FINITE ELEMENT SOLUTION OF UNSTEADY VISCOUS FLOWS PAST MULTIPLE AIRFOILS

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1. Introduction. This work is concerned with the numerical solution of the time-dependent Navier–Stokes equations in the ω-ψ representation for problems in multiply connected 2D regions. A new variational formulation based on a particular decomposition of the stream function space is presented. An uncoupled solution method is thereby derived which extends to transient calculations the method proposed by Glowinski and Pironneau for the biharmonic problem [3], [8]. Similarly to the latter, the proposed uncoupled method leads to a small linear system of equations for determining the additional stream function unknowns on the immersed bodies. After a suitable time discretization, the equations are discretized spatially by means of a mixed finite element method.

The content of the paper is organized as follows. In section 2 we establish the functional setting necessary to formulate the equations for incompressible flows in 2D, using both primitive and nonprimitive variables. We introduce a special decomposition of the stream function space for multiply connected domains. Section 3 addresses the numerical approximation of the ω-ψ equations by means of finite elements. The uncoupled method for enforcing the special conditions induced by the multiple connectedness of the domain is described in detail. Section 4 is devoted to the numerical examples. The complete analysis of the proposed method is given in [6].

2. Preliminaries. In this article Ω is an open bounded domain of $\mathbb{R}^2$; Ω is connected but may be multiply connected; its boundary Γ is smooth, say Γ is $C^{0,1}$. We denote by Γ_0 the exterior boundary of Ω and by Γ_i, $1 \le i \le p$, the other (internal) connected components of Γ, namely, $\Gamma = \partial\Omega = \bigcup_{i=0}^{p} \Gamma_i$. Let $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}})$ be a unit base of $\mathbb{R}^2$ and $(\widehat{\boldsymbol{x}}, \widehat{\boldsymbol{y}}, \widehat{\boldsymbol{z}})$ a right-handed unit base of $\mathbb{R}^3$. To have a convenient notation for the curl operator, let us define $curl\,\phi = \nabla\phi \times \widehat{\boldsymbol{z}}$ and $curl\,\boldsymbol{v} = \widehat{\boldsymbol{z}} \cdot \nabla \times \boldsymbol{v}$. In the sequel, $\boldsymbol{\tau}$ is the oriented unit tangent of Γ so that $(\boldsymbol{n}, \boldsymbol{\tau}, \widehat{\boldsymbol{z}})$ is a right-handed triad of unit vectors, $\boldsymbol{n}$ being the outward normal.

The analysis of the Navier–Stokes equations, supplemented with Dirichlet boundary condition, leads to consider the following Hilbert spaces of solenoidal fields:

$$\begin{aligned} \boldsymbol{J}_0^0(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{L}^2(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{n} \cdot \boldsymbol{v}_{|\Gamma} = 0\}, \\ \boldsymbol{J}_0^1(\Omega) &= \{\boldsymbol{v} \in \boldsymbol{H}^1(\Omega) \mid \nabla \cdot \boldsymbol{v} = 0, \quad \boldsymbol{v}_{|\Gamma} = 0\}. \end{aligned}$$

To introduce the stream function for representing incompressible flows in possible multiply connected domain, we define the following Hilbert spaces:

$$\begin{aligned} \Phi &= \{\varphi \in H^1(\Omega) \mid \varphi_{|\Gamma_0} = 0, \quad \varphi_{|\Gamma_i} = C_i, \ \forall C_i \in \mathbb{R}, \ 1 \le i \le p\}, \\ \Psi &= \{\psi \in H^2(\Omega) \mid \psi_{|\Gamma_0} = 0, \quad \psi_{|\Gamma_i} = C_i, \ \forall C_i \in \mathbb{R}, \ 1 \le i \le p, \quad \partial\psi/\partial n_{|\Gamma} = 0\}. \end{aligned}$$

The relevance of these spaces is brought to light by:

LEMMA 2.1. *(See e.g. [2]) We have the following isomorphisms*

$$(i)\ curl\ :\ \Phi \longrightarrow \boldsymbol{J}_0^0(\Omega); \qquad (ii)\ curl\ :\ \Psi \longrightarrow \boldsymbol{J}_0^1(\Omega).$$

Further, we define the vector space

$$\Psi_0 = \{\psi \in \Psi \mid \psi_{|\cup_{i=1}^p \Gamma_i} = 0\} = H_0^2(\Omega).$$

Finally, we assume to have at hand p functions of Ψ: $k_1, \ldots, k_p$, such that $k_{i|\Gamma_j} = \delta_{ij}$, $i = 1, \ldots, p$, where δ_{ij} is the Kronecker symbol, by means of which we define

$$^{\Gamma}\boldsymbol{K} = \mathrm{span}\langle k_1, \ldots, k_p\rangle.$$

The dimension of $^{\Gamma}\boldsymbol{K}$ is p. We are now able to decompose Ψ as follows.

