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3D VECTOR POISSON-LIKE PROBLEM WITH A TRIPLET OF INTRINSIC SCALAR BOUNDARY CONDITIONS

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In this work, we consider the three-dimensional vector Poisson-like equation supplemented by a nonstandard set of three scalar boundary conditions consisting of the simultaneous specification of the divergence of the unknown, its normal component, and the normal component of its curl on the entire boundary. A weak formulation of this elliptic boundary value problem is proposed. Existence and uniqueness of a solution are established under two compatibility conditions. An uncoupled solution algorithm is introduced together with its finite element approximation. The corresponding error analysis is performed.

Keywords: 3D Poisson-like equation; vector elliptic problem; vector potential in magnetostatics; scalar boundary conditions; weak formulations; uncoupled solution; finite element approximation; error estimates.

1. Introduction

1.1. *Preliminaries*

Let us consider the problem of determining the static magnetic field $\mathbf{H} = \mathbf{H}(\mathbf{r})$ produced by a given electric current density $\mathbf{j}(\mathbf{r})$ in a bounded three-dimensional region Ω embedded within a perfect conductor. The magnetic field is solution to the following boundary value problem

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{H} = \mathbf{j}(\mathbf{r})\,, \\ \boldsymbol{\nabla} \cdot (\mu \mathbf{H}) = 0\,, \\ \mathbf{n} \cdot \mathbf{H}_{|\Gamma} = 0\,, \end{cases}$$

where μ is the magnetic permeability, $\mathbf{n}$ denotes the outward unit normal to $\Gamma = \partial\Omega$, and the source $\mathbf{j}$ is assumed to satisfy the compatibility condition $\boldsymbol{\nabla} \cdot \mathbf{j} = 0$. Henceforth we assume that μ is a positive function bounded from below a.e. by μ_0 and bounded from above a.e. by μ_1.

Expressing the solenoidal field $\mu\mathbf{H}$ in terms of a vector potential $\mathbf{A}$ so that $\mu\mathbf{H} = \boldsymbol{\nabla} \times \mathbf{A}$, the original magnetostatic problem can be rewritten as follows:

$$\begin{cases} \boldsymbol{\nabla} \times \left(\dfrac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{A} \right) = \mathbf{j}(\mathbf{r})\,, \\ \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{A}_{|\Gamma} = 0\,. \end{cases}$$

The solution $\mathbf{A}$ to this problem is non-unique since the solution set is invariant under the so-called gauge transformation

$$\mathbf{A} \longrightarrow \mathbf{A}' = \mathbf{A} + \boldsymbol{\nabla}\,\Psi\,,$$

where $\Psi(\mathbf{r})$ is an arbitrary (sufficiently smooth) function. To eliminate this indeterminacy a supplementary condition must be imposed. For the static magnetic problem considered here, it is classical to assume the (scalar) condition $\boldsymbol{\nabla} \cdot \mathbf{A} = 0$, usually called gauge condition. This condition constitutes an additional scalar equation to be satisfied by the unknown vector field and the above problem for $\mathbf{A}$ can be rewritten as follows:

$$\begin{cases} \boldsymbol{\nabla} \times \left(\dfrac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{A} \right) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{A}) = \mathbf{j}(\mathbf{r})\,, \\ \boldsymbol{\nabla} \cdot \mathbf{A} = 0\,, \\ \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{A}_{|\Gamma} = 0\,. \end{cases}$$

In this new form, the problem has four (scalar) equations and only three scalar unknowns, namely, the three components of $\mathbf{A}$.

As a matter of fact, the supplementary equation is almost satisfied as a consequence of the specific form of the Poisson-like equation and of the presence of the compatibility condition on $\mathbf{j}$. By taking the divergence of the first equation, we obtain $-\nabla^2 \boldsymbol{\nabla} \cdot \mathbf{A} = \boldsymbol{\nabla} \cdot \mathbf{j} = 0$. It follows that the function $\boldsymbol{\nabla} \cdot \mathbf{A}_{|\Gamma}$ is harmonic in

Ω and therefore vanishes in Ω provided that the boundary condition $\boldsymbol{\nabla}\cdot\mathbf{A}_{|\Gamma}=0$ is satisfied. Thus the problem above is formally equivalent to the following system:

$$\begin{cases}\boldsymbol{\nabla}\times\left(\dfrac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{A}\right)-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})=\mathbf{j}(\mathbf{r})\,,\\ \boldsymbol{\nabla}\cdot\mathbf{A}_{|\Gamma}=0\,,\\ \mathbf{n}\cdot\boldsymbol{\nabla}\times\mathbf{A}_{|\Gamma}=0\,.\end{cases}$$

This problem has as many (three) equations as unknowns but only two boundary conditions.

Still, this problem does not determine $\mathbf{A}$ uniquely since the solution set is invariant under the (restricted) gauge transformation

$$\mathbf{A}\longrightarrow\mathbf{A}'=\mathbf{A}+\boldsymbol{\nabla}\eta\,,$$

for any harmonic function $\eta(\mathbf{r})$. To remove the residual arbitrariness in $\mathbf{A}$, it is necessary to impose a third scalar boundary condition, i.e. $\mathbf{n}\cdot\mathbf{A}_{|\Gamma}=0$. In conclusion, we are led to consider the following problem: Find the magnetic potential $\mathbf{A}$ such that

$$\begin{cases}\boldsymbol{\nabla}\times\left(\dfrac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{A}\right)-\boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\mathbf{A})=\mathbf{j}(\mathbf{r})\,,\\ \boldsymbol{\nabla}\cdot\mathbf{A}_{|\Gamma}=0\,,\\ \mathbf{n}\cdot\mathbf{A}_{|\Gamma}=0\,,\\ \mathbf{n}\cdot\boldsymbol{\nabla}\times\mathbf{A}_{|\Gamma}=0\,.\end{cases}$$

Note that formally there are as many equations and boundary conditions as unknowns. The three boundary conditions supplementing the Poisson-like equation for $\mathbf{A}$ are nonclassical in two respects. First, the two conditions involving the divergence and the normal component of the unknown appear to be mutually exclusive if we try to express this problem in a variational form, i.e. if we try to express it as the first-order optimality condition characterizing the minimum of a quadratic functional. Second, the last scalar condition involving the normal component of $\boldsymbol{\nabla}\times\mathbf{A}$ is such that there is no corresponding term at all in the surface integrals involved in the variational formulation of the problem. Thus, this very simple magnetostatic problem, once formulated in terms of vector potential, leads to a Poisson-like equation supplemented with a quite uncomfortable set of boundary conditions. The goal of the present work is to analyze this nonstandard boundary value problem.

1.2. *Scope of the paper*

As an alternative to the approach described above, we could have enforced the homogeneous scalar boundary condition $\mathbf{n}\cdot\boldsymbol{\nabla}\times\mathbf{A}_{|\Gamma}=0$ by imposing the two tangential components of the vector potential to vanish on Γ, i.e. $\mathbf{n}\times\mathbf{A}_{|\Gamma}=0$. When combined with the condition on the divergence, $\boldsymbol{\nabla}\cdot\mathbf{A}_{|\Gamma}=0$, we obtain a problem[a] that fits well in a standard variational formulation. Since this class of problem is quite well understood, we shall not consider this alternative technique any further, and in the rest of the paper we shall enforce $\mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{A}_{|\Gamma} = 0$ directly.

Our aim in this paper is to characterize the solution of the vector Poisson-like equation when the normal component of the unknown, its divergence, and the normal component of its curl are enforced on the boundary.

In a recent paper,[6] we analyzed a vector Poisson equation in two dimensions supplemented with two apparently mutually exclusive scalar boundary conditions: the normal component of the unknown and its divergence. In the present work, we extend our study to the three-dimensional problem nontrivially, for in addition to the two aforementioned scalar boundary conditions, we now enforce the normal component of the curl of the unknown on the entire boundary.

The content of the paper is organized as follows. In Sec. 2 we introduce a weak formulation of our problem with the homogeneous version of the three boundary conditions. The well-posedness of the problem is studied. The kernel of the linear operator associated with the problem is shown to be trivial (contrary to what was found for the 2D problem[6]), while the kernel of the adjoint operator turns out to be nontrivial. As a consequence, the weak problem we started from is modified and reformulated in a well-posed manner. Then, general nonhomogeneous boundary conditions are studied, and the problem is shown to be well-posed if the data satisfy two *compatibility* conditions. In Sec. 3 we introduce a splitting method that leads to an uncoupled numerical algorithm requiring to solve only scalar Dirichlet or Neumann problems for the Poisson or Poisson-like operator and two Quartapelle–Muzzio problems. The finite element approximation and the corresponding error analysis of the split solution are discussed in Sec. 4.

