psi \times \hat{\mathbf{z}}$,
(ii) the viscous term is written as the curl of ω, and

(iii) the velocity test functions in $\mathbf{V} = \{\mathbf{v} \in \mathbf{H}_0^1(\Omega),\ \boldsymbol{\nabla} \cdot \mathbf{v} = 0\}$ are expressed as $\boldsymbol{\nabla}\psi' \times \hat{\mathbf{z}}$ by virtue of the the well-known isomorphism (see Girault and Raviart[4])

$$(\boldsymbol{\nabla} \cdots) \times \hat{\mathbf{z}} :\ H_0^2(\Omega) \to \mathbf{V}\,.$$

At variance with more usual ways of writing the vorticity transport equation, no integration by parts is performed; in other words, the curl of the momentum equation has not been taken in a strong form. In this way the momentum equation plays the role of an evolutionary equation for the stream function (actually for its spatial derivative) whereas the Poisson equation is used as the vorticity definition.

2.1. *The discrete setting in space*

Let W_h and $\Psi_{0,h}$ be two finite-dimensional subspaces of $H^1(\Omega)$ and $H_0^1(\Omega)$, respectively. We also make the hypothesis that $\Psi_{0,h} \subset W_h$. The discrete spaces W_h and $\Psi_{0,h}$ are assumed to have the following approximation and inverse properties:

There is $k \geq 1$ and there exists $c > 0$ such that for $0 \leq r \leq k$,

$$\inf_{w_h \in W_h} [\|w - w_h\|_0 + h\|w - w_h\|_1] \leq ch^{r+1}\|w\|_{r+1}\,, \quad \forall\ w \in H^{r+1}(\Omega)\,,$$

$$\inf_{\psi_h \in \Psi_{0,h}} [\|\psi - \psi_h\|_0 + h\|\psi - \psi_h\|_1] \leq ch^{r+1}\|\psi\|_{r+1}\,, \quad \forall\ \psi \in H^{r+1}(\Omega) \cap H_0^1(\Omega)\,,$$

$$\inf_{\psi_h \in \Psi_{0,h}} \|\psi - \psi_h\|_{1,p} \leq ch^r\|\psi\|_{r+1,p}\,, \quad 2 \leq p \leq \infty\,, \quad \forall\ \psi \in W^{r+1,p}(\Omega) \cap H_0^1(\Omega)\,.$$

There exists $c > 0$ such that, for all v_h in W_h, the following inverse inequality holds

$$\|v_h\|_{n,p} \leq ch^{m-n+\frac{2}{p}-\frac{2}{q}}\|v_h\|_{m,q}\,, \quad 0 \leq m \leq n \leq 1\,, \quad 1 \leq q \leq p \leq \infty\,.$$

These hypotheses hold if W_h and $\Psi_{0,h}$ are finite element spaces based on quasi-uniform triangulation (see e.g. Girault–Raviart,[4] p. 103).

2.2. *Definition of a special set of approximants*

To carry out the error analysis we shall need to use approximants of $\psi(t)$ and $\omega(t)$. Assuming that $\omega \in L^\infty(H^1(\Omega))$, we introduce $\overline{\psi}_h(t) \in \Psi_{0,h}$ and $\overline{\omega}_h(t) \in W_h$ so that

$$\begin{cases} (\overline{\omega}_h(t), v_h) - (\nabla\overline{\psi}_h(t), \nabla v_h) = 0\,, & \forall\ v_h \in W_h\,, \\ (\nabla\overline{\omega}_h(t), \nabla\phi_h) = (\nabla\omega(t), \nabla\phi_h)\,, & \forall\ \phi_h \in \Psi_{0,h}\,. \end{cases} \tag{2.3}$$

We assume hereafter that $\overline{\psi}_h(t)$ and $\overline{\omega}_h(t)$ satisfy the following approximation property: There are $\ell \in [1,k]$, α_ℓ, and $\beta_\ell \geq 1$ such that

$$\forall\ t \geq 0\,,\ \|\psi - \overline{\psi}_h\|_{H^1(\Omega)} + \|\omega - \overline{\omega}_h\|_{L^2(\Omega)} \leq ch^\ell\Big\{\|\psi\|_{H^{\alpha_\ell}(\Omega)} + \|\omega\|_{H^{\beta_\ell}(\Omega)}\Big\}\,. \tag{2.4}$$

Example 2.1. This property holds true for some class of quadrilateral finite elements of degree $k \geq 1$: $\ell = k$, $\alpha_\ell = k+2$, and $\beta_\ell = k+1$. The reader is referred to Girault–Raviart,[4] p. 231, for other details.

Example 2.2. If Ω is convex and the approximation spaces are composed of P_k finite elements based on triangles or convex quadrilaterals, then the approximation property above holds with $\ell = k - \frac{1}{2} - \varepsilon$, $\alpha_\ell = k+2$ and $\beta_\ell = k$; the constant c in (2.4) depends on $\varepsilon > 0$, cf. Girault–Raviart,[4] pp. 226 and 227.

