ERROR ANALYSIS OF A FRACTIONAL TIME-STEPPING TECHNIQUE FOR INCOMPRESSIBLE FLOWS WITH VARIABLE DENSITY*

J.-L. GUERMOND† AND ABNER J. SALGADO‡

Abstract. In this paper we analyze the convergence properties of a new fractional time-stepping technique for the solution of the variable density incompressible Navier–Stokes equations. The main feature of this method is that, contrary to other existing algorithms, the pressure is determined by just solving one Poisson equation per time step. First-order error estimates are proved, and stability of a formally second-order variant of the method is established.

1. Introduction. A fractional time-stepping technique for solving incompressible viscous flows with variable density is introduced and analyzed in this paper. The fluid flows in question are governed by the time-dependent Navier–Stokes equations

$$
(1.1) \qquad
\begin{cases}
\rho_t + \nabla \cdot (\rho u) = 0, \\
\rho \left(u_t + u \cdot \nabla u \right) + \nabla p - \mu \Delta u = f, \\
\nabla \cdot u = 0,
\end{cases}
$$

where the independent variables are the density $\rho > 0$, the velocity field u, and the pressure p. The constant $\mu > 0$ is the dynamic viscosity coefficient, and f is a smooth external force. The fluid occupies a bounded domain Ω in $\mathbb{R}^d$ (with $d = 2$ or 3), and a solution to the above problem is considered over the time interval $[0, T]$. The system (1.1) is supplemented with the following initial and boundary conditions for the density and velocity:

$$
(1.2) \qquad
\begin{cases}
\rho(x,0) = \rho_0(x), & \rho(x,t)|_{\Gamma^-} = a(x,t), \\
u(x,0) = u_0(x), & u(x,t)|_{\Gamma} = b(x,t),
\end{cases}
$$

where $\Gamma = \partial\Omega$ and Γ^- is the inflow boundary, defined by $\Gamma^- = \{x \in \Gamma : u \cdot n < 0\}$, where n is the outward unit normal vector. Throughout this paper we assume that $b = 0$; this means that the boundary is impermeable, i.e., $\Gamma^- = \emptyset$.

Approximating (1.1) and (1.2) can be done by solving the coupled system (1.1), but this approach may sometimes be computer intensive due to the saddle point structure of the problem. We refer the reader to [27], where such a strategy is developed.

Alternative and more efficient approaches usually advocated in the literature consist of using factional time-stepping strategies and exploiting, as far as possible, the techniques and results already established for the solution of constant density incompressible fluid flows. The starting point of most fractional time-stepping algorithms is Chorin and Temam's [7, 32] projection method, which consists of decoupling the incompressibility constraint and the diffusion. Several algorithms extending this idea to variable density fluid flows have been proposed in the literature; see, for example, [1, 4, 19, 29]. However, to the best of our knowledge, [19, 29] are the only papers where projection methods for variable density flows have been proven to be stable, and no rigorous error analysis of these methods has been done yet. Moreover, a common feature of all the projection-like methods referred to above is that at each time step, say t^{k+1}, the pressure or some related scalar unknown, say Φ, must be computed by solving an equation of the following form:

$$-\nabla \cdot \left(\frac{1}{\rho^{k+1}} \nabla \Phi \right) = \Psi, \quad \partial_n \Phi|_\Gamma = 0,$$

where ρ^{k+1} is an approximation of the density at time t^{k+1} and Ψ is a right-hand side that varies at each time step. Solving this problem efficiently is far more technical than just solving a Poisson equation, as it requires assembling and preconditioning a variable-coefficient stiffness matrix at each time step. Note also in passing that it is necessary to have a uniform lower bound on the density for this problem to be solvable. This condition is often overlooked in the literature.

On the basis of the observations above, we have recently started a research program whose objective is to develop a fractional time-stepping strategy that requires only the solution of a Poisson problem to compute the pressure [21, 20]. An adaption of [21, 20] to a phase-field model has been proposed in [31]. The stability of a first-order variant of the method was proved in [21, 20, 31], but no error analysis was proved therein, and the question of whether second-order variants of the proposed method could be proved to be stable was still an open question. The goal of the present paper is to fill these two gaps. First, we provide a rigorous error analysis for the first-order method introduced in [21, 20]. We prove that, provided the density equation is solved correctly, the accuracy of our fractional time-stepping technique is as good as the corresponding schemes for constant density flows. Second, we introduce a second-order version of the method and prove that it is stable.

The paper is organized as follows. Notation, along with space and time discretizations, is introduced in section 2. The first-order algorithm is described in section 3. The error analysis of this algorithm is done is section 4. We show that, provided the density equation is correctly approximated, the method performs as well as its constant density counterpart. A formally second-order version of the algorithm is introduced in section 5, and the stability of the method is proved. As a by-product of our analysis we are able to provide a new proof of stability of the second-order incremental pressure-correction scheme in standard form (see [15]). The novelty of our analysis is that we eliminate the so-called projected velocity from the algorithm. Numerical experiments illustrating the performance of the method are reported in section 6.

2. Preliminaries.

2.1. Notation and assumptions. We consider the time-dependent variable density Navier–Stokes system (1.1)–(1.2) on the finite time interval $[0, T]$ and in an

open, connected, and bounded domain $\Omega \subset \mathbb{R}^d$ ($d = 2$ or 3) with boundary Γ, which we assume to be sufficiently smooth. More precisely, we assume that Ω is such that the Stokes operator possesses the usual regularization properties (see [6, 33]). Moreover, we assume that (1.1)–(1.2) has a unique smooth solution for all $T > 0$ and that all the compatibility conditions required for the solution to be smooth enough are satisfied (see (4.2) below for a more precise statement).

Let $\tau > 0$ be a time step, and let us set $t_k = k\tau$ for $0 \le k \le K := [T/\tau]$. Let E be a normed space equipped with the norm $\| \cdot \|_E$. For any time-dependent function $\phi : [0, T] \longrightarrow E$, we denote $\phi^k := \phi(t^k)$, and the sequence $\phi^0, \phi^1, \ldots, \phi^K$ is denoted by ϕ_τ. To simplify the notation we define the time-increment operator δ by setting

$$(2.1) \qquad \delta\phi^k = \phi^k - \phi^{k-1},$$

and we define the following discrete norms:

$$(2.2) \qquad \|\phi_\tau\|_{\ell^2(E)} := \left(\tau \sum_{k=0}^{K} \|\phi^k\|_E^2 \right)^{1/2}, \qquad \|\phi_\tau\|_{\ell^\infty(E)} := \max_{0 \le k \le K} \left(\|\phi^k\|_E \right).$$

The space of functions $\phi : [0, T] \longrightarrow E$ such that the map $(0, T) \ni t \longrightarrow \|\phi(t)\|_E \in \mathbb{R}$ is L^p-integrable is indifferently denoted by $L^p((0, T); E)$ or $L^p(E)$.

No notational distinction is made between scalar- or vector-valued functions, but spaces of vector-valued functions are identified with bold fonts. The space of functions in $L^2(\Omega)$ that have zero average is denoted by $L_0^2(\Omega)$. We use the standard Sobolev spaces $W^{m,p}(\Omega)$, for $0 \le m \le \infty$ and $1 \le p \le \infty$. The closure with respect to the norm $\| \cdot \|_{W^{m,p}}$ of the space of $\mathcal{C}^\infty$-functions compactly supported in Ω is denoted by $W_0^{m,p}(\Omega)$. To simplify the notation, the Hilbert space $W^{s,2}(\Omega)$ (resp., $W_0^{s,2}(\Omega)$) is denoted by $H^s(\Omega)$ (resp., $H_0^s(\Omega)$). The scalar product of $L^2(\Omega) := H^0(\Omega)$ is denoted by $\langle \cdot, \cdot \rangle$.

Henceforth c denotes a generic constant whose value may change at each occurrence. This constant may depend on the data of the problem and its exact solution, but it does not depend on the discretization parameters or the solution of the numerical scheme.

2.2. The space discretization. To construct a Galerkin approximation of (1.1)–(1.2) we consider three sequences of finite-dimensional spaces $\{W_h\}_{h>0}$, $\{\mathbf{X}_h\}_{h>0}$, $\{M_h\}_{h>0}$, for $h > 0$, with $W_h \subset H^1(\Omega)$, $\mathbf{X}_h \subset \mathbf{H}_0^1(\Omega)$, and $M_h \subset H^1(\Omega) \cap L_0^2(\Omega)$. We use W_h, $\mathbf{X}_h$, and M_h to approximate the density, the velocity, and the pressure, respectively. We assume that the pair of spaces $(\mathbf{X}_h, M_h)$ satisfies a discrete inf-sup condition (cf. [10, 9]); i.e., there is $c > 0$ independent of h such that

$$\inf_{0 \ne q_h \in M_h} \sup_{0 \ne v_h \in \mathbf{X}_h} \frac{\int_\Omega v_h \cdot \nabla q_h}{\|q_h\|_{L^2} \|v_h\|_{\mathbf{H}^1}} \ge c.$$

Moreover, we assume that the following approximation properties hold (cf. [10, 9]): There is $l \in \mathbb{N}^*$ such that for all $\ell \in [0, l]$ the density space satisfies

$$(2.3) \qquad \inf_{r_h \in W_h} \|r - r_h\|_{L^2} \le ch^{\ell+1} \|r\|_{H^{\ell+1}} \quad \forall r \in H^{\ell+1}(\Omega),$$

and the velocity-pressure spaces are such that for all $\ell \in [0, l]$

$$(2.4) \quad \inf_{v_h \in \mathbf{X}_h} \{\|v - v_h\|_{\mathbf{L}^2} + h\|v - v_h\|_{\mathbf{H}^1}\} \le ch^{\ell+1} \|v\|_{\mathbf{H}^{\ell+1}} \quad \forall v \in \mathbf{H}^{\ell+1}(\Omega) \cap \mathbf{H}_0^1(\Omega),$$

$$(2.5) \qquad \inf_{q_h \in M_h} \|q - q_h\|_{L^2} \le ch^\ell \|q\|_{H^\ell} \qquad \forall q \in H^\ell(\Omega) \cap L_0^2(\Omega).$$

For any t in $[0, T]$ we define the Stokes projection of the solution $(u(t), p(t))$ to (1.1)–(1.2) as the pair $(w_h(t), q_h(t)) \in \mathbf{X}_h \times M_h$ that solves

$$(2.6) \quad \begin{cases} \mu \langle \nabla w_h(t), \nabla v_h \rangle + \langle \nabla q_h(t), v_h \rangle = \mu \langle \nabla u(t), \nabla v_h \rangle - \langle p(t), \nabla \cdot v_h \rangle & \forall v_h \in \mathbf{X}_h, \\ \langle w_h(t), \nabla r_h \rangle = 0 & \forall r_h \in M_h. \end{cases}$$

Owing to the regularization properties of the Stokes operator, the following estimates hold.

LEMMA 2.1. *If* $u \in L^\beta \left(\mathbf{H}^{l+1}(\Omega) \cap \mathbf{H}_0^1(\Omega) \right)$ *and* $p \in L^\beta \left(H^l(\Omega) \right)$ *for* $1 \le \beta \le \infty$, *then there exists* $c > 0$ *such that*

$$(2.7) \quad \|u - w_h\|_{L^\beta(\mathbf{L}^2)} + h \left[\|u - w_h\|_{L^\beta(\mathbf{H}^1)} + \|p - q_h\|_{L^\beta(L^2)} \right]$$
$$\le c h^{l+1} \left[\|u\|_{L^\beta(\mathbf{H}^{l+1})} + \|p\|_{L^\beta(H^l)} \right].$$

Moreover, if $u \in L^\beta \left(\mathbf{H}^2(\Omega) \cap \mathbf{H}_0^1(\Omega) \right)$ *and* $p \in L^\beta \left(H^1(\Omega) \right)$, *then*

$$(2.8) \quad \|w_h\|_{L^\beta(\mathbf{L}^\infty \cap \mathbf{W}^{1,3})} + \|q_h\|_{L^\beta(H^1)} \le c \left[\|u\|_{L^\beta(\mathbf{H}^2)} + \|p\|_{L^\beta(H^1)} \right].$$

3. Description of the first-order scheme. We now describe the first-order fractional time-stepping scheme as introduced in [21, 20]. We refer the reader to [21, 20] for the heuristics behind the algorithm.

3.1. Initialization. Given the initial data (ρ_0, u_0), we construct the approximate data $(\rho_h^0, u_h^0, p_h^0) \in W_h \times \mathbf{X}_h \times M_h$, and we assume that u_h^0 satisfies the following estimates:

$$(3.1) \quad \|u_0 - u_h^0\|_{\mathbf{L}^2} + h\|u_0 - u_h^0\|_{\mathbf{H}^1} \le c h^{l+1}.$$

We henceforth assume that $\min_{x \in \bar{\Omega}} \rho_0(x) > 0$ and that the approximate density field ρ_h^0 satisfies the following property:

$$(3.2) \quad \chi \le \rho_h^0 \le \varrho,$$

where the parameters χ and ϱ are assumed to satisfy the following property:

$$(3.3) \quad \chi \le \min_{x \in \bar{\Omega}} \rho_0(x), \qquad \sup_{x \in \bar{\Omega}} \rho_0(x) \le \varrho.$$

The role of the parameters χ and ϱ is clarified in the next subsection. We shall be more specific later about the properties that we expect from ρ_h^0 and p_h^0.