2. Analysis of the Weak Form of the Problem

2.1. *Preliminaries and problem definition*

Throughout this paper, we assume that Ω is a bounded, open, and simply-connected domain of $\mathbb{R}^3$, with a Lipschitz continuous boundary Γ. We suppose that Γ is connected (i.e. Ω is contractible).

The problem we consider in this section consists formally of looking for a vector field $\mathbf{u}$ such that

$$\begin{cases} \boldsymbol{\nabla} \times \left(\dfrac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{u} \right) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) = \mathbf{f} & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \mathbf{u} = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \mathbf{u} = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{u} = 0 & \text{on } \Gamma\,. \end{cases} \tag{2.1}$$

[a] This problem corresponds to a 3D Quartapelle–Muzzio problem with fully homogeneous boundary conditions, see later.

To build a weak formulation to this problem, we introduce the following Hilbert spaces.

$$\mathbf{X}(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,|\, \boldsymbol{\nabla}\cdot\mathbf{v} \in L^2(\Omega), \boldsymbol{\nabla}\times\mathbf{v} \in \mathbf{L}^2(\Omega)\}\,, \tag{2.2}$$

$$\mathbf{X}_N(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega) \,|\, \mathbf{n}\times\mathbf{v}_{|\Gamma} = 0\}\,, \tag{2.3}$$

$$\mathbf{X}_T(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega) \,|\, \mathbf{n}\cdot\mathbf{v}_{|\Gamma} = 0\}\,, \tag{2.4}$$

$$\mathbf{X}_{T,T}(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega) \,|\, \mathbf{n}\cdot\mathbf{v}_{|\Gamma} = 0 \quad \text{and} \quad \mathbf{n}\cdot\boldsymbol{\nabla}\times\mathbf{v}_{|\Gamma} = 0\}\,. \tag{2.5}$$

We introduce the bilinear form

$$a(\mathbf{u},\mathbf{v}) = \left(\frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{u}, \boldsymbol{\nabla}\times\mathbf{v}\right) + (\boldsymbol{\nabla}\cdot\mathbf{u}, \boldsymbol{\nabla}\cdot\mathbf{v})\,, \quad \forall\, \mathbf{u},\mathbf{v} \in \mathbf{X}(\Omega)\,. \tag{2.6}$$

It is clear that $a \in \mathcal{L}(\mathbf{X}(\Omega)\times\mathbf{X}(\Omega);\mathbb{R})$ and a is symmetric positive semidefinite. Furthermore, it can be shown[7,12] that the restriction of a to the Hilbert spaces $\mathbf{X}_N(\Omega)$, $\mathbf{X}_T(\Omega)$ and $\mathbf{X}_{T,T}(\Omega)$ induces a scalar product and that the associated norm is equivalent to the natural norm of $\mathbf{X}(\Omega)$. Hereafter we equip $\mathbf{X}_N(\Omega)$, $\mathbf{X}_T(\Omega)$ and $\mathbf{X}_{T,T}(\Omega)$ with the following scalar product and norm:

$$(\mathbf{u},\mathbf{v})_1 = (\boldsymbol{\nabla}\times\mathbf{u}, \boldsymbol{\nabla}\times\mathbf{v}) + (\boldsymbol{\nabla}\cdot\mathbf{u}, \boldsymbol{\nabla}\cdot\mathbf{v})\,, \quad \text{and} \quad |\cdot|_1 = (\cdot,\cdot)_1^{1/2}\,. \tag{2.7}$$

Given $\mathbf{f} \in \mathbf{L}^2(\Omega)$, problem (2.1) can be reformulated into the following weak form:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{X}_{T,T}(\Omega) \quad \text{such that} \\ a(\mathbf{u},\mathbf{v}) = (\mathbf{f},\mathbf{v}), \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega)\,. \end{cases} \tag{2.8}$$

Determining whether problem (2.8) is well-posed is equivalent to asking whether the operator $A : \mathbf{X}_{T,T}(\Omega) \longrightarrow \mathbf{X}'_N(\Omega)$ defined by

$$\langle A\mathbf{u},\mathbf{v}\rangle = a(\mathbf{u},\mathbf{v})\,, \quad \forall\, \mathbf{u} \in \mathbf{X}_{T,T}(\Omega), \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega)\,, \tag{2.9}$$

is bijective. The answer to this question is rooted in the study of the kernel of A and A^t together with the possibility of deriving *a priori* bounds on the solution.

2.2. *Analysis of the kernel of A*

Let us introduce the following spaces

$$L_0^2(\Omega) = \left\{ q \in L^2(\Omega) \,\middle|\, \int_\Omega q = 0 \right\}\,,$$

$$\tilde{H}^1(\Omega) = H^1(\Omega) \cap L_0^2(\Omega)\,.$$

Since divergence-free vector fields are bound to play an important role in our analysis, we also introduce:

$$\mathbf{J}_T^0(\Omega) = \{\mathbf{v} \in \mathbf{L}^2(\Omega) \,\big|\, \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \quad \text{in } \Omega \;\text{ and }\; \mathbf{n}\cdot\mathbf{v}_{|\Gamma} = 0\}\,,$$

$$\mathbf{J}_N^1(\Omega) = \{\mathbf{v} \in \mathbf{X}_N(\Omega) \,\big|\, \boldsymbol{\nabla}\cdot\mathbf{v} = 0 \quad \text{in } \Omega\}\,.$$

Lemma 2.1. *The operator* $\boldsymbol{\nabla}\times : \mathbf{J}^1_N(\Omega) \longrightarrow \mathbf{J}^0_T(\Omega)$ *is an isomorphism.*

Proof. See the second part of Theorem I.3.6 in Girault–Raviart.[7] □

Corollary 2.1. *We have the orthogonal decomposition*

$$\mathbf{L}^2(\Omega) = \mu^{-1/2}\boldsymbol{\nabla}\times\mathbf{J}^1_N(\Omega) \oplus \mu^{1/2}\boldsymbol{\nabla}\tilde{H}^1(\Omega)\,.$$

Proof. Let $\mathbf{v}$ be a vector field belonging to $\mathbf{L}^2(\Omega)$. Let q in $\tilde{H}^1(\Omega)$ be such that $(\mu^{1/2}\boldsymbol{\nabla}q, \mu^{1/2}\boldsymbol{\nabla}\phi) = (\mathbf{v}, \mu^{1/2}\boldsymbol{\nabla}\phi)$ for all ϕ in $\tilde{H}^1(\Omega)$. Then, set $\mathbf{w} = \mu^{1/2}(\mathbf{v} - \mu^{1/2}\boldsymbol{\nabla}q)$. It is clear that $\mathbf{w}$ belongs to $\mathbf{L}^2(\Omega)$ and, for all ϕ in $\tilde{H}^1(\Omega)$, $(\mathbf{w}, \boldsymbol{\nabla}\phi) = 0$, which proves that $\mathbf{w}$ is in $\mathbf{J}^0_T(\Omega)$. Then, owing to Lemma 2.1, we infer that there is $\boldsymbol{\psi} \in \mathbf{J}^1_N(\Omega)$ such that $\mathbf{w} = \boldsymbol{\nabla}\times\boldsymbol{\psi}$. As a result, we have $\mathbf{v} = \mu^{-1/2}\boldsymbol{\nabla}\times\boldsymbol{\psi} + \mu^{1/2}\boldsymbol{\nabla}q$. Furthermore, it is clear that the decomposition is orthogonal. □

Remark 2.1. Corollary 2.1 is a simple extension of the standard Hodge decomposition $\mathbf{L}^2(\Omega) = \mathbf{J}^0_T(\Omega) \oplus \boldsymbol{\nabla}\tilde{H}^1(\Omega)$.

We are now ready to study the kernel of the operator A.

