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A systematic formula for the asymptotic expansion of singular integrals

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I. Introduction

The formulation of problems in physics often leads to integral equations. The usual difficulties come when inverting those equations. Until now, both analytical and numerical techniques have been used to invert them.

Numerical techniques usually discretize the boundaries of the domain, or the whole domain, where the problem has to be solved. This discretization leads to a system of algebraic equations which can be solved by using matrix inversion techniques. This method is powerful, well adapted to computers and therefore widely used. However, analytical solutions or approximations are useful too. To find such expressions is usually difficult. One way consists of finding an asymptotic approximation of the exact solution. In that respect, in fluid dynamics, theories like the lifting-line (see [8]), the slender body, or the slender ship (see [7]) have been developed. In each case, one dimension of the considered body is substantially smaller than the others and a small parameter like the slenderness or the inverse of the aspect-ratio can be defined. Let ε be that small parameter. That assumption always leads to a line-integral, performed along the "span" or the "length", for which an asymptotic expansion with respect to ε has to be found. Let us consider this integral as:

$$I = \int_D f(x)\, K(x, \varepsilon)\, \mathrm{d}x. \tag{1}$$

Usually the weight function $f(x)$ is unknown and the kernel $K(x, \varepsilon)$ is singular when both ε and x are null. Many useful techniques have been devised to find the asymptotic expansion of I when the domain D and the kernel $K(x, \varepsilon)$ have particular expressions: see N. Bleistein [2] and A. Erdelyi [4]. However, their use is limited in the sense that they generally impose strong restrictive hypotheses on the kernel. When the kernel is too complicated for this set of methods, then the Matched Asymptotic Expansion Method (MAEM) is widely

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used (see ref. [1], [6]). This method consists of dividing the domain into an inner region where the variable of integration x is of the same order as a "boundary layer thickness" to be determined $\delta(\varepsilon)$, and an outer region where x is larger than $\delta(\varepsilon)$. The asymptotic approximation is obtained by matching the two expansions corresponding to each of these regions. First, the MAEM is laborious. As a result, when trying to develop high-order approximations, it is very likely that mistakes will be made. Second, a systematic theory for this method is not well established. Therefore the user might encounter many unexpected difficulties when applying it, especially when he proceeds with the matching of the inner and outer expansions. To avoid these difficulties, a new method is presented when the integral has the following form:

$$I = FP \int_D f(x)\, K(x, \varepsilon)\, \mathrm{d}x\,. \tag{2}$$

D is a domain containing zero, $f(x)$ is a function for which $J + 1$ derivatives can be defined and the $(J + 1)\,st$ is continuous over D. The kernel $K(x, \varepsilon)$ is assumed to be homogeneous of order β, i.e. for any real α the following is true:

$$K(\alpha x, \alpha\varepsilon) = \alpha^{\beta} S(\alpha)\, K(x, \varepsilon) \tag{3}$$

where $S(\alpha)$ is either the unit function or the sign function: $\mathrm{sgn}(\alpha)$. The kernel is also assumed to have $(L + 1)$ derivatives with respect to the second variable: ε, in $D^* \times \{0\}$. Furthermore, the last derivative is assumed to be integrable in the Finite Part sense with respect to ε, and may possess a finite number of singular points in $\mathbf{R}$ with respect to x. The integral is defined by its Finite Part (FP), as introduced by J. Hadamard (see ref. [3], [5]).

Under these conditions which are satisfied in most practical applications, we shall show in this paper that an asymptotic expansion of the integral I in terms of ε^r and $\varepsilon^r \log|\varepsilon|$ can be found, and we shall give it an explicit formulation. If M is the approximation order, then we show that logarithmic terms occur only if the following conditions are all satisfied:

(i) β is an integer.
(ii) $S(x)$ is the sign function $\mathrm{sgn}(x)$. (4)
(iii) M is a positive integer.

In the last section of this paper we shall treat two examples to demonstrate the efficiency of the new formula. In the first example we shall deal with an elliptic kernel. In the second one, we shall present an elegant new method to solve the asymptotic lifting-line problem.

