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NUMERICAL QUADRATURES FOR LAYER POTENTIALS OVER CURVED DOMAINS IN $\mathbb{R}^3$*

JEAN-LUC GUERMOND†

Abstract. Numerical quadratures for calculating simple and double layer potentials over curved domains in $\mathbb{R}^3$ are given. More generally, these numerical quadratures are shown to be useful for evaluating surface integral involving kernels which are pseudohomogeneous of degree -1. In addition to potential flow problems, such kernels are encountered when dealing with Helmholtz problem, Stokes flow problems, or linear elasticity problems.

Key words. boundary integral methods, layer potentials, numerical quadratures, pseudohomogeneity

AMS(MOS) subject classifications. 65D30, 65D32, 65R20

1. Introduction. A large class of linear physical problems can be tackled by means of a boundary integral equation involving a Green function. Among these problems are Stokes flows, potential flows, Helmholtz problems, and linear elasticity problems. Usually the mathematical problem consists of finding a scalar- or vector-valued function s, which is a solution to a Fredholm equation:

$$\lambda s(x) + \int_{\partial\Omega} K(x,y)\cdot s(y)\mathrm{d}\sigma_y = g(x), \tag{1.1}$$

for almost every point x belonging to $\partial\Omega$. $\partial\Omega$ is the boundary of the physical domain Ω, $g(x)$ is a known function, λ is a scalar which may be zero, and $K(x,y)$ is the Green kernel. $K(x,y)$ is either scalar- or matrix-valued. Depending whether λ is zero or not, the equation is said to be of the first kind or of the second kind. A review of the boundary integral formulation of linear problems may be found in [5], [11], [14].

There are numerous means for solving (1.1) numerically. The collocation method is the simplest and consequently one of the most popular amongst engineers. In $\mathbb{R}^3$, it consists of partitioning $\partial\Omega$ into a set of flat panels $(\Gamma_i)_{i\in I}$ and looking for the approximation of s, which is locally constant over the panels Γ_i and satisfies (1.1) when x is a panel centroid. See [2], [9] for a review on this technique. The Galerkin method (cf. Johnson and Scott [10] or Wendland [14]) or the finite element method (cf. Giroire [7] and Nedelec [11]) are other possible approaches for solving (1.1). Actually, a recent work by Johnson and Scott [10] has shown, among other things, that the collocation method reduces to be a Galerkin approximation of the first order.

Whatever method is chosen for solving (1.1), one is always obliged to evaluate the influence of the panel set $(\Gamma_i)_{i\in I}$ on a finite set of points $(x_l)_{l\in L}$, which are either control points or quadrature points. In other words, the following integrals

$$\int_{\Gamma_i} K(x,y)\cdot s(y)\mathrm{d}\sigma_y := \mathcal{K}[s,x,i] \tag{1.2}$$

have to be evaluated. If the panel Γ_i is flat and if s is a polynomial, analytical expressions of (1.2) can be derived for some classes of kernels [9]. In general, if s is a polynomial of degree greater than 1, then it is not consistent to approximate $\partial\Omega$ with flat panels [11], [12]. Hence, when a polynomial approximation of s is desired, $\partial\Omega$

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†Laboratoire d'Informatique pour la Méchanique et les Sciences de l'Ingénieur, Centre National de la Recherche Scientifique, BP 133, 91403 Orsay Cedex, France. (guermond@FRLIM51.bitnet.)

must be approximated by curved panels. In these conditions, analytical expressions of (1.2) cannot be obtained and a numerical approximation is required.

Even though the kernels $K(x,.)$ with which we are concerned are locally integrable, numerical evaluation of (1.2) is hampered by singular or pseudosingular behavior of K, which arises when x belongs to Γ_i or when x is very close to Γ_i. In these two cases, special care is to be given to the way of evaluating (1.2). See Johnson and Scott's paper [10] for examples of numerical approximations of (1.2).

The objective of this paper is to present numerical quadratures that are suitable for approximating surface integrals of type (1.2), whatever the distance from x to Γ_i. The panels Γ_i may be curved at will; it is only assumed that they are defined by regular mappings ψ_i. $K(x,.)$ is assumed to belong to the class of the kernels which are pseudohomogeneous of degree -1.

The outline of the paper is as follows. In §2 some definitions are recalled, the atlas $(\Gamma_i,\psi_i)_{i\in I}$ is defined, and some classical regularity criteria are recalled. In §3, pseudohomogeneity and its consequences are reviewed. A new definition of pseudohomogeneity that emphasizes the importance of polar coordinates is given. It is shown that the Green kernels of the physical problems, which are referred to above, are pseudohomogeneous. Numerical quadratures along with estimates of the quadrature error for the near field, the far field, and the intermediate field are reported in §§4, 5, and 6, respectively.

2. Preliminaries and regularity criteria.

2.1. Regularity of $\partial\Omega$. $\partial\Omega$ must be regular enough to ensure existence and uniqueness of a solution to problem (1.1). In general, it can be shown that $\partial\Omega$ needs only to be Lipschitzian [13]. This condition is realized if $\partial\Omega$ is piecewise C^1 and Ω is locally on one side of its boundary. The second hypothesis means that $\partial\Omega$ may not have cusped edges. In what follows, Ω is an open subset in $\mathbb{R}^3$ whose boundary $\partial\Omega$ is an oriented manifold of dimension 2. $\partial\Omega$ is assumed to be bounded and piecewise C^r, the smoothness degree r is assumed to be as great as needed. The unit of length in $\mathbb{R}^3$ is defined so that $\text{diam}(\Omega)$ is one.

2.2. The panelling definition. Let I be a finite subset of $\mathbb{N}$ and $(\Gamma_i,\psi_i)_{i\in I}$ be a panelling of $\partial\Omega$ so that $\partial\Omega=\cup_{i\in I}\Gamma_i$ and $\text{int}\,(\Gamma_i)\cap\,\text{int}\,(\Gamma_j)=\emptyset$ if $i\neq j$. Let $\widehat{S}$ be the reference simplex of $\mathbb{R}^2$:

$$\widehat{S}=\{(\widehat{x}_1,\widehat{x}_2)\in\mathbb{R}^2\ :\ 0\leq\widehat{x}_1,\ 0\leq\widehat{x}_2,\ 0\leq\widehat{x}_1+\widehat{x}_2\leq 1\}. \tag{2.1}$$

For all i in I, the mapping $\psi_i:\widehat{S}\longrightarrow\Gamma_i$ is assumed to be in $C^r(\text{int}(\widehat{S}))\cap C^0(\widehat{S})$ and to satisfy the continuity condition $\psi_i(\partial\Gamma_i\cap\partial\Gamma_j)=\psi_j(\partial\Gamma_i\cap\partial\Gamma_j)$ if $\partial\Gamma_i\cap\partial\Gamma_j\neq\emptyset$. Note that, at variance with the finite element method, the panelling may not be structured. That is, we do not force the vertices of each panel Γ_i to match the vertices of the panels which have a piece of boundary in common with Γ_i. This degree of freedom may facilitate the panelling generation [8].

2.3. Some definitions. Let $M\subset\mathbb{R}^n$ and $M'\subset\mathbb{R}^{n'}$ be two C^k-manifolds. Let $f:M\longrightarrow M'$ be a C^k mapping. Consider $x\in M$ and denote T_x and $T_f(x)$ the vector spaces which are tangent to M and M' at x and $f(x)$, respectively. Introduce the jth derivative of f, $j\leq k$. Recall that $D^jf(x)$ is a j-linear operator which maps $T_x\times\cdots\times T_x$ to $T_f(x)$. See Avez [1] and Cartan [3] and the references therein for a review on differential calculus.

$\mathbb{R}^n$ and $\mathbb{R}^{n'}$ are equipped with their usual Euclidian structure, $\|.\|$ and $(.,.)$ denote their Euclidian norm and inner product. For $1 \leq q \leq \infty$, we also make use of the q-norms, which will be denoted $\|.\|_q$. The Euclidian structure is obtained when $q = 2$. Recall that the q-norms are all equivalent. It is now possible to define $\|D^j f(x)\|$:

$$\|D^j f(x)\| := \sup\{\|D^j f(x)(\xi_1, \cdots, \xi_j)\| \ : \ \xi_l \in T_x, \|\xi_l\| = 1\}, \tag{2.2}$$

The Sobolev seminorms and norms are introduced. $|f|_{j,q,M}$ denotes the norm of $\|D^j f(\cdot)\|$ in $L^q(M, \mathbb{R})$. $\|f\|_{j,q,M}$ denotes the q-norm of $(|f|_{0,q,M}, \dots, |f|_{j,q,M})$ considered as a vector in $\mathbb{R}^{j+1}$. Recall that $|f|_{0,q,M}$ denotes the norm of f in $L^q(M, \mathbb{R}^{n'})$.

Let $A : \mathbb{R}^n \times \cdots \times \mathbb{R}^n \longrightarrow \mathbb{R}^{n'}$ be a j-linear operator. Define sym(A) the symmetric j-linear operator so that

$$\operatorname{sym}(A)(h_1, \cdots, h_j) = \frac{1}{j!} \sum_{s \in \mathcal{S}_j} A(h_{s(1)}, \cdots, h_{s(j)}), \tag{2.3}$$

where $\mathcal{S}_j$ is the set of the permutations of $\{1, \cdots, j\}$.

LEMMA 2.1. *Let $f : M \longrightarrow E_1$, $g : M \longrightarrow E_2$ where $M \subset \mathbb{R}^n$ is a C^k-manifold and E_1, E_2 are two normed vector spaces. Let G be a normed vector space and $b : E_1 \times E_2 \longrightarrow G$ be a continuous bilinear mapping. Assume f and g are C^j, $j \leq k$, then $x \in M \longmapsto b(f(x), g(x))$ is C^j-differentiable and*

$$D^j b(f(x), g(x)) = \operatorname{sym} \sum_{l=0}^{j} \binom{l}{j} b(D^{j-l} f(x), D^l g(x)). \tag{2.4}$$

Proof. This result is classical, it generalizes Leibnitz's rule. See Avez [1] for details. □

LEMMA 2.2. *Let $M_1 \subset \mathbb{R}^{n_1}$, $M_2 \subset \mathbb{R}^{n_2}$, and $M_3 \subset \mathbb{R}^{n_3}$ be three C^k-manifolds. Let $f : M_1 \longrightarrow M_2$, $g : M_2 \longrightarrow M_3$. Assume f and g are C^j, $j \leq k$, then $g \circ f$ is C^j and*

$$D^j(g \circ f)(x) = \operatorname{sym} \sum_{l=1}^{j} \sum_{r_1 + \cdots + r_l = j} \frac{j!}{l! r_1! \cdots r_l!} D^l g(f(x))(D^{r_1} f(x), \cdots, D^{r_l} f(x)). \tag{2.5}$$

Proof. See Avez [1, p. 38] or Ciarlet and Raviart [4] for more details. □

2.4. The panelling regularity. Let S_{1i}, S_{2i}, and S_{3i} be the images by ψ_i of the reference simplex's vertices $\widehat{S}_1$, $\widehat{S}_2$, and $\widehat{S}_3$. Let $\widetilde{\Gamma}_i$ be the convex hull of (S_{1i}, S_{2i}, S_{3i}). $\widetilde{\Gamma}_i$ is a two-dimensional simplex. Let $\widetilde{h}_i$ be the diameter of $\widetilde{\Gamma}_i$ and $\widetilde{\rho}_i$ the diameter of the circle, which is inscribed in $\widetilde{\Gamma}_i$. In the sequel, the family of simplices $(\widetilde{\Gamma}_i)_{i \in I}$ is regular in accordance with the following.