Remark 3.1. Note that $p_0 := p|_{t=0}$ is not part of the initial data but that this quantity can be computed by solving
$$(3.4)$$
$$\nabla \cdot (\rho_0^{-1} \nabla p_0) = \nabla \cdot (\rho_0^{-1}(f_0 + \mu \Delta u_0) - u_0 \cdot \nabla u_0), \quad \partial_n p_0|_\Gamma = (f_0 + \mu \Delta u_0 - \rho_0 u_0 \cdot \nabla u_0) \cdot n,$$

where we have set $f_0 := f(t{=}0)$. This then requires the initial data to satisfy the following compatibility condition at the boundary $(\rho_0 u_0 \cdot \nabla u_0 - \mu \Delta u_0 + \nabla p_0 - f_0)|_\Gamma = 0$, which we assume to hold. This condition holds, for instance, if $u_0 = 0$ and $f_0 = 0$; i.e., the fluid is a rest at $t = 0$ and the source term is zero at $t = 0$. If the above compatibility condition is not satisfied, the error analysis must be adapted to account for weighted error estimates by proceeding as in [24, 28].

3.2. Time-stepping technique. Given $(\rho_h^k, u_h^k, p_h^k) \in W_h \times \mathbf{X}_h \times M_h$, we now describe how to obtain the next approximations $(\rho_h^{k+1}, u_h^{k+1}, p_h^{k+1}) \in W_h \times \mathbf{X}_h \times M_h$. The algorithm proceeds in three steps: (i) density update, (ii) velocity update, (iii) pressure update.

3.2.1. Density update. The density update is computed using the mass conservation equation, which we recall is hyperbolic. It is well known that Galerkin techniques are not well suited for the solution of hyperbolic problems (see, for instance, [9]). The list of techniques aiming at addressing this issue is endless; among these methods one can cite Galerkin least squares [25], discontinuous Galerkin [25, 35], subgrid viscosity [14], method of characteristics [8], edge stabilization [5], entropy viscosity [16, 17], and many others. We assume that the sequence of approximate densities $\{\rho_h^k\}_{k=0,\dots,K} \subset W_h$ is obtained by one of these stabilization techniques. More precisely, we assume that, given the pair $(\rho_h^k, u_h^k) \in W_h \times \mathbf{X}_h$, the approximation technique that is used to approximate the mass conservation returns ρ_h^{k+1} and that this algorithm satisfies the following stability hypothesis:

$$(3.5) \qquad \chi \le \min_{x \in \bar{\Omega}} \rho_h^{k+1}(x), \qquad \sup_{x \in \bar{\Omega}} \rho_h^{k+1}(x) \le \varrho \quad \forall k \ge 1.$$

Note that this is a natural assumption since, owing to the incompressibility of the velocity field, the density field ρ satisfies properties: $\rho(t) \in [\min_{x \in \bar{\Omega}} \rho_0(x), \sup_{x \in \bar{\Omega}} \rho_0(x)]$ for all $t \ge 0$; cf. Lions [26]. For instance, first-order monotone schemes satisfy (3.5) with $\chi = \min_{x \in \bar{\Omega}} \rho_0(x)$ and $\varrho = \sup_{x \in \bar{\Omega}} \rho_0(x)$.

3.2.2. Velocity update. Having obtained an approximate density, we define

$$(3.6) \qquad \rho_h^\star := \frac{1}{2} \left(\rho_h^{k+1} + \rho_h^k \right),$$

$$(3.7) \qquad p_h^\sharp := p_h^k + \gamma \delta p_h^k, \quad \gamma \in \{0, 1\}.$$

The parameter γ is user-dependent. We say that the method is nonincremental if $\gamma = 0$ and incremental if $\gamma = 1$. The incremental version of the algorithm is more accurate than the nonincremental one. We mention the nonincremental version of the algorithm just for historical reasons: the original algorithm of Chorin and Temam for constant density incompressible flows is nonincremental [7, 32]. When $\gamma = 1$, we define $\delta p_h^0 := 0$.

The next approximation of the velocity field $u_h^{k+1} \in \mathbf{X}_h$ is computed by solving the following problem:

$$(3.8) \qquad \left\langle \frac{\rho_h^\star u_h^{k+1} - \rho_h^k u_h^k}{\tau}, v_h \right\rangle + \left\langle \rho_h^{k+1} u_h^k \cdot \nabla u_h^{k+1}, v_h \right\rangle$$

$$+ \left\langle \frac{1}{2} \nabla \cdot \left(\rho_h^{k+1} u_h^k \right) u_h^{k+1}, v_h \right\rangle + \mu \left\langle \nabla u_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla p_h^\sharp, v_h \right\rangle = \left\langle f^{k+1}, v_h \right\rangle \quad \forall v_h \in \mathbf{X}_h.$$

3.2.3. Pressure update. Finally, we define the pressure approximation $p_h^{k+1} \in M_h$. First, we let $\phi_h^\flat \in M_h$ be the solution of

$$(3.9) \qquad \left\langle \nabla \phi_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau} \left\langle u_h^{k+1}, \nabla r_h \right\rangle \quad \forall r_h \in M_h;$$

then we set

$$(3.10) \qquad p_h^{k+1} = \phi_h^\flat + \gamma p_h^k, \quad \gamma \in \{0, 1\}.$$

Remark 3.2. The algorithm (3.6)–(3.10) unifies the nonincremental ($\gamma = 0$) and incremental ($\gamma = 1$) first-order schemes of [21, 20]. The nonincremental version presented here is a slight simplification of the one presented in the references above.

Remark 3.3. One remarkable feature of the algorithm (3.6)–(3.10) is that, apart from condition (3.5), nothing else is required on the sequence of approximate densities to prove stability.

Remark 3.4. Let us introduce the auxiliary space $\mathbf{Y}_h := \mathbf{X}_h + \nabla M_h$. In view of (3.9), the quantity

$$\bar{u}_h^k := u_h^k - \frac{\tau}{\chi} \nabla \phi_h^\flat \in \mathbf{Y}_h$$

is discretely divergence-free (in the sense that $\langle \bar{u}_h^k, \nabla r_h \rangle = 0$ for all $r_h \in M_h$) and could be used as a solenoidal approximation of the velocity. This particular choice of $\mathbf{Y}_h$ fits into the commutative diagram framework described in [11, 18]. Therefore, it could be possible to develop a much more general theory about fractional time-stepping techniques for variable density incompressible flows that would include our method as a particular instance. More specifically, let us assume that one has at hand a space $\mathbf{Y}_h$ so that $\mathbf{X}_h \subset \mathbf{Y}_h$. Let $B_h : \mathbf{X}_h \longrightarrow M_h$ be the operator defined by $\langle Bv_h, q_h \rangle := \langle \nabla \cdot v_h, q_h \rangle$ for all $v_h \in \mathbf{X}_h$ and for all $q_h \in M_h$. Assume that one can construct an extension of B_h over $\mathbf{Y}_h$, say $C_h : \mathbf{Y}_h \longrightarrow M_h$. The operator C_h being an extension of B_h over $\mathbf{Y}_h$ means that $B_h = C_h i_h$, where i_h is the natural injection $i_h : \mathbf{X}_h \longrightarrow \mathbf{Y}_h$. Then, in this setting, our theory will work by replacing (3.9) by

$$(3.11) \qquad C_h C_h^T \phi_h^\flat = \frac{\chi}{\tau} B_h u_h^{k+1}.$$

We leave the details to the reader.

Remark 3.5. Note that (3.8) can also be rewritten as

$$\left\langle \rho_h^k \frac{u_h^{k+1} - u_h^k}{\tau}, v_h \right\rangle + \left\langle \rho_h^{k+1} u_h^k \cdot \nabla u_h^{k+1}, v_h \right\rangle$$

$$+ \frac{1}{2} \left\langle \left(\frac{\rho_h^{k+1} - \rho_h^k}{\tau} + \nabla \cdot \left(\rho_h^{k+1} u_h^k \right) \right) u_h^{k+1}, v_h \right\rangle$$

$$+ \mu \left\langle \nabla u_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla p_h^\sharp, v_h \right\rangle = \left\langle f^{k+1}, v_h \right\rangle \quad \forall v_h \in \mathbf{X}_h.$$

This form of the momentum equation emphasizes the stabilizing term, which is proportional to the mass conservation equation. This type of stabilization is also used in [23, section 2].

4. Error estimates for the first-order scheme. We prove in this section that the algorithm (3.6)–(3.10) is stable and convergent provided the mass conservation equation is approximated appropriately.

4.1. Consistency analysis. To simplify the notation, we introduce the following functions to represent the errors:

$$(4.1) \qquad \begin{cases} \eta(t) := u(t) - w_h(t), & \mu(t) := p(t) - q_h(t), \\ e_h^k := w_h^k - u_h^k, & \epsilon_h^k := q_h^k - p_h^k. \end{cases}$$

The functions $\eta(t)$ and $\mu(t)$ can be regarded as the interpolation errors, whereas the functions e_h^k and ϵ_h^k represent the approximation errors. In addition to (3.1), we make the following regularity assumptions on the exact solution of problem (1.1):

$$(4.2) \qquad u \in W^{2,\infty}\left(\mathbf{H}_0^1(\Omega) \cap \mathbf{H}^{l+1}(\Omega)\right), \quad p \in W^{1,\infty}\left(H^l(\Omega)\right),$$

where we have already assumed that $l \geq 1$ in (2.3)–(2.5).

Remark 4.1. We recall that the regularity assumption (4.2) requires several compatibility conditions on the data and that the existence of a strong solution in three space dimensions is still an open problem. We refer the reader to [24, 26, 28] for more details on this issue.

Let us now determine the equations that control the errors. By taking the difference between the first equation of (2.6) and (3.8), we obtain the equation that controls e_h^k:

$$(4.3) \qquad \left\langle \frac{\rho_h^\star e_h^{k+1} - \rho_h^k e_h^k}{\tau}, v_h \right\rangle + \mu \left\langle \nabla e_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla\left(q_h^{k+1} - p_h^\sharp\right), v_h \right\rangle$$
$$= \mathcal{R}^{k+1}(v_h) \quad \forall v_h \in \mathbf{X}_h,$$

where the residual $\mathcal{R}^{k+1}(v_h)$ is decomposed as follows:

$$(4.4) \qquad \mathcal{R}^{k+1}(v_h) = R_0^{k+1}(v_h) + R_1^{k+1}(v_h) + R_{nl}^{k+1}(v_h),$$

and

$$(4.5) \qquad R_0^{k+1}(v_h) := \left\langle \rho_h^k \frac{w_h^{k+1} - w_h^k}{\tau} - \rho^{k+1} u_t^{k+1}, v_h \right\rangle,$$

$$(4.6) \qquad R_1^{k+1}(v_h) := \frac{1}{2}\left\langle \frac{\rho_h^{k+1} - \rho_h^k}{\tau} w_h^{k+1} - \rho_t^{k+1} u^{k+1}, v_h \right\rangle,$$

$$(4.7) \qquad R_{nl}^{k+1}(v_h) := \left\langle \rho_h^{k+1} u_h^k \cdot \nabla u_h^{k+1} - \rho^{k+1} u^{k+1} \cdot \nabla u^{k+1}, v_h \right\rangle$$

$$(4.8) \qquad + \frac{1}{2}\left\langle \nabla\cdot(\rho_h^{k+1} u_h^k) u_h^{k+1} - \nabla\cdot(\rho^{k+1} u^{k+1}) u^{k+1}, v_h \right\rangle.$$

To obtain the equation that controls the quantity ϵ_h^k, we use (3.9) along with the property that $\langle w_h, \nabla r_h \rangle = 0$ for all $r_h \in M_h$,

$$(4.9) \qquad \left\langle \nabla \epsilon_h^\flat, \nabla r_h \right\rangle = \frac{\chi}{\tau}\left\langle e_h^{k+1}, \nabla r_h \right\rangle + \left\langle \nabla q_h^\flat, \nabla r_h \right\rangle,$$

where for any sequence ψ_τ we henceforth denote

$$(4.10) \qquad \psi^\flat = \psi^{k+1} - \gamma \psi^k \qquad \text{and} \qquad \psi^\sharp = \psi^k + \gamma \delta \psi^k.$$

Equations (4.3)–(4.9) will be used repeatedly in the error analysis.