II. Preliminary statements

We shall define in this section a set of expressions, which will be useful for the main demonstration in the next section. Under the conditions presented above for $f(x)$ and the Kernel $K(x, \varepsilon)$, the Mac Laurin's formula can be applied to $f(x)$ for any real x belonging in D, and to $K(x, \varepsilon)$ for any real x different from zero:

$$f(x) = \sum_{j=0}^{J} f^{(j)}(0) \frac{x^j}{j!} + \int_0^x \frac{(x-t)^J}{J!} f^{(J+1)}(t)\, dt$$
$$K(x, \varepsilon) = \sum_{l=0}^{J} \frac{\partial^l}{\partial \varepsilon^l} K(x, 0) \frac{\varepsilon^l}{l!} + FP \int_0^\varepsilon \frac{(\varepsilon - v)^L}{L!} \frac{\partial^{L+1}}{\partial v^{L+1}} K(x, v)\, dv. \tag{6}$$

The last integral is defined by its Finite Part, defined below, because $K(x, \varepsilon)$ and its derivatives may have a finite number of singularities in the interval $[0, \varepsilon]$. Let us call $R_{f,J}(x)$ and $R_{K,L}(x, \varepsilon)$ the two remainders. Since the $(J + 1)st$ derivative of $f(x)$ is continuous over D, it is possible to apply the mean value theorem to $R_{f,J}(x)$. That is to say, there exists a function x_1 of x, which takes its values in the open interval: $]0, x[$ and satisfies the following:

$$R_{f,J}(x) = x \frac{(x - x_1)^J}{J!} f^{(J+1)}(x_1). \tag{7}$$

Because of the homogeneity condition (3), $K(x, \varepsilon)$ and its $(L + 1)$ derivatives with respect to ε can be defined almost everywhere over the set: $\mathbf{R} \times \mathbf{R}$. Therefore, all the integrals which will be used with upper and lower limits which do not belong to D will be justified by this argument. Under the homogeneity condition, it is also easy to prove that for any integer l less than or equal to $L + 1$ and x different from zero, the following expressions are true:

$$\frac{\partial^l}{\partial \varepsilon^l} K(x, 0) = x^{\beta - l} S(x)\, \partial_l K(1) \tag{8}$$

where we set the following definition:

$$\partial_l K(1) \overset{\text{def}}{=} \frac{\partial^l}{\partial \varepsilon^l} K(1, 0). \tag{9}$$

Kernels of that kind are very common in physics problems where the homogeneity condition is fundamental. For example, the following kernels satisfy all the conditions

$$K_1(x, \varepsilon) = (x^2 - 2\varepsilon x \cos\alpha + \varepsilon^2)^\lambda$$
$$K_2(x, \varepsilon) = \frac{1}{(x - \varepsilon)\,|x - \varepsilon|}. \tag{10}$$

Finite Parts of integrals are commonly encountered when the Laplace equation is solved by means of the Green function. The solution is usually expressed as a convolution product of the Green function, generally singular, with a "regular" function. The integral is taken over a line, a surface or a more complex domain and sometimes the Green function is so singular that the integral cannot be defined as a "common" integral. Actually it is possible to show that the integration associated with the convolution product has to be performed in the Distributions set, but not in the set of the "Common" functions. As a result, the integral exists almost everywhere in the domain where the kernel, the Green function, is infinite. The physical meaning of this operation is recovered by performing on the integral as many integrations by part as it is necessary for the integrand to be integrable in the Reiman sense or at least in the generalized sense. The Distributions set is also called the Generalized Functions set by some authors. In this very useful set, integration by part is an elementary operation, since every Distribution is infinitely derivable.

Let us illustrate this with some examples. Let us consider a function $h(x)$ decreasing at infinity faster than any monomial, and admitting as many derivatives as necessary. If α is a real non-integer smaller than one, the Finite Part of the integral of the product of $h(x)$ times $1/x^\alpha$ is defined by J. Hadamard as:

$$FP \int_{-\infty}^{+\infty} \frac{h(x)}{x^\alpha} \,\mathrm{d}x = \frac{\Gamma(\alpha - n)}{\Gamma(\alpha)} \int_{-\infty}^{+\infty} x^{n-\alpha} h^{(n)}(x) \,\mathrm{d}x . \tag{11}$$