Remark 2.2. It is important to note that the particular choice of approximants of $\psi(t)$ and $\omega(t)$ made above will allow us to prove near optimal error in space of order h^ℓ for the time-dependent Navier–Stokes equations without resorting to duality arguments, superconvergence arguments, or other intricate arguments that must be invoked to prove (2.4) for the approximation of the Stokes problem (2.3); see for instance the work of Scholz.[11] In other words, by using $\overline{\psi}_h(t)$ and $\overline{\omega}_h(t)$ as approximants of $\psi(t)$ and $\omega(t)$, the error analysis of the Navier–Stokes problem will benefit directly from the sophisticated arguments that come into play in the error analysis of the Stokes problem.

3. The First-Order Scheme

3.1. *The fully discrete algorithm*

To approximate the time derivative we shall use a first-order Euler scheme. Let $[0, T]$ be a finite time interval and N be an integer. We denote $\delta t = T/N$ and $t^n = n\,\delta t$ for $0 \leq n \leq N$. For any function of time, $\varphi(t)$, we denote $\varphi^n = \varphi(t^n)$; furthermore we set the notation $\delta_t \phi^{n+1} = \phi^{n+1} - \phi^n$.

The fully-discrete problem is formulated as follows. The initialization step reads:

$$\begin{cases} \text{find } \psi_h^0 \in \Psi_{0,h} \text{ such that}\,, \\ \forall\ \phi_h \in \Psi_{0,h}\,, \quad (\nabla \psi_h^0, \nabla \phi_h) = (\mathbf{u}_0, \boldsymbol{\nabla} \phi_h \times \hat{\mathbf{z}})\,, \end{cases} \tag{3.1}$$

$$\begin{cases} \text{find } \omega_h^0 \in W_h \text{ such that}\,, \\ \forall\ v_h \in W_h\,, \quad (\omega_h^0, v_h) = (\nabla \psi_h^0, \nabla v_h)\,. \end{cases} \tag{3.2}$$

Then for each $n \geq 0$, carry out the following two steps:

$$\begin{cases} \text{find } \psi_h^{n+1} \in \Psi_{0,h} \text{ such that}\,, \\ \forall\ \phi_h \in \Psi_{0,h}\,, \quad \dfrac{(\nabla(\psi_h^{n+1} - \psi_h^n), \nabla \phi_h)}{\delta t} + b(\omega_h^n, \psi_h^{n+1}, \phi_h) \\ \qquad\qquad = -\nu(\nabla \omega_h^n, \nabla \phi_h) + (\mathbf{f}^{\,n+1}, \boldsymbol{\nabla} \phi_h \times \hat{\mathbf{z}})\,, \end{cases} \tag{3.3}$$

and

$$\begin{cases} \text{find } \omega_h^{n+1} \in W_h \text{ such that}\,, \\ \forall\ v_h \in W_h\,, \quad (\omega_h^{n+1}, v_h) = (\nabla \psi_h^{n+1}, \nabla v_h)\,. \end{cases} \tag{3.4}$$

The nonlinear term is accounted for in a semi-implicit form for the sake of simplicity. All that is said afterwards holds with minor modifications if this term is made explicit. The modifications in question essentially amount to deriving slightly sharper bounds for the nonlinear residuals.

Observe that in the present method the roles of the variables ψ and ω are interchanged with respect to the classical formulation. Here, the dynamical equation for the transport of ω has turned into an equation governing the evolution of the (weak) Laplacian of ψ, whereas the Poisson equation for ψ has become an expression giving the other unknown ω, explicitly.

Remark 3.1. Note that the explicit evaluation of the new vorticity field ω_h^{n+1} through the solution of the mass matrix problem (3.4) does enforce the integral conditions for the vorticity[9] which underlay the Glowinski–Pironneau method. In fact, considering more general, i.e. nonhomogeneous, boundary conditions $\psi_{|\Gamma}^{n+1} = a^{n+1}$ and $(\partial\psi^{n+1}/\partial n)_{|\Gamma} = b^{n+1}$, the vorticity problem would read

$$\begin{cases} \text{find } \omega_h^{n+1} \in W_h \text{ such that }, \\ \forall\, v_h \in W_h\,, \quad (\omega_h^{n+1}, v_h) = (\nabla\psi_h^{n+1}, \nabla v_h) - \displaystyle\int_\Gamma b^{n+1} v_h\,. \end{cases} \tag{3.5}$$

Selecting the functions v_h in the subspace of the discrete harmonic functions $\eta_h \in W_h$ such that $(\nabla\eta_h, \nabla v_h) = 0$, $\forall\, v_h \in \Psi_{0,h}$, the weak equation above gives

$$(\omega_h^{n+1}, \eta_h) \approx \int_\Gamma \left(a^{n+1} \frac{\partial \eta_h}{\partial n} - b^{n+1}\eta_h \right)\,,$$

since it can be shown that $\int_\Gamma a^{n+1}\partial\eta_h/\partial n \approx (\nabla\psi_h^{n+1}, \nabla\eta_h)$. This is indeed the vorticity integral condition for the transient problem at the time level $n+1$. Thus, the proposed method, where the viscous diffusion is made explicit, allows the vorticity integral conditions to be fulfilled *a posteriori*, as already pointed out in Napolitano *et al.*[7]