DEFINITION 2.1. A family of simplices $(\widetilde{\Gamma}_i)_{i \in I}$ is regular if there exists a real number α_0 such that for all indices i in I :

$$0 < \alpha_0 < \frac{\widetilde{\rho}_i}{\widetilde{h}_i}. \tag{2.6}$$

The regularity of the family implies the following result.

THEOREM 2.1. *If the family of simplices $(\widetilde{\Gamma}_i)_{i\in I}$ is regular, for all $i \in I$ there exists a unique invertible linear mapping $\widetilde{\psi}_i : \widehat{S} \longrightarrow \widetilde{\Gamma}_i$ such that:*

$$\forall j \in \{1,2,3\},\ \widetilde{\psi}_i(\widehat{S}_j) = S_{ji}, \tag{2.7}$$

$$\frac{\alpha_0 \widetilde{h}_i}{\widehat{h}} \leq |\widetilde{\psi}_i|_{1,\infty,\widehat{S}} \leq \frac{\widetilde{h}_i}{\widehat{\rho}}, \tag{2.8}$$

$$\frac{\widehat{\rho}}{\widetilde{h}_i} \leq |\widetilde{\psi}_i^{-1}|_{1,\infty,\Gamma_i} \leq \frac{\widehat{h}}{\alpha_0 \widetilde{h}_i}. \tag{2.9}$$

Proof. This result is classical. For a proof, see R. Dautray and J. L. Lions [5, t.II, p. 816]. □

In order to measure the deviation of Γ_i from a plane, it is convenient to define E_i: $\mathrm{int}(\widehat{S}) \longrightarrow \mathbb{R}^3$ so that $E_i := \psi_i - \widetilde{\psi}_i$. The function E_i is C^r. At this point, it is now possible to define regularity criteria for the panelling $(\Gamma_i, \psi_i)_{i\in I}$.

DEFINITION 2.2. The panelling $(\Gamma_i, \psi_i)_{i\in I}$ is regular if

1. the family $(\widetilde{\Gamma}_i)_{i\in I}$ is regular,
2. there are r positive constants $c_1, \cdots, c_r$ so that $c_1 < 1$ and for all j in $\{1, \cdots, r\}$ and for all i in I:

$$|\widetilde{\psi}_i^{-1}|^j_{1,\infty,\Gamma_i} |E_i|_{j,\infty,\widehat{S}} \leq c_j, \tag{2.10}$$

3. there is a constant $\beta_0 \geq 1$ so that for all panels Γ_i and Γ_j, which have a piece boundary in common (i.e., $\partial\Gamma_i \cap \Gamma_j \neq \emptyset$), the diameters $\widetilde{h}_i$ and $\widetilde{h}_j$ satisfy:

$$\widetilde{h}_i/\beta_0 \leq \widetilde{h}_j \leq \beta_0 \widetilde{h}_i. \tag{2.11}$$

Condition 1 implies that simplices $\widetilde{\Gamma}_i$ do not differ very much from equilateral simplices. Condition 2 is rendered more explicit and reads $|E_i|_{j,\infty,\widehat{S}} \leq c_j(\widetilde{h}_i/\widehat{\rho})^j$ when (2.9) is used. This condition says that panels Γ_i must not be too far from simplices $\widetilde{\Gamma}_i$. The third condition imposes two neighbouring panels to have equivalent diameters. It is a kind of quasi-uniformity condition for the mesh, though it is weaker than classical quasi-uniformity conditions for finite element meshes. Note that this condition is automatically satisfied if $(\Gamma_i)_{i\in I}$ is a regular family of finite elements.

Examples of panelling $(\Gamma_i, \psi_i)_{i\in I}$, which are regular when $\sup_{i\in I}\{\widetilde{h}_i\}$ is small enough, are given by Nedelec in [11].

2.5. Preliminary inequalities. In all the sequel of the paper the panelling $(\Gamma_i, \psi_i)_{i\in I}$ is assumed to be regular. The regularity property 2 implies that

LEMMA 2.3. *ψ_i is a C^r-diffeomorphism.*

Proof. We prove first that $D\psi_i$ is invertible. $D\psi_i$ may be written in the form $(Id + DE_i \circ D\widetilde{\psi}_i^{-1}) \circ D\widetilde{\psi}_i$. Condition 2 for $j = 1$ implies that $(Id + DE_i \circ D\widetilde{\psi}_i^{-1})$ is invertible; as a result, $D\psi_i$ is invertible. The lemma results from the fact that $D\psi_i$ is invertible and ψ_i is C^r. □

The regularity properties 1 and 2 have other important consequences to the behavior of ψ_i and ψ_i^{-1}.

THEOREM 2.2. *For all $j \in \{1, \cdots, r\}$ there are constants c so that for all i in I:*

$$|\psi_i|_{j,\infty,\widehat{S}} \leq c|\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}}, \tag{2.12}$$

$$|\psi_i^{-1}|_{j,\infty,\Gamma_i} \leq c|\widetilde{\psi}_i^{-1}|_{1,\infty,\Gamma_i}. \tag{2.13}$$

Proof of the first inequality. For $j = 1$, the inequality comes from the relation $D\psi_i = (Id + DE_i \circ D\widetilde{\psi}_i^{-1}) \circ D\widetilde{\psi}_i$ along with the second regularity hypothesis (2.10) together with regularity inequalities (2.8) and (2.9). For $j > 1$, the inequality is directly derived from $D^j\psi_i = D^jE_i \circ (D\widetilde{\psi}_i^{-1} \circ D\widetilde{\psi}_i)^j$ and hypothesis (2.10). $\square$

Proof of the second inequality. For $j = 1$, we use $D\psi_i^{-1} = D\widetilde{\psi}_i^{-1} \circ (Id + DE_i \circ D\widetilde{\psi}_i^{-1})^{-1}$ together with the inequality $\|(Id+DE_i \circ D\widetilde{\psi}_i^{-1})^{-1}\|_{0,\infty,\Gamma_i} \leq 1/(1-c_1)$, which is derived from the regularity property: $|\widetilde{\psi}_i^{-1}|_{1,\infty,\Gamma_i}|E_i|_{1,\infty,\widehat{S}} \leq c_1 < 1$. For $j > 1$, the inequality is shown by induction on j and by using the relation $D^j(\psi_i \circ \psi_i^{-1}) = 0$ together with Lemma 2.1. $\square$

A direct consequence of Theorem 2.2 together with Theorem 2.1 is that ψ_i and its inverse satisfy the following inequalities:

$$|\psi_i|_{j,\infty,\widehat{S}} \leq c\widetilde{h}_i^j, \tag{2.14}$$

$$|\psi_i^{-1}|_{j,\infty,\widehat{S}} \leq c\widetilde{h}_i^{-1}. \tag{2.15}$$

A result equivalent to that of Theorem 2.2 can be obtained for the Jacobian determinants of the mapping ψ_i and its inverse:

THEOREM 2.3. *Let J_i and J_{-i} be the Jacobian determinants of $D\psi_i$ and $D\psi_i^{-1}$, respectively. For all j in $\{1, \cdots, r\}$ there are constants c so that:*

$$|J_i|_{j,\infty,\widehat{S}} \leq c|\widetilde{\psi}_i|^{j+2}_{1,\infty,\widehat{S}}, \tag{2.16}$$

$$|J_{-i}|_{j,\infty,\Gamma_i} \leq c|\widetilde{\psi}_i^{-1}|^2_{1,\infty,\Gamma_i}. \tag{2.17}$$

Proof of the first inequality. Let $\widehat{e} := (\widehat{e}_1, \widehat{e}_2)$ be the canonical basis of $\mathbb{R}^2$. The square of J_i is equal to $(D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2, D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2)$. Hence, Lemma 2.1 implies that

$$\text{sym} \sum_{l=0}^{j} \binom{l}{j} D^{j-l}J_i.D^lJ_i = D^j(D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2, D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2). \tag{2.18}$$

Applying Lemma 2.1 twice to the right-hand side of (2.18) and using the regularity criteria on ψ_i yields the bound

$$\|D^j(D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2, D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2)\| \leq c|\widetilde{\psi}_i|^{4+j}_{1,\infty,\widehat{S}}.$$

Then, the bound on $|J_i|_{j,\infty,\widehat{S}}$ is obtained by induction on j and by using the inequality

$$\|D^j J_i\| \leq \frac{1}{2}\|J_{-i}\| \left(c\|\widetilde{\psi}_i\|^{4+j}_{1,\infty,\widehat{S}} + \sum_{l=1}^{j-1} \binom{l}{j} \|D^{j-l}J_i\| \, \|D^l J_i\| \right),$$

which is derived from (2.18).

Proof of the second inequality. J_{-i} is the inverse of $J_i \circ \psi_i^{-1}$; hence, Lemma 2.2 yields:
(2.19)

$$D^j J_{-i} = \operatorname{sym} \sum_{l=1}^{j} \sum_{r_1+\cdots+r_l=j} (-1)^l \frac{j!}{r_1!\cdots r_l!} J_{-i}^{l+1} D^{r_1}(J_i \circ \psi_i^{-1}), \cdots, D^{r_l}(J_i \circ \psi_i^{-1}).$$

Using Lemma 2.2 once more together with the regularity criteria on ψ_i and the bounds on $|J_i|_{l,\infty,\widehat{S}}$ yields

$$\|D^l(J_i \circ \psi_i^{-1})\| \leq c|\widetilde{\psi}_i|^2_{1,\infty,\widehat{S}} \cdot$$

Then, the desired inequality is obtained by using the inequality above and (2.5). □

The preceding theorem along with Theorem 2.1 implies that J_i and its inverse satisfy the following inequalities:

$$|J_i|_{j,\infty,\widehat{S}} \leq c\widetilde{h}_i^{j+2}, \tag{2.20}$$

$$|J_{-i}|_{j,\infty,\widehat{S}} \leq c\widetilde{h}_i^{-2}. \tag{2.21}$$

Bounds on the variation of the normal to $\partial\Omega$ will be needed in further developments. These bounds are given by the following theorem.