The integer n is the integer part of α; therefore, the integral used in the right-hand side of (11) is a generalized integral and is obtained by performing on the left-hand side n integrations by part. It is straightforward to see that this definition can be extended to a bounded interval by multiplying the function $h(x)$ by the step function associated with this interval. If α is an integer m, the expression is slightly different, since the last integration leads to $\log|x|$:

$$FP \int_{-\infty}^{+\infty} \frac{h(x)}{x^m} \,\mathrm{d}x = \frac{-1}{(m-1)!} \int_{-\infty}^{+\infty} \log|x| \, h^{(m)}(x) \,\mathrm{d}x . \tag{12}$$

III. General developments

Using the Mac Laurin's formula (5), (6) for $f(x)$ and $K(x, \varepsilon)$, the integral I can be expressed as:

$$\begin{aligned} FP \int_D f(x)\, K(x, \varepsilon) \,\mathrm{d}x &= \sum_{l=0}^{L} \frac{\varepsilon^l}{l!} \partial_l K(1)\; FP \int_D f(x)\, x^{\beta - l} S(x) \,\mathrm{d}x \\ &+ \sum_{j=0}^{J} \frac{f^{(j)}(0)}{j!}\; FP \int_D x^j R_{K,L}(x, \varepsilon) \,\mathrm{d}x \\ &+ FP \int_D R_{f,J}(x)\, R_{K,L}(x, \varepsilon) \,\mathrm{d}x . \end{aligned} \tag{13}$$

In the Appendix the last term is shown to be $\circ(\varepsilon^M)$ where M is an integer which value is discussed farther. A direct consequence is that the first sum over l can be truncated for $l = M$. Now, the problem consists of finding an asymptotic expansion for the second term of (13). For that purpose, the integral is split over four intervals:

$$FP \int_D x^j R_{K,L}(x, \varepsilon)\, dx = FP \int_{-\infty}^{-|\varepsilon|} + \int_{+|\varepsilon|}^{+\infty} + \int_{-|\varepsilon|}^{+|\varepsilon|} - \int_{R-D} x^j R_{K,L}(x, \varepsilon)\, dx. \tag{14}$$

The last integral is proved in the Appendix to be $\circ(\varepsilon^M)$. Using (6), (8) and (9) the previous equation can rewritten:

$$\begin{aligned} FP \int_D x^j R_{K,L}(x, \varepsilon)\, dx = {} & FP \int_{-\infty}^{-|\varepsilon|} + \int_{+|\varepsilon|}^{+\infty} x^j \left[K(x, \varepsilon) - \sum_{l=0}^{L} \frac{\varepsilon^l}{l!} x^{\beta-1} S(x)\, \partial_l K(1) \right] dx \\ & + FP \int_{-|\varepsilon|}^{+|\varepsilon|} x^j K(x, \varepsilon)\, dx \\ & - \sum_{l=0}^{L} \frac{\varepsilon^l}{l!} \partial_l K(1)\, FP \int_{-|\varepsilon|}^{+|\varepsilon|} \frac{S(x)}{x^{l-j-\beta}}\, dz + \circ(\varepsilon^M). \end{aligned} \tag{15}$$

Then a new variable of integration is used: $X = x/\varepsilon$ for all the integrals of the righthand side. However, this rescaling is valid for the two integrals with infinite limits only if the integrand decreases at the infinity faster than $1/|x|$. Therefore the integer J has to be less than $L - \beta$. In the second term of (15) the rescaling is valid for any value of j. This is no longer true for the third term if the exponent $l - j - \beta$ equals 1, because in this case the integrand behaves like $1/x$, and the integration leads to: $(1 - S(-1)) \log|\varepsilon|$. Finally, after the rescaling and some simplifications, the final result is:

$$\begin{aligned} FP \int_D f(x)\, K(x, \varepsilon)\, dx = {} & \sum_{l=0}^{M} \left[\frac{\partial_l K(1)}{l!} FP \int_D \frac{f(x)\, S(x)}{x^{l-\beta}}\, dx \right] \cdot \varepsilon^l \\ & + \sum_{j=0}^{J} \left[\frac{f^{(j)}(0)}{j!} FP \int_{-\infty}^{+\infty} x^j K(x, 1)\, dx \right] \cdot \varepsilon^{j+\beta+1} \\ & - R(\beta) \sum_{\substack{j \geqq -1-\beta \\ j=0}}^{J} \left[[1 - S(-1)] \frac{f^{(j)}(0)}{j!} \frac{\partial_{1+\beta+j} K(1)}{(1+\beta+j)!} \right] \cdot \varepsilon^{j+\beta+1} \log|\varepsilon| \\ & + \circ(\varepsilon^M) \end{aligned} \tag{16}$$