Note also that, in the present formulation, the vorticity boundary value is determined in a way that is very similar to the classical procedure used in the context of finite differences. In fact the vorticity boundary formula used in second-order accurate central differences is obtained by means of a Taylor series expansion:

$$\psi_h(\Delta x) = \psi(0) + \Delta x \frac{\partial\psi(0)}{\partial x} - \frac{\Delta x^2}{2}\omega_h(0) + \mathcal{O}(\Delta x^3)\,.$$

This argument uses the discrete Poisson equation for ψ on the boundary together with the Dirichlet and the Neumann boundary data for ψ. In some sense, the Taylor expansion above mimics the weak Eq. (3.5) for v_h not vanishing on the boundary.

3.2. *The error analysis*

In this section we perform the error analysis of the discrete scheme presented above. Hereafter we shall assume the following regularity properties:

$$
\text{(H)} \quad \begin{cases} \psi \in L^\infty(W^{2,\infty}(\Omega) \cap H^{\ell+2}(\Omega) \cap H^1_0(\Omega))\,, \\ \psi_t \in L^\infty(H^{\alpha_\ell} \cap H^1_0(\Omega))\,, \\ \psi_{tt} \in L^\infty(H^1(\Omega))\,, \\ \omega \in L^\infty(L^\infty(\Omega) \cap H^{\beta_\ell}(\Omega))\,, \\ \omega_t \in L^\infty(H^1(\Omega))\,. \end{cases}
$$

Some of this hypotheses may be weakened, but we shall adopt this stronger set of hypotheses for the sake of simplicity of the arguments to be presented below. Hereafter c will denote a generic positive constant. The main result of this section is as follows:

Theorem 3.1. *Under the hypothesis* (H), *there is* $c_s(\Omega, \psi) > 0$ *and* $c_e(T, \nu, \psi, \Omega) > 0$ *such that, if* $\delta t \leq c_s h^2/\nu$, *then the solution* (ψ_h, ω_h) *of* (3.1)–(3.4) *satisfies*

$$
\|\psi - \psi_h\|_{l^\infty(H^1(\Omega))} + \|\omega - \omega_h\|_{l^2(L^2(\Omega))} \leq c_e(\delta t + h^\ell)\,. \tag{3.6}
$$

Proof. (a) First we introduce the following notations:

$$
\begin{aligned} \overline{\varepsilon}_h^n &= \psi(t^n) - \overline{\psi}_h(t^n)\,, \quad \varepsilon_h^n = \overline{\psi}_h(t^n) - \psi_h^n\,, \\ \overline{e}_h^n &= \omega(t^n) - \overline{\omega}_h(t^n)\,, \quad e_h^n = \overline{\omega}_h(t^n) - \omega_h^n\,. \end{aligned} \tag{3.7}
$$

The terms $\overline{\varepsilon}_h^n$ and $\overline{e}_h^n$ can be viewed as spatial approximation errors, whereas ε_h^n and e_h^n are consistency errors induced by the mixing of the time-stepping and the space approximations.

By using the definition of ψ_h^{n+1} together with that of $\overline{\psi}_h^{n+1}$ we obtain the following equation which controls ε_h^{n+1}:

$$
\forall\ \phi_h \in \Psi_{0,h}\,, \quad \frac{(\nabla(\delta_t \varepsilon_h^{n+1}), \nabla\phi_h)}{\delta t} + \nu(\nabla e_h^n, \nabla\phi_h) = R_L^{n+1}(\phi_h) + R_{NL}^{n+1}(\phi_h)\,, \tag{3.8}
$$

where linear and nonlinear residuals R_L^{n+1} and R_{NL}^{n+1} are defined as follows:

$$
R_L^{n+1}(\phi_h) = \left(\nabla\left(\frac{\delta_t \overline{\psi}_h^{n+1}}{\delta t} - \frac{\partial\psi^{n+1}}{\partial t}\right), \nabla\phi_h\right) - \nu(\nabla\delta_t\omega^{n+1}, \nabla\phi_h)\,,
$$

$$
R_{NL}^{n+1}(\phi_h) = b(\omega_h^n, \psi_h^{n+1}, \phi_h) - b(\omega^{n+1}, \psi^{n+1}, \phi_h)\,.
$$

By using the definition of ω_h^{n+1} and $\overline{\omega}_h^{n+1}$, we deduce that e_h^{n+1} is controlled by the following equation:

$$
\forall\ v_h \in W_h\,, \quad (e_h^{n+1}, v_h) = (\nabla\varepsilon_h^{n+1}, \nabla v_h)\,. \tag{3.9}
$$

(b) Now we take $2\delta t\, \varepsilon_h^{n+1}$ as test function in the momentum equation (3.8).