THEOREM 2.4. *Consider y in* $\operatorname{int}(\Gamma_i)$. *Let n_y be the normal to Γ_i at y, then*

$$|n_y \circ \psi_i|_{j,\infty,\widehat{S}} \leq c|\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \leq c'\widetilde{h}_i^j. \tag{2.22}$$

Proof. The definition of n_y yields that $n_y \circ \psi_i$ is equal to $(D\psi_i.\widehat{e}_1 \times D\psi_i.\widehat{e}_2)/J_i$. The desired bounds are obtained by applying the same technique as above to the preceding expression. □

3. Pseudohomogeneity.

3.1. Definitions and examples. Consider x a point in $\mathbb{R}^n$. Introduce the polar coordinates (r, θ) for y in $\mathbb{R}^n - \{x\}$ so that

$$r := \|y - x\|, \quad \text{and} \quad \theta := \frac{y - x}{r}. \tag{3.1}$$

Denote by P_x the polar mapping $(r, \theta) \mapsto y$. Let Σ be the unit sphere in $\mathbb{R}^n$. P_x is a C^∞-diffeomorphism which maps $]0, \infty[\times\Sigma$ onto $\mathbb{R}^n - \{x\}$. Consider $f : M \longrightarrow \mathbb{R}^{n'}$, where $M \subset \mathbb{R}^n$ is a manifold. When no confusion is possible, the notation $f \circ P_x(r, \theta)$ is hereafter replaced by $f(r, \theta)$ for short. The following result will be needed in further developments.

LEMMA 3.1. *Consider a C^k-manifold $M \subset \mathbb{R}^n$. Assume $f : M \longrightarrow \mathbb{R}^{n'}$ is C^k and its derivatives are bounded in a neighbourhood $B(x,\rho) \cap M$ of x, then $f(r,\theta)$ is C^k in $P_x^{-1}(B(x,\rho) \cap M - \{x\})$, and the k derivatives are bounded in the domain in question.*

Proof. Since P_x is C^∞, the first part of the lemma is a consequence of Lemma 2.2. Furthermore, assume y is in $B(x,\rho)$, then $\|DP_x\|$ is bounded above by $1+\rho$, $\|D^2 P_x\|$ is equal to 2, and for l greater than 2, $\|D^l P_x\|$ is zero. Hence, from Lemma 2.2 we infer that the k derivatives of $f \circ P$ are bounded in $P_x^{-1}(B(x,\rho) \cap M - \{x\})$. □

Let x be a regular point of $\partial\Omega$. The kernels which are considered in this paper are assumed to be pseudohomogeneous of degree -1 in accordance with the following.

DEFINITION 3.1. $K(x,\cdot)$ is said to be pseudohomogeneous of degree -1 at x up to order k, if there is an open neighbourhood U of x so that for all y in $U \cap \partial\Omega - \{x\}$, $rK(x,r,\theta)$ is differentiable with respect to the polar variables up to order k, and the derivatives in question are bounded in $P_x^{-1}(U \cap M - \{x\})$.

Note that there is another definition of pseudohomogeneity; see Johnson and Scott [10], and the following.

DEFINITION 3.2. $K(x,y)$ is pseudohomogeneous of degree -1 at x up to order k, if in a neighbourhood of x on $\partial\Omega$, the kernel admits an expansion of the form:

$$K(x,y) = \sum_{j=-1}^{k-1} K_j(x, \pi_x(y-x)) + R_k(x,y), \tag{3.2}$$

where π_x denotes the projection onto the tangent plane to $\partial\Omega$ at x, $K_j(x,\xi)$ is homogeneous of degree j in the vector ξ, and the remainder R_k is C^k in the neighbourhood in question.

In the sequel of this paper, the kernel K is assumed to be pseudohomogeneous of degree -1 in accordance with the first definition. It may be verified that the first definition generalizes the second one. The equivalence between the two definitions might be true but has not been verified by the author.

In the next subsection, the following kernels will be shown to be pseudohomogeneous of degree -1 at x.

Example 1. The Green functions of Laplace and Helmholtz equations in $\mathbb{R}^3$. Let G_{1L} and G_{1H} be the two kernels in question:

$$G_{1L}(x,y) = -\frac{1}{4\pi\|x-y\|}, \tag{3.3}$$

$$G_{1H}(x,y) = -\frac{e^{ik\|x-y\|}}{4\pi\|x-y\|}. \tag{3.4}$$

G_{1L} and G_{1H} are also called single layer potentials.

Example 2. The double layer potentials of Laplace and Helmholtz equations in $\mathbb{R}^3$. Denote by G_{2L} and G_{2H} these two kernels:

$$G_{2L}(x,y) = \frac{\partial}{\partial n_y} G_{1L}(x,y), \tag{3.5}$$

$$G_{2H}(x,y) = \frac{\partial}{\partial n_y} G_{1H}(x,y), \tag{3.6}$$

where n_y denotes the normal to $\partial\Omega$ at y, and $\partial/\partial n_y$ denotes the derivative at y in the direction n_y.

Example 3. The simple layer potentials of Stokes flow problems and linear elasticity problems.

$$G_S(x,y) = \frac{1}{8\pi\nu}\left(\frac{Id}{\|x-y\|} + \frac{(x-y)\otimes(x-y)}{\|x-y\|^3}\right), \tag{3.7}$$

$$G_E(x,y) = \frac{1}{8\pi\mu}\left(\frac{\lambda+3\mu}{\lambda+2\mu}\frac{Id}{\|x-y\|} + \frac{\lambda+\mu}{\lambda+2\mu}\frac{(x-y)\otimes(x-y)}{\|x-y\|^3}\right), \tag{3.8}$$

where ν is the fluid viscosity and λ, μ are the two elasticity coefficients of the material.

3.2. Review of some kernels. Consider $f : \mathbb{R} \longrightarrow \mathbf{C}$ a complex-valued function which is C^k in a neighbourhood of zero. Assume $\partial\Omega$ is C^k in a neighbourhood $U \cap \partial\Omega$ of x.

PROPOSITION 3.1. *Let $K(x,y) := f(\|y-x\|)/\|y-x\|$, then $K(x,\cdot)$ is pseudohomogeneous of degree -1 at x up to order k.*

Proof. The definition of K implies $rK(x,r,\theta) = f(r)$. Hence, from the local regularity of $\partial\Omega$ at x, we infer that $rK(x,r,\theta)$ is C^k with respect to the polar coordinates. Furthermore, for all $j \le k$, $\|D^j rK\|$ is equal to $f^j(r)$; hence, $D^j(rK)$ is bounded above in a neighbourhood of x. □

As a consequence of Proposition 3.1, we see that the single layer potentials of Laplace and Helmholtz equations are pseudohomogeneous of degree -1 at x up to order k (take $f(r) = -1/4\pi$ or $f(r) = -e^{ikr}/4\pi$).

PROPOSITION 3.2. *Let $K(x,y) := f(\|y-x\|)\partial/\partial n_y\,(1/\,\|y-x\|)$, then $K(x,.)$ is pseudohomogeneous of degree -1 at x up to order $k-2$.*

Proof. The definition of the kernel implies

$$rK(x,r,\theta) = f(r)\frac{(n_y,\theta)}{r}. \tag{3.9}$$

The local regularity of $\partial\Omega$ implies that n_y is C^{k-1} with respect to the polar variables in $U \cap \partial\Omega - \{x\}$. As a result $rK(x,r,\theta)$ is C^{k-1} in $P_x^{-1}(U \cap \partial\Omega - \{x\})$. Let us show now that the derivatives up to order $k-2$ are bounded above in a neighbourhood of x. Note that the normal n_y can be written in the form

$$n_y = n_x + F_x(y),$$

where $F_x(y)$ is C^{k-1} in a neighbourhood of x and $F_x(x)$ is zero. $\partial\Omega$ being locally C^k, there is a real-valued function $G_x(y)$, which is locally C^k at x such that $G_x(x)$ and $DG_x(x)$ are zero, and locally $\partial\Omega$ is defined by

$$y - x = \pi_x(y-x) + G_x(y)n_x,$$

where π_x denotes the projection onto the tangent plane to $\partial\Omega$ at x. As a consequence, we have

$$\theta = \frac{\pi_x(y-x)}{r} + \frac{G_x(y)}{r}n_x,$$

and $rK(x,r,\theta)$ can be cast into the form:

$$rK(x,r,\theta) = f(r)\left[\left(\frac{\pi_x(y-x)}{r},\frac{F_x(y)}{r}\right) + \left(\frac{G_x(y)}{r},\frac{F_x(y)}{r}\right) + \frac{G_x(y)}{r^2}\right].$$

The proposition results from this decomposition and the following lemma. □

LEMMA 3.2. *Let M be a manifold in $\mathbb{R}^n$, let x be a point of M in the vicinity of which M is C^k. Consider $H : M \longrightarrow E$, where E is a Banach space, assume that H is C^k in a neighbourhood $U \cap M$ of x and there is an integer J, $0 \leq J < k$, such that for all $0 \leq j \leq J$, the jth derivative $D^j H(x)$ is zero; then, $H(r,\theta)/r^{J+1}$ is C^k in $P_x^{-1}(U \cap M - \{x\})$ and its derivatives are bounded above up to order $k - J - 1$.*

Proof. Let T_x be the tangent space to M at x. There is an open ball $B(x,\rho)$ in T_x and a function $F : T_x \cap B(x,\rho) \longrightarrow T_x^{\perp}$, which is C^k in $B(x,\rho)$ such that $\partial\Omega$ is locally defined by

$$y - x = \pi_x(y-x) + F(\pi_x(y-x)),$$

where $F(0)$ is zero. Consider y in $M \cap B(x,\rho)$. Define $u : [0,1] \longrightarrow M$ such that

$$u(t) = H\left[x + \pi_x(t(y-x)) + F \circ \pi_x(t(y-x))\right].$$

Note that $u(0) = H(x)$ and $u(1) = H(y)$. According to Lemma 2.2, u is C^k and for all j in $\{0, \cdots, J\}$ $D^j u(0)$ is zero. Furthermore, Lemma 2.2 implies also that there is a function $G : B(x,\rho) \longrightarrow \mathcal{L}_{J+1}(\mathbb{R}^n)$, which is C^{k-J-1} and such that

$$D^{(J+1)}u(t) = G(t(y-x)).(y-x)^{J+1}.$$

As a result, u has the following Taylor expansion:

$$u(1) = H(y) = \frac{1}{J!}\int_0^1 (1-t)^J G(t(y-x)).(y-x)^{J+1}\mathrm{d}t.$$

Hence, $H(r,\theta)/r^{J+1}$ is of class C^k with respect to the polar coordinates in $P_x^{-1}(M \cap B(x,\rho) - \{x\})$, and its derivatives are bounded above up to order $k - J - 1$. □

By taking $f(r) = -1/4\pi$ or $f(r) = (ikr - 1)/4\pi$ and by applying the preceding proposition, we see that the double layer potentials of Laplace and Helmholtz equations in $\mathbb{R}^3$ are pseudohomogeneous of degree -1 up to order $k-2$.