The error analysis is based on energy arguments, and the first of these arguments consists of testing (4.3) with $v_h := 2\tau e_h^{k+1}$. Then, upon observing that $2e_h^{k+1}(\rho_h^\star e_h^{k+1} - \rho_h^k e_h^k) = \rho_h^{k+1}(e_h^{k+1})^2 + \rho_h^k(\delta e_h^{k+1})^2 - \rho_h^k(e_h^k)^2$ (see, e.g., [21, Theorem 3.1]), testing (4.3) with $v_h := 2\tau e_h^{k+1}$ gives

$$(4.11) \quad \|\sigma_h^{k+1} e_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^k \delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\mu\tau \left\|e_h^{k+1}\right\|_{\mathbf{H}^1}^2$$
$$+ 2\tau \left\langle \nabla \epsilon_h^\sharp, e_h^{k+1} \right\rangle = 2\tau \left\langle \nabla(q_h^\sharp - q_h^{k+1}), e_h^{k+1} \right\rangle + 2\tau \mathcal{R}^{k+1}(e_h^{k+1}),$$

where we have introduced the notation $\sigma_h := \sqrt{\rho_h}$.

We finish this section by giving an estimate on the consistency residual $2\tau\mathcal{R}^{k+1}(e_h^{k+1})$; the following lemma provides this estimate.

LEMMA 4.1. *Assume that the solution to* (1.1)–(1.2) *satisfies* (4.2) *and that the sequence of approximate densities* $\{\rho_h^k\}$ *satisfies* (3.5). *Then*

$$(4.12) \quad |\mathcal{R}^{k+1}(e_h^{k+1})| \le c\left[\tau + h^{l+1} + \|\rho_h^k - \rho^k\|_{L^2} + \left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{H^{-1}}\right]^2$$
$$+ \frac{1}{4}\mu\|e_h^{k+1}\|_{\mathbf{H}^1}^2 + c\|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2.$$

Proof. We estimate separately each of the terms that composes $\mathcal{R}^{k+1}(e_h^{k+1})$. For the first term,

$$R_0^{k+1}(e_h^{k+1}) = \left\langle \rho_h^k \frac{1}{\tau}\delta w_h^{k+1} - \rho^{k+1}u_t^{k+1}, e_h^{k+1}\right\rangle$$
$$= \left\langle \rho_h^k\left(\frac{1}{\tau}\delta w_h^{k+1} - u_t^{k+1}\right), e_h^{k+1}\right\rangle$$
$$+ \left\langle (\rho_h^k - \rho^k)u_t^{k+1}, e_h^{k+1}\right\rangle - \left\langle \delta\rho^{k+1}u_t^{k+1}, e_h^{k+1}\right\rangle$$
$$\le c\|e_h^{k+1}\|_{\mathbf{L}^6}\left(\|\rho_h^k\|_{L^\infty}\left\|\frac{1}{\tau}\delta w_h^{k+1} - u_t^{k+1}\right\|_{\mathbf{L}^2} + (\|\rho_h^k - \rho^k\|_{L^2}\right.$$
$$\left. + \|\delta\rho^{k+1}\|_{L^2})\|u_t^{k+1}\|_{\mathbf{L}^3}\right)$$
$$\le c\|e_h^{k+1}\|_{\mathbf{H}^1}\left(\tau + h^{l+1} + \|\rho_h^k - \rho^k\|_{L^2}\right),$$

where we used (2.7), (3.5), and (4.2) to derive the last inequality.

We proceed similarly for the second term:

$$R_1^{k+1}(e_h^{k+1}) = \frac{1}{2}\left\langle \frac{1}{\tau}\delta\rho_h^{k+1}w_h^{k+1} - \rho_t^{k+1}u^{k+1}, e_h^{k+1}\right\rangle$$
$$= \frac{1}{2}\left\langle\left(\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right)w_h^{k+1}, e_h^{k+1}\right\rangle + \frac{1}{2}\left\langle \rho_t^{k+1}(w_h^{k+1} - u^{k+1}), e_h^{k+1}\right\rangle$$
$$\le c\left(\left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{H^{-1}}\|w_h^{k+1}\cdot e_h^{k+1}\|_{\mathbf{H}_0^1}\right.$$
$$\left. + \|e_h^{k+1}\|_{\mathbf{L}^6}\|\rho_t^{k+1}\|_{L^3}\|w_h^{k+1} - u^{k+1}\|_{\mathbf{L}^2}\right)$$
$$\le c\|e_h^{k+1}\|_{\mathbf{H}^1}\left(h^{l+1} + \left\|\frac{1}{\tau}\delta\rho_h^{k+1} - \rho_t^{k+1}\right\|_{H^{-1}}\right),$$

where we used (2.7), (2.8), and (4.2) to derive the last inequality.

The derivation of an estimate for the nonlinear advection component of the residual is done by repeating an argument from [12, 13]; we slightly modify the argument though to account for the fact that the density is not constant. We begin by noticing that, for functions that are smooth enough for the integrals to make sense, the following identity holds:

$$\langle \rho u \cdot \nabla v, v\rangle + \frac{1}{2}\langle \nabla \cdot (\rho u)v, v\rangle = 0.$$

Then, using the above identity with $v = e_h$, we rewrite the term $R_{nl}^{k+1}(e_h^{k+1})$ as follows:

$$
\begin{aligned}
R_{nl}^{k+1}(e_h^{k+1}) = & - \left\langle \rho_h^{k+1} e_h^k \cdot \nabla w_h^{k+1} + \tfrac{1}{2} \nabla \cdot (\rho_h^{k+1} e_h^k) w_h^{k+1}, e_h^{k+1} \right\rangle \\
& + \left\langle (\rho_h^{k+1} - \rho^{k+1}) w_h^k \cdot \nabla w_h^{k+1} + \tfrac{1}{2} \nabla \cdot ((\rho_h^{k+1} - \rho^{k+1}) w_h^k) w_h^{k+1}, e_h^{k+1} \right\rangle \\
& + \left\langle \rho^{k+1} \left(w_h^k \cdot \nabla w_h^{k+1} - u^{k+1} \cdot \nabla u^{k+1} \right) \right. \\
& + \tfrac{1}{2} \left(\nabla \cdot (\rho^{k+1} w_h^k) w_h^{k+1} - \nabla \cdot (\rho^{k+1} u^{k+1}) u^{k+1} \right), \left. e_h^{k+1} \right\rangle \\
:= & \; A_1 + A_2 + A_3,
\end{aligned}
$$

where A_1 is the first term in the right-hand side, A_2 is the second term, and A_3 is the sum of the third and fourth terms. Since the approximate density sequence $\{\rho_h^k\}$ satisfies (3.5) and the approximate velocity sequence $\{w_h^k\}$ satisfies (2.8), we infer

$$
A_1 \leq c \|\sigma_h^k e_h^k\|_{\mathbf{L}^2} \|e_h^{k+1}\|_{\mathbf{H}^1},
$$

where we estimated the second term after integrating it by parts, which is possible given the smoothness of w_h^{k+1} and e_h^{k+1}. Using (2.8) we obtain

$$
A_2 \leq c \|\rho_h^{k+1} - \rho^{k+1}\|_{L^2} \|e_h^{k+1}\|_{\mathbf{H}^1},
$$

where, again, we integrated the second term by parts. Finally, given the smoothness of ρ^{k+1}, an estimate of A_3 is obtained by proceeding as in the constant density case; see, e.g., [12, 13, 24]:

$$
A_3 \leq c(\tau + h^{l+1}) \|e_h^{k+1}\|_{\mathbf{H}^1}.
$$

The estimate (4.12) is obtained by combining the results above. □

4.2. Error estimates. As stated in Remark 3.3, the stability of the algorithm that we are analyzing depends only marginally on the method which is used to approximate the density; the only assumption we make to achieve stability is that the algorithm that solves the mass conservation equation satisfies (3.5). Of course (3.5) is not sufficient to obtain error estimates. Performing the full error analysis would require us to analyze the nonlinear coupling between the mass conservation equation and the momentum conservation equation. This would require us to be specific about the type of approximation which is used to compute the approximate density field and would probably lead to lengthy technicalities of little interest. We are not going to do the full convergence analysis to avoid technicalities and to remain as general as possible on the way the mass conservation equation is approximated. We assume instead that, in some way, we are capable of computing an approximate density sequence $\{\rho_h^k\} \subset W_h$ from the knowledge of the approximated velocity sequence $\{u_h^k\} \subset \mathbf{X}_h$. To be more specific, we assume that there is $m > 0$ such that the following holds:

$$
(4.13) \quad \|(\rho - \rho_h)_\tau\|_{\ell^2(0,t^k;L^2)}^2 + \left\| \left(\rho_t - \frac{\delta \rho_h}{\tau} \right)_\tau \right\|_{\ell^2(0,t^{k+1};H^{-1})}^2 \leq c(\lambda)(\tau + h^m)^2
$$
$$
+ \lambda \|e_{h\tau}\|_{\ell^2(0,t^{k+1};\mathbf{H}^1)}^2 + c(\lambda) \|\sigma_{h\tau} e_{h\tau}\|_{\ell^2(0,t^k;\mathbf{L}^2)}^2,
$$

where $\lambda \geq 0$ can be chosen as small as necessary.

Remark 4.2. We conjecture that (3.5) and (4.13) with $m = 1$ are both realizable by approximating the mass conservation with a linear first-order viscosity or a first-order phase-field approach [31]. We expect that it is possible to satisfy (4.13) with

$m > 1$ by using nonlinear stabilization techniques like discontinuous Galerkin with limiters, entropy viscosity [16, 17], etc.

Given this assumption, the residual term $\mathcal{R}(e_h^{k+1})$ simplifies as follows.

COROLLARY 4.1. *Assume that* (4.13) *holds. Then, the following estimate holds under the regularity assumptions of Lemma* 4.1:

(4.14)
$$2\tau \sum_{n=0}^{k} |\mathcal{R}^{n+1}(e_h^{n+1})| \le c(\tau+h^{\min\{l+1,m\}})^2+\frac{1}{2}\mu\|e_{h\tau}\|_{\ell^2(0,t^{k+1};\mathbf{H}^1)}^2+c\|\sigma_{h\tau}e_{h\tau}\|_{\ell^2(0,t^k;\mathbf{L}^2)}^2.$$

Proof. Use (4.12), where all the terms that involve differences of ρ_h and ρ are bounded in (4.13). The parameter λ is chosen so that $\lambda = \varepsilon\mu$, where ε is chosen small enough. ☐

We now consider the nonincremental and the incremental versions of the algorithm separately.

4.2.1. Nonincremental scheme. The nonincremental version of the method is obtained by setting $\gamma = 0$. We further assume that the algorithm is initialized so that p_h^0 satisfies the following estimate:

(4.15)
$$\|p_h^0\|_{L^2} \le c.$$

Under assumption (4.13), the main error estimate for this algorithm is the following.

THEOREM 4.1. *Assume that the solution to* (1.1)–(1.2) *satisfies* (4.2). *Let* $(u_h)_\tau$ *be the solution of* (3.6)–(3.9) *with* $\gamma = 0$, *and assume that* (3.1), (3.5), (4.13), *and* (4.15) *hold. Then*
(4.16)
$$\|u_\tau-u_{h\tau}\|_{\ell^\infty(\mathbf{L}^2)} \le c\left(h^{\min\{l+1,m\}} + \tau^{\frac{1}{2}}\right), \qquad \|u_\tau-u_{h\tau}\|_{\ell^2(\mathbf{H}^1)} \le c\left(h^{\min\{l,m\}} + \tau^{\frac{1}{2}}\right).$$

CONJECTURE 4.1. *We expect that further regularity assumptions combined with a standard duality argument, e.g., multiplying the error equation by* Se_h^{k+1}, *where* S *is the solution operator to the time-independent Stokes problem, should allow us to conclude that the following estimate holds in addition to* (4.16):

$$\|u_\tau - u_{h\tau}\|_{\ell^2(\mathbf{L}^2)} \le c\left(h^{\min\{l+1,m\}} + \tau\right).$$

The reader is referred to [22, 12, 13] *for more details.*

Remark 4.3. The error estimate (4.16) shows that, at least under assumption (4.13), the time accuracy of the nonincremental fractional time-stepping technique for variable density fluid flows is similar to that of the analogous nonincremental pressure-correction scheme for constant density flows (cf. [15]).