Lemma 2.2. $\operatorname{Ker} A$ *is trivial.*

Proof. Let $\mathbf{u} \in \mathbf{X}_{T,T}(\Omega)$ belong to $\operatorname{Ker} A$. By definition, $\mathbf{u}$ satisfies

$$a(\mathbf{u}, \mathbf{v}) = 0\,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega)\,. \tag{2.10}$$

For any $\mathbf{g} \in \mathbf{J}^0_T(\Omega)$, by Lemma 2.1 we can find a $\mathbf{v} \in \mathbf{J}^1_N(\Omega) \subset \mathbf{X}_N(\Omega)$ such that $\boldsymbol{\nabla}\times\mathbf{v} = \mathbf{g}$. Thus, from (2.10) we have

$$\left(\frac{1}{\mu}\boldsymbol{\nabla}\times\mathbf{u}, \mathbf{g}\right) = 0\,, \quad \forall\, \mathbf{g} \in \mathbf{J}^0_T(\Omega)\,.$$

Since $\boldsymbol{\nabla}\times\mathbf{u} \in \mathbf{J}^0_T(\Omega)$, we deduce $\boldsymbol{\nabla}\times\mathbf{u} = \mathbf{0}$.

On the other hand, for any $\zeta \in L^2(\Omega)$, let $q \in H^1_0(\Omega)$ satisfying $\nabla^2 q = \zeta$ in Ω. Then, by choosing $\mathbf{v} = \boldsymbol{\nabla}q \in \mathbf{X}_N(\Omega)$ in (2.10), we have $(\boldsymbol{\nabla}\cdot\mathbf{u}, \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}q)) = 0$, namely,

$$(\boldsymbol{\nabla}\cdot\mathbf{u}, \zeta) = 0\,, \quad \forall\, \zeta \in L^2(\Omega)\,,$$

which implies that $\boldsymbol{\nabla}\cdot\mathbf{u} = 0$.

Summarizing the analysis above, $|\mathbf{u}|_1 = 0$; hence, $\mathbf{u} = 0$. This completes the proof. □

2.3. *Analysis of the adjoint operator A^t*

Now we turn our attention to the study of the adjoint of A, $A^t : \mathbf{X}_N(\Omega) \longrightarrow (\mathbf{X}_{T,T}(\Omega))'$, which is defined by

$$\langle A^t\mathbf{v}, \mathbf{u}\rangle = a(\mathbf{u}, \mathbf{v})\,, \quad \forall\, \mathbf{u} \in \mathbf{X}_{T,T}(\Omega)\,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega)\,. \tag{2.11}$$

Let us define

$$\mathbf{J}^1_{T,T}(\Omega) = \{\mathbf{v} \in \mathbf{X}_{T,T}(\Omega) \,|\, \boldsymbol{\nabla} \cdot \mathbf{v} = 0 \quad \text{in } \Omega\} \,.$$

Lemma 2.3. *The operator* $\boldsymbol{\nabla}\times :\ \mathbf{J}^1_{T,T}(\Omega) \longrightarrow \mathbf{J}^0_T(\Omega)$ *is an isomorphism.*

Proof. See Theorem I.3.5 in Girault–Raviart.[7] □

Lemma 2.4. *The kernel of* A^t *is such that*

$$\mathrm{Ker}\, A^t = \mathrm{span}(\mathbf{v}_0)\,, \tag{2.12}$$

where $\mathbf{v}_0 \in \mathbf{X}(\Omega)$ *is a unique solution of*

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{v}_0 = \mathbf{0} & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \mathbf{v}_0 = 1 & \text{in } \Omega\,, \\ \mathbf{n} \times \mathbf{v}_0 = \mathbf{0} & \text{on } \Gamma\,. \end{cases} \tag{2.13}$$

Proof. Let $\mathbf{v} \in \mathbf{X}_N(\Omega)$ satisfying

$$a(\mathbf{u}, \mathbf{v}) = 0\,, \quad \forall\, \mathbf{u} \in \mathbf{X}_{T,T}(\Omega)\,. \tag{2.14}$$

For any $\zeta \in L^2_0(\Omega)$, let $q \in \tilde{H}^1(\Omega)$ be the solution of

$$\begin{cases} \nabla^2 q = \zeta & \text{in } \Omega\,, \\ \dfrac{\partial q}{\partial n} = 0 & \text{on } \Gamma\,. \end{cases}$$

Then, by taking $\mathbf{u} = \boldsymbol{\nabla} q \in \mathbf{X}_{T,T}(\Omega)$ in (2.14), we have $(\boldsymbol{\nabla} \cdot \mathbf{v}, \zeta) = 0$, for all $\zeta \in L^2_0(\Omega)$, which implies that

$$\boldsymbol{\nabla} \cdot \mathbf{v} = C_v := \frac{1}{|\Omega|} \int_\Gamma \mathbf{n} \cdot \mathbf{v}\,. \tag{2.15}$$

On the other hand, owing to Lemma 2.3, for any $\mathbf{g} \in \mathbf{J}^0_T(\Omega)$, there exists a unique $\mathbf{w} \in \mathbf{J}^1_{T,T}(\Omega) \subset \mathbf{X}_{T,T}(\Omega)$ such that $\boldsymbol{\nabla} \times \mathbf{w} = \mathbf{g}$. Then, (2.14) gives

$$\left(\frac{1}{\mu}\mathbf{g}, \boldsymbol{\nabla} \times \mathbf{v}\right) = 0\,, \quad \forall\, \mathbf{g} \in \mathbf{J}^0_T(\Omega)\,.$$

Since $\boldsymbol{\nabla} \times \mathbf{v} \in \mathbf{J}^0_T(\Omega)$, we deduce $\boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}$.

Consequently, we obtain that, if $\mathbf{v} \in \mathbf{X}_N(\Omega)$ and $A^t\mathbf{v} = 0$, then $\mathbf{v}$ satisfies the system

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{v} = \mathbf{0}\,, \\ \boldsymbol{\nabla} \cdot \mathbf{v} = C_v\,. \end{cases} \tag{2.16}$$

The converse is also true. □

2.4. A well-posed problem

The analysis above has shown that problem (2.8) is not well-posed, since the corresponding linear operator is not bijective (the operator is injective, but its adjoint is not). Hence we shall now modify the problem accordingly. In order to exclude $\mathrm{span}(\mathbf{v}_0)$ from $\mathbf{X}_N(\Omega)$, we set

$$\mathbf{X}_N^\star(\Omega) = \left\{ \mathbf{v} \in \mathbf{X}_N(\Omega) \,\middle|\, \int_\Gamma \mathbf{n} \cdot \mathbf{v} = 0 \right\}. \tag{2.17}$$

Instead of problem (2.8), we shall hereafter consider the following:

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{X}_{T,T}(\Omega) \quad \text{such that} \\ a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) \,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N^\star(\Omega) \,. \end{cases} \tag{2.18}$$

Theorem 2.1. *Problem* (2.18) *is well-posed.*

Proof. According to the global theory on linear Banach operators (cf. e.g. Nečas,[9] Babuška[2] and Brezzi[3]), problem (2.18) is well-posed iff the following two conditions are satisfied:

(i) there exists a constant $\alpha > 0$ such that

$$\inf_{\mathbf{u} \in \mathbf{X}_{T,T}(\Omega)\backslash\{\mathbf{0}\}} \ \sup_{\mathbf{v} \in \mathbf{X}_N^\star(\Omega)\backslash\{\mathbf{0}\}} \frac{a(\mathbf{u}, \mathbf{v})}{|\mathbf{u}|_1 \, |\mathbf{v}|_1} \geq \alpha \,, \tag{2.19}$$

(ii) for any $\mathbf{v}$ in $\mathbf{X}_N^\star(\Omega)$

$$(\forall\, \mathbf{u} \in \mathbf{X}_{T,T}(\Omega) \,, \quad a(\mathbf{u}, \mathbf{v}) = 0) \Longrightarrow (\mathbf{v} = \mathbf{0}) \,. \tag{2.20}$$

Given $\mathbf{u} \in \mathbf{X}_{T,T}(\Omega)$, let us consider the following problem

$$\begin{cases} \text{Find } \mathbf{w} \in \mathbf{X}_N^\star(\Omega) \quad \text{such that} \\ a(\mathbf{w}, \mathbf{v}') = a(\mathbf{u}, \mathbf{v}') \,, \quad \forall\, \mathbf{v}' \in \mathbf{X}_N^\star(\Omega) \,. \end{cases} \tag{2.21}$$

By the Riesz–Fréchet Theorem, problem (2.21) has a unique solution $\mathbf{w} \in \mathbf{X}_N^\star(\Omega)$, and $\min(1, \mu_1^{-1})|\mathbf{w}|_1 \leq \max(1, \mu_0^{-1})|\mathbf{u}|_1$.