PROPOSITION 3.3. *Kernels of Stokes flow and linear elasticity problems are pseudohomogeneous of degree -1 up to order k.*

Proof. This results from Proposition 3.1 and the polar decomposition:

$$r\frac{(y-x)\otimes(y-x)}{\|y-x\|^3} = \theta \otimes \theta.$$ □

3.3. Three types of approximations. Consider x on $\partial\Omega$ and Γ_i a panel. Even though $K(x,\cdot)$ is locally integrable on $\partial\Omega - \{x\}$, numerical approximations of $\mathcal{K}[s,x,i]$ may present some difficulties. From the definition of the regularity of the kernel,we infer that $K(x,y)$ may not be bounded when y approaches x. Any numerical approximation of $\mathcal{K}[s,x,i]$ must take into account this singular behavior. In order to tackle this difficulty, $\mathbb{R}^3$ is divided into three domains.

The point x is said to belong to the near field of Γ_i if x is in $\mathrm{int}(\Gamma_i)$. If x is at a distance from Γ_i, which is equivalent to the diameter $\widetilde{h}_i$, x is said to belong to the intermediate domain of Γ_i. More precisely, x belongs to the intermediate domain if

$$a\widetilde{h}_i \leq \mathrm{dist}(x,\Gamma_i) \leq b\widetilde{h}_i, \tag{3.10}$$

where the positive constants a and b are yet to be specified. If the distance from Γ_i to x is larger than $b\widetilde{h}_i$, x is said to be in the far field of Γ_i.

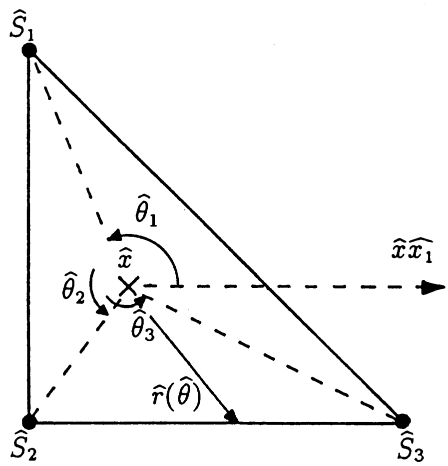

FIG. 1. *Definition of notation in the system of polar coordinates.*

4. Numerical quadrature in the near field. We assume in this section that x is in $\mathrm{int}(\Gamma_i)$. Since $K(x,y)$ may be weakly singular as y approaches x, special care is to be given to the way of approximating (1.2). Since in the polar system of coordinates the function $(r,\theta) \mapsto rK(x,r,\theta)$ is smooth, it seems reasonable that, in the plane of the reference simplex, the function $(\widehat{r},\widehat{\theta}) \mapsto \widehat{r}K(x,\widehat{r},\widehat{\theta})$ should also be smooth. As a result, the polar measure $\widehat{r}\mathrm{d}\widehat{r}\mathrm{d}\widehat{\theta}$ should be suitable for evaluating (1.2). Introduce the polar coordinates $(\widehat{r},\widehat{\theta})$ so that

$$\begin{aligned} \widehat{r}\cos(\widehat{\theta}) &= \widehat{y}_1 - \widehat{x}_1, \\ \widehat{r}\sin(\widehat{\theta}) &= \widehat{y}_2 - \widehat{x}_2. \end{aligned} \tag{4.1}$$

Let $\widehat{S}_1$, $\widehat{S}_2$, and $\widehat{S}_3$ be the three vertices of $\widehat{S}$. Define $\widehat{\theta}_1$, $\widehat{\theta}_2$, and $\widehat{\theta}_3$ as the three angles between the axis $\widehat{x}\widehat{x}_1$ and axes $\widehat{x}\widehat{S}_i$ $(i = 1,2,3)$; see Fig. 1. Then (1.2) can be cast into the form

$$\mathcal{K}[s,x,i] = \sum_{k=1}^{3} \int_{\widehat{\theta}_k}^{\widehat{\theta}_{k+1}} \int_0^{\widehat{r}(\widehat{\theta})} \widehat{r}K(x,\widehat{r},\widehat{\theta}).\widehat{s}_i(\widehat{r},\widehat{\theta})J_i(\widehat{r},\widehat{\theta})\mathrm{d}\widehat{r}\mathrm{d}\widehat{\theta}, \tag{4.2}$$

where $\widehat{\theta}_4 := \widehat{\theta}_1 + 2\pi$ and $\widehat{r}(\widehat{\theta})$ is the distance from $\widehat{x}$ to $\partial\widehat{S}$ along the direction $\widehat{\theta}$. $s \circ \psi_i$ has been replaced by $\widehat{s}_i$ for short. At this point, note that, since ψ_i is smooth, the ratio $\widehat{r}/r$ is also smooth with respect to the polar coordinates. As a result, when $\widehat{\theta}$ is fixed, the function $\widehat{r}K(x,\widehat{r},\widehat{\theta})$ is smooth in $]0,\widehat{r}(\widehat{\theta})]$. Hence, a Gaussian quadrature is suitable for approximating the radial integral:

$$\mathcal{K}[s,x,i] \approx \sum_{k=1}^{3} \int_{\widehat{\theta}_k}^{\widehat{\theta}_{k+1}} \widehat{r}(\widehat{\theta}) \left[\sum_{l=1}^{L} \omega_{1l}\widehat{r}_l(\widehat{\theta})K(x,\widehat{r}_l(\widehat{\theta}),\widehat{\theta}).\widehat{s}_i(\widehat{r}_l(\widehat{\theta}),\widehat{\theta})J_i(\widehat{r}_l(\widehat{\theta}),\widehat{\theta}) \right] \mathrm{d}\widehat{\theta}. \tag{4.3}$$

The quadrature $\int_0^1 g\mathrm{d}\xi \approx \sum_{l=1}^{L} \omega_{1l}g(\xi_{1l})$ is of Gauss–Legendre type with L points, and we have set $\widehat{r}_l(\widehat{\theta}) := \widehat{r}(\widehat{\theta})\xi_{1l}$. Note that L is to be specified so that the quadrature error meets some bound, which will be discussed further.

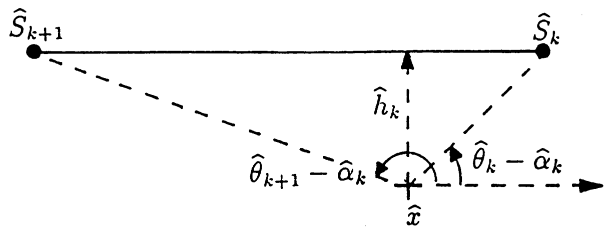

FIG. 2. *Definition of* $\widehat{\alpha}_k$ *and* $\widehat{h}_k$.

At this point, the integration with respect to the azimuthal variable $\widehat{\theta}$ should not pose any difficulty, for the function of $\widehat{\theta}$, which is between the brackets in (4.3), is smooth. However, though the function $\widehat{r}(\widehat{\theta})$ is C^r in $[\widehat{\theta}_k, \widehat{\theta}_{k+1}]$, it may have large variations. Indeed, if $\widehat{\alpha}_k$ denotes the angle $(\widehat{x}\widehat{x}_1, \widehat{S}_{k+1}\widehat{S}_k)$ where $\widehat{S}_4 := \widehat{S}_1$, and if $\widehat{h}_k$ denotes the distance from $\widehat{x}$ to the edge $\widehat{S}_k\widehat{S}_{k+1}$ of $\widehat{S}$, then the distance $\widehat{r}(\widehat{\theta})$ can be written:

$$\widehat{r}(\widehat{\theta}) = \frac{\widehat{h}_k}{\sin(\widehat{\theta} - \widehat{\alpha}_k)}. \tag{4.4}$$

The angle $\widehat{\theta} - \widehat{\alpha}_k$ belongs to the open interval $]0, \pi[$ (cf. Fig. 2). Recall that (1.1) can be solved by using methods like collocation methods, Galerkin methods, or finite elements methods. The common point of all these methods is that $s \circ \psi_i$ is always approximated by polynomials. Let p be the degree of the polynomial approximation. In practice, $\widehat{x}$ belongs to a finite set of control points or quadrature points, which depends on p only; denote by $\widehat{Q}_p$ the finite set in question. Hence, there is a constant $c(p)$, which depends only on p, so that $0 < c(p) \leq \widehat{h}_k$; that is, $\widehat{h}_k$ is bounded below. However, the distance $\widehat{h}_k$ may be numerically small; for instance if $p = 2$, $\widehat{h}_k$ may reach values as small as 5.10^{-2}. In these conditions the angles $\widehat{\theta}_k - \widehat{\alpha}_k$ and $\widehat{\theta}_{k+1} - \widehat{\alpha}_k$ can be very close to zero or π. As a consequence, it is clear that the discretization in $\widehat{\theta}$ must be finer in the vicinity of $\widehat{\theta}_k$ and $\widehat{\theta}_{k+1}$ than in between. The measure that is adapted to the situation is $d\widehat{\theta}/\sin(\widehat{\theta} - \widehat{\alpha}_k)$ rather than $d\widehat{\theta}$. As a result, it is convenient to use the following change of variable:

(4.5)
$$\xi_2 = \frac{1}{\Delta_k}\left(2\ln\left[\tan\left(\frac{\widehat{\theta} - \widehat{\alpha}_k}{2}\right)\right] - \ln\left[\tan\left(\frac{\widehat{\theta}_k - \widehat{\alpha}_k}{2}\right)\right] - \ln\left[\tan\left(\frac{\widehat{\theta}_{k+1} - \widehat{\alpha}_k}{2}\right)\right]\right),$$

where we have set

$$\Delta_k := \ln\left[\tan\left(\frac{\widehat{\theta}_{k+1} - \widehat{\alpha}_k}{2}\right)\right] - \ln\left[\tan\left(\frac{\widehat{\theta}_k - \widehat{\alpha}_k}{2}\right)\right]. \tag{4.6}$$

Consider the following Gauss–Legendre quadrature with N points: $\int_{-1}^{+1} g d\xi \approx \sum_{n=1}^{N} \omega_{2n} g(\xi_{2n})$, where N is yet to be specified. Applying this quadrature to (4.3), where $\widehat{\theta}$ has been changed by ξ_2, we obtain:

$$(4.7)\quad \mathcal{K}[s,x,i] \approx \sum_{k=1}^{3} \frac{\widehat{h}_k}{2}\Delta_k \sum_{n=1}^{N}\sum_{l=1}^{L} \omega_{2n}\omega_{1l}\widehat{r}_{ln} K(x,\widehat{r}_{ln},\widehat{\theta}_n).\widehat{s}_i(\widehat{r}_{ln},\widehat{\theta}_n) J_i(\widehat{r}_{ln},\widehat{\theta}_n),$$

where we have set $\widehat{r}_{ln} := \widehat{r}_l(\widehat{\theta}_n)$.