Proof of Theorem 4.1. In this case $p_h^\sharp = p_h^k$ and $\phi_h^\flat = p_h^{k+1}$. We proceed in two steps:

(i) *Initialization* ($k = 0$): We consider the initial step separately, as it involves the initial quantities. Using (4.11) for $k = 0$ gives

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \le \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau\|\epsilon_h^0\|_{L^2}\|e_h^1\|_{\mathbf{H}^1} + c\tau^2\|e_h^1\|_{\mathbf{H}^1}^2 + 2\tau|\mathcal{R}^1(e_h^1)|.$$

The hypothesis (4.15) together with the assumption that τ is small enough (say, $\tau \le \mu/c$) implies

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + \frac{3}{2}\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \le \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + c\tau + 2\tau|\mathcal{R}^1(e_h^1)|.$$

(ii) *General step* $(k \geq 1)$: Setting $r_h := 2\tau^2 \epsilon_h^k / \chi$ in (4.9), we obtain

(4.17)
$$\frac{\tau^2}{\chi} \left[\left\| \nabla \epsilon_h^{k+1} \right\|_{\mathbf{L}^2}^2 + \left\| \nabla \epsilon_h^k \right\|_{\mathbf{L}^2}^2 - \left\| \nabla \delta \epsilon_h^{k+1} \right\|_{\mathbf{L}^2}^2 \right] - 2\tau \left\langle \nabla \epsilon_h^k, e_h^{k+1} \right\rangle = \frac{2\tau^2}{\chi} \left\langle \nabla q_h^{k+1}, \nabla \epsilon_h^k \right\rangle.$$

Next, apply δ to (4.9) and set $r_h := \tau \delta \epsilon_h^{k+1}$. The Cauchy–Schwarz inequality implies

$$\tau^2 \left\| \nabla \delta \epsilon_h^{k+1} \right\|_{\mathbf{L}^2}^2 \leq \left\| \chi \delta e_h^{k+1} + \tau \nabla \delta q_h^{k+1} \right\|_{\mathbf{L}^2}^2$$
$$= \chi^2 \left\| \delta e_h^{k+1} \right\|_{\mathbf{L}^2}^2 + \tau^2 \left\| \nabla \delta q_h^{k+1} \right\|_{\mathbf{L}^2}^2 + 2\chi\tau \left\langle \nabla \delta q_h^{k+1}, \delta e_h^{k+1} \right\rangle,$$

which, by (3.5), implies

(4.18) $$\frac{\tau^2}{\chi} \left\| \nabla \delta \epsilon_h^{k+1} \right\|_{\mathbf{L}^2}^2 \leq \left\| \sigma_h^k \delta e_h^{k+1} \right\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi} \left\| \nabla \delta q_h^{k+1} \right\|_{\mathbf{L}^2}^2 + 2\tau \left\langle \nabla \delta q_h^{k+1}, \delta e_h^{k+1} \right\rangle.$$

Adding (4.11), (4.17), and (4.18), we obtain

$$\| \sigma_h^{k+1} e_h^{k+1} \|_{\mathbf{L}^2}^2 + 2\mu\tau \| e_h^{k+1} \|_{\mathbf{H}^1}^2$$
$$+ \frac{\tau^2}{\chi} \left[\| \nabla \epsilon_h^{k+1} \|_{\mathbf{L}^2}^2 + \| \nabla \epsilon_h^k \|_{\mathbf{L}^2}^2 \right] \leq 2\tau | \mathcal{R}^{k+1}(e_h^{k+1}) | + \| \sigma_h^k e_h^k \|_{\mathbf{L}^2}^2$$
$$- 2\tau \left\langle \nabla \delta q_h^{k+1}, e_h^k \right\rangle + \frac{\tau^2}{\chi} \| \nabla \delta q_h^{k+1} \|_{\mathbf{L}^2}^2 + \frac{2\tau^2}{\chi} \left\langle \nabla q_h^{k+1}, \nabla \epsilon_h^k \right\rangle.$$

We estimate the last three terms in the right-hand side separately. Integrating by parts and using (2.8), the first one can be estimated as follows:

$$-2\tau \left\langle \nabla \delta q_h^{k+1}, e_h^k \right\rangle \leq 2\tau \| \delta q_h^{k+1} \|_{L^2} \| e_h^k \|_{\mathbf{H}^1} \leq c\tau^3 + \frac{\mu\tau}{2} \| e_h^k \|_{\mathbf{H}^1}^2.$$

Similarly, the second term is estimated as follows:

$$\frac{\tau^2}{\chi} \| \nabla \delta q_h^{k+1} \|_{\mathbf{L}^2}^2 \leq c\tau^3.$$

For the last term, again using (2.8), we obtain

$$\frac{2\tau^2}{\chi} \left\langle \nabla q_h^{k+1}, \nabla \epsilon_h^k \right\rangle \leq c \frac{\tau^2}{\chi} \| \nabla \epsilon_h^k \|_{\mathbf{L}^2} \leq c\tau^2 + \frac{\tau^2}{\chi} \| \nabla \epsilon_h^k \|_{\mathbf{L}^2}^2.$$

Notice that this term is responsible for the loss of optimality; i.e., full first-order accuracy is lost at this point.

Combining the above observations, we finally obtain

$$\| \sigma_h^{k+1} e_h^{k+1} \|_{\mathbf{L}^2}^2 + 2\mu\tau \| e_h^{k+1} \|_{\mathbf{H}^1}^2 \leq 2\tau | \mathcal{R}^{k+1}(e_h^{k+1}) | + \| \sigma_h^k e_h^k \|_{\mathbf{L}^2}^2 + \frac{\mu\tau}{2} \| e_h^k \|_{\mathbf{H}^1}^2 + c\tau^2.$$

By summing over the time steps and using Corollary 4.1, we obtain

$$\sum_{n=0}^{k} \left(\| \sigma_h^{n+1} e_h^{n+1} \|_{\mathbf{L}^2}^2 + \mu\tau \| e_h^{n+1} \|_{\mathbf{H}^1}^2 \right) \leq c(\tau + h^{2\min\{l+1,m\}})$$
$$+ \sum_{n=0}^{k} \left((1 + c\tau) \| \sigma_h^n e_h^n \|_{\mathbf{L}^2}^2 + \frac{\mu\tau}{2} \| e_h^n \|_{\mathbf{H}^1}^2 \right),$$

which, by the discrete Grönwall lemma, implies

$$\|\sigma_{h\tau}e_{h\tau}\|_{\ell^\infty(\mathbf{L}^2)} + \|e_{h\tau}\|_{\ell^2(\mathbf{H}^1)} \le c(\tau^{1/2} + h^{\min\{l+1,m\}}).$$

The claimed error estimates follow from the triangle inequality, the definition

$$u^k - u_h^k = \eta^k + e_h^k,$$

and (in the case of the $\ell^\infty(\mathbf{L}^2)$-norm) assumption (3.5). Notice that it is only at this point that the interpolation error in the $\mathbf{H}^1$-norm, which is of order $\mathcal{O}(h^l)$, is introduced. This is a well-known superconvergence effect induced by our particular choice for the pair (w_h, q_h); see (2.6) and [36]. □

4.2.2. Incremental scheme. The incremental version of the algorithm is obtained by setting $\gamma = 1$. To derive error estimates, we assume that either the initial pressure is properly approximated, i.e.,

$$(4.19) \qquad \|p_h^0 - q_h^0\|_{L^2} \le ch^{\frac{l+1}{2}},$$

or that the approximation of the initial pressure is uniformly bounded in H^1 with respect to h, and the initial approximation of the velocity is discretely divergence-free, i.e.,

$$(4.20) \qquad \|p_h^0\|_{H^1} \le c, \qquad \langle u_h^0, \nabla r_h \rangle = 0 \quad \forall r_h \in M_h.$$

Under these conditions, the main error estimate for this algorithm is stated as follows.

THEOREM 4.2. *Assume that the solution to (1.1)–(1.2) satisfies (4.2). Let $(u_h)_\tau$ be the solution of (3.6)–(3.9) with $\gamma = 1$, and assume that conditions (3.1), (3.5), (4.13) hold. If either (4.19) or (4.20) is satisfied, then*

$$(4.21) \qquad \|u_\tau - u_{h\tau}\|_{\ell^\infty(\mathbf{L}^2)} \le c\left(\tau + h^{\min\{l+1,m\}}\right),$$

$$(4.22) \qquad \|u_\tau - u_{h\tau}\|_{\ell^2(\mathbf{H}^1)} \le c\left(\tau + h^{\min\{l,m\}}\right).$$

Remark 4.4. The error estimates from Theorem 4.2 show that, under the given assumptions on the density approximation, the time accuracy of the incremental pressure-correction algorithm for variable density fluid flows is similar to that of the analogous incremental projection-type pressure-correction scheme for constant density flows (cf. [15]).

Proof of Theorem 4.2. We proceed in two steps:

(i) *Initialization (k = 0):* We consider the first step separately, as it involves the initial quantities. From (4.11) we obtain that

$$(4.23) \quad \|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + \|\sigma_h^0 \delta e_h^1\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \le -2\tau\langle\nabla\epsilon_h^0, e_h^1\rangle - 2\tau\langle\nabla\delta q_h^1, e_h^1\rangle$$
$$+ \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau|\mathcal{R}^1(e_h^1)|.$$

If we assume that condition (4.19) is satisfied, then

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \le c\tau\|\epsilon_h^0\|_{L^2}^2 + c\tau\|\delta q_h^1\|_{L^2}^2 + \frac{\mu\tau}{2}\|e_h^1\|_{\mathbf{H}^1}^2 + \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau|\mathcal{R}^1(e_h^1)|,$$

so that using (2.7) and (4.19) we obtain

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + \mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \le c\tau^2 + h^{2(l+1)} + \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau|\mathcal{R}^1(e_h^1)|.$$

If, on the other hand, we assume that condition (4.20) is satisfied, then (4.23) can be restated as

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + \chi\|\delta e_h^1\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \leq -2\tau\langle\nabla(\epsilon_h^0 + \delta q_h^1), \delta e_h^1\rangle + \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau|\mathcal{R}^1(e_h^1)|,$$

so that using the Cauchy–Schwarz inequality, (2.7), and (4.20) we conclude that

$$\|\sigma_h^1 e_h^1\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^1\|_{\mathbf{H}^1}^2 \leq c\tau^2 + \|\sigma_h^0 e_h^0\|_{\mathbf{L}^2}^2 + 2\tau|\mathcal{R}^1(e_h^1)|.$$

(ii) *General step ($k \geq 1$):* In this case $p_h^\sharp = 2p_h^k - p_h^{k-1}$ and $\phi_h^\flat = \delta p_h^{k+1}$. Setting $r_h := -2\tau^2\delta^2\epsilon_h^{k+1}/\chi$ (which makes sense only for $k \geq 1$) in (4.9), we obtain

$$-\frac{\tau^2}{\chi}\left[\|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\delta\epsilon_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2\right] + 2\tau\left\langle e_h^{k+1}, \nabla\delta^2\epsilon_h^{k+1}\right\rangle$$

$$= -\frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\delta^2\epsilon_h^{k+1}\right\rangle.$$

Setting $r_h := 2\tau^2\epsilon_h^{k+1}/\chi$ in (4.9), we obtain

$$\frac{\tau^2}{\chi}\left[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2\right] = 2\tau\left\langle e_h^{k+1}, \nabla\epsilon_h^{k+1}\right\rangle + \frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\epsilon_h^{k+1}\right\rangle.$$

Adding these two equations we arrive at

$$(4.24) \quad \frac{\tau^2}{\chi}\left[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^k\|_{\mathbf{L}^2}^2\right] - \frac{\tau^2}{\chi}\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2$$

$$- 2\tau\left\langle e_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle = \frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle.$$

Now we apply δ to (4.9) and we set $r_h := \tau\delta^2\epsilon_h^{k+1}$. The Cauchy–Schwarz inequality implies

$$\tau^2\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 \leq \left\|\chi\delta e_h^{k+1} + \tau\nabla\delta^2 q_h^{k+1}\right\|_{\mathbf{L}^2}^2$$

$$= \chi^2\|\delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + \tau^2\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\tau\chi\left\langle\nabla\delta^2 q_h^{k+1}, \delta e_h^{k+1}\right\rangle,$$

and owing to (3.5) we infer

$$(4.25) \quad \frac{\tau^2}{\chi}\|\nabla\delta^2\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 \leq \|\sigma^k\delta e_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\tau\left\langle\nabla\delta^2 q_h^{k+1}, \delta e_h^{k+1}\right\rangle.$$

Adding (4.11), (4.24), and (4.25) and using Corollary 4.1, we arrive at

$$\|\sigma_h^{k+1} e_h^{k+1}\|_{\mathbf{L}^2}^2 + 2\mu\tau\|e_h^{k+1}\|_{\mathbf{H}^1}^2$$

$$+ \frac{\tau^2}{\chi}[\|\nabla\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2 + \|\nabla\delta\epsilon_h^{k+1}\|_{\mathbf{L}^2}^2] \leq 2\tau|\mathcal{R}^{k+1}(e_h^{k+1})| + \|\sigma_h^k e_h^k\|_{\mathbf{L}^2}^2$$

$$+ \frac{\tau^2}{\chi}\|\nabla\epsilon_h^k\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla\delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2$$

$$- 2\tau\left\langle\nabla\delta^2 q_h^{k+1}, e_h^k\right\rangle + \frac{2\tau^2}{\chi}\left\langle\nabla\delta q_h^{k+1}, \nabla\epsilon_h^\sharp\right\rangle.$$

Let us estimate the last three terms separately. Clearly,

$$\tau^2/\chi \|\nabla \delta^2 q_h^{k+1}\|_{\mathbf{L}^2}^2 \le c\tau^3.$$

The second term is bounded from above as follows:

$$-2\tau \left\langle \nabla \delta^2 q_h^{k+1}, e_h^k \right\rangle \le c\tau^3 + \frac{\mu\tau}{2} \|e_h^k\|_{\mathbf{H}^1}^2.$$

Finally, for the third term we have

$$\frac{2\tau^2}{\chi} \left\langle \nabla \delta q_h^{k+1}, \nabla \epsilon_h^\sharp \right\rangle \le c\tau^3 + \frac{\tau^3}{\chi} \|\nabla \epsilon_h^k\|_{\mathbf{L}^2}^2 + \frac{\tau^3}{\chi} \|\nabla \delta \epsilon_h^k\|_{\mathbf{L}^2}^2.$$

We obtain the estimate (4.21)–(4.22) by finishing as in the proof of Theorem 4.1. □

5. A second-order fractional time-stepping method. We have established in the previous section that the incremental version of the scheme (3.5)–(3.10) is first-order accurate in time both for the $\mathbf{L}^2$- and the $\mathbf{H}^1$-norm of the velocity. However, as shown in [12, 13], we expect that the splitting error of the algorithm is second-order since the pressure term

(5.1) $$p_h^\sharp = 2p_h^k - p_h^{k-1}$$

that appears in the approximate momentum equation is a second-order extrapolation of the pressure p_h^{k+1}. This observation is the main motivation for our introducing a variant of the incremental method using a second-order approximation of the time derivative of the velocity.

