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A note on the Stokes operator and its powers

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Abstract The so-called Stokes operator is an important tool in the analysis of the solutions of the Navier-Stokes equations and their numerical approximation. The aim of this note is to clarify certain properties of the fractional powers of this operator which are sometimes misused.

Keywords Navier-Stokes · Stokes operator · Approximation · Numerical analysis · Interpolation spaces

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1 Introduction

The so-called Stokes operator is frequently used in the analysis of the solutions to the Navier-Stokes equations and their numerical approximation. The aim of this note is to clarify certain properties of this operator which are sometimes misused.

Let Ω be an open bounded connected subset of $\mathbb{R}^d$ with boundary $\partial\Omega$. Let $\mathbf{n}$ be the unit outer normal to $\partial\Omega$ and consider

$$\mathbf{H} := \{\mathbf{u} \in \mathbf{L}^2(\Omega) : \nabla\cdot\mathbf{u} = 0, \mathbf{u}\cdot\mathbf{n}|_{\partial\Omega} = 0\}, \tag{1.1}$$

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$$D(A) := \{\mathbf{u} \in \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) : \nabla \cdot \mathbf{u} = 0\}. \tag{1.2}$$

The Stokes operator is defined to be $A := -P_{\mathbf{H}}\Delta$, where $\Delta : \mathbf{H}^2(\Omega) \cap \mathbf{H}^1_0(\Omega) \to \mathbf{L}^2(\Omega)$ denotes the Laplace operator with homogeneous Dirichlet boundary conditions, and $P_{\mathbf{H}}$ is the $\mathbf{L}^2$-projection onto $\mathbf{H}$.

The operator A is positive, self-adjoint, invertible, and its inverse is compact. It is then legitimate to consider A^s for all $s \in \mathbb{R}$ and it turns out that the quantity $(\mathbf{u}, A^s\mathbf{u})^{1/2}$ is a norm, where $(\cdot,\cdot)$ denotes the $\mathbf{L}^2$-scalar product. The question we want to address in this note is to determine whether this norm is equivalent to the Sobolev $\mathbf{H}^s$-norm. In other words, given $s \in \mathbb{R}$ we ask for the existence of constants $c_1, c_2 > 0$ such that

$$c_1\|\mathbf{u}\|_{\mathbf{H}^s} \le (\mathbf{u}, A^s\mathbf{u})^{\frac{1}{2}}, \quad \forall \mathbf{u} \in D(A^{\frac{s}{2}}), \tag{1.3}$$

$$(\mathbf{u}, A^s\mathbf{u})^{\frac{1}{2}} \le c_2\|\mathbf{u}\|_{\mathbf{H}^s}, \quad \forall \mathbf{u} \in D(A^{\frac{s}{2}}). \tag{1.4}$$

The Stokes operator is often used in numerical analysis. It is in particular useful to invoke this operator in the error analysis of the so-called fractional-time stepping techniques for the approximation of the incompressible time-dependent Navier-Stokes equations. This, however, can sometimes lead to erroneous statements. In this respect we want to refer to the recent work [2] where the inequality (1.3) is used with $s = -1$. We are not arguing in this note that the results in [2] are wrong; actually, we are confident that the convergence estimates in [2] will eventually turn out to be true, but the proofs therein are not correct. They can probably be fixed by proceeding as in the corrigendum [14] where earlier statements made on the basis of (1.3) with $s = -1$ (cf. (2.1) in [12] and (2.7) in [13]) were corrected by using alternative arguments.

Our goal in the present note is simply to draw the attention of the community on the fact that (1.3) is false for all $s \le -\frac{1}{2}$ and therefore should not be invoked. Given enough smoothness of the boundary of the domain, the upper bound (1.4) is true for all $s \in \mathbb{R}$. On the contrary, the lower bound (1.3) is false whenever $s \le -\frac{1}{2}$. A counter-example for $s = -1$ was given in [7]. That the paper [7] was written in French probably did not help this result to be publicized.