We are now going to show that $\mathbf{w}$ satisfies

$$\begin{cases} \boldsymbol{\nabla} \times \mathbf{w} = \boldsymbol{\nabla} \times \mathbf{u} & \text{in } \Omega \,, \\ \boldsymbol{\nabla} \cdot \mathbf{w} = \boldsymbol{\nabla} \cdot \mathbf{u} & \text{in } \Omega \,. \end{cases} \tag{2.22}$$

For any $\zeta \in L_0^2(\Omega)$, let $\phi \in H_0^1(\Omega)$ satisfying $\nabla^2 \phi = \zeta$ in Ω, then $\mathbf{v}' = \boldsymbol{\nabla}\phi \in \mathbf{X}_N^\star(\Omega)$. From (2.21), we have

$$(\boldsymbol{\nabla} \cdot \mathbf{w}, \zeta) = (\boldsymbol{\nabla} \cdot \mathbf{u}, \zeta) \,, \quad \forall\, \zeta \in L_0^2(\Omega) \,. \tag{2.23}$$

Since $\mathbf{w} \in \mathbf{X}_N^\star(\Omega)$ and $\mathbf{u} \in \mathbf{X}_{T,T}(\Omega)$, (2.23) implies that the second relation of (2.22) holds.

On one hand, it is easy to see that

$$(\mu^{-1/2} \boldsymbol{\nabla} \times \mathbf{w}, \ \mu^{1/2} \boldsymbol{\nabla} q) = 0 = (\mu^{-1/2} \boldsymbol{\nabla} \times \mathbf{u}, \ \mu^{1/2} \boldsymbol{\nabla} q) \,, \quad \forall\, q \in \tilde{H}^1(\Omega) \,.$$

On the other hand, by (2.21), and $\boldsymbol{\nabla}\cdot\mathbf{w}=\boldsymbol{\nabla}\cdot\mathbf{u}$, we have

$$(\mu^{-1/2}\boldsymbol{\nabla}\times\mathbf{w},\ \mu^{-1/2}\boldsymbol{\nabla}\times\boldsymbol{\phi})=(\mu^{-1/2}\boldsymbol{\nabla}\times\mathbf{u},\ \mu^{-1/2}\boldsymbol{\nabla}\times\boldsymbol{\phi}),\quad\forall\,\boldsymbol{\phi}\in\mathbf{X}_N^\star(\Omega).$$

Thus, owing to Corollary 2.1, we know that any $\boldsymbol{\psi}\in\mathbf{L}^2(\Omega)$ can be written in the following form:

$$\boldsymbol{\psi}=\mu^{1/2}\boldsymbol{\nabla}q+\mu^{-1/2}\boldsymbol{\nabla}\times\boldsymbol{\phi}$$

where $q\in\tilde{H}^1(\Omega)$ and $\boldsymbol{\phi}\in\mathbf{J}_N^1(\Omega)\subset\mathbf{X}_N^\star(\Omega)$. Hence,

$$(\mu^{-1/2}\boldsymbol{\nabla}\times\mathbf{w},\boldsymbol{\psi})=(\mu^{-1/2}\boldsymbol{\nabla}\times\mathbf{u},\boldsymbol{\psi}),\quad\forall\,\boldsymbol{\psi}\in\mathbf{L}^2(\Omega),$$

that is, the first relation of (2.22) holds.

Let us now check conditions (2.19) and (2.20). Noting that (2.22) implies $|\mathbf{u}|_1=|\mathbf{w}|_1$, we have

$$\sup_{\mathbf{v}\in\mathbf{X}_N^\star(\Omega)\backslash\{\mathbf{0}\}}\frac{a(\mathbf{u},\mathbf{v})}{|\mathbf{u}|_1\,|\mathbf{v}|_1}\geq\frac{a(\mathbf{u},\mathbf{w})}{|\mathbf{u}|_1\,|\mathbf{w}|_1}=\frac{a(\mathbf{w},\mathbf{w})}{|\mathbf{w}|_1^2}\geq\min(1,\mu_1^{-1}),\tag{2.24}$$

i.e. condition (2.19) is satisfied.

To establish (2.20), let us assume that $\mathbf{v}\in\mathbf{X}_N^\star(\Omega)$ satisfies:

$$a(\mathbf{u},\mathbf{v})=0,\quad\forall\,\mathbf{u}\in\mathbf{X}_{T,T}(\Omega).$$

Similarly to the arguments used to prove (2.16), we obtain

$$\begin{cases}\boldsymbol{\nabla}\times\mathbf{v}=0 & \text{in }\Omega,\\ \boldsymbol{\nabla}\cdot\mathbf{v}\ \,=0 & \text{in }\Omega.\end{cases}\tag{2.25}$$

Thus, $\mathbf{v}$ should be zero. Therefore, condition (2.20) is also satisfied. This completes the proof. □

Remark 2.2. Theorem 2.1 is equivalent to stating that the operator[b] $A^\star$: $\mathbf{X}_{T,T}(\Omega)\longrightarrow(\mathbf{X}_N^\star(\Omega))'$ defined by

$$\langle A^\star\mathbf{u},\mathbf{v}\rangle=a(\mathbf{u},\mathbf{v}),\quad\forall\,\mathbf{u}\in\mathbf{X}_{T,T}(\Omega),\quad\forall\,\mathbf{v}\in\mathbf{X}_N^\star(\Omega)\tag{2.26}$$

is bijective.

We can now interpret problem (2.18) in strong form.

Theorem 2.2. *If* $\int_\Omega\mathbf{f}\cdot\mathbf{v}_0=0$ *where* $\mathbf{v}_0$ *is defined by* (2.13), *then the solution* $\mathbf{u}$ *of problem* (2.18) *satisfies* (2.1) *in the distribution sense.*

Remark 2.3. The assumption of Theorem 2.2 is a *compatibility* condition for problem (2.1). It can also be expressed in the following alternative form:

$$\int_\Omega\mathbf{f}\cdot\boldsymbol{\nabla}\kappa_0=0,\tag{2.27}$$

[b] Not to be confounded with the adjoint of A, which we denote A^t.

where κ_0 is such that

$$\begin{cases} \nabla^2 \kappa_0 = 1 & \text{in } \Omega\,, \\ \kappa_0 = 0 & \text{on } \Gamma\,. \end{cases} \tag{2.28}$$

2.5. *Nonhomogeneous boundary conditions*

We now consider the case of nonhomogeneous boundary conditions, that is the following boundary value problem:

$$\begin{cases} \boldsymbol{\nabla} \times \left(\dfrac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{u} \right) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}) = \mathbf{f} & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \mathbf{u} = d & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \mathbf{u} = b & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{u} = c & \text{on } \Gamma\,, \end{cases} \tag{2.29}$$

where $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $d \in H^{1/2}(\Gamma)$, b and $c \in H^{-1/2}(\Gamma)$.