where $R(\beta)$ is a function which equals one, if β is an integer, and equals zero, if β is not. For practical cases it is convenient first to choose M, which defines the approximation order; then the value of J is necessarily given by:

$$J = M - [\beta] - 1. \tag{17}$$

Hence, the results stated in the introduction have been proved. Formula (16) gives directly the asymptotic expansion of (2) without any further development. The only difficulty of this procedure consists of evaluating the Finite Parts of the two integrals of the right-hand side. This can be done by using the technique presented in Section II of this paper. To a certain extent, the calculation of these two integrals can be compared to the evaluation of the Mellin transform of $f(x)$ and $K(x, 1)$, which is commonly used in some other methods devoted to the study of asymptotic expansions of integrals (see N. Bleistein [2]).

IV. Examples

IV-1. First example

We shall illustrate the use of formula (16) in this section. Let us first consider an integral I for which both the weight function and the kernel are known. We shall show that formula (16) leads easily to the asymptotic expansion of I, even though the kernel is as complicated as the first one in (10). Let us define $F(\varepsilon)$ as:

$$F(\varepsilon) = \int_{-\infty}^{+\infty} \frac{e^{-x^2}}{\sqrt{x^2 - 2\varepsilon x \cos(\alpha) + \varepsilon^2}} \, \mathrm{d}x. \tag{18}$$

All the required conditions are satisfied for both the weight function and the kernel. The real β equals -1, and if M is the chosen approximation order, then according to (17), J equals M. Thus, the asymptotic expansion of $F(\varepsilon)$ is:

$$\begin{aligned} F(\varepsilon) = & \sum_{l=0}^{M} \left[P_{2l}[\cos(\alpha)] \; FP \int_{-\infty}^{+\infty} \frac{e^{-x^2} \operatorname{sgn}(x)}{x^{2l+1}} \, \mathrm{d}x \right] \cdot \varepsilon^{2l} \\ & + \sum_{j=0}^{M} \left[\frac{(-1)^j}{j!} \, FP \int_{-\infty}^{+\infty} \frac{x^{2j}}{\sqrt{x^2 - 2x\cos(\alpha) + 1}} \, \mathrm{d}x \right] \cdot \varepsilon^{2j} \\ & - 2 \sum_{j=0}^{M} \left[\frac{(-1)^j}{j!} \, P_{2j}[\cos(\alpha)] \right] \cdot \varepsilon^{2j} \log|\varepsilon| + \circ(\varepsilon^M) \end{aligned} \tag{19}$$

where $P_n(x)$ is the Legendre polynomial of degree n. If instead of a square-root, the kernel exponent had been a real λ, then instead of the Legendre polynomials we would have used the Gegenbauer polynomials: $C_n^{\lambda}(x)$ in formula (19). The only difficulty, here, consists of expressing the Finite Parts of the two integrals of the right-hand side. The first one is evaluated by applying to it the integration by part procedure explained in Section II. The following result is found:

$$\begin{aligned} FP \int_{-\infty}^{+\infty} \frac{e^{-x^2} \operatorname{sgn}(x)}{x^{2l+1}} \, \mathrm{d}x &= \frac{-1}{l} \, FP \int_{-\infty}^{+\infty} \frac{e^{-x^2} \operatorname{sgn}(x)}{x^{2l-1}} \, \mathrm{d}x \\ &= 4 \frac{(-1)^l}{l!} \int_{0}^{+\infty} e^{-x^2} \, x \log(x) \, \mathrm{d}x. \end{aligned} \tag{20}$$

The last integral in (20) is a constant independent of both ε and the approximation order, and can be numerically determined. The second Finite Part of (19) is found by replacing the kernel by its series representation in terms of Legendre polynomials:

$$FP \int_{-\infty}^{+\infty} \frac{x^{2j}}{\sqrt{x^2 - 2x\cos(\alpha) + 1}}\, dx = 2\left[\sum_{k=0}^{\infty} \frac{P_{2k}[\cos(\alpha)]}{2j + 2k + 1} - \frac{P_{2k+1}[\cos(\alpha)]}{k-1}\right]. \tag{21}$$

The asymptotic expansion of (18) is now completely determinated and has required a very limited amount of algebra.