This section is organized as follows. In section 5.1 we introduce the second-order algorithm, and we derive some of its immediate properties. In section 5.2 we prove estimates for the solutions of a three term recursion inequality which is used to prove stability of the algorithm. In section 5.3 we reprove stability of a second-order projection scheme for the time-dependent Stokes equations with constant density. Although the result per se is not new (see [12, 13, 15, 9]), to the best of our knowledge the technique that we use is new. The originality of the proof technique is that the so-called projected velocity is totally eliminated from the analysis. This trick enables us to easily extend the proof to the variable density case. The stability proof is reported in section 5.4.

5.1. Description of the algorithm. Keeping the same notation as in the previous sections, the second-order variant of the algorithm is composed of the following steps.

5.1.1. Initialization. First, we choose a penalty parameter χ as in section 3.1. Next, we define the quantity $(\rho_h^0, u_h^0, p_h^0, \phi_h^0 = 0) \in W_h \times \mathbf{X}_h \times M_h \times M_h$ to be a suitable approximation of the initial data of the problem. Then we compute an approximation of the exact solution at time $t = \tau$, say $(\rho_h^1, u_h^1, p_h^1, \phi_h^1 = p_h^1 - p_h^0) \in W_h \times \mathbf{X}_h \times M_h \times M_h$.

5.1.2. Time stepping. Given $(\rho_h^k, u_h^k, p_h^k, \phi_h^k) \in W_h \times \mathbf{X}_h \times M_h \times M_h$ for $1 \le k \le K-1$, we compute the next time-step approximation as follows.

5.1.3. Density update. We are not specific about the way $\rho_h^{k+1} \in W_h$ is computed, but we assume that (3.5) holds and that there is a uniform constant M so that

(5.2) $$\max_{0 \le k \le K-1} \left\| \frac{\rho_h^{k+1} - \rho_h^k}{\tau} \right\|_{L^\infty} \le M\chi.$$

5.1.4. Velocity update. Similarly to section 3.2.2 we define

$$(5.3) \qquad \rho_h^\star := \frac{3}{2}\rho_h^{k+1} - \frac{2}{3}\rho_h^k + \frac{1}{6}\rho_h^{k-1} = \rho_h^{k+1} + \frac{1}{6}(3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}),$$

$$(5.4) \qquad p_h^\sharp := p_h^k + \frac{4}{3}\phi_h^k - \frac{1}{3}\phi_h^{k-1}.$$

Then we compute $u_h^{k+1} \in \mathbf{X}_h$ so that the following holds:

(5.5)
$$\left\langle \frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau}, v_h \right\rangle + \left\langle \rho_h^{k+1}u_h^\star \cdot \nabla u_h^{k+1} + \frac{1}{2}u_h^{k+1}\nabla \cdot (\rho_h^{k+1}u_h^\star), v_h \right\rangle$$

$$+ \mu\left\langle \nabla u_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla p_h^\sharp, v_h \right\rangle = \left\langle f^{k+1}, v_h \right\rangle \quad \forall v_h \in \mathbf{X}_h,$$

where

$$(5.6) \qquad\qquad\qquad\qquad u_h^\star := 2u_h^k - u_h^{k-1}$$

is a second-order extrapolation of the velocity.

5.1.5. Penalty. We compute the pressure correction $\phi_h^{k+1} \in M_h$ so that the following holds:

$$(5.7) \qquad\qquad \left\langle \nabla\phi_h^{k+1}, \nabla r_h \right\rangle = \frac{3\chi}{2\tau}\left\langle u_h^{k+1}, \nabla r_h \right\rangle \quad \forall r_h \in M_h.$$

5.1.6. Pressure update. Finally, the pressure is updated by setting

$$(5.8) \qquad\qquad\qquad\qquad p_h^{k+1} = p_h^k + \phi_h^{k+1}.$$

Remark 5.1. The quantities $(\rho_h^1, u_h^1, p_h^1, \phi_h^1)$ can be computed by using one step of the incremental first-order scheme described in section 3.

Remark 5.2. The term $\left\langle \frac{1}{2}\nabla \cdot (\rho_h^{k+1}u_h^\star)u_h^{k+1}, v_h \right\rangle$ has been added to the equation to obtain unconditional stability with respect to the advection term. As in the proof of Lemma 4.1 we are going to use the following identity:

$$\left\langle \rho_h^{k+1}u_h^\star \cdot \nabla u_h^{k+1} + \frac{1}{2}\nabla \cdot (\rho_h^{k+1}u_h^\star)u_h^{k+1}, u_h^{k+1} \right\rangle$$

$$= \int_\Omega \rho_h^{k+1}u_h^\star \cdot \nabla u_h^{k+1} \cdot u_h^{k+1} + \frac{1}{2}\int_\Omega \nabla \cdot (\rho_h^{k+1}u_h^\star)|u_h^{k+1}|^2 = 0.$$

Remark 5.3. The term

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau} + \frac{1}{2}\nabla \cdot (\rho_h^{k+1}u_h^\star)u_h^{k+1}$$

is a second-order approximation of $[\rho_h u_{h,t}](t^{k+1})$. Indeed, if the involved functions are smooth enough in time, we infer from the definition of $\rho_h^\star$ that

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau} + \frac{1}{2}\nabla \cdot (\rho_h^{k+1}u_h^\star)u_h^{k+1}$$

$$= \frac{\rho_h^{k+1}}{2\tau}(3u_h^{k+1} - 4u_h^k + u_h^{k-1}) + \frac{1}{2}\left(\frac{3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}}{2\tau} + \nabla \cdot (\rho_h^{k+1}u_h^\star)\right)u_h^{k+1}$$

$$= [\rho_h u_{h,t}]^{k+1} + \frac{1}{2}[\rho_{h,t} + \nabla \cdot (\rho_h u_h)]^{k+1}u_h^{k+1} + \mathcal{O}(\tau^2) = [\rho_h u_{h,t}]^{k+1} + \mathcal{O}(\tau^2),$$

which proves the claim.

5.2. Three term recursion inequalities. We prove in this section preliminary results regarding three term recursion inequalities. These results will be needed to prove stability of the algorithm (5.3)–(5.8).

PROPOSITION 5.1. *Assume that the characteristic polynomial of the three term recursion equation*

$$(5.9) \qquad Ax^{k+1} + Bx^k + Cx^{k-1} = g^{k+1}, \qquad k \geq 2,$$

has two (not necessarily distinct) nonzero real roots r_1 and r_2. Then, the generic solution to this equation has the form

$$x^\nu = c_1 r_1^\nu + c_2 r_2^\nu + \frac{1}{A} \sum_{l=2}^\nu r_1^{\nu-l} \sum_{s=2}^l r_2^{l-s} g^s, \quad c_1, c_2 \in \mathbb{R}.$$

Proof. It is sufficient to show that

$$\bar{x}^\nu = \frac{1}{A} \sum_{l=2}^\nu r_1^{\nu-l} \sum_{s=2}^l r_2^{l-s} g^s, \quad \nu \geq 2,$$

with $\bar{x}^1 = \bar{x}^0 = 0$ being a particular solution of (5.9).

Let $n \geq 1$. Multiply (5.9) by r_2^{2n-k-2}, and add all the results for $k = 1, \ldots, n$. Setting $x^1 = x^0 = 0$, we obtain

$$Ar_2^{n-2}x^{n+1} + r_2^{n-2}(Ar_2 + B)x^n + \sum_{k=2}^{n-1} \left[(Ar_2^{2n-k-1} + Br_2^{2n-k-2} + Cr_2^{2n-k-3})x^k \right]$$

$$= \sum_{s=2}^{n+1} r_2^{2n-s-1} g^s,$$

which, since r_2 is a root of the characteristic polynomial, implies

$$(5.10) \qquad Ax^{n+1} + (Ar_2 + B)x^n = \sum_{s=2}^{n+1} r_2^{n+1-s} g^s, \quad n \geq 2.$$

Let $\nu \geq 1$. Multiply (5.10) by $r_1^{\nu-n}$, and add all the results for $n = 1, \ldots, \nu$. We obtain

$$Ax^{\nu+1} + \sum_{l=2}^\nu \left[r_1^{\nu-l}(A(r_1 + r_2) + B)x^l \right] = \sum_{l=2}^{\nu+1} r_1^{\nu+1-l} \sum_{s=2}^l r_2^{l-s} g^s, \quad \nu \geq 1.$$

Since r_1, r_2 are roots of the characteristic polynomial of the recursion equation, we have $B = -(r_1 + r_2)A$, which implies

$$x^{\nu+1} = \frac{1}{A} \sum_{l=2}^{\nu+1} r_1^{\nu+1-l} \sum_{s=2}^l r_2^{l-s} g^s, \quad \nu \geq 1.$$

Hence, $\bar{x}^\nu$ is a particular solution of (5.9). ☐

PROPOSITION 5.2. *Assume that the coefficients of the three term recursion inequality*

$$(5.11) \qquad Ay^{k+1} + By^k + Cy^{k-1} \leq g^{k+1}, \quad k \geq 1,$$

satisfy

$$A > 0, \quad C \geq 0, \quad A + B + C \leq 0.$$

Let $\{y^k\}_{k\geq 0}$ be a solution to (5.11) with initial data y^0 and y^1. If $\{x^k\}_{k\geq 0}$ solves (5.9) with initial data $x^0 = y^0$ and $x^1 = y^1$, then the following estimate holds:

$$y^\nu \leq x^\nu \quad \forall \nu \geq 0.$$

Proof. This is a comparison argument à la Grönwall. Let $\{z^k\}_{k\geq 0}$ be the sequence defined by $z^\nu = y^\nu - x^\nu$. Let us prove by induction that $z^k \leq z^{k-1}$ for all $k \geq 1$. The claim holds true for $k = 1$ since $0 = z^1 \leq z^0 = 0$. Assume now that $z^\nu \leq z^{\nu-1}$ for all $1 \leq \nu \leq k$. The definition of $\{x^k\}_{k\geq 0}$ implies

$$Az^{k+1} + Bz^k + Cz^{k-1} \leq 0 \quad \forall k \geq 1.$$

Hence

$$Az^{k+1} \leq Az^k - (A + B + C)z^k + C(z^k - z^{k-1}) \leq Az^k,$$

which proves the claim. □

The following corollary is a specialization of the two previous results which will be needed in what follows.

COROLLARY 5.1. *The three term recursion equation*

(5.12) $$3x^{k+1} - 4x^k + x^{k-1} = g^{k+1}, \quad k \geq 1,$$

has the following general solution:

$$x^\nu = c_1 + \frac{c_2}{3^\nu} + \sum_{l=2}^\nu \frac{1}{3^{\nu+1-l}} \sum_{s=2}^l g^s, \qquad c_1 \in \mathbb{R}, \ c_2 \in \mathbb{R}.$$

Let $\{y^k\}_{k\geq 0}$ be the solution to the three term recursion inequality

$$3y^{k+1} - 4y^k + y^{k-1} \leq g^{k+1}, \quad k \geq 1,$$

with initial data y^0 and y^1. If $\{x^k\}_{k\geq 0}$ is the solution to (5.12) with initial data $x^0 = y^0$ and $x^1 = y^1$, then the following estimate holds:

$$y^\nu \leq x^\nu \quad \forall \nu \geq 0.$$

Proof. To obtain the generic solution, it is sufficient to notice that the roots of the characteristic polynomial of the equation are $r_2 = 1$ and $r_1 = 1/3$. To obtain the estimate, it is sufficient to notice that $A = 3 > 0$, $C = 1 > 0$, and $A + B + C = 3 - 4 + 1 = 0 \leq 0$. □

5.3. Stability of the incremental pressure-correction algorithm in standard form. The objective of this section is to prove stability estimates for the incremental pressure-correction algorithm in standard form for the approximation of the time-dependent Stokes equation with constant density. This result is not new, but the technique that we use to prove these estimates gives insight on the way to proceed when the density is variable. The main novelty is that the proof does not use

the projected velocity. To the best of our knowledge, this proof technique has never been used before.