The rest of this note is organized as follows. In Sect. 2 we define the Stokes operator and state some of its properties. For the sake of completeness we re-prove that the lower bound (1.3) is true for all $s \in (-\frac{1}{2}, 2]$ and the upper bound (1.4) is true for all $s \in [-2, 2]$. Finally, in Sect. 3 we generalize the counterexample from [7] by showing that the inequality (1.3) is false for $s \le -\frac{1}{2}$ and thus should not be invoked.

2 The Stokes operator

In the sequel, given two Hilbert spaces X and Y, we denote $[X, Y]_s$, $s \in (0, 1)$, the real interpolation between X and Y, i.e., we use the so-called K-method of Lions and Peetre [11], see also [10] or [1, Appendix A]. We start with a standard interpolation result:

Lemma 2.1 (cf. [9]) *Let X be a Hilbert space and let T be a closed positive self-adjoint operator in X. $D(T)$ is the domain of T regarded as a Hilbert space with the graph norm. Then, for $0 < s < 1$ we have*

$$D(T^s) = [D(T), X]_{1-s}.$$

Let us now define the unbounded vector-valued Laplace operator $-\Delta : D(\Delta) := \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega) \to \mathbf{L}^2(\Omega)$. We introduce the so-called Stokes operator $A : D(A) \to \mathbf{H}$ with domain

$$D(A) := \{\mathbf{v} \in \mathbf{H}^2(\Omega) : \nabla \cdot \mathbf{v} = 0\} \cap \mathbf{H}^1_0(\Omega)$$

by setting $A = -P_{\mathbf{H}}\Delta|_{D(A)}$. We henceforth assume that the domain Ω is such that there is $c > 0$

$$\forall \mathbf{v} \in D(A), \quad \|\mathbf{v}\|_{\mathbf{H}^2} \le c\|A\mathbf{v}\|_{\mathbf{L}^2}, \tag{2.1}$$

$$\forall \mathbf{v} \in D(\Delta), \quad \|\mathbf{v}\|_{\mathbf{H}^2} \le c\|\Delta\mathbf{v}\|_{\mathbf{L}^2}. \tag{2.2}$$

These properties hold in two and three space dimensions ($d = 2, 3$) whenever Ω is convex or of class $\mathcal{C}^{1,1}$, see [3, Thm 6.3]. To uniformize notation we set $B := -\Delta$ and $D(B) = \mathbf{H}^1_0(\Omega) \cap \mathbf{H}^2(\Omega)$. Note that (2.2) implies that B is closed.

It follows from (2.1) that A is closed, positive, and self-adjoint. We now define

$$\mathbf{V}^{2s} := [\mathbf{H}, D(A)]_s, \quad s \in (0, 1), \tag{2.3}$$

with $\mathbf{V}^0 := \mathbf{H}$ and $\mathbf{V}^2 := D(A)$. Thanks to Lemma 2.1, we have that

$$D(A^s) = [\mathbf{V}^0, D(A)]_s, \qquad D(B^s) = [\mathbf{L}^2(\Omega), D(B)]_s. \tag{2.4}$$

This leads us to define the following equivalent norm on $\mathbf{V}^{2s}$, $s \in [0, 1]$

$$\|\mathbf{v}\|_{\mathbf{V}^{2s}} := \|A^s\mathbf{v}\|_{\mathbf{L}^2}. \tag{2.5}$$

For $s \in [-1, 0]$ we define $\mathbf{V}^{2s}$ by duality, i.e., $\mathbf{V}^{2s} = (\mathbf{V}^{-2s})'$. The question that we want to address now is to see whether the norms $\|\cdot\|_{\mathbf{V}^s}$ and $\|\cdot\|_{\mathbf{H}^s}$ are equivalent for the members of $\mathbf{V}^s$ (cf. (1.3) and (1.4)).

A first answer to the above question for $s \in [0, 1]$ is given by the following:

Theorem 2.1 (cf. [5]) *Assume that Ω is such that* (2.1) *and* (2.2) *hold. For any $s \in [0, 1]$ we have*

$$D(A^s) = D(B^s) \cap \mathbf{V}^0. \tag{2.6}$$

Proof We reproduce the proof of this standard result for the sake of completeness.