$$|\varepsilon_h^{n+1}|_1^2 + |\delta_t \varepsilon_h^{n+1}|_1^2 + 2\nu\, \delta t(\nabla e_h^n, \nabla \varepsilon_h^{n+1})$$
$$= |\varepsilon_h^n|_1^2 + 2\delta t\, R_L^{n+1}(\varepsilon_h^{n+1}) + 2\delta t\, R_{NL}^{n+1}(\varepsilon_h^{n+1})\,.$$

To obtain a control on the term $(\nabla e_h^n, \nabla \varepsilon_h^{n+1})$, we proceed as follows:

$$\begin{aligned}(\nabla e_h^n, \nabla \varepsilon_h^{n+1}) &= (\nabla e_h^n, \nabla \varepsilon_h^n) + (\nabla e_h^n, \nabla \delta_t \varepsilon_h^{n+1}) \\ &\geq \|e_h^n\|_0^2 - \frac{c}{h}\|e_h^n\|_0\, |\delta_t \varepsilon_h^{n+1}|_1 \\ &\geq \frac{1}{2}\|e_h^n\|_0^2 - \frac{c}{h^2}|\delta_t \varepsilon_h^{n+1}|_1^2\,,\end{aligned}$$

where we have used e_h^n as test function in Eq. (3.9) at time step t^{n-1} to obtain $\|e_h^n\|_0^2 = (\nabla \varepsilon_h^n, \nabla e_h^n)$ and we have used the Cauchy–Schwarz inequality together with an inverse inequality. By replacing this inequality in the equation above, we obtain:

$$|\varepsilon_h^{n+1}|_1^2 + \left(1 - \frac{c_s \nu\, \delta t}{h^2}\right) |\delta_t \varepsilon_h^{n+1}|_1^2 + \nu\, \delta t \|e_h^n\|_0^2$$
$$\leq |\varepsilon_h^n|_1^2 + 2\delta t\, R_L^{n+1}(\varepsilon_h^{n+1}) + 2\delta t\, R_{NL}^{n+1}(\varepsilon_h^{n+1})\,. \qquad (3.10)$$

Now we derive bounds for the two residuals. First we have

$$\begin{aligned}&|R_L^{n+1}(\varepsilon_h^{n+1})| \\ &\quad \leq c\Big\{h^\ell \|\psi_t\|_{L^\infty(H^{\alpha_\ell}(\Omega))} + \delta t \|\psi_{tt}\|_{L^\infty(H^1(\Omega))} + \delta t \|\omega_t\|_{L^\infty(H^1(\Omega))}\Big\} |\varepsilon_h^{n+1}|_1 \\ &\quad \leq c(\delta t + h^\ell)^2 + |\varepsilon_h^{n+1}|_1^2\,.\end{aligned}$$

Second, we split the nonlinear residual as follows:

$$\begin{aligned}-R_{NL}^{n+1}(\varepsilon_h^{n+1}) &= b(\delta_t \omega^{n+1}, \psi^{n+1}, \varepsilon_h^{n+1}) + b(\overline{e}_h^n, \psi^{n+1}, \varepsilon_h^{n+1}) \\ &\quad + b(e_h^n, \psi^{n+1}, \varepsilon_h^{n+1}) + b(\omega_h^n, \overline{\varepsilon}_h^{n+1}, \varepsilon_h^{n+1}) \\ &\quad + b(\omega_h^n, \varepsilon_h^{n+1}, \varepsilon_h^{n+1})\,.\end{aligned}$$

We denote by $R_{NL,\,i}^{n+1}(\varepsilon_h^{n+1})$, $i = 1, \ldots, 5$, the five residuals on the right-hand side. Each residual is bounded from above as follows:

$$\begin{aligned}|R_{NL,\,1}^{n+1}(\varepsilon_h^{n+1})| &\leq c|\varepsilon_h^{n+1}|_{1,2}\, |\psi^{n+1}|_{1,\infty}\, \|\delta_t \omega^{n+1}\|_{0,2} \\ &\leq c\, \delta t |\varepsilon_h^{n+1}|_{1,2}\, \|\psi\|_{L^\infty(W^{1,\infty}(\Omega))}\, \|\omega_t\|_{L^\infty(L^2(\Omega))} \\ &\leq c(\delta t)^2 + |\varepsilon_h^{n+1}|_1^2\,,\end{aligned}$$

$$\begin{aligned}|R_{NL,\,2}^{n+1}(\varepsilon_h^{n+1})| &\leq c|\varepsilon_h^{n+1}|_{1,2}\, |\psi^{n+1}|_{1,\infty}\, \|\overline{e}_h^n\|_{0,2} \\ &\leq ch^\ell |\varepsilon_h^{n+1}|_{1,2}\, \|\psi\|_{L^\infty(W^{1,\infty}(\Omega))} \Big\{\|\omega\|_{L^\infty(H^{\beta_\ell}(\Omega))} + \|\psi\|_{L^\infty(H^{\alpha_\ell}(\Omega))}\Big\}\end{aligned}$$