4.1. Error estimate. Define $h := \sup_{i\in I}\{\widetilde{h}_i\}$, h is the characteristic meshsize. For sake of simplicity $\widehat{s}_i$ is assumed to be a polynomial of degree p. This assumption is not restrictive for, when approximating a solution to (1.1), the unknown function $s \circ \psi_i$ is always approximated by a polynomial.

Define Δ so that

$$\Delta := \sup_{\widehat{x}\in\widehat{Q}_p,\ k=1,2,3} \{\Delta_k\},$$

then we have the following result.

THEOREM 4.1. *Assume $p \leq 2L-1$ and $p \leq 2N-1$. Let $q \in [1,\infty]$; if N is great enough, there are constants c, c', and $\gamma > (\Delta/2)$ such that the quadrature error in the near field satisfies the bound:*

$$(4.8)\qquad E_{LN} \leq c h^{2L-p+1} \|\widehat{s}_i\|_{p,q,\widehat{S}} + c' h (\Delta/2\gamma)^{2N} \|\widehat{s}_i\|_{p,\infty,\widehat{S}}.$$

Proof. Denote by $\widehat{P}_x : (\widehat{r},\widehat{\theta}) \mapsto \widehat{y}$ the polar mapping defined by (4.1). Denote by ϕ_k the mapping $(\xi_1,\xi_2) \mapsto (\widehat{r},\widehat{\theta})$, and denote by ϕ_{1k}, ϕ_{2k} the two components of ϕ_k. Denote by Σ_k the two-dimensional manifold $\phi_k(]0,1[\times]-1,+1[)$. Recall that:

$$\xi_1 = \widehat{r}\frac{\sin(\widehat{\theta}-\alpha_k)}{\widehat{h}_k},$$

$$\xi_2 = \frac{1}{\Delta_k}\left(2\ln\left[\tan\left(\frac{\widehat{\theta}-\widehat{\alpha}_k}{2}\right)\right] - \ln\left[\tan\left(\frac{\widehat{\theta}_k-\widehat{\alpha}_k}{2}\right)\right] - \ln\left[\tan\left(\frac{\widehat{\theta}_{k+1}-\widehat{\alpha}_k}{2}\right)\right]\right).$$

The Jacobian determinant of ϕ_k is equal to $\Delta_k\widehat{h}_k/2$, and it may easily be verified that the product $\Delta_k\widehat{h}_k$ is bounded above by a constant that depends only on the characteristics of the reference simplex: $\widehat{h}$ and $\widehat{\rho}$. Since the quadrature (4.7) is the tensor product of two one-dimensional quadratures, E_{LN} can be bounded by:

$$E_{LN} \leq \sum_{k=1}^{3} \widehat{h}_k \frac{\Delta_k}{2}\left[\sup_{\xi_1\in]0,1[}(E_{1k}(\xi_1)) + \sup_{\xi_2\in]-1,1[}(E_{2k}(\xi_2))\right],$$

where E_{1k} and E_{2k} are the following elementary quadrature errors:

$$E_{1k}(\xi_1) = \left|\int_{-1}^{1} f_k(\xi_1,t)\mathrm{d}t - \sum_{n=1}^{N}\omega_{2n} f_k(\xi_1,\xi_{2n})\right|,$$

$$E_{2k}(\xi_2) = \left|\int_{0}^{1} f_k(t,\xi_2)\mathrm{d}t - \sum_{l=1}^{L}\omega_{1l} f_k(\xi_{1l},\xi_2)\right|,$$

where f_k is short for $\phi_{1k}K(x,\psi_i\circ\widehat{P}_x\circ\phi_k)\cdot\widehat{s}_i(\widehat{P}_x\circ\phi_k)J_i(\widehat{P}_x\circ\phi_k)$.

A bound on $E_{1k}(\xi_1)$ is given by the classical theory of Gaussian quadratures (see Davis [6, p. 344]); that is to say, there is $\eta(\xi_1) \in]-1,1[$ such that:

$$E_{1k}(\xi_1) \leq \frac{2^{2N+1}[N!]^4}{(2N+1)[(2N)!]^3} D_{\xi_2}^{2N}(f_k(\xi_1,\eta(\xi_1))),$$

where D_{ξ_2} denotes the derivative in the direction ξ_2. With the help of Stirling's formula, it is a simple matter of calculus to show that, if N is great enough, there are two constants c and $\gamma > (\Delta/2)$ such that

$$\frac{2^{2N+1}[N!]^4}{(2N+1)[(2N)!]^3} \leq c\gamma^{-2N}.$$

Furthermore, we have

$$D_{\xi_2}^{2N}(f_k(\xi_1,\eta(\xi_1))) \leq \|D^{2N}(f_k(\xi_1,\eta(\xi_1)))\| \leq |f_k|_{2N,\infty,]0,1[\times]-1,1[},$$

and, for $m = 1, 2$, it can be shown by induction on j that $\|D^j\phi_{mk}(\xi_1,\xi_2)\|$ is less than $c(\Delta_k/2)^j$; that is to say, $|\phi_k|_{j,\infty,]0,1[\times]-1,1[}$ is bounded by $c(\Delta_k/2)^j$. As a result, by using Lemma 2.2 and the preceding bound we have:

$$\begin{aligned}
&|f_k|_{2N,\infty,]0,1[\times]-1,1[}\\
&\quad\leq c(\Delta_k/2)^{2N}\|\widehat{r}K(x,\psi_i\circ\widehat{P}_x).\widehat{s}_i(\widehat{P}_x)J_i(\widehat{P}_x)\|_{2N,\infty,\Sigma_k},\\
&\quad\leq c(\Delta_k/2)^{2N}\|\widehat{r}K(x,\psi_i\circ\widehat{P}_x)\|_{2N,\infty,\Sigma_k}\|\widehat{s}_i\circ\widehat{P}_x\|_{2N,\infty,\Sigma_k}\|J_i\circ\widehat{P}_x\|_{2N,\infty,\Sigma_k}.
\end{aligned}$$

In addition, according to Lemma 3.1, Lemma 4.1, and the regularity criteria on J_i we have the following bounds if $\widetilde{h}_i$ is small enough:

$$\begin{aligned}
\|\widehat{r}K(x,\psi_i\circ\widehat{P}_x)\|_{2N,\infty,\Sigma_k} &\leq c/\widetilde{h}_i,\\
\|\widehat{s}_i\circ\widehat{P}_x\|_{2N,\infty,\Sigma_k} &\leq c\|\widehat{s}_i\|_{p,\infty,\widehat{S}},\\
\|J_i\circ\widehat{P}_x\|_{2N,\infty,\Sigma_k} &\leq c\|J_i\|_{2N,\infty,\widehat{S}} \leq c'\widetilde{h}_i^2.
\end{aligned}$$

As a result, we obtain the final bound:

$$\sum_{k=1}^{3}\widehat{h}_k\frac{\Delta_k}{2}\sup_{\xi_1\in]0,1[}(E_{1k}(\xi_1)) \leq ch(\Delta/2\gamma)^{2N}\|\widehat{s}_i\|_{p,\infty,\widehat{S}}.$$

For the second quadrature error, $E_{2k}(\xi_2)$, we can apply the Bramble–Hilbert lemma:

$$E_{2k}(\xi_2) \leq c|f_k|_{2L,q,S_{\xi_2}},$$

where S_{ξ_2} is short for the segment $[0,1]\times\xi_2$, and the derivatives are to be taken in the direction ξ_1. Note that $\phi_k(\cdot,\xi_2)$, where ξ_2 is fixed, is linear with respect to ξ_1 and $|\phi_k(\cdot,\xi_2)|_{1,\infty,S_{\xi_2}}$ is bounded above by $\widehat{h}$. Hence, by using Lemma 2.2 we obtain:

$$E_{2k}(\xi_2) \leq c|\widehat{r}K(x,\psi_i\circ\widehat{P}_x).\widehat{s}_i(\widehat{P}_x)J_i(\widehat{P}_x)|_{2L,q,S_\theta},$$

where the derivatives are to be taken with respect to the radial coordinate $\widehat{r}$, and $S_\theta \subset \Sigma_k$ is the segment $\widehat{r} \in]0, \widehat{h}_k/\sin(\widehat{\theta}-\alpha_k)[$ where $\widehat{\theta} := \phi_{2k}(\xi_2)$. By using Lemma 2.1 together with various forms of Hölder inequality we have:

$$E_{2k}(\xi_2) \leq \sum_{j=0}^{2L} \left|\widehat{s}_i \circ \widehat{P}_x\right|_{j,q,S_\theta} \sum_{l=0}^{2L-j} c_{jl} \left|\widehat{r}K(x,\psi_i \circ \widehat{P}_x)\right|_{l,\infty,S_\theta} \left|J_i \circ \widehat{P}_x\right|_{2L-j-l,\infty,S_\theta}.$$

Furthermore, the definition of the radial derivative yields:

$$D^j_{\widehat{r}}(f \circ \widehat{P}_x)(\widehat{r},\widehat{\theta}) = D^j f(\widehat{y}) \cdot (\widetilde{\theta})^j,$$

where we have set $\widetilde{\theta} = (\widehat{y}-\widehat{x})/\widehat{r}$. Hence, by using the equality above and the regularity criteria on J_i, we have the following bounds:

$$|\widehat{s}_i \circ \widehat{P}_x|_{j,q,S_\theta} \leq |\widehat{s}_i|_{p,\infty,\widehat{S}},$$

$$\|J_i \circ \widehat{P}_x\|_{2L-j-l,\infty,\Sigma_k} \leq \|J_i\|_{2L-j-l,\infty,\widehat{S}} \leq c\widetilde{h}_i^{2L-j-l+2}.$$