To solve problem (2.29), one has to show how its boundary conditions can be lifted. For this purpose, let $q \in \tilde{H}^1(\Omega)$ be the solution to the Neumann problem

$$\begin{cases} \nabla^2 q = \dfrac{|\Gamma|}{|\Omega|}\, \bar{b} & \text{in } L^2(\Omega)\,, \\ \dfrac{\partial q}{\partial n} = b & \text{in } H^{-1/2}(\Gamma)\,, \end{cases} \tag{2.30}$$

where

$$\bar{b} = \frac{1}{|\Gamma|} \langle b, 1 \rangle = \frac{1}{|\Gamma|} \int_\Gamma b \tag{2.31}$$

and $b \in H^{-1/2}(\Gamma)$ guarantees that (2.31) is well defined. Then $\mathbf{u}_b = \boldsymbol{\nabla} q$ is such that $\boldsymbol{\nabla} \cdot \mathbf{u}_b \in L^2(\Omega)$, $\boldsymbol{\nabla} \times \mathbf{u}_b = \mathbf{0} \in \mathbf{L}^2(\Omega)$, and

$$\begin{cases} \boldsymbol{\nabla} \times \left(\dfrac{1}{\mu} \boldsymbol{\nabla} \times \mathbf{u}_b \right) - \boldsymbol{\nabla}(\boldsymbol{\nabla} \cdot \mathbf{u}_b) = \mathbf{0} & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \mathbf{u}_b = \dfrac{|\Gamma|}{|\Omega|}\, \bar{b} & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \mathbf{u}_b = b & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{u}_b = 0 & \text{on } \Gamma\,. \end{cases} \tag{2.32}$$

Furthermore, if

$$\int_\Gamma c = 0\,, \tag{C1}$$

then there exists a unique $p \in \tilde{H}^1(\Omega)$ such that

$$\begin{cases} \nabla \cdot (\mu \nabla p) = 0 & \text{in } L^2(\Omega)\,, \\ \mu \dfrac{\partial p}{\partial n} = c & \text{in } H^{-1/2}(\Gamma)\,. \end{cases} \tag{2.33}$$

By Theorem I.3.5 in Girault–Raviart,[7] we can find a unique $\mathbf{u}_c$ such that

$$\begin{cases} \nabla \times \mathbf{u}_c = \mu \nabla p & \text{in } \Omega\,, \\ \nabla \cdot \mathbf{u}_c = 0 & \text{in } \Omega\,, \\ \mathbf{n} \cdot \mathbf{u}_c = 0 & \text{on } \Gamma\,, \end{cases} \tag{2.34}$$

which implies that $\nabla \cdot \mathbf{u}_c \in L^2(\Omega)$, $\nabla \times \mathbf{u}_c \in \mathbf{L}^2(\Omega)$, and

$$\begin{cases} \nabla \times \left(\dfrac{1}{\mu} \nabla \times \mathbf{u}_c \right) - \nabla(\nabla \cdot \mathbf{u}_c) = \mathbf{0} & \text{in } \Omega\,, \\ \nabla \cdot \mathbf{u}_c = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \mathbf{u}_c = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \nabla \times \mathbf{u}_b = c & \text{on } \Gamma\,. \end{cases} \tag{2.35}$$

Therefore, setting $\boldsymbol{\phi} = \mathbf{u} - \mathbf{u}_b - \mathbf{u}_c$, we have

$$\begin{cases} \nabla \times \left(\dfrac{1}{\mu} \nabla \times \boldsymbol{\phi} \right) - \nabla(\nabla \cdot \boldsymbol{\phi}) = \mathbf{f} & \text{in } \Omega\,, \\ \nabla \cdot \boldsymbol{\phi} = d - \dfrac{|\Gamma|}{|\Omega|}\, \bar{b} & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \boldsymbol{\phi} = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \cdot \nabla \times \boldsymbol{\phi} = 0 & \text{on } \Gamma\,. \end{cases} \tag{2.36}$$

The weak form of problem (2.36) corresponding to the setting developed in this paper can be written as:

$$\begin{cases} \text{Find } \boldsymbol{\phi} \in \mathbf{X}_{T,T}(\Omega) \quad \text{such that} \\ a(\boldsymbol{\phi}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \displaystyle\int_\Gamma \left(d - \frac{|\Gamma|}{|\Omega|} \bar{b} \right) (\mathbf{n} \cdot \mathbf{v})\,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N^\star(\Omega)\,. \end{cases} \tag{2.37}$$

Since

$$\begin{aligned} \left| \int_\Gamma \left(d - \frac{|\Gamma|}{|\Omega|}\, \bar{b} \right) (\mathbf{n} \cdot \mathbf{v}) \right| &\le \left| d - \frac{|\Gamma|}{|\Omega|}\, \bar{b} \right|_{1/2,\Gamma} |\mathbf{n} \cdot \mathbf{v}|_{-1/2,\Gamma} \\ &\le \left| d - \frac{|\Gamma|}{|\Omega|}\, \bar{b} \right|_{1/2,\Gamma} |\mathbf{v}|_{\mathbf{H}(\mathrm{div},\Omega)} \\ &\le \left| d - \frac{|\Gamma|}{|\Omega|}\, \bar{b} \right|_{1/2,\Gamma} |\mathbf{v}|_1\,, \end{aligned}$$

the linear form $\mathbf{v} :\longrightarrow (\mathbf{f}, \mathbf{v}) + \int_\Gamma (d - \frac{|\Gamma|}{|\Omega|}\,\bar{b})(\mathbf{n} \cdot \mathbf{v})$ is continuous on $\mathbf{X}_N^\star(\Omega)$. Then, problem (2.37) is well-posed.

Let us assume now that

$$(\mathbf{f}, \mathbf{v}_0) + \int_\Gamma \left(d - \frac{|\Gamma|}{|\Omega|}\,\bar{b}\right)(\mathbf{n} \cdot \mathbf{v}_0) = 0\,,$$

with $\mathbf{v}_0 = \boldsymbol{\nabla}\kappa_0$, which is equivalent to

$$(\mathbf{f}, \boldsymbol{\nabla}\kappa_0) - |\Gamma|\,\bar{b} + \int_\Gamma d\,\frac{\partial \kappa_0}{\partial n} = 0\,. \tag{C2}$$

Then, the solution $\boldsymbol{\phi}$ of problem (2.37) also satisfies

$$a(\boldsymbol{\phi}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \int_\Gamma \left(d - \frac{|\Gamma|}{|\Omega|}\,\bar{b}\right)(\mathbf{n} \cdot \mathbf{v})\,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega)\,. \tag{2.38}$$

If we introduce the space:

$$\mathbf{X}_{b,c}(\Omega) = \{\mathbf{v} \in \mathbf{X}(\Omega) \mid \mathbf{n} \cdot \mathbf{v}_{|\Gamma} = b \quad \text{and} \quad \mathbf{n} \cdot \boldsymbol{\nabla} \times \mathbf{v}_{|\Gamma} = c\}\,, \tag{2.39}$$

then the weak form of problem (2.29) can be written as

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{X}_{b,c}(\Omega) \quad \text{such that} \\ a(\mathbf{u}, \mathbf{v}) = (\mathbf{f}, \mathbf{v}) + \langle d, \mathbf{n} \cdot \mathbf{v} \rangle_{1/2,\Gamma}\,, \quad \forall\, \mathbf{v} \in \mathbf{X}_N(\Omega). \end{cases} \tag{2.40}$$

Therefore, we have

Theorem 2.3. *Assume that* $\mathbf{f} \in \mathbf{L}^2(\Omega)$, $d \in H^{1/2}(\Gamma)$, b *and* $c \in H^{-1/2}(\Gamma)$. *And that* compatibility *conditions* (C1) *and* (C2) *hold. Then, problem* (2.40) *has a unique solution.*

Remark 2.4. When Ω is not simply-connected or Γ is not connected, the results obtained in this section have to be modified to account for finite dimensional vector spaces of vector fields with zero divergence, zero curl, and either zero normal or tangential trace at the boundary. More details on this aspect of the question can be found, e.g., in Refs. 1, 5 and 12.

3. A Split Solution Method and Its Variational Formulation

In this section, we split problem (2.29) into a sequence of uncoupled simple problems.

First, for $\mathbf{f} \in \mathbf{L}^2(\Omega)$ and $d \in H^{1/2}(\Gamma)$, we consider the following Dirichlet problem (to get $\phi = \boldsymbol{\nabla} \cdot \mathbf{u}$)

$$\begin{cases} -\nabla^2 \phi = \boldsymbol{\nabla} \cdot \mathbf{f} & \text{in } \Omega\,, \\ \phi = d & \text{on } \Gamma\,, \end{cases} \tag{3.1}$$

which has a unique solution $\phi \in H^1(\Omega)$.