Step 1. ($D(A^s) \subset D(B^s) \cap \mathbf{V}^0$). We consider the injection operator $i : \mathbf{V}^0 \to \mathbf{L}^2(\Omega)$ and its restriction to $\mathbf{V}^2$, $i : \mathbf{V}^2 \to D(B)$. The following holds

$$\|i\mathbf{v}\|_{\mathbf{L}^2} = \|\mathbf{v}\|_{\mathbf{L}^2} = \|\mathbf{v}\|_{\mathbf{V}^0}, \quad \forall v \in \mathbf{V}^0.$$

The following also holds owing to (2.1)

$$\|i\mathbf{v}\|_{D(B)} = \|\mathbf{v}\|_{\mathbf{H}^2} \leq c\|A\mathbf{v}\|_{\mathbf{L}^2} = c\|\mathbf{v}\|_{D(A)}, \quad \forall v \in D(A).$$

We conclude using the Riesz-Thorin interpolation theorem; i.e., there is $c > 0$ so that for all

$$\|\mathbf{v}\|_{D(B^s)} = \|i\mathbf{v}\|_{D(B^s)} \leq c\|\mathbf{v}\|_{D(A^s)}, \quad \forall \mathbf{v} \in D(A^s).$$

This proves that $D(A^s) \subset D(B^s)$ with continuous embedding. This also implies $D(A^s) \subset D(B^s) \cap \mathbf{V}^0$ since $D(A^s) \subset \mathbf{V}^0$.

Step 2. Let us introduce the operator $K : D(B) \to D(A)$ defined by

$$K\mathbf{w} = A^{-1}P_{\mathbf{H}}B\mathbf{w}, \quad \forall \mathbf{w} \in D(B).$$

Since by definition $B = -\Delta$ and $A\mathbf{v} = -P_{\mathbf{H}}\Delta\mathbf{v}$ for all $\mathbf{v} \in D(A)$, it follows that

$$K\mathbf{v} = -A^{-1}P_{\mathbf{H}}\Delta\mathbf{v} = A^{-1}A\mathbf{v} = \mathbf{v}, \quad \forall \mathbf{v} \in D(A). \tag{2.7}$$

We want to extend K to a bounded operator from $\mathbf{L}^2(\Omega)$ to $\mathbf{V}^0$. Since $D(B)$ is dense in $\mathbf{L}^2(\Omega)$ and $D(A)$ is dense in $\mathbf{V}^0$ it suffices to show that there is a constant c such that the inequality $\|K\mathbf{w}\|_{\mathbf{L}^2} \leq c\|\mathbf{w}\|_{\mathbf{L}^2}$ holds for all $\mathbf{w} \in D(B)$. Recall that given the regularity of the domain, the following estimate holds (see (2.1)):

$$\|BA^{-1}\mathbf{w}\|_{\mathbf{L}^2} \leq c\|A^{-1}\mathbf{w}\|_{\mathbf{H}^2} \leq c\|\mathbf{w}\|_{\mathbf{L}^2}, \quad \forall \mathbf{w} \in \mathbf{V}^0. \tag{2.8}$$

Then the desired estimate is obtained as follows. Let $\mathbf{w} \in D(B)$

$$\begin{aligned}
\|K\mathbf{w}\|_{\mathbf{L}^2} &= \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(K\mathbf{w},\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} = \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(A^{-1}P_{\mathbf{H}}B\mathbf{w},\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} \\
&= \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(A^{-1}P_{\mathbf{H}}B\mathbf{w}, P_{\mathbf{H}}\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} \\
&= \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(P_{\mathbf{H}}B\mathbf{w}, A^{-1}P_{\mathbf{H}}\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} = \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(B\mathbf{w}, A^{-1}P_{\mathbf{H}}\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} \\
&= \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{(\mathbf{w}, BA^{-1}P_{\mathbf{H}}\mathbf{y})}{\|\mathbf{y}\|_{\mathbf{L}^2}} \\
&\leq \|\mathbf{w}\|_{\mathbf{L}^2} \sup_{0\neq\mathbf{y}\in\mathbf{L}^2(\Omega)} \frac{\|BA^{-1}P_{\mathbf{H}}\mathbf{y}\|_{\mathbf{L}^2}}{\|\mathbf{y}\|_{\mathbf{L}^2}}.
\end{aligned}$$

Since $P_{\mathbf{H}}\mathbf{y} \in \mathbf{V}^0$, (2.8) implies the estimate. As a result K can be extended continuously and we abuse the notation by using the same symbol to denote the extension in question $K : \mathbf{L}^2(\Omega) \to \mathbf{V}^0$. Note that $K\mathbf{v} = \mathbf{v}$ for all $\mathbf{v} \in \mathbf{V}^0$ owing to (2.7).