In addition, Lemma 4.2 yields

$$|\widehat{r}K(x,\psi_i \circ \widehat{P}_x)|_{l,\infty,S_\theta} \leq c\widetilde{h}_i^{l-1}.$$

In conclusion, by using the equivalence of norms in finite-dimensional, normed vector space, we obtain the final bound:

$$\sum_{k=1}^{3} \widehat{h}_k \frac{\Delta_k}{2} \sup_{\xi_1 \in]0,1[} (E_{2k}(\xi_2)) \leq ch^{2L-p+1} \|\widehat{s}_i\|_{p,q,\widehat{S}},$$

which completes the proof. □

Assume the quadrature error must be of $\mathcal{O}(h^k)$ so that it is of the same order as that induced by the approximation of the solution to (1.1) by some particular scheme. Then, integer L must be chosen so that:

$$L = \sup\{(p+1)/2, (k+p-1)/2\}. \tag{4.9}$$

Define constants c and $C_\gamma > 1$, and define the set S_Δ so that

$$S_\Delta := \left\{N : \frac{2^{2N+1}[N!]^4}{(2N+1)[(2N)!]^3} \leq c(C_\gamma \Delta/2)^{-2N}\right\}. \tag{4.10}$$

Then integer N must be chosen so that

$$N = \sup\left\{\inf\{S_\Delta\}, \frac{(p+1)}{2}, \frac{(k-1)}{2}\frac{\ln(1/h)}{\ln(C_\gamma)}\right\}. \tag{4.11}$$

If h is small enough the ratio of N/L behaves like $\ln(1/h)$; that is to say, more quadrature points must be put in the azimuthal direction than in the radial direction. This fact has been well observed on numerical tests (cf. [8]).

Note that error estimate (4.8) is consistent with that of Johnson and Scott's [10, Lemma 3.1, p. 1365]. The present approach, though, may be more natural than that referred to above since it emphasizes the polar coordinates' role.

Now, we prove Lemmas 4.1 and 4.2, which have been used in the demonstration above.

LEMMA 4.1. *With the same notation as that of the preceding theorem, if $\widetilde{h}_i$ is small enough, we have the following bound:*

$$\|\widehat{r}K(x, \psi_i \circ \widehat{P}_x)\|_{j,\infty,\Sigma_k} \leq c/\widetilde{h}_i. \tag{4.12}$$

Proof. According to Lemma 2.1 we have

$$|\widehat{r}K|_{j,\infty,\Sigma_k} \leq \sum_{l=0}^{j} c_{jl} \left|\frac{\widehat{r}}{r}\right|_{j-l,\infty,\Sigma_k} |r(\psi_i \circ \widehat{P}_x)K(x, \psi_i \circ \widehat{P}_x)|_{l,\infty,\Sigma_k}. \tag{4.13}$$

In a first step we give a bound on $|\widehat{r}/r|_{j-l,\infty,\Sigma_k}$. By using the Taylor expansion of ψ_i up to order one, we have

$$\left(\frac{r(\widehat{r},\widehat{\theta})}{\widehat{r}}\right)^2 = \left(\int_0^1 D\psi_i(\widehat{x} + t(\widehat{y} - \widehat{x})).\widehat{\theta}\mathrm{d}t, \int_0^1 D\psi_i(\widehat{x} + t(\widehat{y} - \widehat{x})).\widehat{\theta}\mathrm{d}t\right). \tag{4.14}$$

This relation shows that the ratio $r/\widehat{r}$ is bounded above by $c\widetilde{h}_i$. By induction on l and by using the relation above, it is possible to prove that $|r/\widehat{r}|_{l,\infty,\Sigma_k}$ is bounded above by $c\widetilde{h}_i$ if $\widetilde{h}_i$ is small enough. By using Lemma 2.2 with $f(\widehat{r},\widehat{\theta}) = r(\widehat{r},\widehat{\theta})$ and $g(t) = 1/t$ we obtain

$$|\widehat{r}/r|_{l,\infty,\Sigma_k} \leq c\widetilde{h}_i^{-1}. \tag{4.15}$$

In a second step we find a bound to $|rK|_{l,\infty,\Sigma_k}$. Denote by ϕ_i the mapping $P_x^{-1} \circ \psi_i \circ \widehat{P}_x$. Then Lemma 2.2 yields

$$\begin{aligned} D^l r(\psi_i \circ \widehat{P}_x)K(x, \psi_i \circ \widehat{P}_x) = \operatorname{sym} \sum_{l=1}^{j} \sum_{r_1+\cdots+r_l=j} \frac{j!}{l! r_1! \cdots r_l!} \\ \cdot D^l(rK(r,\theta)) \cdot D^{r_1}\phi_i(\widehat{r},\widehat{\theta}) \cdots D^{r_l}\phi_i(\widehat{r},\widehat{\theta}). \end{aligned} \tag{4.16}$$

Denote by ϕ_{i1} and ϕ_{i2} the two components of ϕ_i, i.e., $r = \phi_{i1}(\widehat{r},\widehat{\theta})$ and $\theta = \phi_{i2}(\widehat{r},\widehat{\theta})$. For all integer l we have

$$D^l\phi_{i1} = D^l\left(\widehat{r}.\frac{r}{\widehat{r}}\right).$$

Hence, by using Lemma 2.1 and the bound on $|\widehat{r}/r|_{l,\infty,\Sigma_k}$ that has been found above, we prove that $|\phi_{i1}|_{l,\infty,\Sigma_k}$ is bounded by $c\widetilde{h}_i$. For the second component of ϕ_i we have

$$D^l\phi_{i2} = D^l\left(\frac{\widehat{r}}{r}\int_0^1 D\psi_i(\widehat{x} + t(\widehat{y} - \widehat{x})).\widehat{\theta}\mathrm{d}t\right).$$

By using Lemma 2.1 and the bounds that have been found above we obtain that $|\phi_{i2}|_{l,\infty,\Sigma_k}$ is bounded above by a constant. As a result $|\phi_i|_{l,\infty,\Sigma_k}$ is bounded above by a constant if $\widetilde{h}_i$ is small enough. Furthermore, since K is pseudohomogeneous of degree -1, it comes that $|rK|_{l,\infty,\phi_i(\Sigma_k)}$ is bounded above by a constant. In conclusion, if $\widetilde{h}_i$ is small enough, we have:

$$|r(\psi_i \circ \widehat{P}_x)K(\psi_i \circ \widehat{P}_x)|_{l,\infty,\Sigma_k} \leq c. \tag{4.17}$$

Substituting the bounds (4.15) and (4.17) into (4.13) we obtain the desired bound (4.12). □

LEMMA 4.2. *With the same notation as that of the preceding theorem we have the following bound:*

$$\|\widehat{r}K(x,\psi_i\circ\widehat{P}_x)\|_{j,\infty,S_\theta}\le c\widetilde{h}_i^{j-1}. \tag{4.18}$$

Proof. We proceed as for the proof of the preceding lemma. According to Lemma 2.1 we have

$$|\widehat{r}K|_{j,\infty,S_\theta}\le\sum_{l=0}^{j}c_{jl}\left|\frac{\widehat{r}}{r}\right|_{j-l,\infty,S_\theta}|r(\psi_i\circ\widehat{P}_x)K(x,\psi_i\circ\widehat{P}_x)|_{l,\infty,S\theta}. \tag{4.19}$$

In a first step we find a bound to $|\widehat{r}/r|_{j-l,\infty,S_\theta}$. By using the Taylor expansion of ψ_i up to order one as in (4.14) we can show by induction on l that $|r/\widehat{r}|_{l,\infty,S_\theta}$ is bounded above by $c\widetilde{h}_i^{l+1}$. By using Lemma 2.2 with $f(\widehat{r},\widehat{\theta})=r(\widehat{r},\widehat{\theta})$ and $g(t)=1/t$ we obtain:

$$|\widehat{r}/r|_{l,\infty,S_\theta}\le c\widetilde{h}_i^{l-1}. \tag{4.20}$$

In a second step we find a bound to $|rK|_{l,\infty,S_\theta}$. As in the preceding lemma, denote by ϕ_i the mapping $P_x^{-1}\circ\psi_i\circ\widehat{P}_x$. In order to apply (4.16) we look for a bound on $|\phi_i|_{l,\infty,S_\theta}$. Denote by D_1 the derivative with respect to $\widehat{r}$; then, for all integer l we have

$$D_1^l\phi_{i1}=D_1^l\left(\widehat{r}.\frac{r}{\widehat{r}}\right).$$

Hence, by using Lemma 2.1 and the bound on $|\widehat{r}/r|_{l,\infty,\Sigma_k}$ that has been found above, we prove that $|\phi_{i1}|_{l,\infty,S_\theta}$ is bounded by $c\widetilde{h}_i^l$ if $\widetilde{h}_i$ is small enough. For the second component of ϕ_i we have:

$$D_1^l\phi_{i2}=D_1^l\left(\frac{\widehat{r}}{r}\int_0^1 D\psi_i(\widehat{x}+t(\widehat{y}-\widehat{x})).\widehat{\theta}\mathrm{d}t\right).$$

By using Lemma 2.1 and the bounds that have been found above we obtain that $|\phi_{i2}|_{l,\infty,S_\theta}$ is bounded above by $c\widetilde{h}_i^l$. As a result $|\phi_i|_{l,\infty,S_\theta}$ is bounded above by $c\widetilde{h}_i^l$. Furthermore, since K is pseudohomogeneous of degree -1, it comes that $|rK|_{l,\infty,\phi_i(\Sigma_k)}$ is bounded above by a constant. In conclusion we have:

$$|r(\psi\circ\widehat{P}_x)K(\psi_i\circ\widehat{P}_x)|_{l,\infty,S_\theta}\le c.\widetilde{h}_i^l \tag{4.21}$$

Substituting the bounds (4.20) and (4.21) into (4.19) we obtain the desired bound (4.18). □

5. Numerical quadrature in the far field.

5.1. The quadrature rule. Assume here that x belongs to the far field of Γ_i, i.e., x is such that the distance from x to Γ_i is larger than $b\widetilde{h}_i$ where b is independent of i and is yet to be specified.