We restrict ourselves for the time being to the time-dependent Stokes equations:

$$\begin{cases} u_t - \Delta u + \nabla p = f, & u|_{t=0} = u_0, \quad u|_\Gamma = 0, \\ \nabla{\cdot}u = 0. \end{cases}$$

Without going into the details (for which we refer the reader to [9, 12, 13, 15]), we now describe the incremental pressure-correction projection in standard form. After proper initialization, given $(u_h^k, p_h^k, \phi_h^k) \in \mathbf{X}_h \times M_h \times M_h$, the next iterate is computed in three steps:

(i) Find $u_h^{k+1} \in \mathbf{X}_h$ so that

$$(5.13) \qquad \left\langle \frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau}, v_h \right\rangle + \left\langle \nabla u_h^{k+1}, \nabla v_h \right\rangle + \left\langle \nabla p_h^\sharp, v_h \right\rangle = \left\langle f^{k+1}, v_h \right\rangle,$$

where $p_h^\sharp$ is defined in (5.4).

(ii) Find $\phi_h^{k+1} \in M_h$ so that

$$(5.14) \qquad \left\langle \nabla \phi_h^{k+1}, \nabla r_h \right\rangle = \frac{3}{2\tau} \left\langle u_h^{k+1}, \nabla r_h \right\rangle.$$

(iii) Finally, update the pressure by

$$(5.15) \qquad p_h^{k+1} = p_h^k + \phi_h^{k+1}.$$

We now prove that the above algorithm is stable. To avoid irrelevant technicalities, we assume that $f \equiv 0$.

THEOREM 5.1. *The solution $\{(u_h^k, p_h^k)\}_{k \geq 0} \subset \mathbf{X}_h \times M_h$ to (5.13)–(5.15) satisfies the following estimate:*

$$\|u_h^k\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2 + \sum_{l=2}^{k} \left[\tau \|u_h^l\|_{\mathbf{H}^1}^2 + \tau^2 \|\nabla \delta p_h^{l-1}\|_{\mathbf{L}^2}^2 \right]$$

$$\leq c \left(\|u_h^0\|_{\mathbf{L}^2}^2 + \|u_h^1\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^1\|_{\mathbf{L}^2}^2 \right) \quad \forall k \geq 2.$$

Proof. We proceed in two steps:

(i) *Initialization:* We consider the steps $k = 1, 2$ separately, as they involve the initial quantities. Let us begin by noticing that the definition of $p_h^\sharp$ involves only terms from the previous time steps. For $k = 1$ or 2 we set $v_h := 4\tau u_h^{k+1}$ in (5.13). Then using the identity

$$2a^{k+1} \left(3a^{k+1} - 4a^k + a^{k-1} \right) = \left| a^{k+1} \right|^2 + \left| 2a^{k+1} - a^k \right|^2 + \left| \delta^2 a^{k+1} \right|^2$$
$$- \left| a^k \right|^2 - \left| 2a^k - a^{k-1} \right|^2$$

and the Cauchy–Schwarz inequality we obtain

$$\frac{1}{2} \|u_h^{k+1}\|_{\mathbf{L}^2}^2 + \|2u_h^{k+1} - u^k\|_{\mathbf{L}^2}^2 + 4\tau \|u_h^{k+1}\|_{\mathbf{H}^1}^2 \leq \|u_h^k\|_{\mathbf{L}^2}^2 + \|2u_h^k - u_h^{k-1}\|_{\mathbf{L}^2}^2 + 8\tau^2 \|\nabla p_h^\sharp\|_{\mathbf{L}^2}^2,$$

which implies

$$\|u_h^{k+1}\|_{\mathbf{L}^2}^2 + \tau \|u_h^{k+1}\|_{\mathbf{H}^1}^2 \leq c \left(\|u_h^0\|_{\mathbf{L}^2}^2 + \|u_h^1\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \tau^2 \|\nabla p_h^1\|_{\mathbf{L}^2}^2 \right)$$

for $k = 1$ or 2. The estimate on the pressure is obtained using (5.14)–(5.15) as follows:

$$\frac{4\tau^2}{9}\|\nabla \delta p_h^{k+1}\|_{\mathbf{L}^2}^2 \leq \|u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

The triangle inequality and the estimates obtained above imply the claimed estimate for the first two steps $k = 1, 2$.

(ii) *General step:* For $k \geq 3$ notice that, by (5.15),

$$p_h^\sharp = \frac{7p_h^k - 5p_h^{k-1} + p_h^{k-2}}{3} = \frac{3p_h^{k+1} - 3\delta^2 p_h^{k+1} + \delta^2 p_h^k}{3}.$$

Setting $v_h := 4\tau u_h^{k+1}$ in (5.13) and using the identity

$$2a^{k+1}\left(3a^{k+1} - 4a^k + a^{k-1}\right) = 3\left|a^{k+1}\right|^2 - 4\left|a^k\right|^2 + \left|a^{k-1}\right|^2$$
$$+ 2\left|\delta a^{k+1}\right|^2 - 2\left|\delta a^k\right|^2 + \left|\delta^2 a^{k+1}\right|^2,$$

we obtain

$$3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2$$
$$+ 4\tau\left\langle \nabla p_h^{k+1}, u_h^{k+1}\right\rangle - 4\tau\left\langle \nabla \delta^2 p_h^{k+1}, u_h^{k+1}\right\rangle + \frac{4\tau}{3}\left\langle \nabla \delta^2 p_h^k, u_h^{k+1}\right\rangle = 0.$$

From the projection equation (5.14) and the pressure update equation (5.15) we deduce that

$$\left\langle \nabla r_h, u_h^{k+1}\right\rangle = \frac{2\tau}{3}\left\langle \nabla r_h, \nabla \delta p_h^{k+1}\right\rangle, \quad r_h \in M_h.$$

Using this property together with the identity $2a(a - b) = a^2 - b^2 + (a - b)^2$, we infer

$$3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2$$
$$+ \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla \delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2\right] + \frac{8\tau^2}{9}\left\langle \nabla \delta^2 p_h^k, \nabla \delta p_h^{k+1}\right\rangle = 0.$$

Now we use the following identity:

$$\|\delta u_h\|_{\mathbf{L}^2}^2 = \left\|\delta u_h - \frac{2\tau}{3}\nabla \delta^2 p_h\right\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9}\|\nabla \delta^2 p_h\|_{\mathbf{L}^2}^2,$$

which we apply at time steps t^{k+1} and t^k (note that it is critical to have $k \geq 3$ here), and we obtain

$$3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}^2}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2$$
$$+ 2\left\|\delta u_h^{k+1} - \frac{2\tau}{3}\nabla \delta^2 p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\delta u_h^k - \frac{2\tau}{3}\nabla \delta^2 p_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2\right]$$
$$- \frac{4\tau^2}{9}\|\nabla \delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9}\|\nabla \delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9}\left\langle \nabla \delta^2 p_h^k, \nabla \delta p_h^{k+1}\right\rangle = 0.$$

We observe from this inequality that we need to control the last three terms. We rewrite these as follows:

$$-\frac{4\tau^2}{9}\|\nabla\delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9}\|\nabla\delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle$$

$$= -\frac{4\tau^2}{9}\|\nabla\delta^3 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{4\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\left(\delta^2 p_h^k + 2\delta^2 p_h^{k+1} - 2\delta p_h^{k+1}\right)\right\rangle.$$

Applying δ^2 to (5.14) and using the Cauchy–Schwarz inequality, we obtain, for $k \geq 3$,

$$\frac{4\tau^2}{9}\|\nabla\delta^3 p_h^{k+1}\|_{\mathbf{L}^2}^2 \leq \|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

Observing that $\delta^2 p_h^k + 2\delta^2 p_h^{k+1} - 2\delta p_h^{k+1} = -\delta p_h^k - \delta p_h^{k-1}$ and using the inequality above, we obtain the following bound:

$$-\frac{4\tau^2}{9}\|\delta^2 p_h^{k+1}\|^2 - \frac{8\tau^2}{9}\|\delta^2 p_h^k\|^2 + \frac{8\tau^2}{9}\left\langle\nabla\delta^2 p_h^k, \nabla\delta p_h^{k+1}\right\rangle$$

$$\geq -\|\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9}\left[\|\delta p_h^k\|^2 - \|\delta p_h^{k-1}\|^2\right],$$

from which we finally deduce the following energy estimate:

$$\begin{align}
(5.16) \quad & 3\|u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|u_h^k\|_{\mathbf{L}}^2 + \|u_h^{k-1}\|_{\mathbf{L}^2}^2 + 4\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \\
& + 2\left\|\delta u_h^{k+1} - \frac{2\tau}{3}\nabla\delta^2 p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\delta u_h^k - \frac{2\tau}{3}\nabla\delta^2 p_h^k\right\|_{\mathbf{L}^2}^2 \\
& + \frac{4\tau^2}{3}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2\right] + \frac{4\tau^2}{9}\left[\|\nabla\delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla\delta p_h^{k-1}\|_{\mathbf{L}^2}^2\right] \leq 0.
\end{align}$$

We are now going to use the stability estimates proved in section 5.2. Let us define the quantities

$$a^s := \|u_h^s\|_{\mathbf{L}^2}^2,$$

$$b^s := 4\tau\|u_h^s\|_{\mathbf{H}^1}^2 + \frac{4\tau^2}{3}\|\nabla\delta p_h^{s-1}\|_{\mathbf{L}^2}^2,$$

$$d^s := 2\left\|\delta u_h^s - \frac{2\tau}{3}\nabla\delta^2 p_h^s\right\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{3}\|\nabla p_h^s\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9}\|\nabla\delta p_h^{s-1}\|_{\mathbf{L}^2}^2.$$

Then (5.16) can be rewritten as

$$3a^{k+1} - 4a^k + a^{k-1} \leq -\left(b^{k+1} + d^{k+1} - d^k\right), \quad k \geq 3.$$

Setting $g^{k+1} := -\left(b^{k+1} + d^{k+1} - d^k\right)$, this three term recursion inequality satisfies the hypotheses of Corollary 5.1 for $k \geq 3$. Hence

$$a^\nu \leq c\left(a^1 + a^2\right) - \sum_{l=3}^\nu \frac{1}{3^{\nu+1-l}} \sum_{s=3}^l \left(b^s + d^s - d^{s-1}\right), \quad \nu \geq 3,$$

or

$$a^\nu + \sum_{l=3}^\nu \frac{1}{3^{\nu+1-l}}d^l + \sum_{l=3}^\nu \frac{1}{3^{\nu+1-l}} \sum_{s=3}^l b^s \leq c\left(a^1 + a^2 + d^2\right), \quad \nu \geq 3.$$

Dropping some positive terms in the left-hand side, we deduce

$$a^\nu + \frac{1}{3}d^\nu + \frac{1}{3}\sum_{s=2}^{\nu} b^s \leq c\left(a^1 + a^2 + d^2\right).$$

Given the bounds obtained in the initialization step, this inequality implies the claimed result. $\square$

5.4. Stability of the second-order fractional time-stepping scheme for variable density flows. We now establish stability for the algorithm (5.5)–(5.8). Again, assume that $f \equiv 0$ to avoid irrelevant technicalities. The main result of this section is the following.

THEOREM 5.2. *Assume that the sequence of approximate densities $\{\rho_h^k\}_{k\geq 0} \subset W_h$ satisfies (3.5) and (5.2). Then, for τ small enough, the sequence $\{(u_h^k, p_h^k)\}_{k\geq 0} \subset \mathbf{X}_h \times M_h$ obtained by the algorithm (5.5)–(5.8) satisfies the following estimate:*

$$(5.17) \quad \|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + \mu\tau\|u_h^k\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla \delta p_h^{k-1}\|^2$$

$$\leq \mathcal{K}(1 + e^{cT})\left(\|\sigma_h^0 u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1 u_h^1\|_{\mathbf{L}^2}^2 + \|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \|\nabla p_h^1\|_{\mathbf{L}^2}^2\right) \quad \forall k \geq 2$$

for some constants c and $\mathcal{K}$.

Proof. Note first that, as already mentioned in Remark 5.3, the time derivative can be rewritten as follows:

$$\frac{3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1}u_h^k + \rho_h^{k+1}u_h^{k-1}}{2\tau} = \rho^{k+1}\frac{3u_h^{k+1} - 4u_h^k + u_h^{k-1}}{2\tau}$$

$$+ \frac{1}{2}u_h^{k+1}\frac{3\rho^{k+1} - 4\rho^k + \rho^{k-1}}{2\tau},$$

which is an approximation of $\rho\partial_t u + \frac{1}{2}u\partial_t\rho$. Once tested with u, the expression $(\rho\partial_t u + \frac{1}{2}u\partial_t\rho)u$ gives $\partial_t(\frac{1}{2}\rho u^2)$, and after integration over Ω and over the time interval $(0,T)$ this yields kinetic energy conservation. We have reproduced this argument at the discrete level for the first-order time stepping described in section 4; see (4.11). To avoid unnecessary technicalities associated with BDF2, we are now going to simplify the argument and content ourselves with a suboptimal stability analysis which will yield the growth constant $(1 + e^{cT})$ in (5.17). We refer the reader to Remark 5.4 for a discussion on how to proceed in full generality.