Step 3. $(D(B^s) \cap \mathbf{V}^0 \subset D(A^s))$. Let $\mathbf{u} \in D(B^s) \cap \mathbf{V}^0$, then

$$\|K\mathbf{u}\|_{D(A)} = \|A^{-1}P_{\mathbf{H}}B\mathbf{u}\|_{D(A)} = \|AA^{-1}P_{\mathbf{H}}B\mathbf{u}\|_{\mathbf{L}^2} \leq c\|B\mathbf{u}\|_{\mathbf{L}^2} = c\|\mathbf{u}\|_{D(B)}.$$

Moreover, since the operator K is bounded in $\mathbf{L}^2(\Omega)$

$$\|K\mathbf{u}\|_{\mathbf{L}^2} \leq c\|\mathbf{u}\|_{\mathbf{L}^2}, \quad \forall \mathbf{u} \in \mathbf{L}^2(\Omega).$$

We conclude by using the Riesz-Thorin interpolation theorem. There exist c so that the following holds

$$\|\mathbf{v}\|_{D(A^s)} = \|K\mathbf{v}\|_{D(A^s)} \leq c\|\mathbf{v}\|_{D(B^s)}, \quad \forall \mathbf{v} \in D(B^s) \cap \mathbf{V}^0,$$

which proves that the embedding $D(B^s) \cap \mathbf{V}^0 \subset D(A^s)$ is continuous. □

Let us now introduce the notation

$$\mathbf{H}_0^s(\Omega) := \left[\mathbf{L}^2(\Omega), \mathbf{H}_0^1(\Omega)\right]_s, \quad \forall s \in [0, 1]. \tag{2.9}$$

This notation is slightly different from what is usually done in textbooks at $s = \frac{1}{2}$; what we denote $\mathbf{H}_0^{\frac{1}{2}}(\Omega)$ is usually denoted $\mathbf{H}_{00}^{\frac{1}{2}}(\Omega)$ (see [10, Theorem 11.7]).

Corollary 2.1 *Assume that the hypotheses of Theorem 2.1 hold, then*

$$\mathbf{V}^s = \mathbf{H}^s(\Omega) \cap \mathbf{V}^0, \quad \forall s \in \left[0, \frac{1}{2}\right), \tag{2.10}$$

$$\mathbf{V}^s = \mathbf{H}_0^s(\Omega) \cap \mathbf{V}^0, \quad \forall s \in \left[\frac{1}{2}, 1\right], \tag{2.11}$$

$$\mathbf{V}^s = \mathbf{H}^s(\Omega) \cap \mathbf{V}^1, \quad \forall s \in [1, 2]. \tag{2.12}$$

Proof It is known that $D(B^{\frac{s}{2}}) = \mathbf{H}_0^s(\Omega)$ for all $s \in [0, 1]$ and $D(B^{\frac{s}{2}}) = \mathbf{H}^s(\Omega) \cap \mathbf{H}_0^1(\Omega)$ for all $s \in [1, 2]$, (see e.g. [6]). The characterization (2.6) thus establishes that

$$\mathbf{V}^s = D(A^{\frac{s}{2}}) = D(B^{\frac{s}{2}}) \cap \mathbf{V}^0 = \mathbf{H}_0^s(\Omega) \cap \mathbf{V}^0, \quad \forall s \in [0, 1],$$
$$\mathbf{V}^s = D(A^{\frac{s}{2}}) = D(B^{\frac{s}{2}}) \cap \mathbf{V}^0 = \mathbf{H}^s(\Omega) \cap \mathbf{V}^1, \quad \forall s \in [1, 2].$$

The statement (2.10) is a consequence of the fact that the spaces $\mathbf{H}^s(\Omega)$ and $\mathbf{H}_0^s(\Omega)$ coincide for $0 \leq s < \frac{1}{2}$ with uniformly equivalent norms (see [10, Theorems 11.1, 11.6]). □