$$
\begin{aligned}
&\le ch^{2\ell} + |\varepsilon_h^{n+1}|_1^2 , \\
|R_{NL,3}^{n+1}(\varepsilon_h^{n+1})| &\le c|\varepsilon_h^{n+1}|_{1,2}\, |\psi^{n+1}|_{1,\infty}\, \|e_h^n\|_{0,2} \\
&\le c|\varepsilon_h^{n+1}|_{1,2}\, \|\psi\|_{L^\infty(W^{1,\infty}(\Omega))}\, \|e_h^n\|_{0,2} \\
&\le \gamma \|e_h^n\|_0^2 + c_\gamma |\varepsilon_h^{n+1}|_1^2 ,
\end{aligned}
$$

where γ is a positive constant that may be chosen arbitrarily small, and c_γ is a constant that depends on γ.

$$
\begin{aligned}
|R_{NL,4}^{n+1}(\varepsilon_h^{n+1})| &\le |b(\overline{e}_h^n, \overline{\varepsilon}_h^{n+1}, \varepsilon_h^{n+1})| + |b(e_h^n, \overline{\varepsilon}_h^{n+1}, \varepsilon_h^{n+1})| + |b(\omega^n, \overline{\varepsilon}_h^{n+1}, \varepsilon_h^{n+1})| \\
&\le c_1 |\varepsilon_h^{n+1}|_{1,4}\, |\overline{\varepsilon}_h^{n+1}|_{1,4} \Big\{ \|\overline{e}_h^n\|_{0,2} + \|e_h^n\|_{0,2} \Big\} \\
&\quad + c_2 |\varepsilon_h^{n+1}|_{1,2}\, \|\omega\|_{L^\infty(L^\infty(\Omega))}\, |\overline{\varepsilon}_h^{n+1}|_{1,2} .
\end{aligned}
$$

By using the inverse inequality $\|\phi_h\|_{1,4} \le ch^{-1/2}\|\phi_h\|_{1,2}$, which holds in 2-D for all ϕ_h in $\Psi_{0,h}$, and using the existence of an appropriate interpolation operator on $\Psi_{0,h}$, we can prove the estimate

$$
\begin{aligned}
|\overline{\varepsilon}_h|_{1,4} \le c_1 h^{\ell-1/2} &\Big\{ \|\psi\|_{L^\infty(H^{\alpha_\ell}(\Omega))} + \|\omega\|_{L^\infty(H^{\beta_\ell}(\Omega))} \Big\} \\
&+ c_2 h^{1/2} \Big\{ \|\psi\|_{L^\infty(W^{3/2,4}(\Omega))} + \|\psi\|_{L^\infty(H^2(\Omega))} \Big\} .
\end{aligned}
$$

Since we have assumed $\ell \ge 1$, we obtain

$$
|R_{NL,4}^{n+1}(\varepsilon_h^{n+1})| \le ch^{2\ell} + \gamma \|e_h^n\|_0^2 + c_\gamma |\varepsilon_h^{n+1}|_1^2 .
$$

The fifth residual $R_{NL,5}^{n+1}$ is zero, given the rotational form we have adopted for the advection term.

In summary, the nonlinear residual is bounded from above as follows:

$$
2\delta t |R_{NL}^{n+1}(\varepsilon_h^{n+1})| \le c\,\delta t(\delta t + h^\ell)^2 + \gamma\, \delta t \|e_h^n\|_0^2 + c_\gamma\, \delta t |\varepsilon_h^{n+1}|_1^2 ,
$$

where γ is a generic positive constant that will be chosen small enough hereafter. By inserting this bound into (3.10) we obtain

$$
\begin{aligned}
(1 - c_\gamma\, \delta t)|\varepsilon_h^{n+1}|_1^2 + \left(1 - \frac{c_1 \nu\, \delta t}{h^2}\right) |\delta_t \varepsilon_h^{n+1}|_1^2 + \nu\, \delta t \left(1 - \frac{\gamma}{\nu}\right) \|e_h^n\|_0^2 \\
\le |\varepsilon_h^n|_1^2 + c_2\, \delta t(\delta t + h^\ell)^2 .
\end{aligned}
$$

Now we set $\gamma = \nu/2$ and we choose δt so that $\delta t < \min(c_1 h^2/\nu, 1/c_\gamma)$. Then, the discrete Gronwall lemma together with the Poincaré inequality yields

$$
\|\varepsilon_h\|_{l^\infty(H^1(\Omega))} + \|e_h\|_{l^2(L^2(\Omega))} \le c(T, \nu, \psi, \Omega)(\delta t + h^\ell + |\varepsilon_h^0|_1) .
$$

To obtain a bound on $|\varepsilon_h^0|_1$ we observe that

$$
\forall\ \phi_h \in \Psi_{0,h} , \quad (\nabla \psi_h^0 - \overline{\psi}_h(0), \nabla \phi_h) = (\omega(0) - \overline{\omega}_h(0), \phi_h) .
$$