THEOREM 2.2. *We have the decomposition*

$$\Psi = \Psi_0 \oplus {}^{\Gamma}\boldsymbol{K}.$$

Remark 2.1. Note that this decomposition is quite arbitrary since the functions $k_1, \ldots, k_p$ are arbitrary. Such a decomposition is *not* orthogonal unless the functions k_i are carefully chosen. A decomposition orthogonal with respect to a suitable scalar product will be considered in the following to derive an uncoupled solution method. □

3. The Navier–Stokes problem in $\boldsymbol{u}$-p and ω-ψ formulations. For sake of simplicity, no-slip boundary condition on the entire boundary are considered. Let us introduce $a \in \mathcal{L}(\boldsymbol{J}_0^1(\Omega)^2, \mathbb{R})$ so that $a(\boldsymbol{u}, \boldsymbol{v}) = \nu\,(\boldsymbol{\nabla u}, \boldsymbol{\nabla v})$ and $b \in \mathcal{L}(\boldsymbol{J}_0^1(\Omega)^3, \mathbb{R})$ so that $b(\boldsymbol{u}, \boldsymbol{v}, \boldsymbol{w}) = ((\boldsymbol{u} \cdot \boldsymbol{\nabla})\boldsymbol{v}, \boldsymbol{w})$. Consider $\boldsymbol{f}$ in $\boldsymbol{H}^{-1}(\Omega)$ (a body force) and $\boldsymbol{u}_0$ in $\boldsymbol{J}_0^0(\Omega)$. We hereafter consider the following Navier–Stokes problem:

$$\mathcal{P}_0 \begin{cases} \text{Find } \boldsymbol{u} \in L^2(0,T; \boldsymbol{J}_0^1(\Omega)) \cap C(0,T; \boldsymbol{J}_0^0(\Omega)) \text{ with } \boldsymbol{u}_{|t=0} = \boldsymbol{u}_0 \text{ such that} \\ \forall \boldsymbol{v} \in \boldsymbol{J}_0^1(\Omega), \qquad (\boldsymbol{u}_t, \boldsymbol{v}) + a(\boldsymbol{u}, \boldsymbol{v}) + b(\boldsymbol{u}, \boldsymbol{u}, \boldsymbol{v}) = (\boldsymbol{f}, \boldsymbol{v}). \end{cases}$$

To derive the ω-ψ formulation we proceed as follows (see also [4] and [6]). Since $curl : \Psi \longrightarrow \boldsymbol{J}_0^1(\Omega)$ is an isomorphism (lemma 2.1), we can replace the test functions of $\boldsymbol{J}_0^1(\Omega)$ in $\mathcal{P}_0$ by that of $curl\,\Psi$. Furthermore, thanks to the decomposition of ψ introduced in theorem 2.2, we can separate the action of the test functions of Ψ_0 from that of $^{\Gamma}\boldsymbol{K}$ to obtain the following problem:

$$\mathcal{P}_1 \begin{cases} \text{Find } \omega \in L^2(0,T; L^2(\Omega)) \cap C(0,T; H^{-1}(\Omega)) \text{ and} & \\ \psi \in L^2(0,T; \Psi) \cap C(0,T; \Phi), \ \text{ such that} & \\ \forall \phi \in \Phi, \quad ((\boldsymbol{\nabla}\psi)_{|t=0}, \boldsymbol{\nabla}\phi) = (\hat{\boldsymbol{z}} \times \boldsymbol{u}_0, \boldsymbol{\nabla}\phi), \text{ and for all } t > 0 & \\ \omega_t - \nu\nabla^2\omega + J(\omega, \psi) = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{f}, & \text{in } H^{-2}(\Omega), \\ -\omega - \nabla^2\psi = 0, & \text{in } L^2(\Omega), \\ (\boldsymbol{\nabla}\psi_t, \boldsymbol{\nabla}k_i) - \nu\,(\omega, \nabla^2 k_i) + (J(\omega, \psi),\ k_i) = (\hat{\boldsymbol{z}} \times \boldsymbol{f}, \boldsymbol{\nabla}k_i), & 1 \le i \le p, \end{cases}$$

where $J(\omega, \psi)$ denotes the Jacobian determinant.