We now assume that Ω is such that the transformation $T : \mathbf{L}^2(\Omega) \to H^1(\Omega)/\mathbb{R}$ defined so that

$$\Delta(T\mathbf{v}) = \nabla \cdot \mathbf{v}, \qquad \partial_n(T\mathbf{v})|_{\partial\Omega} = 0, \tag{2.13}$$

can be extended from $\mathbf{H}^1(\Omega)$ to $H^2(\Omega)/\mathbb{R}$. This property is known to be true if Ω is convex or of class $\mathcal{C}^{1,1}$. The Riesz-Thorin interpolation theorem implies that $T : \mathbf{H}^s(\Omega) \to H^{1+s}(\Omega)/\mathbb{R}$ is continuous for any $s \in [0, 1]$. The following result is a consequence of this property.

Lemma 2.2 *Assume that Ω is such that* (2.1), (2.2), *and* (2.13) *hold, then $P_{\mathbf{H}}$ can be continuously extended as an operator from $\mathbf{H}^s(\Omega)$ to $\mathbf{V}^s$ for all $s \in [0, \frac{1}{2})$.*

Proof We proceed by density. Let $s \in [0, \frac{1}{2})$ and let $\mathbf{v}$ be a member of $D(B)$, then the property (2.13) implies that there exists c, independent of $\mathbf{v}$, so that

$$\|T\mathbf{v}\|_{H^{1+s}} \leq c\|\mathbf{v}\|_{\mathbf{H}^s}.$$

The definition of $T\mathbf{v}$ implies that $P_{\mathbf{H}}\mathbf{v} = \mathbf{v} - \nabla(T\mathbf{v})$, which in turn gives

$$\|P_{\mathbf{H}}\mathbf{v}\|_{\mathbf{H}^s} \leq \|\mathbf{v}\|_{\mathbf{H}^s} + \|\nabla(T\mathbf{v})\|_{\mathbf{H}^s} \leq \|\mathbf{v}\|_{\mathbf{H}^s} + \|T\mathbf{v}\|_{H^{1+s}} \leq c\|\mathbf{v}\|_{\mathbf{H}^s}.$$

This means that $P_{\mathbf{H}}\mathbf{v} \in \mathbf{H}^s(\Omega) \cap \mathbf{V}_0$. The conclusion follows from the characterization (2.10). □

Theorem 2.2 *The norms $\|\cdot\|_{\mathbf{V}^s}$ and $\|\cdot\|_{\mathbf{H}^s}$ are equivalent in $\mathbf{V}^s$ for $s \in (-\frac{1}{2}, 0)$ and $\mathbf{V}^0 \cap \mathbf{H}^s(\Omega)$ is continuously embedded in $\mathbf{V}^s$ for all $s \in [-1, 0]$. In other words*

$$\mathbf{V}^s = \mathbf{H}^s(\Omega) \cap \mathbf{V}^0, \quad \forall s \in \left(-\frac{1}{2}, 0\right], \tag{2.14}$$

$$\mathbf{H}^s(\Omega) \cap \mathbf{V}^0 \subset \mathbf{V}^s, \quad \forall s \in [-1, 0]. \tag{2.15}$$

Proof We proceed by duality.

Step 1. ($\|\mathbf{v}\|_{\mathbf{V}^s} \leq c\|\mathbf{v}\|_{\mathbf{H}^s}$). Let $s \in [-1, 0]$. We proceed by density; let $\mathbf{v} \in \mathbf{V}^0$, then

$$\|\mathbf{v}\|_{\mathbf{V}^s} = \sup_{0 \neq \mathbf{w} \in \mathbf{V}^{-s}} \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|_{\mathbf{V}^{-s}}} \leq c \sup_{0 \neq \mathbf{w} \in \mathbf{V}^{-s}} \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|_{\mathbf{H}^{-s}}} \leq c \sup_{0 \neq \mathbf{w} \in \mathbf{H}_0^{-s}(\Omega)} \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|_{\mathbf{H}^{-s}}} = c\|\mathbf{v}\|_{\mathbf{H}^s}.$$

The first equality is the definition of the norm. The first inequality is a consequence of the norm equivalence expressed in (2.10)–(2.11) for $s \in [-1, 0]$. The second inequality is a consequence of $\mathbf{V}^{-s}$ being a subset of $\mathbf{H}_0^{-s}(\Omega)$, see (2.10)–(2.11) again. Finally, the last equality holds by definition. Conclude by density.