By the compatibility condition (C2), we can check that

$$\int_\Omega \phi = \int_\Gamma b\,.$$

This implies that the following Neumann problem

$$\begin{cases} \nabla^2 q = \phi & \text{in } \Omega\,, \\ \dfrac{\partial q}{\partial n} = b & \text{on } \Gamma\,, \end{cases} \tag{3.2}$$

has a unique solution $q \in \tilde{H}^1(\Omega)$ (exactly as in the two-dimensional case[6]). Then, by (C1), there exists a unique function $p \in \tilde{H}^1(\Omega)$ satisfying

$$\begin{cases} \boldsymbol{\nabla} \cdot (\mu \boldsymbol{\nabla} p) = 0 & \text{in } \Omega\,, \\ \mu \dfrac{\partial p}{\partial n} = c & \text{on } \Gamma\,. \end{cases} \tag{3.3}$$

Finally, consider the following two vector elliptic problems

$$\begin{cases} -\nabla^2 \boldsymbol{\psi} = \mathbf{f} + \boldsymbol{\nabla}\phi & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \boldsymbol{\psi} = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \times \boldsymbol{\psi} = \mathbf{0} & \text{on } \Gamma\,, \end{cases} \tag{3.4}$$

and

$$\begin{cases} -\nabla^2 \mathbf{w} = \mu(\boldsymbol{\nabla} \times \boldsymbol{\psi} + \boldsymbol{\nabla} p) & \text{in } \Omega\,, \\ \boldsymbol{\nabla} \cdot \mathbf{w} = 0 & \text{on } \Gamma\,, \\ \mathbf{n} \times \mathbf{w} = \mathbf{0} & \text{on } \Gamma\,. \end{cases} \tag{3.5}$$

These vector problems are the 3D version of the boundary value problem introduced by Quartapelle and Muzzio[10] in the particular case of homogeneous boundary values. By the theory developed recently[13,14] or by (3.11) and (3.12) (see below), we know that they have unique solutions $\boldsymbol{\psi} \in \mathbf{X}_N(\Omega)$ and $\mathbf{w} \in \mathbf{X}_N(\Omega)$.

Finally, one can verify that the vector field

$$\boldsymbol{\nabla} \times \mathbf{w} + \boldsymbol{\nabla} q = \mathbf{u} \in \mathbf{X}(\Omega) \tag{3.6}$$

is the solution of problem (2.29).

Remark 3.1. The above splitting process provides another proof of Theorem 2.3. In fact, the existence is obtained by (3.6). For the uniqueness, we just use Lemma 2.2.

The solution of problem (2.29) can be determined by solving the following sequence of variational forms of (3.1) to (3.6):

For

$$H^1_d(\Omega) = \{\varphi \in H^1(\Omega) \,|\, \varphi = d \quad \text{on } \Gamma\}\,, \tag{3.7}$$

$$\begin{cases} \text{Find } \phi \in H^1_d(\Omega) \quad \text{such that} \\ (\boldsymbol{\nabla}\phi, \boldsymbol{\nabla}\varphi) = -(\mathbf{f}, \boldsymbol{\nabla}\varphi)\,, \quad \forall\, \varphi \in H^1_0(\Omega)\,, \end{cases} \tag{3.8}$$

$$\begin{cases} \text{Find } q \in \tilde{H}^1(\Omega) \quad \text{such that} \\ (\nabla q, \nabla \varphi) = -(\phi, \varphi) + \langle \varphi, b \rangle_{1/2,\Gamma}, \quad \forall\, \varphi \in \tilde{H}^1(\Omega)\,, \end{cases} \tag{3.9}$$

$$\begin{cases} \text{Find } p \in \tilde{H}^1(\Omega) \quad \text{such that} \\ (\mu \nabla p, \nabla \varphi) = \langle \varphi, c \rangle_{1/2,\Gamma}, \quad \forall\, \varphi \in \tilde{H}^1(\Omega)\,, \end{cases} \tag{3.10}$$

$$\begin{cases} \text{Find } \boldsymbol{\psi} \in \mathbf{X}_N(\Omega) \quad \text{such that} \\ (\boldsymbol{\psi}, \boldsymbol{\varphi})_1 = (\mathbf{f} + \nabla \phi, \boldsymbol{\varphi})\,, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_N(\Omega)\,, \end{cases} \tag{3.11}$$

$$\begin{cases} \text{Find } \mathbf{w} \in \mathbf{X}_N(\Omega) \quad \text{such that} \\ (\mathbf{w}, \boldsymbol{\varphi})_1 = (\mu(\nabla \times \boldsymbol{\psi} + \nabla p), \boldsymbol{\varphi})\,, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{X}_N(\Omega)\,, \end{cases} \tag{3.12}$$

$$\begin{cases} \text{Find } \mathbf{u} \in \mathbf{L}^2(\Omega) \quad \text{such that} \\ (\mathbf{u}, \boldsymbol{\varphi}) = (\nabla \times \mathbf{w} + \nabla q, \boldsymbol{\varphi})\,, \quad \forall\, \boldsymbol{\varphi} \in \mathbf{L}^2(\Omega)\,. \end{cases} \tag{3.13}$$

Note that, for the 3D Quartapelle–Muzzio problem with homogeneous conditions, the decomposition of the solution[13,14] is still valid with a minor modification when the domain Ω is Lipschitz. Thus, the solutions of problems (3.11) and (3.12) can be split into

$$\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \boldsymbol{\psi}_{\mathcal{H}} \quad \text{and} \quad \mathbf{w} = \mathbf{w}_0 + \mathbf{w}_{\mathcal{H}}\,, \tag{3.14}$$

respectively, where $\boldsymbol{\psi}_0,\ \mathbf{w}_0 \in \mathbf{H}_0^1(\Omega)$ and $\boldsymbol{\psi}_{\mathcal{H}},\ \mathbf{w}_{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_N(\Omega)$, having introduced the following space of harmonic vector fields:

$$\boldsymbol{\mathcal{H}}_N(\Omega) = \{\mathbf{v} \in \mathbf{X}_N(\Omega) \,\big|\, (\mathbf{v}, \mathbf{w})_1 = 0, \quad \forall\, \mathbf{w} \in \mathbf{H}_0^1(\Omega)\}\,. \tag{3.15}$$

Then, problems (3.11) and (3.12) can be written respectively as:

$$\begin{cases} \text{Find } (\boldsymbol{\psi}_0, \boldsymbol{\psi}_{\mathcal{H}}) \in \mathbf{H}_0^1(\Omega) \times \boldsymbol{\mathcal{H}}_N(\Omega) \text{ such that } \forall\, \boldsymbol{\varphi}_0 \in \mathbf{H}_0^1(\Omega)\,,\ \forall\, \boldsymbol{\varphi}_{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_N(\Omega)\,, \\ \text{(i) } (\boldsymbol{\psi}_0, \boldsymbol{\varphi}_0)_1 = (\mathbf{f} + \nabla \phi, \boldsymbol{\varphi}_0)\,, \\ \text{(ii) } (\boldsymbol{\psi}_{\mathcal{H}}, \boldsymbol{\varphi}_{\mathcal{H}})_1 = -(\boldsymbol{\psi}_0, \boldsymbol{\varphi}_{\mathcal{H}})_1 + (\mathbf{f} + \nabla \boldsymbol{\phi}, \boldsymbol{\varphi}_{\mathcal{H}})\,, \end{cases} \tag{3.16}$$

and, with $\boldsymbol{\psi} = \boldsymbol{\psi}_0 + \boldsymbol{\psi}_{\mathcal{H}}$,

$$\begin{cases} \text{Find } (\mathbf{w}_0, \mathbf{w}_{\mathcal{H}}) \in \mathbf{H}_0^1(\Omega) \times \boldsymbol{\mathcal{H}}_N(\Omega) \text{ such that } \forall\, \boldsymbol{\varphi}_0 \in \mathbf{H}_0^1(\Omega)\,,\ \forall\, \boldsymbol{\varphi}_{\mathcal{H}} \in \boldsymbol{\mathcal{H}}_N(\Omega)\,, \\ \text{(i) } (\mathbf{w}_0, \boldsymbol{\varphi}_0)_1 = (\mu(\nabla \times \boldsymbol{\psi} + \nabla p), \boldsymbol{\varphi}_0)\,, \\ \text{(ii) } (\mathbf{w}_{\mathcal{H}}, \boldsymbol{\varphi}_{\mathcal{H}})_1 = -(\mathbf{w}_0, \boldsymbol{\varphi}_{\mathcal{H}})_1 + (\mu(\nabla \times \boldsymbol{\psi} + \nabla p), \boldsymbol{\varphi}_{\mathcal{H}})\,. \end{cases} \tag{3.17}$$

Subproblems (3.16.i) and (3.17.i) are classical vector Dirichlet problems and are easily solved as three independent scalar Dirichlet problems. Subproblems (3.16.ii) and (3.17.ii) are vector problems which cannot be solved componentwise since, by definition, the three components of the vector fields in $\boldsymbol{\mathcal{H}}_N(\Omega)$ are coupled.