IV-2. Second example

Let us now illustrate the present method with the asymptotic lifting-line problem. This model, introduced by M. D. Van-Dyke [8], is now very classical in incompressible aerodynamics. It is mainly a matter of finding the asymptotic approximation of the pressure distribution on an unswept zero-thickness wing of large aspect ratio. We shall show that formula (16) leads to a very elegant solution of this problem. Let us call S the wing surface in the plan xOy, x being the downstream direction, and y the spanwise direction. If $\alpha(x, y)$ is the camber distribution of this wing (known function), and $\gamma(\xi, \eta)$ the loading distribution (unknown function), then the integral equation to invert is:

$$\alpha(x, y) = \frac{1}{4\pi} FP \iint_S \frac{\gamma(\xi, \eta)}{(y-\eta)^2}\left[1 + \frac{x - \xi}{\sqrt{(y-\eta)^2 + (x-\xi)^2}}\right] d\eta\, d\xi. \tag{22}$$

In order to solve asymptotically this problem, M. D. Van-Dyke proposed the concept of a high aspect-ratio wing for which the span-scale is very large compared with the chord-scale. If ε is the ratio of the chord to the span, then he found that the beginning of the asymptotic expansion of the loading function is:

$$\gamma(\xi, \eta) = \gamma_0(\xi, \eta) + \varepsilon\gamma_1(\xi, \eta) + \varepsilon^2\gamma_2(\xi, \eta) + \varepsilon^2 \log(\varepsilon)\,\gamma_3(\xi, \eta) + \circ(\varepsilon^2). \tag{23}$$

He did not use (22) to prove this result, but applied the MEAM to the velocity potential, and worked on the set of differential equations and boundary conditions. In their paper [6], T. Kida and Y. Miyai showed that the procedure used by M. D. Van-Dyke was unnecessarily complicate and had probably led the author to a miscalculation. They proved that the solution can be simply recovered by expanding Eq. (22) with respect to ε. They defined the new set of variables $\xi - x = u$ and $\eta - y = v$; then they rewrote (22) with these new variables:

$$\alpha(x, y) = \frac{1}{4\pi} FP \int_{\text{Chord}} du\, FP \int_{\text{Span}} \frac{\gamma(u, v)}{v^2}\left[1 - \frac{u}{\sqrt{v^2 + u^2}}\right] dv. \tag{24}$$

Here, the point is that the ratio: u over v, is order of ε almost everywhere on the wing, as a result, in the integral over the span u can be considered as a small parameter. Therefore, after having integrated by part the integral over the span, T. Kida and Y. Miyai expanded it with respect to the small parameter u. To carry out the expansion they used MAEM. They reached the order $u^2 \log(u)$ after a great deal of algebra and were limited to this order by the calculation complexity. But, as we shall see, by applying formula (16) to this integral, we can reach any approximation order we want in only one step. The application of (16) is straightforward and yields to:

$$FP \int_{\text{Span}} \frac{\gamma(u, v)}{v^2}\left[1 - \frac{u}{\sqrt{v^2 + u^2}}\right] \mathrm{d}v = -B_{-2}\frac{\gamma(u, 0)}{u} + FP \int_{\text{Span}} \frac{\partial \gamma}{\partial v}(u, v)\frac{\mathrm{d}v}{v}$$
$$- \sum_{l=0}^{M}\left[A_{2l} FP \int_{\text{Span}} \frac{\gamma(u, v)}{v^{2l+3}} \operatorname{sgn}(v)\, \mathrm{d}v + B_{2l}\frac{\partial^{2(l+1)}}{\partial v^{2(l+1)}}\gamma(u, 0)\right] u^{2l}$$
$$- \sum_{l=0}^{M}\left[C_{2l}\frac{\partial^{2(l+1)}}{\partial v^{2(l+1)}}\gamma(u, 0)\right] u^{2l} \log|u| \tag{25}$$

where the constants A_{2l}, B_{2l}, and C_{2l} are given by:

$$A_{2l} = P_{2l}[0] = \frac{(-1)^l (2l-1)!!}{2^l l!} \tag{26}$$

$$B_{2l} = \frac{2}{(2l+2)!}\sum_{k=0}^{\infty}\frac{P_{2k}[0]}{(2l+2k+1)} \tag{27}$$

$$C_{2l} = -\frac{2}{(2l+2)!}P_{2l}[0] \tag{28}$$

$$B_{-2} = -2 \tag{29}$$

These results are obtained by using the series representation of the kernel in terms of the Legendre polynomials. After setting the new variable $u = \varepsilon(\Xi - X)$, and performing the chordwise integration on (25), we come up with an original asymptotic representation of (22)

$$\alpha(x, y) = \frac{1}{2\pi} FP \int_{\text{Chord}} \frac{\gamma(\Xi, y)}{\Xi - X}\, \mathrm{d}E - \frac{\varepsilon}{4\pi} FP \int_{\text{Span}} \frac{\dot{\Gamma}(y)}{y - \eta}\, \mathrm{d}\eta$$
$$- \frac{1}{4\pi}\sum_{l=0}^{M} \varepsilon^{2l+1}\left[A_{2l} FP \iint_{S} \frac{\gamma(\Xi, y)}{(\eta - y)^{2l+3}} \operatorname{sgn}(\Xi - X)\, \mathrm{d}\eta\, \mathrm{d}\Xi\right.$$
$$\left. + B_{2l}\int_{\text{Chord}} \frac{\partial^{2(l+1)}}{\partial \eta^{2(l+1)}}\gamma(\Xi, y)\,(\Xi - X)^{2l+1}\, \mathrm{d}\Xi\right]$$
$$- \frac{1}{4\pi}\sum_{l=0}^{M} \varepsilon^{2l+1}\log(\varepsilon)\left[\int_{\text{Chord}} \frac{\partial^{2(l+1)}}{\partial \eta^{2(l+1)}}\gamma(\Xi, y)(\Xi - X)^{2l+1}\log|\Xi - X|\, \mathrm{d}\Xi\right] \tag{30}$$

where $\Gamma(y)$ is the circulation, that is to say the integral of $\gamma(\Xi, y)$ over the chord. Our goal, in the present paper, is not to give a complete solution to this problem; however, we can briefly outline it. By following the same procedure as the one proposed in [6], it is possible to asymptotically solve this equation, step by step. Basically, for each step of approximation of $\gamma(\Xi, y)$ we have to solve the following Cauchy problem:

$$\text{R.H.S}\,(\gamma_0, \gamma_1, \ldots, \gamma_{n-1}) = \frac{1}{2\pi} FP \int_{\text{Chord}} \frac{\gamma_n(\Xi, y)}{\Xi - X}\, d\Xi . \tag{31}$$

The left-hand side depends only on the former approximations, and the right-hand side is the dominant term of (30). In this particular case, the Finite Part of the integral is its Cauchy Principal value. This 2-D problem is very classical insofar as the n^{th} approximation of the loading distribution is analytic in the vertical plan. It is straightforward to check, up to the second order, that (31) yields to the same solution as the one obtained by Eqs. (7) and (8) in [6]. The present result confirms the small miscalculation found by T. Kida and Y. Miayai in M. D. Van-Dyke's work (ref. [8], formula (9.15)). Moreover, the asymptotic expansion can be calculated up to any order M, step by step, and it is possible to show that the general solution at this order can be written as follows:

$$\gamma^{(M)} = \sum_{k=0}^{M} \gamma_k \varepsilon^k + \sum_{n=2}^{M} \gamma_{n,1} \varepsilon^n \log(\varepsilon) + \sum_{n=4}^{M} \sum_{\substack{m=2 \\ m \leq n-2}}^{M-2} \gamma_{n,m} \varepsilon^n \log^m(\varepsilon) . \tag{32}$$

V. Conclusions

The present method is useful in expanding an integral with a singular homogeneous kernel in terms of a small parameter, up to any order where the derivatives of the weight function are defined. The formula (16) which gives the asymptotic expansion is simple and does not need any further mathematical development. Moreover, the occurrence of logarithmic terms is explained and can be predicted under the conditions stated in the introduction.