PROPOSITION 3.1. *Problem $\mathcal{P}_1$ is equivalent to $\mathcal{P}_0$ provided one sets $\boldsymbol{u} = \boldsymbol{\nabla}\psi \times \hat{\boldsymbol{z}}$ and $\omega = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{u}$.*

Remark 3.1. Actually, the advection–diffusion equation for the vorticity (understood in the distributional sense) can be obtained by "taking the curl" of the momentum equation; but by doing so (to "get rid of" the pressure) we forget something. The decomposition of Ψ enables one to recover the missing pieces of information by testing the original momentum equation against the curl of the p functions k_i. □

Remark 3.2. If the source term is smooth and the solution (ω, ψ) is smooth enough, the additional conditions due to the multiple connectedness can be interpreted in the classical sense. If $\boldsymbol{f}$ is in $\boldsymbol{H}^1(\Omega)$, the vorticity equation $\omega_t - \nu\nabla^2\omega + J(\omega, \psi) = \hat{\boldsymbol{z}} \cdot \boldsymbol{\nabla} \times \boldsymbol{f}$ holds in $L^2(\Omega)$. It follows that, after using integration by parts in the weak vorticity equation, the p extra conditions above yield:

$$1 \leq i \leq p, \qquad \nu \oint_{\Gamma_i} \frac{\partial \omega}{\partial n} = - \oint_{\Gamma_i} \boldsymbol{f} \cdot \boldsymbol{\tau}.$$

These well-known "strong" relations are frequently used in finite difference codes. □

4. Numerical approximation of the ω-ψ problem. We now assume that the time derivative is approximated by means of some standard finite differencing scheme, with the viscous diffusion term taken into account implicitly and the nonlinear term explicitly, and restrict the analysis to the corresponding linear semi-discrete problem. Setting $\gamma = 1/\Delta t$, the variational problem reads: Find ψ in $H^2(\Omega)$ such that

$$\begin{cases} \nu\,(\nabla^2\psi, \nabla^2\psi') + \gamma\,(\boldsymbol{\nabla}\psi, \boldsymbol{\nabla}\psi') = r(\psi'), & \forall \psi' \in \Psi = \Psi_0 \oplus {}^{\Gamma}\boldsymbol{K}, \\ \partial\psi/\partial n_{|\Gamma} = 0, & \\ \psi_{|\Gamma_0} = 0, \qquad\qquad \psi_{|\Gamma_i} = \Xi_i, & 1 \leq i \leq p, \end{cases}$$

Here Ξ_i, $1 \leq i \leq p$, represent p constants to be determined jointly with the unknown functions ω and ψ. The linear form $r(\psi')$ contains the information from the past and the source terms of the problem, for details see [6].

The semi-discrete problem above is now recast to be approximated by means of classical mixed finite element techniques developed for problems in simply connected domains. Let $\mathcal{F}_h$ be a regular triangulation of Ω. Let P_k be the space of polynomials of two variables of degree $\leq k$; we introduce the following finite dimensional spaces in which we will seek an approximation of ψ and ω, respectively:

$$W_h = \{\phi_h \in C^0(\overline{\Omega}) \mid \phi_{h|T} \in P_k,\ \forall T \in \mathcal{F}_h\},$$
$$\Psi_h = \{\phi_h \in C^0(\overline{\Omega}) \mid \phi_{h|T} \in P_k,\ \forall T \in \mathcal{F}_h, \quad \phi_{h|\Gamma_0} = 0, \quad \phi_{h|\Gamma_i} = C_i,\ \ 1 \leq i \leq p\}.$$

A natural approximation of Ψ_0 consists in $\Psi_{0,h} = \{\phi_h \in \Psi_h \mid \phi_{h|\cup_{i=1}^{p}\Gamma_i} = 0\}$.

Now, setting $\sigma = \gamma/\nu$, we build an approximation of ${}^{\Gamma}\boldsymbol{K}_\sigma$ by defining the functions $\omega_h^i \in W_h$ and $\psi_h^i \in \Psi_h$, $1 \leq i \leq p$, as follows

$$\begin{cases} (\boldsymbol{\nabla}\omega_h^i, \boldsymbol{\nabla}\phi_h) + \sigma\,(\omega_h^i, \phi_h) = 0, & \forall \phi_h \in \Psi_{0,h}, \\ -(\omega_h^i, v_h) + (\boldsymbol{\nabla}\psi_h^i, \boldsymbol{\nabla}v_h) = 0, & \forall v_h \in W_h, \\ \psi^i_{h|\Gamma_j} = \delta_{ij}, & 1 \leq j \leq p. \end{cases}$$

We denote by ${}^{\Gamma}\boldsymbol{K}_{\sigma,h}$ the finite dimensional linear space $\mathrm{span}\langle\psi_h^1, \ldots, \psi_h^p\rangle$. These problems are very classical and can be solved by using the Glowinski–Pironneau method

[3], which is valid irrespective of the multiple connectedness of Ω; for details on this technique, see also [8].