Summarizing the above results, we have:

Theorem 3.1. *Problem* (2.29) *can be solved by the sequence of variational problems* (3.8)–(3.10), (3.16), (3.17) *and* (3.13).

4. Finite Element Approximation and Its Convergence Analysis

4.1. *The discrete problem*

Throughout this section, for simplicity, Ω is assumed to be a convex polyhedral domain. Furthermore, we assume that μ is a smooth function. The case corresponding to discontinuous values of μ can also be treated provided the mesh matches the discontinuities of μ.

Let $\mathcal{T}^h$ be a partition of $\bar{\Omega}$ that fits the boundary, and denote by Σ^h the set of all the boundary faces. We assume that $\mathcal{T}^h$ is composed of tetrahedra, and the normal $\mathbf{n}$ is constant on each face $s \in \Sigma^h$. For any given integer $\ell \geq 1$, we introduce the following finite element spaces:

$$S^{h,\ell} = \{\varphi^h \in C^0(\bar{\Omega}) \,|\, \varphi^h_{|\kappa} \in \mathbb{P}_\ell, \quad \forall \kappa \in \mathcal{T}^h\}\,, \tag{4.1}$$

$$S_0^{h,\ell} = S^{h,\ell} \cap H_0^1(\Omega)\,, \tag{4.2}$$

$$S_d^{h,\ell} = \{\varphi^h \in S^{h,\ell} \,|\, \varphi^h_{|s} = \mathcal{I}_\ell d, \quad \forall s \in \Sigma^h\}\,, \tag{4.3}$$

$$\tilde{S}^{h,\ell} = \left\{\varphi^h \in S^{h,\ell} \,\middle|\, \int_\Omega \varphi^h = 0\right\}\,, \tag{4.4}$$

$$\mathbf{S}^{h,\ell} = [S^{h,\ell}]^3\,, \tag{4.5}$$

$$\mathbf{S}_0^{h,\ell} = [S_0^{h,\ell}]^3\,, \tag{4.6}$$

$$\mathbf{S}_N^{h,\ell} = \{\boldsymbol{\varphi}^h \in \mathbf{S}^{h,\ell} \,|\, \mathbf{n} \times \boldsymbol{\varphi}^h_{|s} = \mathbf{0}, \quad \forall s \in \Sigma^h\}\,, \tag{4.7}$$

$$\boldsymbol{\mathcal{H}}_N^{h,\ell} = \{\boldsymbol{\varphi}^h \in \mathbf{S}_N^{h,\ell} \,|\, (\boldsymbol{\varphi}^h, \mathbf{v}^h)_1 = 0, \quad \forall \mathbf{v}^h \in \mathbf{S}_0^{h,\ell}\}\,, \tag{4.8}$$

where $\mathbb{P}_\ell$ denotes the space of all polynomials defined in $\mathbb{R}^3$, of degree less than or equal to $\ell \geq 1$, $\mathcal{I}_\ell$ denotes the Clément $\mathbb{P}_\ell$-interpolation operator over Σ^h.

Let $k \geq 1$ be an integer and let j be another integer such that either $j = k$ or $j = k + 1$. Then, finite element approximations to the sequence of problems (3.8)–(3.10), (3.16), (3.17) and (3.13) can be proposed as follows:

$$\begin{cases} \text{Find } \phi^h \in S_d^{h,k} \quad \text{such that} \\ (\boldsymbol{\nabla}\phi^h, \boldsymbol{\nabla}\varphi^h) = -(\mathbf{f}, \boldsymbol{\nabla}\varphi^h)\,, \quad \forall\, \varphi^h \in S_0^{h,k}\,, \end{cases} \tag{4.9}$$

$$\begin{cases} \text{Find } q^h \in \tilde{S}^{h,j} \quad \text{such that} \\ (\boldsymbol{\nabla}q^h, \boldsymbol{\nabla}\varphi^h) = -(\phi^h, \varphi^h) + \langle \varphi^h, b\rangle_{1/2,\Gamma}\,, \quad \forall\, \varphi^h \in \tilde{S}^{h,j}\,, \end{cases} \tag{4.10}$$

$$\begin{cases} \text{Find } p^h \in \tilde{S}^{h,k} \quad \text{such that} \\ (\mu \boldsymbol{\nabla} p^h, \boldsymbol{\nabla} \varphi^h) = \langle \varphi^h, c \rangle_{1/2,\Gamma}, \quad \forall\, \varphi^h \in \tilde{S}^{h,k}, \end{cases} \tag{4.11}$$

$$\begin{cases} \text{Find } (\boldsymbol{\psi}_0^h, \boldsymbol{\psi}_{\mathcal{H}}^h) \in \mathbf{S}_0^{h,j} \times \boldsymbol{\mathcal{H}}_N^{h,j} \text{ s.t.}, \quad \forall\, \boldsymbol{\varphi}_0^h \in \mathbf{S}_0^{h,j}, \quad \forall\, \boldsymbol{\varphi}_{\mathcal{H}}^h \in \boldsymbol{\mathcal{H}}_N^{h,j} \\ \text{(i)} \quad (\boldsymbol{\psi}_0^h, \boldsymbol{\varphi}_0^h)_1 = (\mathbf{f} + \boldsymbol{\nabla}\phi^h, \boldsymbol{\varphi}_0^h), \\ \text{(ii)} \quad (\boldsymbol{\psi}_{\mathcal{H}}^h, \boldsymbol{\varphi}_{\mathcal{H}}^h)_1 = -(\boldsymbol{\psi}_0^h, \boldsymbol{\varphi}_{\mathcal{H}}^h)_1 + (\mathbf{f} + \boldsymbol{\nabla}\phi^h, \boldsymbol{\varphi}_{\mathcal{H}}^h), \end{cases} \tag{4.12}$$

$$\begin{cases} \text{Find } (\mathbf{w}_0^h, \mathbf{w}_{\mathcal{H}}^h) \in \mathbf{S}_0^{h,j} \times \boldsymbol{\mathcal{H}}_N^{h,j} \text{ s.t.}, \quad \forall\, \boldsymbol{\varphi}_0^h \in \mathbf{S}_0^{h,j}, \quad \forall\, \boldsymbol{\varphi}_{\mathcal{H}}^h \in \boldsymbol{\mathcal{H}}_N^{h,j} \\ \text{(i)} \quad (\mathbf{w}_0^h, \boldsymbol{\varphi}_0^h)_1 = (\mu(\boldsymbol{\nabla} \times \boldsymbol{\psi}^h + \boldsymbol{\nabla} p^h), \boldsymbol{\varphi}_0^h), \\ \text{(ii)} \quad (\mathbf{w}_{\mathcal{H}}^h, \boldsymbol{\varphi}_{\mathcal{H}}^h)_1 = -(\mathbf{w}_0^h, \boldsymbol{\varphi}_{\mathcal{H}})_1 + (\mu(\boldsymbol{\nabla} \times \boldsymbol{\psi}^h + \boldsymbol{\nabla} p^h), \boldsymbol{\varphi}_{\mathcal{H}}^h), \end{cases} \tag{4.13}$$

$$\begin{cases} \text{Find } \mathbf{u}^h \in \mathbf{S}^{h,k} \quad \text{such that} \\ (\mathbf{u}^h, \boldsymbol{\varphi}^h) = (\boldsymbol{\nabla} \times (\mathbf{w}_0^h + \mathbf{w}_{\mathcal{H}}^h) + \boldsymbol{\nabla} q^h, \boldsymbol{\varphi}^h), \quad \forall\, \boldsymbol{\varphi}^h \in \mathbf{S}^{h,k}. \end{cases} \tag{4.14}$$

In problem (4.13), $\boldsymbol{\psi}^h = \boldsymbol{\psi}_0^h + \boldsymbol{\psi}_{\mathcal{H}}^h$. Subproblems (4.9)–(4.11), (4.12.i), (4.13.i) and (4.14) can be easily solved in an uncoupled way. The coupled subproblems (4.12.ii) and (4.13.ii) can be solved either by the uncoupled direct method of Glowinski and Pironneau[13,14] or by one of its iterative variants, for instance the conjugate gradient method.[8,10]

Remark 4.1. In the particular case when the domain is a box, the Quartapelle–Muzzio problems (3.4) and (3.5) can be written naturally as a system of three independent scalar Poisson equations each one supplemented with mixed Dirichlet–Neumann conditions. In this case an uncoupled solution of problem (2.29) is obtained directly.