Assume as above that $\widehat{s}_i$ is a polynomial of degree p. Let t be a positive integer such that $p\le t$. Consider the following quadrature rule

$$\int_{\widehat{S}} \widehat{q}(\widehat{x})\mathrm{d}\widehat{x} = \sum_{q=1}^{Q(t)} \omega_q^{\text{far}} \widehat{q}(\widehat{x}_q), \tag{5.1}$$

which is assumed to be exact for all polynomials $\widehat{q}$ of degree less than or equal to t. A compilation of quadrature rules of this kind is given in [5, t. II, pp. 780–870]. An approximation of $\mathcal{K}[s,x,i]$ is given by:

$$\mathcal{K}[s,x,i] \approx \sum_{q=1}^{Q(t)} \omega_q^{\text{far}} K(x, \psi_i(\widehat{x}_q)).\widehat{s}_i(\widehat{x}_q) J_i(\widehat{x}_q) := \mathcal{K}_t[s,x,i]. \tag{5.2}$$

The quadrature error $|\mathcal{K}[s,x,i] - \mathcal{K}_t[s,x,i]|$ is denoted by E_t for short. Bounds on E_t will be obtained once some preliminary bounds on $|K(x,\psi_i)|_{l,\infty,\widehat{S}}$ are derived.

5.2. Preliminary bounds. Assume x is in the intermediate or in the far field of Γ_i. In the rest of the paper, $P_l(z)$ denotes polynomials of degree less than or equal to l, and whose coefficients are positive and independent of i.

LEMMA 5.1. *Consider $\widehat{y}$ in $\operatorname{int}(\widehat{S})$ and $j \geq 1$, then*

$$\|D^j(\| x - \psi_i(\widehat{y}) \|)\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \frac{P_{j-1}(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{j-1}}, \tag{5.3}$$

Proof. Let $\widehat{e}$ be a vector in $\mathbb{R}^2$ whose norm is one. The equality

$$\| x - \psi_i(\widehat{y}) \| \, D(\| x - \psi_i(\widehat{y}) \|) \cdot \widehat{e} = (x - \psi_i(\widehat{y}), D\psi_i(\widehat{y}).\widehat{e}) \tag{5.4}$$

yields the bound $\|D(\| x - \psi_i(\widehat{y}) \|)\| \leq c|\widetilde{\psi}_i|_{1,\infty,\widehat{S}}$.

For j greater than one, the inequality is obtained by induction on j and by applying Lemma 2.1 to both sides of (5.4). □

PROPOSITION 5.1. *Let j be a nonnegative integer. Let $K(x,y)$ be the single layer potential of Laplace or Helmholtz equation, or that of the Stokes flow problem or the linear elasticity problem. Then the following bound holds:*

$$\|D^j K(x, \psi_i(\widehat{y}))\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \frac{P_j(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{j+1}}. \tag{5.5}$$

Proof. For the single layer potential of Laplace equation the result is a direct consequence of the following inequality with $n = 1$:

$$\|D^j(\| x - \psi_i(\widehat{y}) \|^{-n})\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \frac{P_{j-1}(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{j+n}}, \tag{5.6}$$

where j and n are two positive integers. This inequality results from Lemma 5.1 and Lemma 2.2 where $g(z) = z^{-n}$ and $f(\widehat{y}) = \| x - \psi_i(\widehat{y}) \|$.

For the simple layer of Helmholtz equation, the inequality results from Lemma 2.1, inequality (5.6) with $n = 1$, and the following bound:

$$\|D^j(\mathrm{e}^{ik\|x-\psi_i(\widehat{y})\|})\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \frac{P_{j-1}(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{j-1}}, \tag{5.7}$$

where j is an integer greater than or equal to one. To prove this inequality apply Lemma 2.2 with $g(z) = e^{ikz}$ and $f(\widehat{y}) = \| x - \psi_i(\widehat{y}) \|$, and use Lemma 5.1.

For the kernels of Stokes flows or elasticity problems, the inequality results from Lemma 2.1, inequality (5.6) for $n = 3$, and the following bound:

$$\|D^j((y-x)\otimes(y-x))\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}}(c \| x - \psi_i(\widehat{y}) \| + c'), \tag{5.8}$$

where j is an integer greater than or equal to one. This inequality is a consequence of Lemma 2.1. □

PROPOSITION 5.2. *Let $K(x,y)$ be the double layer potential of Laplace or Helmholtz equation, then the following bound holds*:

$$\|D^j K(x,\psi_i(\widehat{y}))\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}} \frac{P_j(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{j+2}}. \tag{5.9}$$

Proof. For the double layer potential of Laplace equation use Lemma 2.1 together with bound (5.6) for $n = 3$ and the following bound:

$$\|D^j(n_y, x - \psi_i(\widehat{y}))\| \leq |\widetilde{\psi}_i|^j_{1,\infty,\widehat{S}}(c \| x - \psi_i(\widehat{y}) \| + c'), \tag{5.10}$$

where j is an integer greater than or equal to one. To prove (5.10), apply Lemma 2.1, use bound (2.22) on $|n_y \circ \psi_i|_{l,\infty,\widehat{S}}$ together with Lemma 5.1.

For the double layer potential of Helmholtz equation use the bounds that have been obtained for the double layer potential of Laplace equation together with (5.7) and Lemma 2.1. □

5.3. Error estimate. Before estimating the quadrature error, it is necessary to obtain a bound on $|K(x,\psi_i)|_{l,\infty,\widehat{S}}$, which is independent of x. For this purpose we have the following result.

PROPOSITION 5.3. *Let b be a constant so that $b\widetilde{h}_i \leq \mathrm{dist}(x,\Gamma_i)$. Assume there is a constant α so that*

$$\|D^l K(x,\psi_i(\widehat{y}))\| \leq |\widetilde{\psi}_i|^l_{1,\infty,\widehat{S}} \frac{P_l(\| x - \psi_i(\widehat{y}) \|)}{\| x - \psi_i(\widehat{y}) \|^{l+\alpha}}, \tag{5.11}$$

where P_l is a polynomial of degree less than or equal to l and whose coefficients are positive. If $\widetilde{h}_i$ is small enough there exists a constant $\gamma_l \geq 1$, independent of i, so that we have the following:

$$|K(x,\psi_i)|_{l,\infty,\widehat{S}} \leq c \frac{\widetilde{h}_i^{-\alpha}}{(\gamma_l b)^{l+\alpha}}, \tag{5.12}$$

Proof. Define $X := \| x - \psi_i(\widehat{y}) \|$, then X belongs to $[b\widetilde{h}_i, \infty[$. If $X > 1$ the ratio $P_l(X)/X^{l+\alpha}$ is bounded by a constant c. If $X \leq 1$, there exists X_0 in $[b\widetilde{h}_i, 1]$ so that $P_l(X)/X^{l+\alpha}$ is bounded by $P_l(X_0)/X_0^{l+\alpha}$. Furthermore, since $P_l(X)$ increases as X increases, $P(X_0)$ is bounded by $P(1)$. Denote by γ_{il} the ratio $X_0/b\widetilde{h}_i$, note that γ_{il} is greater or equal to one. Let $\gamma_l := \inf_{i\in I}\{\gamma_{il}\}$, then $P_l(X)/X^{l+\alpha}$ is bounded by $P_l(1)/(\gamma_l b\widetilde{h}_i)^{\alpha+l}$. Recall that $h := \sup_{i\in I}\{\widetilde{h}_i\}$; if h is small enough, the bound which has been found for $X \leq 1$ is also valid for $X > 1$. Inequality (5.12) results from the bound above, inequality (5.11), and the fact that $|\widetilde{\psi}_i|_{1,\infty,\widehat{S}}$ is bounded by $c'\widetilde{h}_i$. □

Note that the kernels that have been studied in §3.2 satisfy condition (5.11); indeed, $\alpha = 1$ if K is a single layer potential, and $\alpha = 2$ if K is a double layer potential.

THEOREM 5.1. *Assume x and K satisfy conditions of the preceding proposition. Let $q \in [1, \infty]$; there is a constant $\gamma \geq 1$ so that, if h is small enough, the quadrature error in the far field satisfies*

$$E_t \leq c \frac{h^{2-\alpha}}{(\gamma b)^{t+1-p+\alpha}} \|\widehat{s}_i\|_{p,q,\widehat{S}}. \tag{5.13}$$

Proof. From the Bramble–Hilbert lemma we infer that

$$E_t \leq c|K(x, \psi_i).\widehat{s}_i J_i|_{t+1,q,\widehat{S}}.$$

Using the fact that $\widehat{s}_i$ is a polynomial of degree p, where $p \leq t$, and applying Lemma 2.1 together with various forms of Hölder inequality yields:

$$E_t \leq \sum_{j=0}^{p} |\widehat{s}_i|_{j,q,\widehat{S}} \sum_{l=0}^{t+1-j} c_{jl} |K(x, \psi_i)|_{l,\infty,\widehat{S}} |J_i|_{t+1-j-l,\infty,\widehat{S}}.$$

where c_{jl} are positive constants that do not depend on i. From Proposition 5.3 and Theorem 2.3, it results that

$$E_t \leq \sum_{j=0}^{p} |\widehat{s}_i|_{j,q,\widehat{S}} \sum_{l=0}^{t+1-j} c_{jl} (\widetilde{h}_i \gamma_l b)^{-l-\alpha} \widetilde{h}_i^{t+1-j+2}.$$

Setting $\gamma := \inf\{\gamma_l\}$ and using the equivalence of norms in finite dimensional, normed vector spaces, we obtain the desired result if $h := \sup_{i \in I}\{\widetilde{h}_i\}$ is small enough. □

Assume as in the previous section that the quadrature error must be of $\mathcal{O}(h^k)$ so that it is of the same order as that induced by the approximation of the solution to (1.1) by some particular scheme. Then, the integer t and the far field constant b must be chosen so that:

$$t = \sup\left\{p, p - 1 - \alpha + (k + \alpha - 2)\frac{\ln(1/h)}{\ln(\gamma b)}\right\}. \tag{5.14}$$

The constant b must be chosen great enough so that $\gamma b > 1$. Numerical tests performed on the simple and double layer potentials of the Laplace equation have shown that for $0 \leq p \leq 4$, an optimal choice for b may be $0.2 \leq b \leq 0.4$ (see [8]).

6. The intermediate field.

6.1. The quadrature rule. If x is in the intermediate field of Γ_i, a compound rule is used. For sake of simplicity $\widehat{s}_i$ is still assumed to be a polynomial of degree p. As explained before, this is not a restrictive hypothesis.