Now, we introduce the discrete counterpart of the Laplace operator $A_h \in \mathcal{L}(W_h)$ as follows. For $\psi_h \in W_h$, $A_h\psi_h$ is the unique element of W_h which satisfies

$$\forall v_h \in W_h, \qquad (A_h\psi_h, v_h) = (\nabla\psi_h, \nabla v_h).$$

Now we define the scalar product $((\cdot\,,\cdot))_h$ in W_h

$$((\psi_h, \psi_h'))_h = (A_h\psi_h, A_h\psi_h') + \sigma\,(\nabla\psi_h, \nabla\psi_h').$$

PROPOSITION 4.1. *We have the decomposition, orthogonal with respect to* $((\cdot\,,\cdot))_h$,

$$\Psi_h = \Psi_{0,h} \oplus {}^{\Gamma}\boldsymbol{\mathcal{K}}_{\sigma,h}.$$

According to this decomposition, it is natural to set at each time level:

$$\omega_h = \omega_h^0 + \sum_{j=1}^{p} \Xi_j\, \omega_h^j, \qquad \psi_h = \psi_h^0 + \sum_{j=1}^{p} \Xi_j\, \psi_h^j.$$

Thanks to the orthogonality of the decomposition with respect to $((\cdot\,,\cdot))_h$, the functions $\omega_h^0 \in W_h$ and $\psi_h^0 \in \Psi_{0,h}$ are solution to the following linear (uncoupled) problem

$$\begin{cases} \nu\,(\nabla\omega_h^0, \nabla\phi_h) + \gamma\,(\omega_h^0, \phi_h) = s(\phi_h), & \forall \phi_h \in \Psi_{0,h}, \\ -(\omega_h^0, v_h) + (\nabla\psi_h^0, \nabla v_h) = 0, & \forall v_h \in W_h. \end{cases}$$

The set of equations controlling the constants Ξ_i, $1 \le i \le p$, are obtained by testing the momentum equation against the curl of functions ${}^{\Gamma}\boldsymbol{\mathcal{K}}_{\sigma,h}$, which yields the following $p \times p$ linear system

$$B\Xi = g,$$

where the coefficients of matrix B and vector g are given by

$$\begin{aligned} B_{ij} &= \nu\,(\nabla\omega_h^j, \nabla\psi_h^i) + \gamma\,(\omega_h^j, \psi_h^i), \\ g_i &= -\nu\,(\nabla\omega_h^0, \nabla\psi_h^i) - \gamma\,(\omega_h^0, \psi_h^i) + s(\psi_h^i). \end{aligned}$$

From the practical viewpoint, we have

PROPOSITION 4.2. *B is symmetric definite positive and*

$$B_{ij} = \nu\,(\omega_h^j, \omega_h^i) + \gamma\,(\nabla\psi_h^j, \nabla\psi_h^i).$$

Remark 4.1. In practice, the matrix B is calculated and inverted once and for all at the preprocessing stage. The calculation of the right-hand side g can be greatly accelerated by setting $\psi_h^i = \psi_{0,h}^i + \mu_h^i$, where $\psi_{0,h}^i$ is zero on Γ_i and μ_h^i is zero on the degrees of freedom which do not belong to Γ_i. Thanks to this decomposition, we obtain:

$$g_i = -\nu\,(\nabla\omega_h^0, \nabla\mu_h^i) - \gamma\,(\omega_h^0, \mu_h^i) + s(\mu_h^i).$$

As a result, only the functions μ_h^i need to be stored for the calculation of g_i. □

5. Numerical examples. The validity of the proposed formulation is demonstrated by presenting some numerical results for unsteady flow past airfoils at moderate Reynold numbers R. The variables vorticity and stream function are approximated by means of a piecewise linear interpolation over a Delaunay triangular mesh, generated by the method of Rebay [9]. The large sparse symmetric systems of linear equations are solved by Choleski's method after an internal reordering of the unknowns by means of Sparspack [1]. For example, results for the unsteady flow past a NACA 0012 airfoil at an angle of incidence of 34° for $R = 1000$ ($\Delta t = 0.02$) are reported in Figure 1. In this figure, the streamlines obtained from the present method at $t = 3.6$ are compared with those calculated at the same time by means of a new fractional-step projection method [5]. The contour lines of pressure at $t = 1.6$ provided by the ω-ψ uncoupled method are compared in Figure 2 with those of the solution calculated by means of the fractional-step projection method [5].