Using Assumption (5.2), we have the following estimate:

$$\left\langle\left(3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1}\right)u_h^{k+1}, u_h^{k+1}\right\rangle$$

$$= 3\int_\Omega \left(\rho_h^{k+1} - \rho_h^k\right)|u_h^{k+1}|^2 - \int_\Omega \left(\rho_h^k - \rho_h^{k-1}\right)|u_h^{k+1}|^2$$

$$\geq -\left(3\left\|\frac{\rho_h^{k+1} - \rho_h^k}{\chi}\right\|_{L^\infty} + \left\|\frac{\rho_h^k - \rho_h^{k-1}}{\chi}\right\|_{L^\infty}\right)\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2$$

$$\geq -4M\tau\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

A similar treatment gives

$$2\left\langle\rho_h^{k+1}\left(3u_h^{k+1} - 4u_h^k + u_h^{k-1}\right), u_h^{k+1}\right\rangle \geq 3\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4 + 8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2$$

$$+ (1 - 6M\tau)\|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k\delta u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k+1}\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

Combining the above two inequalities gives

$$(5.18) \quad 2 \left\langle 3\rho_h^\star u_h^{k+1} - 4\rho_h^{k+1} u_h^k + \rho_h^{k+1} u_h^{k-1}, u_h^{k+1} \right\rangle \geq (3 - 4M\tau)\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2$$
$$- (4 + 8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + (1 - 6M\tau)\|\sigma_h^{k-1} u_h^{k-1}\|_{\mathbf{L}^2}^2$$
$$+ 2\|\sigma_h^{k+1} \delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k \delta u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k+1} \delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2.$$

This estimate will be used repeatedly.

Now we proceed in two steps, as in the proof of Theorem 5.1: First we investigate the time steps $k = 1, 2$; then we investigate the cases $k \geq 3$.

(i) *Initialization:* Let $k \in \{1, 2\}$, and set $v_h := 4\tau u_h^{k+1}$ in (5.5). Using (5.18) and the Cauchy–Schwarz inequality, we get

$$(3 - 4M\tau)\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 \leq \frac{8\tau^2}{\chi}\|\nabla p_h^\sharp\|_{\mathbf{L}^2}^2 + \frac{\chi}{2}\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2,$$

which by (3.5) implies that if τ is small enough,

$$\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2$$
$$\leq c \left(\|\sigma_h^0 u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1 u_h^1\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^1\|_{\mathbf{L}^2}^2 \right).$$

The estimate on the pressure is obtained mutatis mutandis the argument in the initialization step of the proof of Theorem 5.1. Hence

$$\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla \delta p_h^{k+1}\|_{\mathbf{L}^2}^2$$
$$\leq c \left(\|\sigma_h^0 u_h^0\|_{\mathbf{L}^2}^2 + \|\sigma_h^1 u_h^1\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^0\|_{\mathbf{L}^2}^2 + \frac{\tau^2}{\chi}\|\nabla p_h^1\|_{\mathbf{L}^2}^2 \right), \quad k = 1, 2.$$

(ii) *General step:* For $k \geq 3$ we proceed as in the general step for the constant density case. Using (5.18) we obtain the estimate

$$(3 - 4M\tau)\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4 + 8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2$$
$$+ (1 - 6M\tau)\|\sigma_h^{k-1} u_h^{k-1}\|_{\mathbf{L}^2}^2 + 2\|\sigma_h^{k+1} \delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - 2\|\sigma_h^k \delta u_h^k\|_{\mathbf{L}^2}^2$$
$$+ \|\sigma_h^{k+1} \delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 + \frac{4\tau^2}{3\chi} \left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla \delta p_h^k\|^2 \right]$$
$$- \frac{4\tau^2}{3\chi}\|\nabla \delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle \nabla \delta^2 p_h^k, \nabla \delta p_h^{k+1} \rangle \leq 0.$$

Add and subtract to this inequality the terms $2\chi\|\delta u_h\|_{\mathbf{L}^2}^2$ taken at time steps t^{k+1} and t^k. Now, as in the constant density case, use the identity

$$\chi\|\delta u_h\|_{\mathbf{L}^2}^2 = \left\| \chi^{1/2}\delta u_h - \frac{2\tau}{3\chi^{1/2}}\nabla \delta^2 p_h \right\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\|\nabla \delta^2 p_h\|_{\mathbf{L}^2}^2$$

to deduce

$$(5.19) \quad (3 - 4M\tau)\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4 + 8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + (1 - 6M\tau)\|\sigma_h^{k-1} u_h^{k-1}\|_{\mathbf{L}^2}^2$$
$$+ \|\sigma_h^{k+1} \delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2$$
$$+ 2\left\|(\rho_h^{k+1} - \chi)^{1/2} \delta u_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|(\rho_h^k - \chi)^{1/2} \delta u_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ 2\left\|\chi^{1/2} \delta u_h^{k+1} - \frac{2\tau}{3\chi^{1/2}} \nabla \delta^2 p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\chi^{1/2} \delta u_h^k - \frac{2\tau}{3\chi^{1/2}} \nabla \delta^2 p_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ \frac{4\tau^2}{3\chi}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2\right]$$
$$- \frac{4\tau^2}{9\chi}\|\nabla \delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9\chi}\|\nabla \delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle \nabla \delta^2 p_h^k, \nabla \delta p_h^{k+1}\rangle \le 0,$$

where we used assumption (3.5).

By assumption (3.5), the control on the last three pressure terms is obtained in a way similar to the proof of Theorem 5.1, thus giving

$$- \frac{4\tau^2}{9\chi}\|\nabla \delta^2 p_h^{k+1}\|_{\mathbf{L}^2}^2 - \frac{8\tau^2}{9\chi}\|\nabla \delta^2 p_h^k\|_{\mathbf{L}^2}^2 + \frac{8\tau^2}{9\chi}\langle \nabla \delta^2 p_h^k, \nabla \delta p_h^{k+1}\rangle$$
$$\ge -\|\sigma_h^{k+1} \delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\left[\|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2\right].$$

Applying this estimate to (5.19) we arrive at the energy estimate

$$(5.20) \quad (3 - 4M\tau)\|\sigma_h^{k+1} u_h^{k+1}\|_{\mathbf{L}^2}^2 - (4 + 8M\tau)\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + (1 - 6M\tau)\|\sigma_h^{k-1} u_h^{k-1}\|_{\mathbf{L}^2}^2$$
$$+ 4\mu\tau\|u_h^{k+1}\|_{\mathbf{H}^1}^2 + 2\left\|(\rho_h^{k+1} - \chi)^{1/2} \delta u_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|(\rho_h^k - \chi)^{1/2} \delta u_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ 2\left\|\chi^{1/2} \delta u_h^{k+1} - \frac{2\tau}{3\chi^{1/2}} \nabla \delta^2 p_h^{k+1}\right\|_{\mathbf{L}^2}^2 - 2\left\|\chi^{1/2} \delta u_h^k - \frac{2\tau}{3\chi^{1/2}} \nabla \delta^2 p_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ \frac{4\tau^2}{3\chi}\left[\|\nabla p_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2\right]$$
$$+ \frac{4\tau^2}{9\chi}\left[\|\nabla \delta p_h^k\|_{\mathbf{L}^2}^2 - \|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2\right] \le 0.$$

Introducing the notation

$$A := 3 - 4M\tau, \quad B = -(4 + 8M\tau), \quad C = 1 - 6M\tau,$$
$$a^k := \|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2, \quad k \ge 0,$$
$$b^k := 4\mu\tau\|u_h^k\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{3\chi}\|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2, \quad k \ge 1,$$
$$d^k := 2\left\|(\rho_h^k - \chi)^{1/2} \delta u_h^k\right\|_{\mathbf{L}^2}^2 + 2\left\|\chi^{1/2} \delta u_h^k + \frac{2\tau}{3\chi^{1/2}} \nabla \delta^2 p_h^k\right\|_{\mathbf{L}^2}^2$$
$$+ \frac{4\tau^2}{3\chi}\|\nabla p_h^k\|_{\mathbf{L}^2}^2 + \frac{4\tau^2}{9\chi}\|\nabla \delta p_h^{k-1}\|_{\mathbf{L}^2}^2, \quad k \ge 2,$$

inequality (5.20) can be rewritten as

$$Aa^{k+1} + Ba^k + Ca^{k-1} \le -\left(b^{k+1} + d^{k+1} - d^k\right), \quad k \ge 3.$$

Define $g^{k+1} := -(b^{k+1} + d^{k+1} - d^k)$. If τ is small enough, this three term recursion inequality satisfies the assumptions of Proposition 5.2. The roots of the characteristic polynomial are

$$r_1 := \frac{2 + 4M\tau - \sqrt{1 + 38M\tau - 8M\tau^2}}{3 - 4M\tau} = \frac{1}{3}\left(1 - \frac{41M\tau}{3} + \mathcal{O}(\tau^2)\right),$$

$$r_2 := \frac{2 + 4M\tau + \sqrt{1 + 38M\tau - 8M\tau^2}}{3 - 4M\tau} = 1 + 9M\tau + \mathcal{O}(\tau^2).$$

Both roots are positive; the first one is strictly less than one third, and the second is greater but close to one. Hence, for $\nu \geq 3$,

$$a^\nu \leq c(a^1 + a^2)(r_1^\nu + r_2^\nu) - \frac{1}{3 - 4M\tau} \sum_{l=3}^{\nu} r_1^{\nu-l} \sum_{s=3}^{l} r_2^{l-s}(b^s + d^s - d^{s-1}),$$

which, since τ is small, can be rewritten as

$$(5.21) \quad a^\nu + \frac{1}{3}b^\nu \leq \mathcal{K}(1 + e^{cT})(a^1 + a^2) - \frac{1}{3 - 4M\tau} \sum_{l=3}^{\nu} r_1^{\nu-l} \sum_{s=3}^{l} r_2^{l-s}(d^s - d^{s-1})$$

for some constants c and $\mathcal{K}$.

Notice that

$$\sum_{s=3}^{l} r_2^{l-s}(d^s - d^{s-1}) = d^l + (r_2 - 1)\sum_{s=3}^{l-1} r_2^{l-s-1} d^s.$$

Hence (5.21) implies

$$a^\nu + \frac{1}{3}b^\nu + \frac{1}{3}d^\nu \leq \mathcal{K}(1 + e^{cT})(a^1 + a^2).$$

This inequality, combined with the estimates obtained at the initialization step, implies the result. □

Remark 5.4. Hypothesis (5.2) is suboptimal and possibly difficult to ascertain in general. We introduced it to uncouple the analysis of the present algorithm from the analysis of the mass conservation equation so as to isolate the difficulties and focus mainly on the effect of the pressure splitting. Hypothesis (5.2) could be weakened by making use of the following identity:

$$(5.22) \quad 2\langle\rho_h^{k+1}(3u_h^{k+1} - 4u_h^k + u_h^{k-1}), u_h^{k+1}\rangle + \langle(3\rho_h^{k+1} - 4\rho_h^k + \rho_h^{k-1})u_h^{k+1}, u_h^{k+1}\rangle$$
$$= 3\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2$$
$$+ 2(\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 - \|\sigma_h^k \delta u_h^k\|_{\mathbf{L}^2}^2) + \|\sigma_h^{k+1}\delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2 + \mathcal{I}^{k+1},$$

where the remainder $\mathcal{I}^{k+1}$ has the following form:

$$(5.23) \quad \mathcal{I}^{k+1} = \int_\Omega \left((\rho_h^{k+1} - \rho_h^{k-1})\delta^2|u_h^{k+1}|^2 + 2\delta^2\rho_h^{k+1}\delta|u_h^{k+1}|^2 - 2\delta\rho_h^{k+1}|\delta u_h^k|^2\right).$$

In (5.22), the group $3\|\sigma_h^{k+1}u_h^{k+1}\|_{\mathbf{L}^2}^2 - 4\|\sigma_h^k u_h^k\|_{\mathbf{L}^2}^2 + \|\sigma_h^{k-1}u_h^{k-1}\|_{\mathbf{L}^2}^2$ must be understood as the discrete time derivative of the kinetic energy, and the group $2(\|\sigma_h^{k+1}\delta u_h^{k+1}\|_{\mathbf{L}^2}^2 -$

$\|\sigma_h^k \delta u_h^k\|_{\mathbf{L}^2}^2) + \|\sigma_h^{k+1} \delta^2 u_h^{k+1}\|_{\mathbf{L}^2}^2$ is the numerical dissipation that controls the pressure splitting. The identity (5.22) is a refinement of (5.18). The proof of Theorem 5.2 can be rewritten by using (5.22) instead of (5.18) and by weakening hypothesis (5.2) appropriately. We believe that it is possible to weaken (5.2) since $\mathcal{I}^{k+1}$ is formally $\mathcal{O}(\tau^3)$.