As a result, $|\varepsilon_h^0|_1 \le c\|\overline{e}_h(0)\|_0$; i.e. $|\varepsilon_h^0|_1 \le ch^\ell$ according to (2.4). The final result is a consequence of the relations

$$\psi^n - \psi_h^n = \overline{\varepsilon}_h^n + \varepsilon_h^n\,,$$

$$\omega^n - \omega_h^n = \overline{e}_h^n + e_h^n\,.$$

The proof is complete. □

4. Second-Order Scheme

The present technique is not restricted to first order; it can be modified to obtain higher order accuracy in time. This can be done simply by approximating the time derivative by a high order finite differencing (Crank–Nicolson, three-level backward differencing, etc.) and by extrapolating the terms that involve ω, accordingly. To illustrate this possibility we present in the following a second-order scheme based on the three-level backward differencing of the time derivative.

Initialize the scheme by evaluating (ψ_h^0, ω_h^0) and (ψ_h^1, ω_h^1). ψ_h^0 and ω_h^0 are evaluated from the initial data through (3.1) and (3.2). ψ_h^1 can be obtained by many means; for instance, it can be calculated by using a second-order Runge–Kutta technique; from ψ_h^1 one evaluates ω_h^1 easily. Then, for each $n \ge 1$, carry out the following two steps:

$$\begin{cases} \text{find } \psi_h^{n+1} \in \Psi_{0,h} \text{ such that}\,,\ \forall\ \phi_h \in \Psi_{0,h}\,, \\ \dfrac{(\nabla(3\psi_h^{n+1} - 4\psi_h^n + \psi_h^{n-1}), \nabla\phi_h)}{2\delta t} + b(2\omega_h^n - \omega_h^{n-1}, \psi_h^{n+1}, \phi_h) \\ \qquad = -\nu(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\phi_h) + (\mathbf{f}^{\,n+1}, \boldsymbol{\nabla}\phi_h \times \hat{\mathbf{z}})\,, \end{cases} \tag{4.1}$$

and

$$\begin{cases} \text{find } \omega_h^{n+1} \in W_h \text{ such that}\,, \\ \forall\ v_h \in W_h\,, \quad (\omega_h^{n+1}, v_h) = (\nabla\psi_h^{n+1}, \nabla v_h)\,. \end{cases} \tag{4.2}$$

This scheme is second-order accurate in time. In this paper we shall not carry out the error analysis of this scheme since it follows the same ideas as those that have been used to analyze the first-order scheme. The main technical difficulty (classical though) consists in deriving bounds for the nonlinear residuals. Nevertheless, to give an idea on the stability mechanism, we give a stability result. Let c_p be the Poincaré constant:

$$\forall\ \phi \in H_0^1(\Omega)\,, \quad c_p\|\phi\|_0 \le |\phi|_1\,, \tag{4.3}$$

and let c_i be the constant such that

$$\forall\ v_h \in W_h\,, \quad |v_h|_1 \le c_i h^{-1}\|v_h\|_0\,. \tag{4.4}$$

Let us assume that $\mathbf{f} \in C^0(0,+\infty;L^2(\Omega)) \cap L^\infty(0,+\infty;L^2(\Omega))$ and let us denote $\|\mathbf{f}\| = \sup_{0<t<+\infty} \|\mathbf{f}(t)\|_0$; then we can prove

Theorem 4.1. *If $\nu\,\delta t/h^2 \le 1/[2(c_i^2 + c_p^2 h^2)]$ and $\nu\,\delta t \le 2/c_p^2$, then ψ_h solution of the second-order scheme* (4.1) *and* (4.2) *satisfies*

$$\limsup_{n\to+\infty} \left[|\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 \right]^{1/2} \le 16 \frac{\|\mathbf{f}\,\|}{\nu c_p^2}\,. \tag{4.5}$$

Proof. First we recall the relation

$$\begin{aligned} 2(a^{k+1}, 3a^{k+1} - 4a^k + a^{k-1}) &= |a^{k+1}|^2 + |2a^{k+1} - a^k|^2 + |\delta_{tt} a^{k+1}|^2 \\ &\quad - |a^k|^2 - |2a^k - a^{k-1}|^2\,, \end{aligned}$$

where we have set $\delta_{tt} a^{k+1} = a^{k-1} - 2a^k + a^{k+1}$. Now we take $4\delta t\, \psi_h^{n+1}$ as test function in (4.1)

$$\begin{aligned} &|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + |\delta_{tt}\psi_h^{n+1}|_1^2 + 4\nu\,\delta t(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\psi_h^{n+1}) \\ &\quad \le |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + \gamma\,\delta t|\psi^{n+1}|_1^2 + 4\gamma^{-1}\,\delta t\|\mathbf{f}^{\,n+1}\|_0^2\,, \end{aligned} \tag{4.6}$$

where the contribution of the advection term is zero given the rotational form we have chosen for it. We obtain some control on the term $(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\psi_h^{n+1})$ by proceeding as follows:

$$\begin{aligned} (\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\psi_h^{n+1}) &= (\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\delta_{tt}\psi_h^{n+1}) \\ &\quad + (\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla(2\psi_h^n - \psi_h^{n-1}))\,. \end{aligned}$$