Let N be a positive integer. The reference simplex $\widehat{S}$ is divided into N^2 geometrically equivalent simplices $(\widehat{S}_n)_{n=1,\cdots,N^2}$. Let $\widehat{\psi}_n$ be the unique linear mapping, which maps $\widehat{S}_n$ into $\widehat{S}$, and whose Jacobian determinant is positive. Let t be an integer, yet to be specified, so that $p \leq t$. Consider on $\widehat{S}$ a quadrature rule that is assumed to be exact for polynomials of degree less than or equal to t. This quadrature rule is applied on each elementary simplex $\widehat{S}_n$. As a result, (1.2) can approximated by

$$(6.1)\quad \mathcal{K}[s,x,i] \approx \sum_{n=1}^{N^2}\sum_{q=1}^{Q(t)} \frac{1}{N^2}\omega_q^{\text{inter}} K(x,\psi_i(\widehat{x}_q^n))\cdot \widehat{s}_i(\widehat{x}_q^n)J_i(\widehat{x}_q^n) := \mathcal{K}_{tN}[s,x,i],$$

where we have set $\widehat{x}_q^n := \widehat{\psi}_n(\widehat{x}_q)$ for short.

6.2. Error estimate.

LEMMA 6.1. *Consider $f : \widehat{S}_n \longrightarrow \mathbb{R}^{n'}$. Assume that $f \in C^j(\text{int}(\widehat{S}_n))$ then*

$$(6.2)\quad |f\circ\widehat{\psi}_n|_{j,\infty,\widehat{S}} \leq \frac{1}{N^j}|f|_{j,\infty,\widehat{S}_n}.$$

Proof. $\widehat{\psi}_n$ is the combination of a translation, a rotation, and the scaling mapping Id/N. As a result $|\widehat{\psi}_n|_{1,\infty,\widehat{S}_n}$ is equal to $1/N$; note also that the Jacobian determinant of $\widehat{\psi}_n$ is equal to $1/N^2$. Furthermore, Lemma 2.2 implies that

$$D^j(f\circ\widehat{\psi}_n) = \text{sym}[D^jf(\widehat{\psi}_n)(D\widehat{\psi}_n)^j].$$

The lemma results from this equality and the value of $|\widehat{\psi}_n|_{1,\infty,\widehat{S}_n}$. □

Let E_{tN} be the quadrature error in the intermediate field.

THEOREM 6.1. *Assume x is in the intermediate field and K satisfies bound (5.11). There is a constant $\beta \geq 1$ so that, if h is small enough, the quadrature error is bounded by*

$$(6.3)\quad E_{tN} \leq c\frac{h^{2-\alpha}}{(\beta a N_t)^{t+1-p+\alpha}}\|\widehat{s}_i\|_{p,q,\widehat{S}},$$

where $N_t := N^{t+3/t+1-p+\alpha}$.

Proof. Let E_n be the quadrature error on the elementary simplex $\widehat{S}_n$:

$$\begin{aligned} E_n := \frac{1}{N^2}\Bigg| &\int_{\widehat{S}_n} K(x,\psi_i(\widehat{\psi}_n(\widehat{y}))).\widehat{s}_i(\widehat{\psi}_n(\widehat{y}))J_i(\widehat{\psi}_n(\widehat{y}))d\widehat{y} \\ &-\sum_{q=1}^{Q(t)}\omega_q^{\text{inter}}K(x,\psi_i(\widehat{x}_q^n))\ .\widehat{s}_i(\widehat{x}_q^n)J_i(\widehat{x}_q^n)\Bigg|. \end{aligned}$$

Then, E_{tN} is bounded by $\sum_{n=1}^{N^2} E_n$. As in the demonstration of Theorem 5.1, we infer from the Bramble–Hilbert lemma that

$$E_n \leq \frac{1}{N^2}\sum_{j=0}^{p}|\widehat{s}_i(\widehat{\psi}_n)|_{j,q,\widehat{S}}\sum_{l=0}^{t+1-j} c_{jl}|K(x,\psi_i(\widehat{\psi}_n))|_{l,\infty,\widehat{S}}|J_i(\widehat{\psi}_n)|_{t+1-j-l,\infty,\widehat{S}},$$

where we have used the fact that $\widehat{s}_i$ is a polynomial of degree $p \leq t$, and we have applied Lemma 2.1 together with various forms of Hölder inequality. Note that the positive constants c_{jl} do not depend on i. According to Lemma 6.1, we have

$$|K(x,\psi_i(\widehat{\psi}_n))|_{l,\infty,\widehat{S}} \leq \frac{1}{N^l}|K(x,\psi_i)|_{l,\infty,\widehat{S}_n} \leq \frac{1}{N^l}|K(x,\psi_i)|_{l,\infty,\widehat{S}}.$$

Furthermore, from Proposition 5.3 we infer that, if h is small enough, there are constants $\beta_l \geq 1$ so that:

$$|K(x,\psi_i)|_{l,\infty,\widehat{S}} \leq c\frac{\widetilde{h}_i^{-\alpha}}{(\beta_l a)^{l+\alpha}}.$$

Hence, applying Lemma 6.1 to $\widehat{s}_i(\widehat{\psi}_n)$ and $J_i(\widehat{\psi}_n)$ and using the preceding results yield:

$$E_n \leq \frac{1}{N^2}\sum_{j=0}^{p}\frac{1}{N^j}|\widehat{s}_i|_{j,q,\widehat{S}}\sum_{l=0}^{t+1-j} c_{jl}\frac{\widetilde{h}_i^{-\alpha}}{N^l(\beta_l a)^{l+\alpha}}\left(\frac{\widetilde{h}_i}{N}\right)^{t+1-j-l}\widetilde{h}_i^2.$$

Setting $\beta := \inf\{\beta_l\}$ and using the equivalence of norms in finite dimensional, normed vector spaces, we obtain the desired result if $h := \sup_{i\in I}\{\widetilde{h}_i\}$ is small enough. □

Assume the quadrature error must be of $\mathcal{O}(h^k)$ so that it is of the same order as that induced by the approximation of the solution to (1.1) by some particular scheme. Assume also that N is chosen so that $N_t\beta a$ is greater than a specified constant C_N. Then, the integer t must be chosen so that:

$$t = \sup\left\{p, p-1-\alpha+(k+\alpha-2)\frac{\ln(1/h)}{\ln(C_N)}\right\}. \tag{6.4}$$

We have yet to specify N and the intermediate field constant a. Note that

$$N = \left(\frac{C_N}{\beta a}\right)^{t+1-p+\alpha/t+3}. \tag{6.5}$$

Furthermore, x belongs to a finite set of control or quadrature points. Let Q_{Ip} be the set in question. The intermediate field constant can be chosen so that

$$a = \inf\left\{\frac{\operatorname{dist}(x,\Gamma_i)}{\widetilde{h}_i}\ :\ x\in Q_{Ip}-\Gamma_i,\ i\in I\right\}. \tag{6.6}$$

Note that the present scheme is of interest only if we can be sure that N is bounded as h converges to zero. For this matter we have to show that a does not converge to zero as h decreases to zero. Actually, it will be shown that, thanks to the regularity condition 3, a is bounded below when h converges to zero.

Recall that on each panel, Γ_j, the set of control or quadrature points is the image by ψ_j of a unique set of points of $\widehat{S}$ that is denoted by $\widehat{Q}_p$. We have assumed also that no point of $\widehat{Q}_p$ belongs to the boundary of the simplex $\widehat{S}$. Then, we have the following result.

PROPOSITION 6.1. *a is bounded below as h converges to zero.*

Proof. Let x_0 and Γ_{i0} be the control point and the panel for which $\operatorname{dist}(x_0,\Gamma_{i0})$ is equal to $a\widetilde{h}_{i0}$. Let Γ_{j0} the panel to which x_0 belongs. Since $\partial\Omega$ is Lipschitzian, there is a constant c such that

$$\operatorname{dist}(x,\partial\Gamma_{j0}) \leq c\inf_{y\notin\Gamma_{j0}}\{\|y-x\|\} \leq c\operatorname{dist}(x,\Gamma_{i0}).$$

Furthermore, x_0 being a control or a quadrature point, there is a constant $c(p)$ such that

$$c(p) \leq \operatorname{dist}(\widehat{x}_0,\partial\widehat{S}) = \inf_{\widehat{y}\in\partial\widehat{S}}\{\|\widehat{y}-\widehat{x}_0\|\} = \inf_{y\in\partial\Gamma_{j0}}\{\|\psi_{j0}^{-1}(y)-\psi_{j0}^{-1}(x_0)\|\}.$$

If h is small enough, it comes that

$$c(p) \leq \inf_{y \in \partial\Gamma_{j0}} \{ |\psi_{j0}^{-1}|_{1,\infty,\Gamma_{j0}} \|y - x_0\| + o(\|y - x_0\|) \} \leq \frac{c'}{\widetilde{h}_{j0}} \mathrm{dist}(x, \partial\Gamma_{j0}).$$

If h is small enough Γ_{j0} has necessarily a piece of boundary in common with Γ_{i0}. Hence, Γ_{i0} and Γ_{j0} are to neighbouring panels. It is at this point that the regularity condition 3 is needed; this condition implies that there is a constant β_0 such that $\widetilde{h}_i \leq \beta_0 \widetilde{h}_{j0}$. As a result, the desired bound is obtained:

$$\forall i \in I, \forall x \in Q_{Ip} - \Gamma_i, \qquad \frac{\beta_0 c(p)}{cc'} \leq \frac{\mathrm{dist}(x, \Gamma_i)}{\widetilde{h}_i}. \qquad \square$$

As a consequence, N is bounded above as h converges to zero. Note also that the fact that Ω is locally on one side of its boundary has been used in the preceding proof.

7. Conclusion. Numerical quadratures for approximating integrals of type (1.2) over curved domains in $\mathbb{R}^3$ along with estimates on the quadrature errors have been presented: (4.7), (5.2), and (6.1). A new definition of pseudohomogeneity that emphasizes the role of polar coordinates has been given. The numerical quadratures (4.7), (5.2), and (6.1) are suitable whenever the kernel of the physical problem that is considered is pseudohomogeneous of degree -1. The simple layer potentials of Laplace and Helmholtz equations along with that of the Stokes flow problem and the linear elasticity problem have been shown to be pseudohomogeneous of degree -1. The same conclusion has been drawn on double layer potentials of Laplace and Helmholtz equations. The numerical quadrature presented here may be useful when an approximation of the solution to (1.1) is sought. No approximation of the surface $\partial\Omega$ is needed. The present approach only requires that $\partial\Omega$ is defined by a regular chart of the form $(\Gamma_i, \psi_i)_{i \in I}$, which may be provided by a standard CAD system.

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