Another example is the flow past a multiple-body profile with high-lift devices, consisting of a slat, the main airfoil and a flap [7], with angle of incidence of 25°. In Figure 3 the streamlines of the solutions at time 3.6 for $R = 500$ ($\Delta t = 0.001$) obtained by the two methods are compared. The comparisons show that the nonprimitive variable method is capable of predicting the dynamics of the flow field in multiply connected domains quite correctly, even in the presence of sharp geometrical singularities of the boundary, like those at the trailing edges of the considered airfoils.

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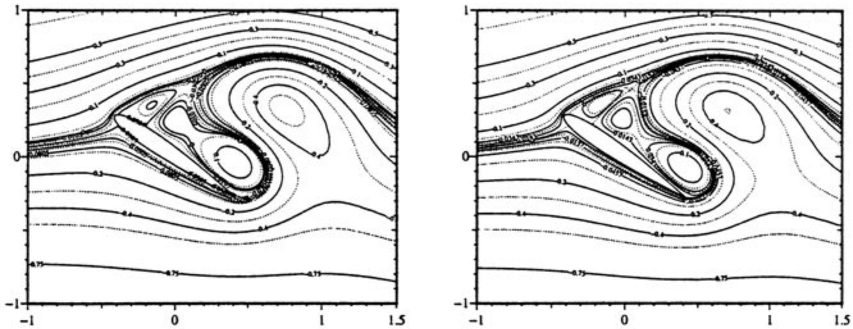

FIG. 1. *NACA 0012 airfoil at $\alpha = 34°$ and $R = 1000$. Comparison of streamlines of the ω-ψ (left) and u-p (right) solutions at $t = 3.6$.*

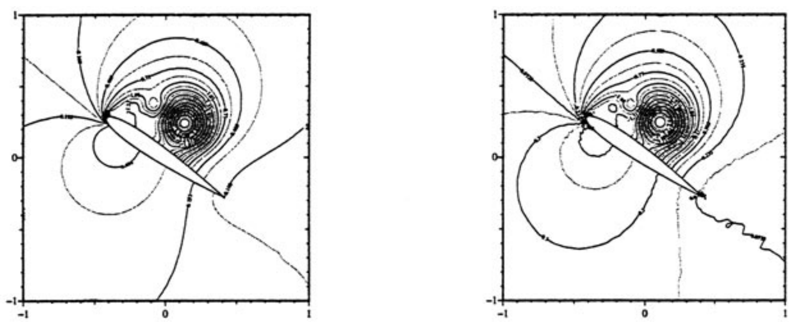

FIG. 2. *NACA 0012 airfoil at $\alpha = 34°$ and $R = 1000$. Comparison of pressure fields of the ω-ψ (left) and u-p (right) solutions at $t = 1.6$.*

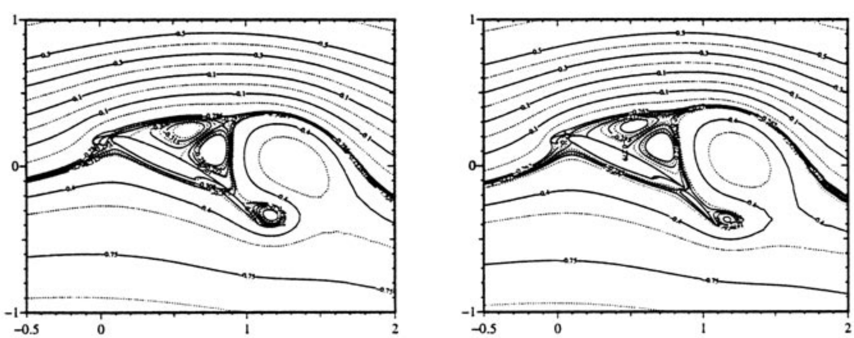

FIG. 3. *Multibody airfoil at $\alpha = 25°$ and $R = 500$. Streamlines for the solution of vorticity/stream function equations at $t = 2.8$ (left) and the solution of the projection method at the same time (right).*