By using $2\omega_h^n - \omega_h^{n-1}$ as test function in $2 \times (4.2)^n - (4.2)^{n-1}$, we obtain

$$(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla(2\psi_h^n - \psi_h^{n-1})) = \|2\omega_h^n - \omega_h^{n-1}\|_0^2\,.$$

As a result, by using the inverse inequality (4.4) together with the Cauchy–Schwarz inequality we have

$$(\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\psi_h^{n+1}) \ge \frac{1}{2}\|2\omega_h^n - \omega_h^{n-1}\|_0^2 - \frac{c_i^2}{2h^2}|\delta_{tt}\psi_h^{n+1}|_1^2\,.$$

From (4.2) and (4.3) we infer

$$c_p|2\psi_h^n - \psi_h^{n-1}|_1 \le \|2\omega_h^n - \omega_h^{n-1}\|_0\,.$$

As a result, we obtain

$$\begin{aligned} (\nabla(2\omega_h^n - \omega_h^{n-1}), \nabla\psi_h^{n+1}) &\ge \frac{c_p^2}{2}|2\psi_h^n - \psi_h^{n-1}|_1^2 - \frac{c_i^2}{2h^2}|\delta_{tt}\psi_h^{n+1}|_1^2 \\ &\ge \frac{c_p^2}{2}(|\psi_h^{n+1}|_1 - |\delta_{tt}\psi_h^{n+1}|_1)^2 - \frac{c_i^2}{2h^2}|\delta_{tt}\psi_h^{n+1}|_1^2 \end{aligned}$$

$$\geq \frac{c_p^2}{4}(|\psi_h^{n+1}|_1 - 2|\delta_{tt}\psi_h^{n+1}|_1)^2 - \frac{c_i^2}{2h^2}|\delta_{tt}\psi_h^{n+1}|_1^2$$

$$\geq \frac{c_p^2}{4}|\psi_h^{n+1}|_1^2 - \left(\frac{c_p^2}{2} + \frac{c_i^2}{2h^2}\right)|\delta_{tt}\psi_h^{n+1}|_1^2 .$$

By inserting this inequality in (4.6), we deduce that

$$|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + \left[1 - 2\nu\,\delta t\left(c_p^2 + \frac{c_i^2}{h^2}\right)\right]|\delta_{tt}\psi_h^{n+1}|_1^2 + (\nu c_p^2 - \gamma)\,\delta t|\psi_h^{n+1}|_1^2$$

$$\leq |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + 4\gamma^{-1}\,\delta t\|\mathbf{f}^{\,n+1}\|_0^2 .$$

If $\nu\,\delta t/h^2 \leq 1/2(c_i^2 + c_p^2 h^2)$, by setting $\gamma = \nu c_p^2/2$ we obtain

$$|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + \frac{1}{2}\nu c_p^2\,\delta t|\psi_h^{n+1}|_1^2$$

$$\leq |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + \frac{8\delta t}{\nu c_p^2}\|\mathbf{f}^{\,n+1}\|_0^2 .$$

Hereafter we denote $c_0 = \nu\,\delta t\,c_p^2/2$. By using the triangular inequality, we deduce that

$$-\frac{3}{7}|\psi_h^n|_1^2 + \frac{1}{7}\left(|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2\right) \leq |\psi_h^{n+1}|_1^2 ,$$

from which we infer

$$|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + (c_0 - c_1)|\psi_h^{n+1}|_1^2 + \frac{c_1}{7}\left(|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2\right)$$

$$\leq |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + \frac{3c_1}{7}|\psi_h^n|_1^2 + \frac{8\delta t}{\nu c_p^2}\|\mathbf{f}^{\,n+1}\|_0^2 ,$$

$$|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + \left(c_0 - c_1 - \frac{3c_1^2}{49}\right)|\psi_h^{n+1}|_1^2$$

$$+ \frac{c_1}{7}\left(|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + \frac{3c_1}{7}|\psi_h^{n+1}|_1^2\right)$$

$$\leq |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + \frac{3c_1}{7}|\psi_h^n|_1^2 + \frac{8\delta t}{\nu c_p^2}\|\mathbf{f}^{\,n+1}\|_0^2 .$$

Now we choose c_1 such that $c_0 - c_1 - \frac{3}{49}c_1^2 \geq 3c_1/7$; i.e. $c_1 = 35[(1 + \frac{3}{25}c_0)^{1/2} - 1]/3$. We obtain finally

$$\left(1 + \frac{c_1}{7}\right)\left(|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 + \frac{3c_1}{7}|\psi_h^{n+1}|_1^2\right)$$

$$\leq |\psi_h^n|_1^2 + |2\psi_h^n - \psi_h^{n-1}|_1^2 + \frac{3c_1}{7}|\psi_h^n|_1^2 + \frac{8\delta t}{\nu c_p^2}\|\mathbf{f}^{\,n+1}\|_0^2 .$$