4.2. *Error estimates*

Let $\mathcal{T}^h$ belong to a regular family of partitions[4] and denote below by $\|\cdot\|_s$ and $|\cdot|_s$ the standard norm and semi-norm of the Sobolev space $H^s(\Omega)$, $s > 0$. We denote by C a generic constant independent of h.

Estimating $\phi - \phi^h$. Since ϕ^h is a conforming $\mathbb{P}_k$ finite element approximation of ϕ, this estimate is classical. If $\phi \in H^{k+1}(\Omega)$, we then have (cf. Ciarlet[4] or Strang and Fix[11]):

$$\|\phi - \phi^h\|_0 + h|\phi - \phi^h|_1 \leq Ch^{k+1}|\phi|_{k+1}. \tag{4.15}$$

Estimating $q - q^h$. Similarly to the analysis of the 2D problem,[6] if $q \in H^{j+1}(\Omega)$ and $\phi \in H^{k+1}(\Omega)$, we have, for $j = k$ or $k + 1$

$$\|\boldsymbol{\nabla}(q - q^h)\|_0 \leq Ch^j\{|q|_{j+1} + |\phi|_{k+1}\}. \tag{4.16}$$

Estimating $p - p^h$. This estimate is also classical. If $p \in H^{k+1}(\Omega)$, we then have (cf. Ciarlet[4] or Strang and Fix[11]):

$$\|p - p^h\|_0 + h|p - p^h|_1 \le Ch^{k+1}|p|_{k+1}\,. \tag{4.17}$$

Estimating $\boldsymbol{\psi}_0 - \boldsymbol{\psi}_0^h$ *and* $\boldsymbol{\psi}_{\mathcal{H}} - \boldsymbol{\psi}_{\mathcal{H}}^h$. Similar to the analysis,[6] if $\boldsymbol{\psi}_0, \boldsymbol{\psi}_{\mathcal{H}} \in \mathbf{H}^{j+1}(\Omega)$ and $\phi \in H^{k+1}(\Omega)$, we can get, for $j = k$ or $k+1$,

$$|\boldsymbol{\psi}_0 - \boldsymbol{\psi}_0^h|_1 \le Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\phi|_{k+1}\}\,, \tag{4.18}$$

$$|\boldsymbol{\psi}_{\mathcal{H}} - \boldsymbol{\psi}_{\mathcal{H}}^h|_1 \le Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\phi|_{k+1}\}\,. \tag{4.19}$$

Estimating $\mathbf{w}_0 - \mathbf{w}_0^h$ *and* $\mathbf{w}_{\mathcal{H}} - \mathbf{w}_{\mathcal{H}}^h$. Similarly, we can get

$$|\mathbf{w}_0 - \mathbf{w}_0^h|_1 \le Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\mathbf{w}_0|_{j+1} + |\phi|_{k+1} + |p|_{k+1}\}\,, \tag{4.20}$$

$$|\mathbf{w}_{\mathcal{H}} - \mathbf{w}_{\mathcal{H}}^h|_1 \le Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\mathbf{w}_0|_{j+1} + |\mathbf{w}_{\mathcal{H}}|_{j+1} + |\phi|_{k+1} + |p|_{k+1}\}\,, \tag{4.21}$$

if $\boldsymbol{\psi}_0, \boldsymbol{\psi}_{\mathcal{H}}, \mathbf{w}_0, \mathbf{w}_{\mathcal{H}} \in \mathbf{H}^{j+1}(\Omega)$, ϕ and $p \in H^{k+1}(\Omega)$, $j = k$ or $k+1$.

Estimating $\mathbf{u} - \mathbf{u}^h$. By (3.13) and (4.14), noting that (4.16), (4.20) and (4.21), we finally infer

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}^h\|_0 \le Ch^j\{&|q|_{j+1} + |\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\mathbf{w}_0|_{j+1} \\ &+ |\mathbf{w}_{\mathcal{H}}|_{j+1} + |\mathbf{u}|_{k+1} + |\phi|_{k+1} + |p|_{k+1}\}\,,\end{aligned} \tag{4.22}$$

if $q \in H^{j+1}(\Omega)$, $\boldsymbol{\psi}_0$, $\boldsymbol{\psi}_{\mathcal{H}}, \mathbf{w}_0, \mathbf{w}_{\mathcal{H}} \in \mathbf{H}^{j+1}(\Omega)$, $\mathbf{u} \in \mathbf{H}^{k+1}(\Omega)$, ϕ and $p \in H^{k+1}(\Omega)$, $j = k$ or $k+1$.

Summarizing the analysis above, we obtain:

Theorem 4.1. *Assume that the solutions*

$$\begin{pmatrix} \phi \\ q \\ p \\ \boldsymbol{\psi}_0 \\ \boldsymbol{\psi}_{\mathcal{H}} \\ \mathbf{w}_0 \\ \mathbf{w}_{\mathcal{H}} \\ \mathbf{u} \end{pmatrix} \quad \textit{to problems} \quad \begin{pmatrix} (3.8) \\ (3.9) \\ (3.10) \\ (3.16.\mathrm{i}) \\ (3.16.\mathrm{ii}) \\ (3.17.\mathrm{i}) \\ (3.17.\mathrm{ii}) \\ (3.13) \end{pmatrix} \quad \textit{belong to} \quad \begin{pmatrix} H^{k+1}(\Omega) \\ H^{j+1}(\Omega) \\ H^{k+1}(\Omega) \\ \mathbf{H}^{j+1}(\Omega) \\ \mathbf{H}^{j+1}(\Omega) \\ \mathbf{H}^{j+1}(\Omega) \\ \mathbf{H}^{j+1}(\Omega) \\ \mathbf{H}^{j+1}(\Omega) \end{pmatrix}.$$

Then there exists a constant C *independent of* h *such that the following error estimates hold*:

$$\|\phi - \phi^h\|_0 + h|\phi - \phi^h|_1 \le Ch^{k+1}|\phi|_{k+1}\,, \tag{4.23}$$

$$|q - q^h|_1 \leq Ch^j\{|q|_{j+1} + |\phi|_{k+1}\}\,, \tag{4.24}$$

$$\|p - p^h\|_0 + h|p - p^h|_1 \leq Ch^{k+1}|p|_{k+1}\,, \tag{4.25}$$

$$|\boldsymbol{\psi}_0 - \boldsymbol{\psi}_0^h|_1 \leq Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\phi|_{k+1}\}\,, \tag{4.26}$$

$$|\boldsymbol{\psi}_{\mathcal{H}} - \boldsymbol{\psi}_{\mathcal{H}}^h|_1 \leq Ch^j\{|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\phi|_{k+1}\}\,, \tag{4.27}$$

$$\begin{aligned}|\mathbf{w}_0 - \mathbf{w}_0^h|_1 \leq Ch^j\{&|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\mathbf{w}_0|_{j+1}\\ &+ |\phi|_{k+1} + |p|_{k+1}\}\,,\end{aligned} \tag{4.28}$$

$$\begin{aligned}|\mathbf{w}_{\mathcal{H}} - \mathbf{w}_{\mathcal{H}}^h|_1 \leq Ch^j\{&|\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1}\\ &+ |\mathbf{w}_0|_{j+1} + |\mathbf{w}_{\mathcal{H}}|_{j+1} + |\phi|_{k+1} + |p|_{k+1}\}\,,\end{aligned} \tag{4.29}$$

$$\begin{aligned}\|\mathbf{u} - \mathbf{u}^h\|_0 \leq Ch^j\{&|q|_{j+1} + |\boldsymbol{\psi}_0|_{j+1} + |\boldsymbol{\psi}_{\mathcal{H}}|_{j+1} + |\mathbf{w}_0|_{j+1}\\ &+ |\mathbf{w}_{\mathcal{H}}|_{j+1} + |\mathbf{u}|_{k+1} + |\phi|_{k+1} + |p|_{k+1}\}\,.\end{aligned} \tag{4.30}$$

Remark 4.2. By Theorem 4.1, if $j = k + 1$, then we get an optimal convergence result for approximating $\mathbf{u}$ with order of $\mathcal{O}(h^{k+1})$. If we choose $j = k$, then the convergence result only can reach $\mathcal{O}(h^k)$. In the special case $k = 1$, the linear finite elements can be applied to all approximations.

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