Step 2. ($\|\mathbf{v}\|_{\mathbf{H}^s} \leq c\|\mathbf{v}\|_{\mathbf{V}^s}$). Let $s \in (-\frac{1}{2}, 0]$. Let $\mathbf{v} \in \mathbf{V}^0$, then

$$\begin{aligned} \|\mathbf{v}\|_{\mathbf{H}^s} &= \sup_{0 \neq \mathbf{w} \in \mathbf{H}^{-s}(\Omega)} \frac{(\mathbf{v}, \mathbf{w})}{\|\mathbf{w}\|_{\mathbf{H}^{-s}}} = \sup_{0 \neq \mathbf{w} \in \mathbf{H}^{-s}(\Omega)} \frac{(\mathbf{v}, P_{\mathbf{H}}\mathbf{w})}{\|\mathbf{w}\|_{\mathbf{H}^{-s}}} \leq c \sup_{0 \neq \mathbf{w} \in \mathbf{H}^{-s}} \frac{(\mathbf{v}, P_{\mathbf{H}}\mathbf{w})}{\|P_{\mathbf{H}}\mathbf{w}\|_{\mathbf{H}^{-s}}} \\ &\leq c \sup_{0 \neq \mathbf{z} \in \mathbf{V}^{-s}} \frac{(\mathbf{v}, \mathbf{z})}{\|\mathbf{z}\|_{\mathbf{H}^{-s}}} \leq c \sup_{0 \neq \mathbf{z} \in \mathbf{V}^{-s}} \frac{(\mathbf{v}, \mathbf{z})}{\|\mathbf{z}\|_{\mathbf{V}^{-s}}} = \|\mathbf{v}\|_{\mathbf{V}^s}. \end{aligned}$$

The first equality holds by definition since $\mathbf{H}_0^{-s}(\Omega) = \mathbf{H}^{-s}(\Omega)$ for all $s \in (-\frac{1}{2}, 0]$. The second holds because $P_{\mathbf{H}}$ is a projection and $\mathbf{v} \in \mathbf{V}^0$. The first inequality holds because $P_{\mathbf{H}} : \mathbf{H}^{-s}(\Omega) \to \mathbf{H}^{-s}(\Omega)$ is bounded for all $s \in (-\frac{1}{2}, 0]$, see Lemma 2.2. The second inequality holds because $P_{\mathbf{H}}(\mathbf{H}^{-s}(\Omega))$ is a subset of $\mathbf{V}^{-s}$, see Lemma 2.2 again. The last inequality holds because of the norm equivalence which has already been proved. The last equality is a definition. Conclude by density. □

3 A density result and a counterexample

Let us now show that (1.3) is false for $-1 \leq s < -\frac{1}{2}$. To this end, let $s \in [-1, -\frac{1}{2})$ and define the closed subspace of $H^{1-s}(\Omega)/\mathbb{R}$

$$D_{\mathrm{ND}} := \left\{ p \in H^{1-s}(\Omega)/\mathbb{R} : \nabla p \in \mathbf{H}_0^{-s}(\Omega) \right\}.$$

We will also need the following subspaces of $\mathbf{H}^s(\Omega)$:

$$\mathbf{S} := \{\mathbf{u} \in \mathbf{H}^s(\Omega) : \nabla \cdot \mathbf{u} = 0\},$$

$$\mathbf{S}_0 := \{\mathbf{u} \in \mathbf{S} : (\mathbf{u}, \nabla p) = 0,\ \forall p \in D_{\mathrm{ND}}\}.$$

S can be seen as the space of solenoidal distributions of $\mathbf{H}^s(\Omega)$ and, if it were possible to define the normal trace of a distribution on $\partial\Omega$, $\mathbf{S}_0$ could be seen as the space of distributions such that for any connected component of $\partial\Omega$, say Γ, the following holds

$$\int_\Gamma \mathbf{u}\cdot\mathbf{n} = 0.$$

Let $\mathcal{V}$ be the subspace of $\mathcal{D}(\Omega)$ consisting of $\mathcal{C}^\infty$ solenoidal functions with compact support in Ω. To construct the counterexample, we need the following density result.