Two examples have been presented. In the first one, the weight function has an explicit expression. The asymptotic expansion of the integral has been found up to any order. As long as the kernel is homogeneous, this method generalizes all the previous methods dealing with such a class of kernels (see [2], [4]). The amount of algebra is restricted to the evaluation of the Finite Part of two integrals. In the second example the weight function is unknown, and we have shown that formula (16) is useful for finding its asymptotic expansion. Such a method can be helpful to elegantly solve asymptotic problems like lifting-line, slender wing or body, or slender ship, where most of the analytical work is concentrated on the asymptotic expansion of a line-integral with respect to the small parameter introduced.

It certainly would be of great interest to investigate in what direction this method could be generalized, and to determine the radius of convergence of the series given by formula (16). It is likely that more general answers are underlying every problems in which MAEM is used.

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Appendix

A.1 It is proved below that the Finite Part of the integral of the product of the two remainders $R_{f,J}$ and $R_{K,L}$ is $\circ(\varepsilon^M)$.

Let us call R_1 this term; then, using (5) to (7), a new expression is found for R_1:

$$R_1 = FP \int_D x \frac{(x-x_1)^J}{J!} f^{(J+1)}(x_1)\, FP \int_0^{\varepsilon} \frac{(\varepsilon - v)^L}{L!} \frac{\partial^{L+1}}{\partial v^{L+1}} K(x, v)\, dv\, dx. \quad \text{(A-1)}$$

Let $\{x_i\}$, $i = 1, \ldots, N$ be the finite set of singularities of $\frac{\partial^{L+1}}{\partial v^{L+1}} K(x, 1)$. The infinity is eventually excluded from this set. Then, according to the homogeneity of the $(L+1)$th derivative of K, the set $\{1/x_i\}$, $i = 1, \ldots, N$ represents the singularities of $\frac{\partial^{L+1}}{\partial v^{L+1}} K(1, x)$. Let $x_{\max}$ be the real defined by the expression:

$$\begin{cases} x_{\max} = \sup\limits_{i=1,N} |x_i| + 1 & \text{if } N \neq 0; \\ x_{\max} = 1 & \text{otherwise}. \end{cases} \quad \text{(A-2)}$$

Hence, it is valid to define the real S_L as:

$$S_L = \sup_{v \in 0,\, 1/x_{\max}} \left| \frac{\partial^{L+1}}{\partial v^{L+1}} K(1, v) \right|. \quad \text{(A-3)}$$

If ε is small enough for the interval $I_\varepsilon = [-\varepsilon x_{\max}, \varepsilon x_{\max}]$ to be included in D, then the integral in (A-1) can be divided into two parts:

$$R_1 = \frac{\varepsilon^{L+1}}{L!\,J!} FP \int_{D-I_\varepsilon} S(x)\,(x-x_1)^J x^{\beta - L} f^{(J+1)}(x_1) \cdot \int_0^1 (1-t)^L \frac{\partial^{L+1}}{\partial v^{L+1}} K\left(1, \frac{\varepsilon t}{x}\right) dt\, dx$$

$$+ S(\varepsilon)\frac{\varepsilon^{J+\beta+2}}{L!J!} FP \int_{-x_{\max}}^{+x_{\max}} (u-u_1)^J u f^{(J+1)}(\varepsilon u_1)$$

$$\cdot FP \int_0^1 (1-v)^L \frac{\partial^{L+1}}{\partial v^{L+1}} K(u,v)\, \mathrm{d}v \mathrm{d}u. \tag{A-4}$$

Let I_0 be a bounded interval including I_ε, independent of ε and belonging to D. Let S_J be the real as:

$$S_J = \sup_{x \in I_0} |f^{(J+1)}(x_1)|. \tag{A-5}$$

Then $|R_1|$ can be bounded as:

$$|R_1| \leqq \frac{|\varepsilon|^{L+1}}{L!J!}\left| FP \int_{D-I_0} (x-x_1)^J x^{\beta-L} f^{(J+1)}(x_1) \int_0^1 (1-t)^L \frac{\partial^{L+1}}{\partial v^{L+1}} K\left(1, \frac{\varepsilon t}{x}\right) \mathrm{d}t\, \mathrm{d}x \right|$$

$$+ S_L S_J \frac{|\varepsilon|^{L+1}}{L!J!} \int_{I_0 - I_\varepsilon} |x|^{J+\beta-L} \mathrm{d}x \tag{A-6}$$

$$+ \frac{|\varepsilon|^{J+\beta+2}}{L!J!}\left| \int_{-x_{\max}}^{+x_{\max}} (u-u_1)^J u f^{(J+1)}(\varepsilon u_1) FP \int_0^1 (1-v)^J \frac{\partial^{L+1}}{\partial v^{L+1}} K(u,v)\, \mathrm{d}v \mathrm{d}u \right|.$$