CONJECTURE 5.1. *Since the algorithm* (5.5)–(5.8) *is formally second-order consistent and stable, we conjecture that the following error estimates hold:*

$$\|\sigma_\tau u_\tau - \sigma_{h\tau} u_{h\tau}\|_{\ell^\infty(\mathbf{L}^2)} \le c(\tau^2 + h^{l+1})$$

and

$$\|u_\tau - u_{h\tau}\|_{\ell^2(\mathbf{H}^1)} \le c(\tau + h^l).$$

The techniques presented here, together with those of [12, 13, 22, 30], *may provide a proof of these facts. This conjecture is substantiated by the numerical experiments reported in section* 6. *Recall also that the above conjectured error estimates are actually known to hold when the density is constant; cf., e.g.,* [12, 13, 22, 30].

Remark 5.5. In full analogy with the constant density case, it is possible to construct a rotational version (see [22, 34]) of the algorithm introduced above by replacing the pressure update (5.8) by the following: Find $p_h^{k+1} \in M_h$ so that

$$(5.24) \qquad \langle p_h^{k+1}, r_h \rangle = \langle p_h^k + \phi_h^{k+1}, r_h \rangle + \mu \langle u_h^{k+1}, \nabla r_h \rangle.$$

The numerical experiments reported in section 6 show that the algorithm (5.5)–(5.24) is stable and accurate. We have not been able to prove this fact yet.

6. Numerical experiments. To test the accuracy of the second-order algorithm proposed in section 5, both in standard and rotational forms, we solve problem (1.1)–(1.2) using an analytical solution defined on the unit disk

$$(6.1) \qquad \Omega = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 < 1\}.$$

The exact solution is

$$(6.2) \qquad \rho(\mathbf{r}, t) = 2 + r\cos(\theta - \sin t),$$

$$(6.3) \qquad u(\mathbf{r}, t) = (-y, x)^\top \cos t,$$

$$(6.4) \qquad p(\mathbf{r}, t) = \sin x \sin y \sin t,$$

and the corresponding right-hand side in the momentum equation is

$$(6.5) \qquad f(\mathbf{r}, t) = \begin{pmatrix} (y \sin t - x \cos^2 t)\rho(\mathbf{r}, t) + \cos x \sin y \sin t \\ -(x \sin t + y \cos^2 t)\rho(\mathbf{r}, t) + \sin x \cos y \sin t \end{pmatrix}.$$

The computations are performed using the library `deal.II` (see [3, 2]). The mass conservation equation is discretized in space using $\mathbb{Q}_2$ continuous finite elements, which we stabilize using an entropy viscosity technique similar to the one described in [17]. In time, we use a second-order backward difference formula (BDF2). To approximate the velocity and pressure, we use Taylor–Hood (i.e., $(\mathbb{Q}2, \mathbb{Q}1)$) elements. We perform the accuracy tests with respect to τ on a mesh consisting of 5120 quadrangular cells. The dimensions of the vector spaces W_h, $\mathbf{X}_h$, and M_h are as follows:

$$(6.6) \qquad \dim W_h = 20609,$$

$$(6.7) \qquad \dim \mathbf{X}_h = 41218,$$

$$(6.8) \qquad \dim M_h = 5185.$$

TABLE 1
Error in time for standard scheme.

τ	ρ–L^2	Rate	u–$\mathbf{L}^2$	Rate	u–$\mathbf{H}^1$	Rate	p–L^2	Rate
0.1	9.15E-003	—	6.93E-003	—	3.29E-002	—	4.34E-002	—
0.05	1.27E-003	2.84	1.70E-003	2.03	9.93E-003	1.73	1.21E-002	1.84
0.03	2.10E-004	2.60	4.20E-004	2.02	3.20E-003	1.64	3.62E-003	1.74
0.01	4.18E-005	2.33	1.05E-004	2.00	1.11E-003	1.52	1.19E-003	1.60
0.01	8.65E-006	2.27	2.61E-005	2.00	3.63E-004	1.62	3.78E-004	1.66

TABLE 2
Error in time for rotational scheme.

τ	ρ–L^2	Rate	u–$\mathbf{L}^2$	Rate	u–$\mathbf{H}^1$	Rate	p–L^2	Rate
0.1	3.70E-003	—	3.90E-003	—	1.59E-002	—	1.12E-002	—
0.05	6.38E-004	2.54	1.18E-003	1.73	4.89E-003	1.70	3.31E-003	1.76
0.03	1.35E-004	2.24	3.34E-004	1.82	1.43E-003	1.78	9.34E-004	1.83
0.01	3.21E-005	2.07	9.03E-005	1.89	4.03E-004	1.82	2.53E-004	1.88
0.01	7.85E-006	2.03	2.37E-005	1.93	1.12E-004	1.84	6.71E-005	1.92

We measure the maximum over the time interval $[0, 10]$ of the errors measured in various norms. This mesh is chosen so that the discretization error in space is significantly smaller than that induced by the time discretization. The convergence with respect to τ is verified in the range $5\times10^{-3} \leq \tau \leq 1\times10^{-1}$.

6.1. The standard formulation. We test the second-order standard formulation described in section 5.1. The results are shown in Table 1. The error on the velocity and the density in the L^2-norm is $\mathcal{O}(\tau^2)$, and the error on the velocity in the $\mathbf{H}^1$-norm and on the pressure in the L^2-norm is $\mathcal{O}(\tau)$. This is consistent with Conjecture 5.1.

6.2. The rotational formulation. Next we test the rotational version of the method which consists of using the pressure update (5.24), introduced in Remark 5.5, instead of (5.8). The results are shown in Table 2. We observe that all the errors are fully second-order with respect to τ. It is likely that there is a superconvergence effect due to the regularity of the domain. We recall that a similar superconvergence effect is observed for the rotational variant of the pressure-correction algorithm for constant density flows (see [22]). We conjecture that in general domains the error on the velocity measured in the $\mathbf{L}^2$-norm is $\mathcal{O}(\tau^2)$, and the error on the velocity in the $\mathbf{H}^1$-norm and on the pressure in the L^2-norm is $\mathcal{O}(\tau^{3/2})$.

REFERENCES

[1] A. S. ALMGREN, J. B. BELL, P. COLELLA, L. H. HOWELL, AND M. L. WELCOME, *A conservative adaptive projection method for the variable density incompressible Navier-Stokes equations*, J. Comput. Phys., 142 (1998), pp. 1–46.

[2] W. BANGERTH, R. HARTMANN, AND G. KANSCHAT, *deal.II: A Finite Element Differential Equations Analysis Library*, http://www.dealii.org.

[3] W. BANGERTH, R. HARTMANN, AND G. KANSCHAT, *deal.II—a general-purpose object-oriented finite element library*, ACM Trans. Math. Softw., 33 (2007), article 24.

[4] J. B. BELL AND D. L. MARCUS, *A second-order projection method for variable-density flows*, J. Comput. Phys., 101 (1992), pp. 334–348.

[5] E. BURMAN AND P. HANSBO, *Edge stabilization for Galerkin approximations of convection-diffusion-reaction problems*, Comput. Methods Appl. Mech. Engrg., 193 (2004), pp. 1437–1453.

[6] L. CATTABRIGA, *Su un problema al contorno relativo al sistema di equazioni di Stokes*, Rend. Sem. Mat. Univ. Padova, 31 (1961), pp. 308–340.

[7] A. CHORIN, *Numerical solution of the Navier-Stokes equations*, Math. Comp., 22 (1968), pp. 745–762.

[8] J. DOUGLAS, JR., AND T. F. RUSSELL, *Numerical methods for convection-dominated diffusion problems based on combining the method of characteristics with finite element or finite difference procedures*, SIAM J. Numer. Anal., 19 (1982), pp. 871–885.

[9] A. ERN AND J.-L. GUERMOND, *Theory and Practice of Finite Elements*, Appl. Math. Sci. 159, Springer-Verlag, New York, 2004.

[10] V. GIRAULT AND P.-A. RAVIART, *Finite Element Methods for Navier-Stokes Equations: Theory and Algorithms*, Springer Ser. Comput. Math., Springer-Verlag, Berlin, 1986.

[11] J.-L. GUERMOND, *Some practical implementations of projection methods for Navier–Stokes equations*, M2AN Math. Model. Numer. Anal., 30 (1996), pp. 637–667.

[12] J.-L. GUERMOND, *Un résultat de convergence d'ordre deux pour l'approximation des équations de Navier–Stokes par projection incrémentale*, C. R. Acad. Sci. Paris Sér. I Math., 325 (1997), pp. 1329–1332.

[13] J.-L. GUERMOND, *Un résultat de convergence d'ordre deux en temps pour l'approximation des équations de Navier-Stokes par une technique de projection incrémentale*, M2AN Math. Model. Numer. Anal., 33 (1999), pp. 169–189.

[14] J.-L. GUERMOND, A. MARRA, AND L. QUARTAPELLE, *Subgrid stabilized projection method for 2D unsteady flows at high Reynolds number*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 5857–5876.

[15] J. L. GUERMOND, P. MINEV, AND J. SHEN, *An overview of projection methods for incompressible flows*, Comput. Methods Appl. Mech. Engrg., 195 (2006), pp. 6011–6045.

[16] J.-L. GUERMOND AND R. PASQUETTI, *Entropy-based nonlinear viscosity for Fourier approximations of conservation laws*, C. R. Math. Acad. Sci. Paris, 346 (2008), pp. 801–806.

[17] J.-L. GUERMOND, R. PASQUETTI, AND B. POPOV, *Entropy viscosity method for nonlinear conservation laws*, J. Comput. Phys., 230 (2011), pp. 4248–4267.

[18] J.-L. GUERMOND AND L. QUARTAPELLE, *On the approximation of the unsteady Navier–Stokes equations by finite element projection methods*, Numer. Math., 80 (1998), pp. 207–238.

[19] J.-L. GUERMOND AND L. QUARTAPELLE, *A projection FEM for variable density incompressible flows*, J. Comput. Phys., 165 (2000), pp. 167–188.

[20] J.-L. GUERMOND AND A. SALGADO, *A fractional step method based on a pressure Poisson equation for incompressible flows with variable density*, C. R. Math. Acad. Sci. Paris, 346 (2008), pp. 913–918.

[21] J.-L. GUERMOND AND A. SALGADO, *A splitting method for incompressible flows with variable density based on a pressure Poisson equation*, J. Comput. Phys., 228 (2009), pp. 2834–2846.

[22] J.-L. GUERMOND AND J. SHEN, *On the error estimates for the rotational pressure-correction projection methods*, Math. Comp., 73 (2004), pp. 1719–1737.

[23] F. GUILLÉN-GONZÁLEZ AND J. V. GUTIÉRREZ-SANTACREU, *Unconditional stability and convergence of fully discrete schemes for 2D viscous fluids models with mass diffusion*, Math. Comp., 77 (2008), pp. 1495–1524.

[24] J. G. HEYWOOD AND R. RANNACHER, *Finite element approximation of the nonstationary Navier–Stokes problem. I. Regularity of solutions and second-order error estimates for spatial discretization*, SIAM J. Numer. Anal., 19 (1982), pp. 275–311.

[25] C. JOHNSON, U. NÄVERT, AND J. PITKÄRANTA, *Finite element methods for linear hyperbolic equations*, Comput. Methods Appl. Mech. Engrg., 45 (1984), pp. 285–312.

[26] P.-L. LIONS, *Mathematical Topics in Fluid Mechanics: Volume 1: Incompressible Models*, Oxford Lecture Ser. Math. Appl. 3, The Clarendon Press, Oxford University Press, New York, 1996.

[27] C. LIU AND N. J. WALKINGTON, *Convergence of numerical approximations of the incompressible Navier–Stokes equations with variable density and viscosity*, SIAM J. Numer. Anal., 45 (2007), pp. 1287–1304.

[28] A. PROHL, *Projection and Quasi-compressibility Methods for Solving the Incompressible Navier-Stokes Equations*, Adv. Numer. Math., B. G. Teubner, Stuttgart, 1997.

[29] J.-H. PYO AND J. SHEN, *Gauge-Uzawa methods for incompressible flows with variable density*, J. Comput. Phys., 221 (2007), pp. 181–197.

[30] J. SHEN, *On error estimates of projection methods for the Navier-Stokes equations: Second-order schemes*, Math. Comp., 65 (1996), pp. 1039–1065.

[31] J. SHEN AND X. YANG, *A phase-field model and its numerical approximation for two-phase incompressible flows with different densities and viscosities*, SIAM J. Sci. Comput., 32 (2010), pp. 1159–1179.

[32] R. Temam, *Sur l'approximation de la solution des équations de Navier–Stokes par la méthode de pas fractionnaires*, Arch. Rational Mech. Anal., 33 (1969), pp. 377–385.

[33] R. Temam, *Navier-Stokes Equations: Theory and Numerical Analysis*, North–Holland, Amsterdam, 1984.

[34] L. Timmermans, P. Minev, and F. van de Vosse, *An approximate projection scheme for incompressible flow using spectral elements*, Internat. J. Numer. Methods Fluids, 22 (1996), pp. 673–688.

[35] N. J. Walkington, *Convergence of the discontinuous Galerkin method for discontinuous solutions*, SIAM J. Numer. Anal., 42 (2005), pp. 1801–1817.

[36] M. F. Wheeler, *A priori L_2 error estimates for Galerkin approximations to parabolic partial differential equations*, SIAM J. Numer. Anal., 10 (1973), pp. 723–759.