Theorem 3.1 *Let the domain Ω be star shaped with respect to a ball. Let $s \in [-1, -\frac{1}{2})$. Then $\mathcal{V}$ is dense in $\mathbf{S}_0$ in the norm of $\mathbf{H}^s(\Omega)$.*

Proof Let $\mathbf{f}$ be a continuous linear form in $\mathbf{S}_0$ such that $\langle \mathbf{f}, \mathbf{v}\rangle = 0$ for all $v \in \mathcal{V}$. If we show that this form is zero on $\mathbf{S}_0$, a well-known corollary of the Hahn-Banach theorem will allow us to conclude.

Since $\mathbf{S}_0$ is a subspace of $\mathbf{H}^s(\Omega)$, by the Hahn-Banach theorem the form $\mathbf{f}$ can be extended to $\mathbf{H}^s(\Omega)$. Denote by $\tilde{\mathbf{f}}$ one of the possible extensions. It is possible to identify $\tilde{\mathbf{f}}$ with a function in $\mathbf{H}_0^{-s}(\Omega)$. Since

$$\langle \tilde{\mathbf{f}}, \mathbf{v}\rangle = \langle \mathbf{f}, \mathbf{v}\rangle = 0, \quad \forall \mathbf{v} \in \mathcal{V},$$

De Rham's Theorem (cf. [4] or [15, Proposition 1.1]) allows us to conclude that $\tilde{\mathbf{f}}$ is the gradient of a distribution, that is

$$\exists p \in \mathcal{D}'(\Omega) : \tilde{\mathbf{f}} = \nabla p.$$

Given that $\tilde{\mathbf{f}} \in \mathbf{H}_0^{-s}(\Omega)$, and the fact that the operator $\nabla : H^{1-s}(\Omega)/\mathbb{R} \to \mathbf{H}^{-s}(\Omega)$ is bounded, injective, and has closed range (see [8, Theorem 2.4], since Ω is star-shaped with respect to a ball) allows us to conclude that $p \in H^{1-s}(\Omega)/\mathbb{R}$ and therefore $p \in D_{\mathrm{ND}}$.

Given $\mathbf{u} \in \mathbf{S}_0$, we have

$$\langle \mathbf{f}, \mathbf{u}\rangle = \langle \tilde{\mathbf{f}}, \mathbf{u}\rangle = \langle \mathbf{u}, \nabla p\rangle = 0,$$

by the definition of D_{ND}. This shows that $\mathbf{f}$ is zero everywhere in $\mathbf{S}_0$. □

Remark 3.1 This density result is a generalization of [7, Theorem 3.1], where it is proved in the case $s=-1$ only.

From the above density result we are able to prove that the inequality (1.3) is false for any $s \in [-1, -\frac{1}{2})$ as stated in the following:

Theorem 3.2 *Let the domain Ω be star shaped with respect to a ball. Let $s \in [-1, -\frac{1}{2})$. Then, there is a sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}} \subset \mathbf{V}^s$ and a constant c such that*

$$\|\mathbf{u}_k\|^2_{\mathbf{H}^s} \geq ck(\mathbf{u}_k, A^s\mathbf{u}_k), \quad \forall k \in \mathbb{N}.$$

Proof To fix the ideas, let us assume that $\partial\Omega$ has $m+1$ connected components Γ_i, $i=0,\dots,m$ with Γ_0 being the boundary of the unbounded component of $\Omega^{\complement} := \mathbb{R}^d \setminus \bar{\Omega}$. Notice that if $q \in D_{\mathrm{ND}}$, then $\partial q/\partial n|_{\partial\Omega} = 0$ and $q|_{\Gamma_i} = a_i$ with $a_i \in \mathbb{R}$ arbitrary.