Then choosing the integers J, L, and M such as $M = L$ and $J = M - [\beta] + 1$, it is straightforward to see that $\lim_{\varepsilon \to 0} \frac{|R_1|}{|\varepsilon|^M} = 0$; hence we have the expected result:

$$R_1 = \circ(\varepsilon^M). \tag{A-7}$$

A.2 Let R_2 be the second remainder, as:

$$R_2 = \int_{R-D} x^j R_{K,L}(x, \varepsilon)\, \mathrm{d}x. \tag{A-8}$$

Using the same method as before, it is easy to obtain the following inequality for R_2:

$$|R_2| \leqq \frac{\varepsilon^{L+1}}{L!J!}\left| FP \int_{R-D} x^j \int_0^1 (1-t)^L \frac{\partial^{L+1}}{\partial v^{L+1}} K\left(1, \frac{\varepsilon t}{x}\right) \mathrm{d}t\, \mathrm{d}x \right|. \tag{A-9}$$

The final result is now obvious:

$$|R_2| = \circ(\varepsilon^M). \tag{A-10}$$

Nomenclature

D	domain of integration
$f(x)$	weight function
FP	Finite Part

$h(x)$	weight function
I_ε, I_0	bounded intervals
j, J	integers
$K(x, \varepsilon)$	singular kernel
l, L	integers
M	integer defining the approximation order
$P_k(x)$	Legendre polynomial
$\mathbf{R}$	set of real numbers
$R(\beta)$	equals 1 if β is an integer and 0 if not
$R_{f,J}, R_{K,L}$	remainders of Taylor developments
$S(\alpha)$	equals either 1 or the sign function: $sgn(\alpha)$
t, u, v, x	variable of integration
α, λ	real numbers
β	homogeneity order of the kernel
$\Gamma(\alpha)$	Euler's integral (gamma function)
ε	"small" parameter
$[\,.\,]$	integer part of.

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Abstract

Asymptotic theories like the lifting-line, the slender body or the slender ship lead to line-integrals with singular kernels. Sometimes these integrals are "improper", that is to say that they are defined only by their Finite Part. To find asymptotic expansions of these integrals, the Matched Asymptotic Expansion Method is widely used along with other more specific methods depending on the kernel type. The first method is laborious and not systematic, and the other methods are sometimes too much specific to treat general cases. Moreover, all of them are not well adapted to deal with Finite Part integrals.

Here, a new method is proposed to avoid the previous difficulties. This method is systematic for homogeneous kernels and gives approximations up to any order, as long as the derivative of the weight function exists at this given order. Moreover the occurrence of logarithmic terms in the expansion is explained and easily predictable. An elliptic integral and the classical lifting-line theory are treated to illustrate the ease of this method.

Résumé

Les théories asymptotiques telles que la ligne portante, le corps élancé ou le navire de grand allongement conduisent à des intégrales curvilignes à noyaux singuliers. Parfois, ces intégrales sont "impropres" c'est à dire qu'elles sont définies en Parties Finies. Différentes méthodes ont été mises au point pour trouver les développements asymptotiques de ces intégrales. Généralement elles dépendent fortement de la nature du noyau, et c'est finalement la méthode des développements raccordés qui est utilisées quand le noyau est trop compliqué. Cependant, cette méthode est laborieuse et comme les précèdentes non adaptée aux intégrales défines par leur Partie Finie.

Une nouvelle méthode est proposée pour surmonter ces difficultés. Cette méthode est systématique pour les noyaux homogènes et donne les approximations à tout ordre pourvu que les dérivées de la fonction poids existent jusqu'à cet ordre. De plus la présence de termes logarithmiques dans le développement est expliquée et aisément prédictible.

Une intégrale elliptique, ainsi que la fameuse théorie de la ligne portante sont traités pour illustrer les possibilités de la méthode.

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