By induction we easily infer the following bound

$$|\psi_h^{n+1}|_1^2 + |2\psi_h^{n+1} - \psi_h^n|_1^2 \leq \alpha^n \left(|\psi_h^1|_1^2 + |2\psi_h^1 - \psi_h^0|_1^2 + \frac{3c_1}{7}|\psi_h^1|_1^2 \right) + \frac{8\delta t}{\nu c_p^2}\frac{1-\alpha^n}{1-\alpha}\|\mathbf{f}\|^2 .$$

where $\alpha = 1/(1 + c_1/7)$. If $c_0 \leq 1$, then one can prove $1 \geq c_0 \geq c_1 \geq c_0/2$. The conclusion follows easily. □

Remark 4.1. Note that the bound (4.5) shows that the L^2-norm of the approximate velocity is uniformly bounded in time. This *a priori* bound is similar to the one that is satisfied by the continuous velocity field. This bound is necessary to prove that the discrete semigroup possesses an attractor; see Temam,[12] pp. 26 and 27.

5. Conclusions

In this paper we have presented a variational formulation for solving the time-dependent Navier–Stokes equations expressed in terms of the stream function and the vorticity. The solution of the two equations is uncoupled owing to an explicit treatment of the viscous diffusion together with a non-standard writing of the evolutionary term in the weak form of the momentum equation.

While making the viscous diffusion explicit to derive the vorticity boundary value by a Taylor expansion is a classical procedure within the finite differences context (see E and Liu[2]), the extension of this technique within a variational setting seems to have been overlooked in the literature, to the authors' knowledge.

The main advantage of this method is its extreme algorithmic simplicity, especially when compared to the Glowinski–Pironneau method and related techniques. The error analysis of this scheme has been performed; the explicit treatment of the viscous term implies a stability condition of the type: $\nu\,\delta t/h^2 \leq c$. Hence, the gain in simplicity is paid by a loss of stability. This stability constraint may be severe for creeping flows, but the matter improves for convection dominated flows since the stability limit scales with the Reynolds number.

For convection dominated flows, the most important issue is the spatial discretization which must be fine enough to resolve the thin structures of the flow, as expressed by the well known condition for the cell Reynolds number to be $O(1)$. When combining the cell Reynolds number condition and the stability condition referred to above, we obtain $\delta t \leq ch$. The method presented in this paper can accommodate time discretizations of high order of accuracy. As an illustration of this property, we have proposed a second-order accurate scheme based on the three-level backward differencing combined with a linear extrapolation in time of the vorticity in the viscous term as well as in the nonlinear term.

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References

1. Achdou and O. Pironneau, *A fast solver for Navier–Stokes equations in the laminar regime using mortar finite element and boundary element methods*, *SIAM J. Numer. Anal.* **32** (1995) 985–1016.
2. W. E and J.-G. Liu, *Vorticity boundary conditions and related issues for finite differences schemes*, *J. Comput. Phys.* **124** (1996) 368–382.
3. M. Fortin and F. Thomasset, *Mixed finite-element methods for incompressible flow problems*, *J. Comput. Phys.* **31** (1979) 113–145.
4. V. Girault and P.-A. Raviart, **Finite Element Methods for Navier–Stokes Equations**, *Springer Series in Computational Mathematics*, *Vol. 5* (Springer-Verlag, 1986).
5. R. Glowinski and O. Pironneau, *Numerical methods for the first biharmonic equation and for the two-dimensional Stokes problem*, *SIAM Rev.* **12** (1979) 167–212.
6. J.-L. Guermond and L. Quartapelle, *FEM solution of the ψ–ω equations with explicit viscous diffusion*, **Computational Fluid Dynamics**, eds. K. D. Papailiou *et al.*, Proceedings of the Fourth European Fluid Dynamics Conference, Athens, 7–11 September, Greece, **1**, part II (1998) 1258–1263.
7. M. Napolitano, G. Pascazio and L. Quartapelle, *A reivew of vorticity conditions in the numerical solution of the ζ–ψ equations*, *Comput. Fluids* **28** (1999) 139–186.
8. R. Peyret and T. D. Taylor, **Computational Methods for Fluid Flow** (Springer-Verlag, 1983).
9. L. Quartapelle, **Numerical Solution of the Incompressible Navier–Stokes Equations**, *ISNM*, *Vol. 113* (Birkhäuser, 1993).
10. V. Ruas, *Iterative solution of the stationary incompressible Navier–Stokes equations in stream function–vorticity formulation*, *C. R. Acad. Sci. Paris*, Série I **321** (1995) 381–386.
11. R. Scholz, *A mixed method for fourth order problems using linear finite elements*, *R.A.I.R.O. Anal. Numer.* **12** (1978) 85–90.
12. R. Temam, **Infinite-Dimensional Dynamical Systems in Mechanics and Physics**, *Applied Mathematical Sciences*, *Vol. 68* (Springer-Verlag, 1988).