For any $i \in \{0,\dots,m\}$ we consider an arbitrary nonzero function g_i in $L^2(\Gamma_i)$ satisfying the constraint $\int_{\Gamma_i} g_i = 0$. Let $\Phi \in H^1(\Omega)/\mathbb{R}$ solve

$$\begin{cases} \Delta\Phi = 0, \\ \frac{\partial\Phi}{\partial n}|_{\Gamma_i} = g_i. \end{cases}$$

It is well known that this problem has a unique solution. By construction, $\nabla\Phi$ is in $\mathbf{S}$ and we have

$$\int_\Omega \nabla\Phi\nabla q = \int_{\partial\Omega} \frac{\partial\Phi}{\partial n} q = \sum_{i=0}^m \int_{\Gamma_i} q g_i, \quad \forall q \in H^1(\Omega).$$

In particular, if $q \in D_{\mathrm{ND}}$ we have

$$(\nabla\Phi, \nabla q) = 0, \quad \forall q \in D_{\mathrm{ND}}.$$

Set $\mathbf{u} = \nabla\Phi$. Note that $\mathbf{u}$ is nonzero since Φ cannot be constant. The reasoning above implies $\mathbf{u} \in \mathbf{S}_0$. By Theorem 3.1, there exists a sequence $\{\mathbf{u}_k\}_{k\in\mathbb{N}}$ in $\boldsymbol{\mathcal{V}} \subset \mathbf{V}^s$ such that $\mathbf{u}_k \to \mathbf{u}$ in $\mathbf{H}^s(\Omega)$ as $k \to +\infty$. In particular, this sequence can be chosen so that

$$\|\mathbf{u} - \mathbf{u}_k\|_{\mathbf{H}^s} \leq \frac{1}{k}\|\mathbf{u}\|_{\mathbf{H}^s}.$$

Set $\mathbf{v}_k = A^{-1}\mathbf{u}_k$. This means that $\mathbf{v}_k$ solves the Stokes problem

$$\begin{cases} -\Delta\mathbf{v}_k + \nabla p_k = \mathbf{u}_k, \\ \nabla\cdot\mathbf{v}_k = 0, \quad \mathbf{v}_k|_{\partial\Omega} = 0. \end{cases}$$

In particular,

$$\begin{cases} -\Delta\mathbf{v}_k + \nabla(p_k - \Phi) = \mathbf{u}_k - \mathbf{u}, \\ \nabla\cdot\mathbf{v}_k = 0, \quad \mathbf{v}_k|_{\partial\Omega} = 0. \end{cases}$$

A simple energy argument implies that

$$\|\mathbf{v}_k\|_{\mathbf{H}^1} \leq \|\mathbf{u}_k - \mathbf{u}\|_{\mathbf{H}^{-1}}.$$

Moreover, given the smoothness of the boundary of the domain (cf. (2.1))

$$\|\mathbf{v}_k\|_{\mathbf{H}^2} \leq c\|\mathbf{u}_k - \mathbf{u}\|_{\mathbf{H}^0}.$$

Therefore, by interpolation,

$$\|\mathbf{v}_k\|_{\mathbf{H}^{2+s}} \leq c\|\mathbf{u}_k - \mathbf{u}\|_{\mathbf{H}^s}.$$

By definition,

$$(\mathbf{u}_k, A^s\mathbf{u}_k) = (A\mathbf{v}_k, A^{1+s}\mathbf{v}_k) = (A^{2+s}\mathbf{v}_k, \mathbf{v}_k) = \|\mathbf{v}_k\|^2_{\mathbf{V}^{2+s}}.$$

We have shown the equivalence of norms for $s + 2 > -1/2$, therefore

$$(\mathbf{u}_k, A^s\mathbf{u}_k) \leq c\|\mathbf{v}_k\|^2_{\mathbf{H}^{2+s}} \leq c\|\mathbf{u}_k - \mathbf{u}\|^2_{\mathbf{H}^s} \leq \frac{c}{k}\|\mathbf{u}_k\|^2_{\mathbf{H}^s},$$

which proves the result. □

Remark 3.2 This is a generalization of [7, Theorem 4.1], where this result is proved for $s = -1$. Notice that both in the density result (Theorem 3.1) and the proof of Theorem 3.2, it is essential to have $s < -1/2$ so that we can properly define, e.g. the normal derivative of functions in D_{ND}.

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